Discontinuous homomorphisms without Hamel bases

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Abstract

We produce a model of $\mathsf{ZF} + \mathsf{DC}$ in which there exists a discontinuous homomorphism from $(\mathbb{R}, +)$ to itself but no Hamel basis for $\mathbb{R}$, and prove a generalization of this result in terms of internal direct sums.

1 Introduction

We say that a subset of a vector space S is a *Hamel basis* for S if it is a maximal linearly independent subset of S over the scalar field $\mathbb{Q}$. A permutation of a Hamel basis for $\mathbb{R}$ naturally extends to an automorphism of the group $(\mathbb{R}, +)$; if the permutation moves some elements and fixes others the induced homomorphism is discontinuous. In this paper we show (Theorem 3.3) that the existence of a discontinuous homomorphism from $(\mathbb{R}, +)$ to itself does not, in $\mathsf{ZF} + \mathsf{DC}$, imply the existence of a Hamel basis for any vector space over $\mathbb{Q}$ containing a copy of $(\mathbb{R}, +)$.

Our interest in this problem was inspired by two related results regarding selectors for the Vitali equivalence relation $\mathbb{E}_0$. In [6] it was shown that the

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existence of a discontinuous homomorphism from $\mathbb{R}$ to itself induces (in ZF) a selector for this equivalence relation (this fact is used in Remark 6.1 below). In [7] it was shown that the existence of a selector for $\mathbb{E}_0$ does not imply the existence of a discontinuous homomorphism from $\mathbb{R}$ to itself (assuming the consistency of a strongly inaccessible cardinal, which can be removed by combining the proof in [7] with the approach used here). These results naturally lead to the questions answered in this paper.

In the second part of the paper we consider ways of writing $\mathbb{R}$ as an internal direct sum of subspaces. The existence of a Hamel basis for $\mathbb{R}$ is equivalence to $\mathbb{R}$ being an internal direct sum of one-dimensional subspaces (see Remark 6.1). The real line being equal to a nontrivial internal direct sum gives rise to a discontinuous homomorphism. This leaves a range of questions asking when $\mathbb{R}$ being an internal direct sum of one type implies that it is an internal direct sum of another type. In Theorem 5.3 we answer one such question by showing that $\mathbb{R}$ being a nontrivial internal direct sum does not imply that it is an internal direct sum of countable-dimensional subspaces.

While the main result of the second part of the paper subsumes the original form of the question answered in the first part of the paper, we include both arguments, in part because the argument from the first part can be modified to produce discontinuous homomorphisms of various kinds. The nonexistence result proved in the first part is also generalized in a different way than the corresponding part of the second result. Many similar variations are possible and we leave these to the interested reader.

The arguments here are similar to earlier ones to due to Horowitz and Shelah [4, 9]. They can also be easily adapted to the methods of [7].

2 Amalgamations

In this section we prove the key lemma of the paper (Lemma 2.5), which will be applied in Sections 3 and 4. We first review some standard facts about amalgamating homomorphisms.

If $(G, +)$ and $(H, +)$ are abelian groups, and $f\colon G \to H$ is a partial function, then we say that f is *additive* (or a *partial homomorphism*) if the equation $f(x + y) = f(x) + f(y)$ holds whenever x, y and $x + y$ are in the domain of f.

A group $(G, +)$ is said to be *divisible* if for each $x \in G$ and $n \in \mathbb{N}$ there is a $y \in G$ such that $ny = x$ (where ny represents y added to itself n times). The additive group of any vector space over a field containing $\mathbb{Q}$ is clearly divisible. We repeatedly and implicitly use the following standard fact.

Lemma 2.1. *If $(G, +)$ is a divisible abelian group, G' is a subgroup of G and $h\colon G' \to G$ is a homomorphism, then h extends to a homomorphism from G to itself.*

Suppose that $\pi_1, \ldots, \pi_n$ are partial homomorphisms from one abelian group $(G, +)$ to another abelian group $(H, +)$, such that the domain of each π_i is a subgroup of G. We say that $\pi_1, \ldots, \pi_n$ can be *amalgamated* if there is a partial homomorphism $\pi\colon (G, +) \to (H, +)$ containing each π_i whose domain is also a subgroup. It is a standard fact (and easy to see) that $\pi_1, \ldots, \pi_n$ can be amalgamated if and only if, for all $a_i, b_i \in \mathrm{dom}(\pi_i)$ $(1 \leq i \leq n)$, if $a_1 + \cdots + a_n = b_1 + \cdots + b_n$ then $\pi_1(a_1) + \cdots + \pi_n(a_n) = \pi_1(b_1) + \cdots + \pi_n(b_n)$. Letting $c_i = a_i - b_i$, we get the following equivalent condition: whenever $c_i \in \mathrm{dom}(\pi_i)$ $(1 \leq i \leq n)$ are such that $c_1 + \cdots + c_n = 0$, $\pi_1(c_1) + \cdots + \pi_n(c_n) = 0$. We state two specific versions of this fact, both of which will be used in the proof of Lemma 2.5.

Lemma 2.2. *Suppose that h_1 and h_2 are partial homomorphisms from an abelian group $(G, +)$ to another abelian group $(H, +)$, and that the domain of each h_i is a subgroup of G. Then h_1 and h_2 can be amalgamated if and only if they agree on the intersection of their domains.*

Lemma 2.3. *Suppose that h_1, h_2 and h_3 are partial homomorphisms from an abelian group $(G, +)$ to another abelian group $(H, +)$, and that the domain of each h_i is a subgroup of G. Then h_1, h_2 and h_3 can be amalgamated if and only if the equation $h_1(a_1) + h_2(a_2) = h_3(a_3)$ holds whenever a_i $(1 \leq i \leq 3)$ are such that $a_1 + a_2 = a_3$ and each a_i is in the domain of the corresponding h_i.*

It is a classical fact that any Borel (or even Baire-measurable) homomorphism between Polish groups is continuous (see [3, 8]). We will use the following variation of this fact.

Theorem 2.4. *If $\epsilon > 0$ and $f\colon (-\epsilon, \epsilon) \to \mathbb{R}$ is Borel and additive, then there is a real number a such that $f(x) = ax$ for all $x \in (-\epsilon, \epsilon)$.*

Proof. For each real number x, define $g(x)$ to be $nf(x/n)$, for any positive integer n such that $|x/n| < \epsilon/2$. The assumptions imply that g is well-defined, additive and Borel, and that it extends f. $\square$

We will let P denote Cohen forcing, in the form $B(\mathbb{R})/I$, where $B(\mathbb{R})$ denotes the set of Borel subsets of $\mathbb{R}$ and I the ideal of meager sets. It follows then that P is a c.c.c. forcing adding a real number and having continuous reading of names (so every real number in the extension is the result of applying a ground-model Borel function to the generic real), and that, for any Borel set $B \subseteq \mathbb{R}^{n+1}$, the set of $\bar{x} \in \mathbb{R}^n$ for which $(\bar{x}, y) \in B$ for an I-positive set of y is also Borel (see Theorem 16.1 of [5]). Since I is shift-invariant, it follows that if x is a V-generic real added by P, and y is real number in the ground model, then $x + y$ is also V-generic for P. We fix the terms P and I for the rest of the paper, and refer the reader to [1] for more information about Cohen forcing and its products.

Real numbers x_1 and x_2 are *mutually P-generic* over M if they are the two P-generic reals over M produced by forcing over M with the product forcing $P \times P$.

The following is the key lemma in the proofs of our main theorems.

