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A permutation group acting transitively on certain collections of models ☆

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ABSTRACT

It is shown, from σ-centered Martin's Axiom, that there exists a proper dense subgroup of the symmetric group on a countably infinite set whose natural action on sufficiently flexible relational structures is transitive. This allows us to give consistent positive answers to some questions of Peter M. Neumann from the 1980s.

1. Introduction

Let Ω be a countably infinite set and $\mathrm{Sym}(\Omega)$ be the group of all bijections on Ω. Recall that $\mathrm{Sym}(\Omega)$ has the natural topology of pointwise convergence, under which it is a Polish group. In this paper, we will consider relational models (in the sense of universal algebra) $\mathbb{M} = (\Omega, \{R_i\}_{i\in I})$ whose universe is Ω and collection $\{R_i\}_{i\in I}$ of relations is countable (possibly finite or empty). Each permutation $f \in \mathrm{Sym}(\Omega)$ naturally gives an isomorphism

$$f : \mathbb{M} = (\Omega, \{R_i\}_{i\in I}) \to \mathbb{M}' = (\Omega, \{R'_i\}_{i\in I})$$

where for each $i \in I$, $R'_i(f(a_0), \ldots, f(a_{n-1}))$ if and only if $R_i(a_0, \ldots, a_{n-1})$. Thus the action of $\mathrm{Sym}(\Omega)$ on Ω extends to a transitive action on the collection $\overline{\mathbb{M}}$ of those elements of the isomorphism class of a model $\mathbb{M}$ whose universe is Ω (recall that a group action $G \curvearrowright X$ is *transitive* if for each $x, x' \in X$ there exists $g \in G$ such that $gx = x'$). It is natural to ask whether the restriction of this action to some proper (dense) subgroup $G < \mathrm{Sym}(\Omega)$ is also transitive.

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To manage the reader's expectations, we consider a simple situation in which this can fail. Take $\mathbb{M} = (\Omega, \{<\})$ where $<$ is a linear order on Ω which is order isomorphic to the order on the set ω of natural numbers. It is straightforward to see that if $G \leq \mathrm{Sym}(\Omega)$ has transitive action on $\overline{\mathbb{M}}$ then $G = \mathrm{Sym}(\Omega)$. So, to allow a proper subgroup to have a transitive action on $\overline{\mathbb{M}}$ there should be some reasonable flexibility in the automorphisms of $\mathbb{M}$. The following definition conveys such a notion.

Definition 1. We will say that a model $\mathbb{M} = (\Omega, \{R_i\}_{i\in I})$ is *inductively flexible* if there exists a collection $\mathcal{F}$ of partial automorphisms of $\mathbb{M}$ (i.e. each element of $\mathcal{F}$ is a restriction of an automorphism on $\mathbb{M}$) such that

(1) each $f \in \mathcal{F}$ has finite domain;
(2) $\emptyset \in \mathcal{F}$;
(3) $f \in \mathcal{F}$ implies $f^{-1} \in \mathcal{F}$; and
(4) for each $f \in \mathcal{F}$ and $a \in \Omega \setminus \mathrm{dom}(f)$, the set $\{b \in \Omega : f \cup \{(a, b)\} \in \mathcal{F}\}$ is infinite.

Under a mild set-theoretic assumption we can find a *single* proper dense subgroup of $\mathrm{Sym}(\Omega)$ whose action on $\overline{\mathbb{M}}$ is transitive for *every* such $\mathbb{M}$.

Theorem 1 (*ZFC + MA(σ-centered)*)**.** *There exists a proper, dense subgroup $G < \mathrm{Sym}(\Omega)$ such that if $\mathbb{M}$ is inductively flexible then G acts transitively on $\overline{\mathbb{M}}$.*

The assertion MA(σ-centered) is equivalent to the equality $\mathfrak{p} = 2^{\aleph_0}$ [1], and is a consequence of Martin's Axiom, which holds under the continuum hypothesis. Thus Theorem 1 gives the consistency, with ZFC, of the existence of such a subgroup G. One can think of many relational structures which are inductively flexible, especially among Fraïssé limits. Two examples of inductively flexible models are an order on Ω isomorphic to $\mathbb{Q}$ and a partition of Ω into infinitely many infinite sets. Theorem 1 therefore allows us to derive consistent positive solutions to some old Kourovka Notebook questions [3, 9.41 (c) and 9.42] of Peter M. Neumann (see Corollaries 9 and 10).

We give some technical preliminary arguments in Section 2 and then derive the main results in Section 3.