Lemma 2.5. *Suppose that x_1 and x_2 are mutually P-generic reals over V, and let $x_3 = x_1 - x_2$.*

1. *If y is an element of $(\mathbb{R}^{V[x_1]} + \mathbb{R}^{V[x_2]}) \cap \mathbb{R}^{V[x_3]}$, then there exist $a, b \in \mathbb{R}^V$ such that $y = ax_3 + b$.*
2. *Let*
    - *h_0 be a homomorphism from $(\mathbb{R}, +)$ to itself in V,*
    - *c be a real number in V, and,*
    - *for each $i \in \{1, 2, 3\}$, h_i be, in $V[x_i]$, a homomorphism from $(\mathbb{R}, +)^{V[x_i]}$ to itself extending h_0, with $h_i(dx_i) = cdx_i$ for all $d \in \mathbb{R}^V$.*

    Then h_1, h_2 and h_3 can be amalgamated in $V[x_1, x_2]$.

Proof. For any x_1, x_2, x_3 and y as given there exist Borel functions r, s and t in V such that $y = r(x_1) + s(x_2) = t(x_3)$. There exists then a condition $(B_1, B_2) \in P \times P$ (with $x_1 \in B_1$ and $x_2 \in B_2$) such that, letting $\dot{g}_1$ and $\dot{g}_2$ be the standard names for the reals produced by each copy of P, (B_1, B_2) forces that $r(\dot{g}_1) + s(\dot{g}_2) = t(\dot{g}_1 - \dot{g}_2)$. We will show that for any such B_1, B_2, r, s and t, there exist $a, b \in \mathbb{R}^V$ and a condition $(B'_1, B'_2) \leq (B_1, B_2)$ forcing that $t(\dot{g}_1 - \dot{g}_2)$ is equal to $a(\dot{g}_1 - \dot{g}_2) + b$.

Shrinking B_1 and B_2 if necessary, we may assume that there exist real numbers c_1, c_2 and $\epsilon > 0$ such that (for each $i \in \{1,2\}$) B_i is subset of the interval $(c_i - \epsilon, c_i + \epsilon)$ whose complement in this interval is in I. For each $i \in \{1,2\}$ and $j \in \omega$, let $B_i^j = B_i \cap (c_i - \epsilon/2^j, c_i + \epsilon/2^j)$.

For each $d \in (-\epsilon/2, \epsilon/2) \cap V$, (B_1^1, B_2^1) forces that $r(\dot{g}_1 + d) + s(\dot{g}_2 + d)$ is equal to $t((\dot{g}_1 + d) - (\dot{g}_2 + d)) = t(\dot{g}_1 - \dot{g}_2)$. This implies that (B_1^1, B_2^1) forces that $r(\dot{g}_1 + d) + s(\dot{g}_2 + d)$ is the same for all values of $d \in (-\epsilon/2, \epsilon/2) \cap V$. It follows then that (B_1^1, B_2^1) forces that $r(\dot{g}_1 + d) - r(\dot{g}_1) = s(\dot{g}_2) - s(\dot{g}_2 + d)$ for all such values of d. Since the realizations of $\dot{g}_1$ and $\dot{g}_2$ will be mutually generic, this value must be in V, and must depend only on d and not on the generic filter. Let $u(d)$ be the corresponding value as forced by B_1^1 (for r) and B_2^2 (for s). Then u is also a Borel function, since $u(d)$ is the unique value taken by the Borel function $r(x + d) - r(x)$ on an I-large subset of B_1^1 (here we use the fact that the "comeagerly many" quantifier preserves Borelness).

We claim that $u(d + e) = u(d) + u(e)$ for all $d, e \in (-\epsilon/4, \epsilon/4)$. To see this, note that B_1^1 forces that $r(\dot{g}_1 + (d+e)) = r(\dot{g}_1) + u(d+e)$, and B_1^2 forces that

$$r(\dot{g}_1 + (d + e)) = r((\dot{g}_1 + d) + e) = r(\dot{g}_1 + d) + u(e) = r(\dot{g}_1) + u(d) + u(e).$$

Since the restriction of u to $(-\epsilon/4, \epsilon/4)$ is an additive Borel function, it follows from Lemma 2.4 that it is multiplication by some real number a.

It follows that there exists a real number $b_1 \in V$ such that B_1^3 forces that $r(\dot{g}_1) - a\dot{g}_1 = b_1$. To see this, suppose that two conditions $C, D \leq B_1^3$ forced the value of $r(\dot{g}_1) - a\dot{g}_1$ to be, respectively, less and more than some rational number q. We may assume that $D = C + d$, for some $d \in (-\epsilon/4, \epsilon/4)$. Then C forces that

$$q < r(\dot{g}_1 + d) - a(\dot{g}_1 + d) = r(\dot{g}_1) + ad - a\dot{g}_1 - ad < q.$$

An analogous argument shows that B_2^3 forces that $s(\dot{g}_2) + a\dot{g}_2 = b_2$, for some $b_2 \in \mathbb{R}$. It follows that (B_1^3, B_2^3) forces that

$$t(\dot{g}_1 - \dot{g}_2) = r(\dot{g}_1) + s(\dot{g}_2) = a(\dot{g}_1 - \dot{g}_2) + (b_1 + b_2).$$

For the second part, by Lemma 2.3 above, it suffices to show that

$$h_1(y_1) + h_2(y_2) = h_3(y_3)$$

holds whenever $y_i \in \mathbb{R}^{V[x_i]}$ $(1 \leq i \leq 3)$ are such that $y_1 + y_2 = y_3$. Fixing such y_1, y_2 and y_3, we have from the first part of the lemma that $y_3 = ax_3 + b$, for some $a, b \in \mathbb{R}^V$. Then

$$\begin{aligned} h_3(y_3) &= h_3(ax_3 + b) \\ &= h_3(ax_3) + h_3(b) \\ &= acx_3 + h_0(b) \\ &= ac(x_1 - x_2) + h_0(b) \\ &= acx_1 + h_1(b) - acx_2 \\ &= h_1(ax_1 + b) - h_2(ax_2). \end{aligned}$$

We also have that $y_1 + y_2 = y_3 = ax_3 + b = a(x_1 - x_2) + b = (ax_1 + b) - ax_2$. Since h_1 and h_2 are extensions of h_0 existing in mutually generic extensions of V (so the intersection of their domains is $\mathbb{R}^V$) they can be amalgamated, which implies that

$$h_1(y_1) + h_2(y_2) = h_1(ax + b) - h_2(ax).$$

□

3 The first model

Let κ be a cardinal of uncountable cofinality such that $\kappa^{<\kappa} = \kappa$. For each $d \subseteq \kappa$ let P_d be the finite support product of our fixed partial order P, indexed by the elements of d. So a condition p in P_d has the form

$$\{B^p_\alpha : \alpha \in \mathrm{supp}(p)\},$$

where $\mathrm{supp}(p)$ is a finite subset of d and each B^p_α is a Borel I-positive subset of $\mathbb{R}$. Then $p_2 \leq p_1$ in P_d if and only if $\mathrm{supp}(p_1) \subseteq \mathrm{supp}(p_2)$ and $B^{p_2}_\alpha \setminus B^{p_1}_\alpha \in I$ for each $\alpha \in \mathrm{supp}(p_1)$. We write G_d for a generic filter for P_d, and $\dot{g}_\alpha$ for the natural $P_{\{\alpha\}}$-name for the generic real added by $P_{\{\alpha\}}$.

Let Q be the following partial order. Conditions are pairs $(d, \dot{h})$ such that d is a countable subset of κ and $\dot{h}$ is a P_d-name for a homomorphism from $(\mathbb{R}, +)$ to itself. We allow $d = \emptyset$, in which case $\dot{h}$ is a check-name for a homomorphism in the ground model. The order is: $(d_1, \dot{h}_1) \leq (d_0, \dot{h}_0)$ if $d_0 \subseteq d_1$ and $1_{P_{d_1}}$ forces that $\dot{h}_{0,G_{d_0}} \subseteq \dot{h}_{1,G_{d_1}}$. Given $q \in Q$ we write d_q and $\dot{h}_q$

for the first and second coordinates of q, respectively. We write Q_d for the set of $q \in Q$ for which $d_q \subseteq d$.

We adopt the convention that undertilded names (like $\underset{\sim}{B}$) are Q-names and dotted names (like $\dot{h}$) are P_d-names (for some $d \subseteq \kappa$), often existing in the ground model V. Combining the two notations, we get things like $\underset{\sim}{\dot{H}}$, the natural Q-name for the P_κ-name $\bigcup\{\dot{h} : (d, \dot{h}) \in K\}$, where K denotes the Q-generic filter.

The two following lemmas imply that $\underset{\sim}{\dot{H}}_K$ is, in the Q-extension $V[K]$, a P_κ-name for a homomorphism from $(\mathbb{R}^{V[K,G_\kappa]}, +)$ to itself. Both lemmas follow from the remarks in the previous section, specifically Lemma 2.1 for $\mathbb{R}$, which says that any partial homomorphism from a subgroup of $(\mathbb{R}, +)$ to $(\mathbb{R}, +)$ can be extended to a total homomorphism (in the ground model and in any forcing extension). The genericity of K will guarantee that the realization of $\underset{\sim}{\dot{H}}_K$ is discontinuous.

Lemma 3.1. *For each $\alpha \in \kappa$, the set of $q \in Q$ with $\alpha \in d_q$ is dense.*

Lemma 3.2. *Every descending ω-sequence in Q has a lower bound.*

We note several consequences of Lemmas 3.1 and 3.2.

- The partial order P_κ is the same in V and in any Q-extension $V[K]$, from which it follows that the forcing iteration $Q * P_\kappa$ and the product forcing $Q \times P_\kappa$ are forcing-equivalent.

- Densely many conditions $(q, p) \in Q \times P_\kappa$ have the property that $\mathrm{supp}(p) \subseteq d_q$. We will call such conditions *normal*.

- Letting (K, G_κ) denote a V-generic filter for $Q * P_\kappa$, every every element of $\mathbb{R}^{V[K,G_\kappa]}$ is an element of $\mathbb{R}^{V[G_d]}$, for some countable $d \subseteq \kappa$. In particular, $\mathbb{R}^{V[K,G_\kappa]} = \mathbb{R}^{V[G_\kappa]}$.

Theorem 3.3 below shows that forcing with $Q * P_\kappa$ produces a model with a discontinuous homomorphism from $\mathbb{R}$ to $\mathbb{R}$ and no Hamel basis for $\mathbb{R}$. We strengthen the nonexistence part of the theorem by replacing $\mathbb{R}$ with an arbitrary Polish vector space containing a copy of $\mathbb{R}$. Most of the rest of this section is a proof of the theorem.

Theorem 3.3. *Suppose that (K, G_κ) is a V-generic filter for $Q * P_\kappa$, and let W be the model*

$$\mathrm{HOD}_{V,\mathbb{R}^{V[G_\kappa]},\underset{\sim}{\dot{H}}_{K,G_\kappa}}$$

as defined in $V[K, G_\kappa]$. *Then* $\underset{\sim}{\dot{H}}_{K,G_\kappa}$ *is a discontinuous homomorphism from* $(\mathbb{R}, +)^{V[K,G_\kappa]}$ *to itself, and* DC *holds in* W. *Furthermore, if, in* V, $(S, +_S)$ *is a Polish vector space over* $\mathbb{Q}$ *in for which there exists a continuous injective homomorphism in* V *from* $(\mathbb{R}, +)$ *to* $(S, +_S)$*, then* W *does not contain a Hamel basis for* $(S, +_S)$.

That W satisfies DC follows from standard arguments, using the fact that W is an inner model of $V[K, G_\kappa]$ (a model of Choice), that it contains the reals of this model, and is closed under ordinal definability. Briefly, the point is that, given a tree T in W without terminal nodes, there exist a set v in V (a wellordering of a sufficiently large set, for instance), a sequence of real numbers $\langle x_n : n \in \omega\rangle$ and a branch b through T such that, for each $n \in \omega$, the nth element of b is definable from x_n, v and $\underset{\sim}{\dot{H}}_{K,G_\kappa}$.

The following amalgamation lemma follows from Lemma 2.2 and the fact that mutually generic extensions have no new real numbers in common.

Lemma 3.4. *Suppose that* (q_1, p_1) *and* (q_2, p_2) *are conditions in* $Q \times P_\kappa$. *Suppose that* $q \in Q$ *is weaker than both* q_1 *and* q_2*, with* $d_q = d_{q_1} \cap d_{q_2}$*, and that* $B^{p_1}_\alpha \cap B^{p_2}_\alpha \in I^+$ *for all* $\alpha \in \mathrm{supp}(p_1) \cap \mathrm{supp}(p_2)$. *Then* (q_1, p_1) *and* (q_2, p_2) *are compatible.*

Remark 3.5 shows that the indices of the conditions in $Q \times P_\kappa$ play no role in the creation of the model W, and that any partial injection from κ to κ maps conditions to isomorphic conditions.

Remark 3.5. *If* d *is a subset of* κ*, and* $f : d \to \kappa$ *is injective, then* f *induces isomorphisms* $f_Q : Q_d \to Q_{f[d]}$ *and* $f_P : P_d \to P_{f[d]}$ *such that for each* $q \in Q_d$ *and* $p \in P_d$,

- $\mathrm{supp}(f_P(p)) = f[\mathrm{supp}(p)]$;
- $B^{f_P(p)}_{f(\alpha)} = B^p_\alpha$ *for all* $\alpha \in \mathrm{supp}(p)$;
- $d_{f_Q(q)} = f[d_q]$;
- *for any* V*-generic filter* $G \subseteq P_{q_d}$, $f_P[G]$ *is a* V*-generic filter for* $P_{f[d_q]}$*, and* $\dot{h}_{q,G} = \dot{h}_{f_Q(q), f_P[G]}$.

In the case where $d = \kappa$*, we get that, if* (K, G_κ) *is* V*-generic for* $Q * P_\kappa$*, then* $(f_Q[K], f_P[G_\kappa])$ *is also* V*-generic, and the two generic extensions give rise to the same model* W.

Suppose toward a contradiction that $\dot{\underset{\sim}{B}}$ is a Q-name for a P_κ-name for a Hamel basis for $(S,+_S)$ in $\mathrm{HOD}_{V,\mathbb{R}^{V[G_\kappa]},\dot{\underset{\sim}{H}}_{K,G_\kappa}}$, as forced by some $Q\times P_\kappa$-condition (q_0,p_0). Replacing (q_0,p_0) with a stronger condition if necessary, we may assume that it is normal, and that there exist a $v\in V$, a formula φ, a cardinal λ and $P_{d_{q_0}}$-names $\dot{r}_1,\ldots,\dot{r}_k$ for elements of $\mathbb{R}$ such that

$$(q_0,p_0)\Vdash_{Q\times P_\kappa}\dot{\underset{\sim}{B}}=\{y\in S: V_\lambda[K,G_\kappa]\models\varphi(y,v,\dot{\underset{\sim}{H}}_{K,G_\kappa},\dot{r}_{1,G_{d_{q_0}}},\ldots,\dot{r}_{k,G_{d_{q_0}}})\}.$$

Below we write $\tau\in\dot{\underset{\sim}{B}}$ as an abbreviation for the statement

$$V_\lambda[K,G_\kappa]\models\varphi(\tau_{G_\kappa},v,\dot{\underset{\sim}{H}}_{K,G_\kappa},\dot{r}_{1,G_{d_{q_0}}},\ldots,\dot{r}_{k,G_{d_{q_0}}}).$$

The first key point is that, for each $d\subseteq\kappa$, the restriction of $\dot{\underset{\sim}{B}}_{K,G_\kappa}$ to $S^{V[G_d]}$ is decided by the restrictions of K and G_κ to d.

Lemma 3.6. *For each normal $(q,p)\leq(q_0,p_0)$, for each P_{d_q}-name τ for an element of S, the statement $\tau\in\dot{\underset{\sim}{B}}$ is decided by (q,p') for densely many $p'\in P_{d_q}$ below p. Moreover, for each such τ there is a P_{d_q}-name σ for a finite subset of $\dot{\underset{\sim}{B}}$ such that p_0 forces that the realization of τ will be a linear combination of the realization of σ.*

Proof. If the first part of lemma failed for some (q,p) and τ we could find conditions $(q_1,p_1),(q_2,p_2)\leq(q,p)$ forcing opposite truth values for the statement $\tau\in\dot{\underset{\sim}{B}}$, with $p_1\restriction q_d=p_2\restriction q_d$. Applying Remark 3.5 to an injection $f\colon d_{q_2}\to\kappa$ fixing d_q pointwise and mapping $d_{q_2}\setminus d_q$ to a set disjoint from $d_{q_1}\setminus d_q$, we may assume in addition that $d_{q_1}\cap d_{q_2}=d_q$. This is impossible since by Lemma 3.4, (q_1,p_1) and (q_2,p_2) would also be compatible.

The second part of the lemma is proved similarly, by letting σ be any name for the unique finite subset of $\dot{\underset{\sim}{B}}$ of which the realization of τ will be a linear combination. An argument like the one just given shows that G_{d_q} must decide the size of σ and the members of σ, using (for instance) the fact that any finite set S is coded by the set of basic open sets it intersects. $\square$

We can then let, for each $q\leq q_0$, $\dot{B}^q$ be the P_{d_q}-name for the value of $\dot{\underset{\sim}{B}}_{K,G_\kappa}\cap S^{V[G_{d_q}]}$, for any $Q\times P_\kappa$-generic filter (K,G_κ) containing (q_0,p_0). Then p_0 forces in P_{d_q} that the realization of $\dot{B}^q$ will be a maximal linearly independent subset of the space S as interpreted in the P_{d_q}-extension.

We now show that the realization of $\dot{B}^q$ depends only on the realization of $\dot{h}_q$, which gives us one half of our desired contradiction.

Lemma 3.7. *Let d be a countable subset of κ, and suppose that q_1 and q_2 are conditions below q_0 with $d_{q_1} = d_{q_2} = d$. If G^1_d and G^2_d are V-generic filters for P_d such that*

- $p_0 \in G^1_d \cap G^2_d$,
- $V[G^1_d] = V[G^2_d]$ *and*
- $\dot{h}_{q_1,G^1_d} = \dot{h}_{q_2,G^2_d}$,

then $\dot{B}^{q_1}_{G^1_d} = \dot{B}^{q_2}_{G^2_d}$.

Proof. Let G^* be $V[G^1_d]$-generic for $P_{\kappa\setminus d}$. Then (G^1_d, G^*) and (G^2_d, G^*) naturally induce V-generic filters G_1 and G_2 for P_κ, and $V[G_1] = V[G_2]$. It suffices to see that there exist $V[G_1]$-generic filters K_1 and K_2 for Q^V, with $q_1 \in K_1$ and $q_2 \in K_2$ such that the models W induced by (K_1, G_1) and (K_2, G_2) are the same. We can do this in a forcing extension of $V[G_1]$ in which $\mathcal{P}(Q)^V$ is countable. Say that a pair $(q'_1, q'_2) \in Q^V \times Q^V$ is *good* if $q'_1 \leq q_1$, $q'_2 \leq q_2$, $d_{q'_1} = d_{q'_2}$ and $\dot{h}_{q'_1,G_1\restriction d_{q'_1}} = \dot{h}_{q'_2,G_2\restriction d_{q'_2}}$. Then (q_1, q_2) is good, and it suffices to show that whenever (q'_1, q'_2) is good and $q''_1 \leq q'_1$, there exists a $q''_2 \leq q'_2$ such that (q''_1, q''_2) is good (since the corresponding fact with 1 and 2 reversed will follow by symmetry).

Given such (q'_1, q'_2) and q''_1, let $d_{q''_2}$ be $d_{q''_1}$, and call this set d''. Since $V[G^1_d] = V[G^2_d]$,

$$V[G_1 \restriction d''] = V[G_2 \restriction d''],$$

so there exists a $P_{d''}$-name $\dot{h}$ such that $\dot{h}_{G_2\restriction d''} = \dot{h}_{q''_1,G_1\restriction d''}$. Since $\dot{h}_{q''_1,G_1\restriction d''}$ extends $\dot{h}_{q'_2,G_2\restriction d_{q'_2}}$, we may assume that every condition in $P_{d''}$ forces that $\dot{h}$ extends $\dot{h}_{q'_2}$, and we can let $\dot{h}_{q''_2}$ be this $\dot{h}$. □

We now derive the other half of the contradiction. Fix two ordinals, $\alpha_1 < \alpha_2$ in $\kappa \setminus d_{q_0}$ and let

- $d_0 = d_{q_0}$,
- $d_1 = d_0 \cup \{\alpha_1\}$,
- $d_2 = d_0 \cup \{\alpha_2\}$ and
- $d = d_{q_0} \cup \{\alpha_1, \alpha_2\}$.

We will find generic filters G_d and G'_d for P_d, such that

1. $p_0 \in G_d \restriction P_{d_0} = G'_d \restriction P_{d_0}$,
2. $G_d \restriction P_{\{\alpha_1\}} = G'_d \restriction P_{\{\alpha_1\}}$ and
3. $V[G_d] = V[G'_d]$,

and a homomorphism $h \in V[G_d]$ from $(\mathbb{R}^{V[G_d]}, +)$ to itself such that

- h extends $\dot{h}_{q_0, G_d \restriction P_{d_0}}$,
- $h \restriction V[G_d \restriction P_{d_1}] \in V[G_d \restriction P_{d_1}]$,
- $h \restriction V[G_d \restriction P_{d_2}] \in V[G_d \restriction P_{d_2}]$ and
- $h \restriction V[G'_d \restriction P_{d_2}] \in V[G'_d \restriction P_{d_2}]$,

in such a way that no Hamel basis B for $S^{V[G_d]}$ can have the property that

- $B \cap V[G_d \restriction P_{d_1}]$ is a Hamel basis for S in $V[G_d \restriction P_{d_1}]$,
- $B \cap V[G_d \restriction P_{d_2}]$ is a Hamel basis for S in $V[G_d \restriction P_{d_2}]$ and
- $B \cap V[G'_d \restriction P_{d_2}]$ is a Hamel basis for S in $V[G'_d \restriction P_{d_2}]$.

There will exist then P_d-names $\dot{h}$, $\dot{h}'$, $\dot{h}_1$, $\dot{h}_2$ and $\dot{h}'_2$ such that

- $(d, \dot{h})$, $(d, \dot{h}')$, $(d_1, \dot{h}_1)$, $(d_2, \dot{h}_2)$ and $(d_2, \dot{h}'_2)$ are conditions in Q,
- $(d, \dot{h}), (d, \dot{h}') \leq (d_1, \dot{h}_1) \leq q_0$,
- $(d, \dot{h}) \leq (d_2, \dot{h}_2) \leq q_0$,
- $(d, \dot{h}') \leq (d_2, \dot{h}'_2) \leq q_0$ and
- $\dot{h}_{G_d} = \dot{h}'_{G'_d} = h$.

This will finish the proof, by contradicting Lemma 5.7.

Let $G_{d_{q_0}}$ be a V-generic filter for $P_{d_{q_0}}$ containing p_0. Let x_1 and x_2 be mutually P-generic reals over $V[G_{d_{q_0}}]$, and let $x_3 = x_1 - x_2$. Let G_d be the P_d-generic filter extending $G_{d_{q_0}}$ giving rise to x_1 in coordinate α_1 and x_2 in coordinate α_2. Let G'_d be the P_d-generic filter extending $G_{d_{q_0}}$ giving rise to x_1

in coordinate α_1 and x_3 in coordinate α_2. Then G_d and G'_d satisfy conditions (1)-(3) above.

Fix a real number $c \in V[G_{d_{q0}}]$, and for each $i \in \{1,2,3\}$ let h_i in $V[G_{d_{q0}}][x_i]$ be a homomorphism from $(\mathbb{R},+)^{V[G_{d_{q0}}][x_i]}$ to itself extending the homomorphism $\dot{h}_{q0,G_{d_{q0}}}$, with $h_i(dx_i) = cdx_i$ for all $d \in \mathbb{R}^{V[G_{d_{q0}}]}$. Applying Lemma 2.5, let h be an amalgamation of h_1, h_2 and h_3 in $V[G_d]$.

If $\dot{B}^q_{G_{d_q}} = \dot{B}^{q'}_{G_{d_{q'}}}$, then the restriction of this set to each of the models $V[G_{d_{q0}}][x_1]$, $V[G_{d_{q0}}][x_2]$ and $V[G_{d_{q0}}][x_3]$ would be a Hamel basis in the corresponding model for the corresponding version of S. The remaining point is that there cannot be a Hamel basis B for $(S,+_S)$ with this property. To see this, note first that since x_1, x_2 and x_3 are pairwise mutually generic over $V[G_{d_{q0}}]$, the intersection of any two of these models is $V[G_{d_{q0}}]$. Now let π be (in V), a continuous injective homomorphism from $(\mathbb{R},+)$ to $(S,+_S)$. If a B as above did exist, then each of $\pi(x_1)$, $\pi(x_2)$ and $\pi(x_3)$ would be a linear combination of elements of the corresponding set $B \cap V[G_{d_{q0}}][x_i]$ (with rational coefficients), and a linear combination of elements of $B \cap V[G_d]$ in a unique way. The equation $\pi(x_3) +_S \pi(x_2) = \pi(x_1)$ however then gives two different linear combinations for $\pi(x_1)$. This finishes the proof of Theorem 3.3.

Remark 3.8. *The generic homomorphism given by the construction above is surjective but not injective. The approach above leads to several possible variations, including the following. For each one has to prove the corresponding versions of Lemma 2.1, 2.2 and 2.3, and apply the second part of Lemma 2.5 with the appropriate value of c if necessary. The corresponding version of Q in each case is the natural one, except possibly for the first case below.*

1. *Requiring h to be injective but not surjective. This seems to be the most subtle of the five variations listed here. Naturally, one should pick a real to keep out of the range of the generic homomorphism. However, letting $\dot{h}$ be any name for an injective homomorphism missing this real will not work. Instead, one can naturally associate to each real x in the P_κ-extension the minimal $a \subseteq \kappa$ for which $x \in V[G_a]$, and require that $h(x)$ has the same minimal a. Additional conditions should be added to implement Lemma 2.5. Alternately one decompose $\mathbb{R}$ into an internal direct sum of countably-infinite-dimensional subspaces, the sense of the next section. We leave the details to the interested reader.*

2. *Requiring h to be bijective. Here it is natural to require each $\dot{h}$ to be a name for an automorphism of $(\mathbb{R}, +)$.*

3. *Requiring the range of h to be $(\mathbb{Q}, +)$ (or to have dimension n over $\mathbb{Q}$, for some $n \leq \omega$). Here the adaptation is straightforward, letting c be 0 in the application of Lemma 2.5.*

4. *Requiring that $h(h(x)) = x$ (or $h^n(x) = x$ for some integer $n \geq 2$) for all $x \in \mathbb{R}$. Here again the adaptation is straightforward, letting c be 1 in the application of Lemma 2.5.*

5. *Requiring $\mathbb{R}/\{x : h(x) = x\}$ to have dimension n, for some positive $n \leq \omega$. Here is is natural to start with an an automorphism h in the ground model derived from an appropriate countably supported permutation of a Hamel basis, and work below a condition of the form $(\emptyset, \check{h})$. Again, let $c = 1$.*

An interesting question (asked of us by Asaf Karagila) is whether the existence of discontinuous homomorphisms of the types listed above can be separated. For each fixed n, the existence of a homomorphism of type 3 appears to be equivalent to existence of one of type 5, and to imply (in the case where n is finite) the existence of one of type 4. A homomorphism of type 4 is also bijective, and one of type 5 in the case $n = \omega$ gives one of type 1. Otherwise these questions are open as far as we know. The following related question also appears to be open.

Question 3.9 (Schilhan, Zapletal)**.** *Does the existence of a Hamel basis for $\mathbb{R}^2$ imply the existence of a Hamel basis for $\mathbb{R}$?*

We note that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a discontinuous homomorphism, then for some i, j the function $x \mapsto f_i(x\mathbf{e}_j)$ is a discontinuous homomorphism from $\mathbb{R}$ to $\mathbb{R}$, where $f_1, \ldots, f_m$ are the component functions of f and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors for $\mathbb{R}^n$. It follows that a negative answer to Question 3.9 would give another proof of the main result of this paper.

4 Internal direct sums

We say that collection of subspaces $\mathcal{W} = \{W_i : i \in I\}$ of a vector space S is *linearly independent* if the trivial subspace $\{0\}$ is not in $\mathcal{W}$ and 0 cannot be written as a sum of nonzero elements of a nonempty set of distinct W_i's. If in addition every element of S can be written as a sum of elements of the W_i's (i.e., if $\mathcal{W}$ spans S), S is said to be equal to the *internal direct sum* of $\mathcal{W}$ (in symbols, $S = \bigoplus \mathcal{W}$), and $\mathcal{W}$ is said to be a *maximal* linearly independent family. We write $\sum \mathcal{W}$ or $\sum_{i \in I} W_i$ for the span of $\mathcal{W}$. In this section and the next we generalize the results above by rephrasing them in terms of internal direct sums.

Remark 4.1. *Both Hamel bases and discontinuous homomorphisms are related to internal direct sums.*

- *If B is a Hamel basis for a vector space $\mathcal{S}$ (over $\mathbb{Q}$), then S is the internal direct sum of the one-dimensional spaces $\{bq : q \in \mathbb{Q}\}$ ($b \in B$).*
- *If a topological vector space S (again over $\mathbb{Q}$) is the internal direct sum of $\{W_i : i \in I\}$ with $|I| \geq 2$, then one gets a discontinuous homomorphism from S to S by (for instance) sending one of these two classes to 0 and the members of the other classes to themselves.*

We will be concerned only with the case where S is the vector space $\mathbb{R}$ over $\mathbb{Q}$.

Internal direct sums can be categorized by the dimensions of the corresponding subspaces. We write $\Phi_{\mathrm{c}}(\mathbb{R})$ for the statement that $\mathbb{R}$ is equal to an internal direct sum of countable-dimensional spaces (this notation will be generalized in Section 6). The following generalizes one version of Theorem 3.3.

Theorem 4.2. *If* ZFC *is consistent then so is* $\mathsf{ZF} + \mathsf{DC} + \neg\Phi_{\mathrm{c}}(\mathbb{R})$ + *"$\mathbb{R}$ is equal to an internal direct sum of an uncountable family of subspaces."*

Theorem 5.3 below is a more explicit version of Theorem 4.2. We use the following notion of extension for linearly independent families.

Definition 4.3. *Given two families $\mathcal{W}$ and $\mathcal{U}$ of subspaces of some vector space S, we say that $\mathcal{U}$* extends *$\mathcal{W}$ if $\sum \mathcal{W} \subseteq \sum \mathcal{U}$ and for each $W \in \mathcal{W}$ there is a $U \in \mathcal{U}$ such that*

$$U \cap \sum \mathcal{W} = W.$$

If $\mathcal{U}$ extends $\mathcal{W}$ and $\mathcal{U}$ is linearly independent, then so is $\mathcal{W}$, and there is a unique U for each W as Definition 4.3. It may be the case, however, that $U \cap \sum \mathcal{W} = \{0\}$ for multiple $U \in \mathcal{U}$.

Definition 4.4. *Given a family of subspaces $\mathcal{W}$ of some vector space S, and a subspace U, the* restriction *of $\mathcal{W}$ to U is $\{W \cap U : W \in \mathcal{W}\} \setminus \{0\}$.*

It is straightforward to verify that extension is a transitive relation on linearly independent families, and that a family $\mathcal{W}$ extends its restriction to a subspace U.

Definition 4.5. *Suppose that $\mathcal{U}_0$ and $\mathcal{U}_1$ are linearly independent extensions of $\mathcal{W}$, and that*

$$\sum \mathcal{U}_0 \cap \sum \mathcal{U}_1 = \sum \mathcal{W}.$$

The natural amalgamation *of $\mathcal{U}_0$ and $\mathcal{U}_1$ is the collection of sets U of one of the two following forms.*

- *U is in some $\mathcal{U}_i$ and $U \cap \sum \mathcal{U}_{1-i} = \{0\}$.*
- *$U = U_0 + U_1$ for some $U_0 \in \mathcal{U}_0$, $U_1 \in \mathcal{U}_1$ such that*

$$U_0 \cap \sum \mathcal{W} = U_1 \cap \sum \mathcal{W} \neq \emptyset.$$

Remark 4.6. *It is straightforward to check that the natural amalgamation of $\mathcal{U}_0$ and $\mathcal{U}_1$ is an independent extension of both $\mathcal{U}_0$ and $\mathcal{U}_1$.*

Remark 4.6 gives the following.

- If $V[G_0]$ and $V[G_1]$ are mutually generic extensions of V, $\mathbb{R}^V = \bigoplus \mathcal{W}$, $\mathbb{R}^{V[G_0]} = \bigoplus \mathcal{U}_0$, $\mathbb{R}^{V[G_1]} = \bigoplus \mathcal{U}_1$, and $\mathcal{U}_0$ and $\mathcal{U}_1$ both extend $\mathcal{W}$, then the natural amalgamation of $\mathcal{U}_0$ and $\mathcal{U}_1$ is a linearly independent extension of $\mathcal{U}_0$ and $\mathcal{U}_1$.
- Suppose that $\mathcal{W} = \{W_i : i \in I\}$ is a linearly independent family, and v is a vector not in $\sum_{i \in I} W_i$. Then $\mathcal{W} \cup \operatorname{span}\{v\}$ is a linearly independent family extending $\mathcal{W}$, as is the set obtained from $\mathcal{W}$ by replacing any one W_i with $W_i \cup \operatorname{span}\{v\}$.
- The Axiom of Choice implies that every linearly independent family of subspaces of a vector space extends to one spanning the space.
- If $V \subseteq V'$ are models of ZFC and $\mathbb{R}^V = \bigoplus \mathcal{W}$, then there is a U such that $\mathbb{R}^{V'} = \bigoplus(\mathcal{W} \cup \{U\})$, $U = \{ax : a \in \mathbb{R}^V, x \in U\}$ and $U \cap \mathbb{R}^V = \{0\}$.

The following is a version of the second part of Lemma 2.5, adapted to internal direct sums.

Lemma 4.7. *Suppose that x_1 and x_2 are mutually P-generic reals over V, and let $x_3 = x_1 - x_2$. Suppose that,*

- *in V, $\mathbb{R} = \oplus\mathcal{W}$;*
- *for each $i \in \{1, 2, 3\}$, U_i is, in $V[x_i]$ a subspace of $\mathbb{R}^{V[x_i]}$ such that $x_i \in U_i$, $U_i = \{ax : a \in \mathbb{R}^V, x \in U_i\}$, $U_i \cap \mathbb{R}^V = \{0\}$ and $\mathbb{R}^{V[x_i]} = \bigoplus(\mathcal{W} \cup \{U_i\})$.*

Then, letting U be the span of $\bigcup_{i=1}^{3} U_i$, $\mathcal{W} \cup \{U\}$ is linearly independent and extends $\mathcal{W} \cup \{U_i\}$ for each i in $\{1, 2, 3\}$.

Proof. We need to see that $\mathcal{W} \cup \{U\}$ is linearly independent, and that the intersection of U with each $\mathbb{R}^{V[x_i]}$ is the corresponding U_i. The second of these follows easily from the first.

To see that $\mathcal{W} \cup \{U\}$ is linearly independent, suppose that $c \in \mathbb{R}^V$ and $y_i \in U_i$ ($i \in \{1, 2, 3\}$) are such that $c + y_1 + y_2 + y_3 = 0$. The first part of Lemma 2.5 then gives that $y_3 = a(x_1 - x_2) + b$, for some $a, b \in \mathbb{R}^V$. Since $b \in U_3 \cap \mathbb{R}^V$, $b = 0$. Then $c + (y_1 + ax_1) + (y_2 - ax_2) = 0$. Since $\mathcal{W} \cup \{U_1, U_2\}$ is linearly independent (by Remark 4.6), this implies that c, $y + ax_1$ and $y_2 - ax_2$ are all 0. From this it follows that $y_1 + y_2 + y_3$ is 0. □

5 The second model

The second model is like the first, except that we add a family of linearly independent subspaces instead of a generic homomorphism. We let κ and P_κ be exactly as before. For the current proof Q is the following partial order. Conditions are pairs $(d, \dot{\mathcal{W}})$ such that d is a countable subset of κ and $\dot{\mathcal{W}}$ is a P_d-name for a linearly independent set of subspaces of $\mathbb{R}$ whose span is the set of reals of the forcing extension. The order is: $(d_1, \dot{\mathcal{W}}_1) \leq (d_0, \dot{\mathcal{W}}_0)$ if $d_0 \subseteq d_1$ and $1_{P_{d_1}}$ forces that $\dot{\mathcal{W}}_{1,G_{d_1}}$ extends $\dot{\mathcal{W}}_{0,G_{d_0}}$. Given $q \in Q$ we write d_q and $\dot{\mathcal{W}}_q$ for the first and second coordinates of q, respectively. Given $d \subseteq \kappa$, we let Q_d denote the set of $q \in Q$ for which $d_q \subseteq d$.

We let $\underset{\sim}{\dot{W}}$ be the natural Q-name for the P_κ-name for the unique set of linearly independent subspaces of $\mathbb{R}$ extending the realization $\dot{\mathcal{W}}$ for each $(d, \dot{\mathcal{W}})$ in the Q-generic filter K. The two following lemmas show that, in the $Q * P_\kappa$-extension, $\mathbb{R}$ is the internal direct sum of the realization of $\dot{\underset{\sim}{\mathcal{W}}}$. Both lemmas follow from the remarks in the previous section.

Lemma 5.1. *For each $\alpha \in \kappa$, the set of $q \in Q$ with $\alpha \in d_q$ is dense.*

Lemma 5.2. *Every descending ω-sequence in Q has a lower bound.*

Lemmas 5.1 and 5.2 give the following facts, which are unchanged from the first forcing construction of this paper.

- The partial order P_κ is the same in V and in any Q-extension $V[K]$, from which it follows that the forcing iteration $Q * P_\kappa$ and the product forcing $Q \times P_\kappa$ are forcing-equivalent.
- Densely many conditions $(q, p) \in Q \times P_\kappa$ have the property that $d_q \subseteq \mathrm{supp}(p)$. We will call such conditions *normal*.
- Letting (K, G_κ) denote a V-generic filter for $Q * P_\kappa$, every every element of $\mathbb{R}^{V[K,G_\kappa]}$ is an element of $\mathbb{R}^{V[G_d]}$, for some countable $d \subseteq \kappa$. In particular, $\mathbb{R}^{V[K,G_\kappa]} = \mathbb{R}^{V[G_\kappa]}$.

A linearly independent collection $\mathcal{W}$ of subspaces of $\mathbb{R}$ induces the equivalence relation E of being in the same member of $\mathcal{W}$, where the domain of E is the set of nonzero vectors in $\bigcup \mathcal{W}$. In this way the name $\dot{\underset{\sim}{\mathcal{W}}}$ induces a $Q \times P_\kappa$-name $\dot{\underset{\sim}{E}}$ for the corresponding equivalence relation. Since the equivalence relation corresponding to a linearly independent family is a subset of $\mathbb{R}^{V[G]} \times \mathbb{R}^{V[G]}$ (as opposed to a set of subsets of $\mathbb{R}^{V[G]}$) it is easier to work with equivalence relations for some of the arguments below, and in particular for defining the model V^*.

The rest of this section is a proof of the following theorem.

Theorem 5.3. *Suppose that (K, G_κ) is a V-generic filter for $Q * P_\kappa$, and let V^* be the model*

$$\mathrm{HOD}_{V, \mathbb{R}^{V[G_\kappa]}, \dot{\underset{\sim}{E}}_{K,G_\kappa}}.$$

Then DC *holds in V^*, and $(\mathbb{R}, +)^{V[K,G_\kappa]}$ is the internal direct sum of $\dot{\underset{\sim}{\mathcal{W}}}_{K,G_\kappa}$. Furthermore, in V^*, $\mathbb{R}^{V[K,G_\kappa]}$ is not equal to an internal direct sum of countable dimensional subspaces.*

As before, V^* satisfies DC because it is an inner model of the model $V[K, G_\kappa]$, which satisfies ZFC and has the same real numbers, and every element of V^* is ordinal definable in $V[K, G_\kappa]$ from a real number, a member of V and the realization of $\dot{\underset{\sim}{E}}$.

The following amalgamation lemma follows from Remark 4.6.

Lemma 5.4. *Suppose that (q_1, p_1) and (q_2, p_2) are conditions in $Q \times P_\kappa$. Suppose that $q \in Q$ is weaker than both q_1 and q_2, with $d_q = d_{q_1} \cap d_{q_2}$, and that $B^{p_1}_\alpha \cap B^{p_2}_\alpha \in I^+$ for all $\alpha \in \operatorname{supp}(p_1) \cap \operatorname{supp}(p_2)$. Then (q_1, p_1) and (q_2, p_2) are compatible.*

The following remark carries over verbatim from our first construction.

Remark 5.5. *If d is a subset of κ, and $f \colon d \to \kappa$ is injective, then f induces isomorphisms $f_Q \colon Q_d \to Q_{f[d]}$ and $f_P \colon P_d \to P_{f[d]}$ such that for each $q \in Q_d$ and $p \in P_d$,*

- $\operatorname{supp}(f_P(p)) = f[\operatorname{supp}(p)]$*;*
- $B^{f_P(p)}_{f(\alpha)} = B^p_\alpha$ *for all* $\alpha \in \operatorname{supp}(p)$*;*
- $d_{f_Q(q)} = f[d_q]$*;*
- *for any V-generic filter $G \subseteq P_{q_d}$, $f_P[G]$ is a V-generic filter for $P_{f[d_q]}$, and $\dot{\mathcal{W}}_{q,G} = \dot{\mathcal{W}}_{f_Q(q), f_P[G]}$.*

If $a = \kappa$ and (K, G_κ) is V-generic for $Q * P_\kappa$, then $(f_Q[K], f_P[G_\kappa])$ is also V-generic, and the two generic extensions give rise to the same model V^*.

Suppose toward a contradiction that $\dot{\underset{\sim}{\mathcal{V}}}$ is a Q-name for a P_κ-name for a linearly independent set of countable-dimensional subspaces of $(\mathbb{R}, +)$ in V^* whose span is $\mathbb{R}$, as forced by some $Q \times P_\kappa$-condition (q_0, p_0). Let $\dot{\underset{\sim}{F}}$ be a $Q * P_\kappa$-name for the equivalence relation corresponding to $\dot{\underset{\sim}{\mathcal{V}}}$.

Replacing (q_0, p_0) with a stronger condition if necessary, we may assume that it is normal, and that there exist a $v \in V$, a formula φ, a cardinal λ and $P_{d_{q_0}}$-names $\dot{r}_1, \ldots, \dot{r}_k$ for elements of $\mathbb{R}$ such that

$$(q_0, p_0) \Vdash_{Q \times P_\kappa} \dot{\underset{\sim}{F}} = \{y \in \mathbb{R}^2 : V_\lambda[K, G_\kappa] \models \varphi(y, v, \dot{\underset{\sim}{E}}_{K,G_\kappa}, \dot{r}_{1,G_{d_{q_0}}}, \ldots, \dot{r}_{k,G_{d_{q_0}}})\}.$$

We will write $\tau \in \dot{\underset{\sim}{F}}$ for $V_\lambda[K, G_\kappa] \models \varphi(\tau_{G_\kappa}, v, \dot{\underset{\sim}{E}}_{K,G_\kappa}, \dot{r}_{1,G_{d_{q_0}}}, \ldots, \dot{r}_{k,G_{d_{q_0}}})$.

As before, the first key point is that, for each $d \subseteq \kappa$, the restriction of $\dot{\underset{\sim}{F}}_{K,G_\kappa}$ is decided by the restrictions of K and G_κ to d.

Lemma 5.6. *Each normal $(q,p) \leq (q_0,p_0)$ forces that for each P_{d_q}-name τ for a real number in the domain of $\dot{\underset{\sim}{F}}$, the $\dot{\underset{\sim}{F}}$-equivalence class of τ will be contained in the P_{d_q}-extension of V. Moreover, for each P_{d_q}-name τ for a real number there is a P_{d_q}-name σ for a finite subset of the domain of $\dot{\underset{\sim}{F}}$ such that p_0 forces that the realization of τ will be a linear combination of the members of σ.*

Proof. This follows from Lemma 5.4 and Remark 5.5. Suppose that the lemma failed for some (q,p) and τ. Then there exist a condition $(q',p') \leq (p,q)$ and $P_{d_{q'}}$-name σ for a real number which is forced by (q',p') to be $\dot{\underset{\sim}{F}}$-equivalent to τ and not a member of the P_{d_q}-extension of V. Given any $(q'',p'') \leq (q',p')$ and any $\alpha < \omega_1$, letting $f\colon d_{q'} \to \omega_1$ be an injection which is the identity function on d_q and which maps the rest of $d_{q'}$ to elements of $\omega_1 \setminus \alpha \cup d_{q''}$, we get a condition $(f_Q(q'), f_P(p'))$ compatible with (q'',p'') mapping σ to a $P_{f[d_{q'}]}$-name σ_f for a real $\dot{\underset{\sim}{F}}$-equivalent to τ and not a member of the P_{d_q}-extension of V. By mutual genericity the realization of σ_f will not be in the $P_{\alpha\cup d_{q''}}$-extension. It follows by genericity that there is no countable $d \subsetneq \kappa$ such that the $\dot{\underset{\sim}{F}}$-equivalence class of τ will be contained in the P_d-extension.

The second part of the lemma is proved in the same way as the second part of the proof of Lemma 3.6, using the fact that the realization of τ will be a finite sum of members of $\dot{\underset{\sim}{F}}$ in only countably many ways. □

We can then let, for each $q \leq q_0$, $\dot{F}^q$ be the P_{d_q}-name for the value of $\dot{\underset{\sim}{F}}_{K,G_\kappa} \restriction \mathbb{R}^{V[G_{d_q}]}$, for any $Q \times P_\kappa$-generic filter (K,G_κ) containing (q_0,p_0). Then p_0 forces in P_{d_q} that the realization of $\dot{F}^q$ will be the equivalence relation corresponding to a decomposition of $\mathbb{R}$ into countable-dimensional subspaces in the P_{d_q}-extension.

We have the following version of Lemma 5.7, which has essentially the same proof.

Lemma 5.7. *If $q,q' \leq q_1$, and G_{d_q} and $G_{d_{q'}}$ are (respectively), V-generic filters for P_{d_q} and $P_{d_{q'}}$ with $p_1 \in G_{d_q} \cap G_{d_{q'}}$ such that*

$$\mathbb{R}^{V[G_{d_q}]} = \mathbb{R}^{V[G_{d_{q'}}]}$$

and $\dot{\mathcal{W}}_{q,G_{d_q}} = \dot{\mathcal{W}}_{q',G_{d_{q'}}}$, then $\dot{F}^q_{G_{d_q}} = \dot{F}^{q'}_{G_{d_{q'}}}$.

We now derive a contradiction, as before. Fix two ordinals, $\alpha_1 < \alpha_2$ in $\kappa \setminus d_{q0}$ and let $d = d_{q0} \cup \{\alpha_1, \alpha_2\}$. We will find generic filters G_d and G'_d for P_d, such that

1. $p_0 \in G_d \restriction P_{d_{q0}} = G'_d \restriction P_{d_{q0}}$,
2. $G_d \restriction P_{\{\alpha_1\}} = G'_d \restriction P_{\{\alpha_1\}}$ and
3. $V[G_d] = V[G'_d]$,

and a maximal linearly independent family of subspaces $\mathcal{W} \in V[G_d]$ of $\mathbb{R}^{V[G_d]}$ into subspaces, extending $\dot{\mathcal{W}}_{q0,G_d \restriction P_{q_{d_0}}}$, whose restriction to each of $V[G_{d_{q0} \cup \{\alpha_1\}}]$, $V[G_{d_{q0} \cup \{\alpha_2\}}]$ and $V[G'_{d_{q0} \cup \{\alpha_2\}}]$ is a maximal linearly independent family in that model. We will have furthermore that $\dot{F}^q_{G_{d_q}} \neq \dot{F}^{q'}_{G_{d_{q'}}}$. This will finish the proof.

Let $G_{d_{q0}}$ be a V-generic filter for $P_{d_{q0}}$ containing p_0. Let x_1 and x_2 be mutually P-generic reals over $V[G_{d_{q0}}]$, and let $x_3 = x_1 - x_2$. Let G_d be the P_d-generic filter extending $G_{d_{q0}}$ giving rise to x_1 in coordinate α_1 and x_2 in coordinate α_2. Let G'_d be the P_d-generic filter extending $G_{d_{q0}}$ giving rise to x_1 in coordinate α_1 and x_3 in coordinate α_2. Then G_d and G'_d satisfy conditions (1)-(3) above.

For each $i \in \{1, 2, 3\}$ let U_i be, in $V[G_{d_{q0}}][x_i]$, a subspace of $\mathbb{R}^{V[G_{d_{q0}}][x_i]}$ such that

- $x_i \in U_i$,
- $U_i = \{ax : a \in \mathbb{R}^{V[G_{d_{q0}}]}, x \in U_i\}$,
- $U_i \cap \mathbb{R}^{V[G_{d_{q0}}]} = \{0\}$ and
- $\mathbb{R}^{V[G_{d_{q0}}][x_i]}$ is the internal direct sum of $\dot{\mathcal{W}}_{q0,G_d \restriction P_{q_{d_0}}} \cup \{U_i\}$.

By Lemma 4.7, letting U be the span of $\bigcup_{i=1}^{3} U_i$, $\dot{\mathcal{W}}_{q0,G_d \restriction P_{q_{d_0}}} \cup \{U\}$ extends $\dot{\mathcal{W}}_{q0,G_d \restriction P_{q_{d_0}}} \cup \{U_i\}$ for each i in $\{1, 2, 3\}$, as desired.

If $\dot{F}^q_{G_{d_q}} = \dot{F}^{q'}_{G_{d_{q'}}}$, then this set induces a set $\mathcal{V}$ such that

- in $V[G_{d_q}]$, $\mathcal{V}$ is a maximal linearly independent family,

- the intersection of $\mathcal{V}$ with each of the models $V[G_{d_{q_0}\cup\{\alpha_1\}}]$, $V[G_{d_{q_0}\cup\{\alpha_2\}}]$ and $V[G'_{d_{q_0}\cup\{\alpha_2\}}]$ is a maximal linearly independent family in the corresponding model and
- for each $S \in \mathcal{V}$ and each of the models $V[G_{d_{q_0}\cup\{\alpha_1\}}]$, $V[G_{d_{q_0}\cup\{\alpha_2\}}]$ and $V[G'_{d_{q_0}\cup\{\alpha_2\}}]$ if S has a nonzero member in the model then it is contained in the model.

If a $\mathcal{V}$ as above did exist, then each of x_1, x_2 and x_3 would be a sum of elements of distinct spaces from $\mathcal{V}$ in a unique way. The equation $x_3+x_2 = x_1$ however then gives two different sums for x_1.

6 Questions

The results above induce the following questions (and many other natural variations) which we do not know the answers to.

We write $\Phi_n(\mathcal{S})$ for the assertion that a vector space $\mathcal{S}$ is an internal direct sum of n-dimensional subspaces, and $\Phi_{\mathrm{f}}(S)$ for the assertion that $\mathcal{S}$ is the internal direct sum of finite-dimensional subspaces.

Remark 6.1. *Strengthening the first part of Remark 4.1 above, we note that the existence of a Hamel basis for $\mathbb{R}$ is equivalent to $\mathbb{R}$ being an internal direct sum of one-dimensional subspaces (over $\mathbb{Q}$). The forward direction of this is the first part of the remark. For the reverse direction, the existence of such an internal direct sum gives a discontinuous homomorphism from $\mathbb{R}$ to itself, as noted above, and therefore an $\mathbb{E}_0$-selector (by [6]). Since the nonzero rational numbers form an abelian group under multiplication, this gives a selector for associated equivalence relation (i.e., a choice of a nonzero vector from each space), by [2], which is then a Hamel basis.*

Question 6.2. *For which positive integers n and m does $\Phi_n(\mathbb{R})$ imply $\Phi_m(\mathbb{R})$?*

Question 6.3. *Does $\Phi_{\mathrm{c}}(\mathbb{R})$ imply $\Phi_{\mathrm{f}}(\mathbb{R})$ or does $\Phi_{\mathrm{f}}(\mathbb{R})$ imply $\Phi_n(\mathbb{R})$ for some or any positive integer n?*

Question 6.4. *Does the existence of a nontrivial homomorphism from $\mathbb{R}$ to itself imply that $\mathbb{R}$ is equal to an internal direct sum consisting of at least two subspaces?*

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