2. Strongly independent sets

In this section we consider elements in the free group $F = F(\{x_n\}_{n<\omega})$ of countably infinite rank. We use the convention that an element $W \in F$ is a *reduced* word, meaning that if x_i appears in W then x_i^{-1} does not appear immediately before or after it in W. Thus an element of F is *nontrivial* if it does not represent the trivial element, i.e. if it is a word of length at least one. We write $W(x_0, \ldots, x_n)$ to mean that the only letters which appear in W are in the set $\{x_0^{\pm 1}, \ldots, x_n^{\pm 1}\}$, but possibly only a proper subset appears. We will say that a reduced word $W(x_0, \ldots, x_n)$ *uses* x_i if either x_i or x_i^{-1} appears in W. Note that if $W(x_0, x_1, \ldots, x_n)$ uses x_0 then W can be decomposed uniquely as an alternating concatenation of maximal nontrivial subwords $U(x_1, \ldots, x_n)$ which do not use x_0 and maximal nontrivial words x_0^k with $k \in \mathbb{Z} \setminus \{0\}$:

$$W \equiv (U_{m+1}(x_1, \ldots, x_n))x_0^{k_m} \cdots x_0^{k_1} U_1(x_1, \ldots, x_n)x_0^{k_0}(U_0(x_1, \ldots, x_n)) \tag{†}$$

where the rightmost term U_0 and/or the leftmost term U_{m+1} might not exist. For such a decomposition we write $L'(W) = (\sum_{i=0}^{m} |k_m|) + J(W)$ where $J(W) = m + 2$ in case both U_0 and U_{m+1} exist, $J(W) = m + 1$ in case exactly one of U_0 and U_{m+1} exist, and $J(W) = m$ in case neither U_{m+1} nor U_0 exist. Thus, $L'(W)$ adds the number of occurrences of x_0 and of x_0^{-1} and of subwords of form U_i in the decomposition (†). Thus, for a word $W(x_0, \ldots, x_n)$ which uses x_0 we can write

$$W \equiv V_{L'(W)-1} V_{L'(W)-2} \cdots V_0 \tag{††}$$

where each V_i is x_0, or x_0^{-1}, or one of the U_i in the decomposition (†).

For the next definition, we take c to be a symbol such that $c \notin \Omega$. Also, a *partial bijection on Ω* is a function f such that for some $X \subseteq \Omega$ and $g \in \mathrm{Sym}(\Omega)$ we have $f = g \upharpoonright X$ (we allow $X = \Omega$).

Definition 2. Suppose that $W(x_0, \ldots, x_n)$ is a word which uses x_0 and $f_0, f_1, \ldots, f_n$ is a list of partial bijections on Ω. We say that a tuple $(t_{L'(W)}, \ldots, t_1, t_0)$ is an *itinerary of W with respect to $f_0, \ldots, f_n$* provided

(1) for each $0 \le i \le L'(W)$ we have $t_i \in \Omega \sqcup \{c\}$, and $t_i \in \Omega$ for some i;
(2) if $t_i \in \Omega$ and $0 \le i < L'(W)$ then

$$t_{i+1} = \begin{cases} V_i(f_0, \ldots, f_n)t_i & \text{if } t_i \in \mathrm{dom}(V_i(f_0, \ldots, f_n)), \\ c & \text{otherwise ;} \end{cases}$$

(3) if $t_{i+1} \in \Omega$ and $0 \le i < L'(W)$ then

$$t_i = \begin{cases} (V_i(f_0, \ldots, f_n))^{-1} t_{i+1} & \text{if } t_{i+1} \in \mathrm{ran}(V_i(f_0, \ldots, f_n)), \\ c & \text{otherwise .} \end{cases}$$

(4) if $t_j = c$ and $t_i \in \Omega$ for some $i < j$ then $t_\ell = c$ for all $L'(W) \ge \ell \ge j$;
(5) if $t_j = c$ and $t_i \in \Omega$ for some $j < i$ then $t_\ell = c$ for all $0 \le \ell \le j$.

The idea of an itinerary is to take an element of Ω and place it somewhere in the word and move it through the word backwards and forwards as much as possible, with a symbol c showing where the movement will not be defined. The following is clear.

Lemma 2. *Suppose $W(x_0, \ldots, x_n)$ uses x_0, the functions $f_0, \ldots, f_n$ are partial bijections on Ω and $(t_{L'(W)}, \ldots, t_0)$, and $(t'_{L'(W)}, \ldots, t'_0)$ are two itineraries of W with respect to $f_0, \ldots, f_n$ with $t_i = t'_i \in \Omega$ for some $0 \le i \le L'(W)$. Then the two itineraries are equal.*

Notation 3. *Suppose that $W(x_0, \ldots, x_n)$ uses x_0 and the functions $f_0, \ldots, f_n$ are partial bijections on Ω. We use $\mathrm{It}(W, f_0, \ldots, f_n)$ to denote the set of all itineraries of $W(x_0, \ldots, x_n)$ with respect to $f_0, \ldots, f_n$. For $L'(W) \ge i \ge 0$ and $a \in \Omega$ we let $\mathrm{It}(W, f_0, \ldots, f_n, i, a)$ be the unique (by Lemma 2) element of $\mathrm{It}(W, f_0, \ldots, f_n)$ which has $t_i = a$. If $\bar{t} = (t_{L'(W)}, \ldots, t_0) \in \mathrm{It}(W, f_0, \ldots, f_n)$ then let $\mathrm{set}(\bar{t})$ be the set $\{t_{L'(W)}, \ldots, t_0\} \setminus \{c\}$.*

Definition 3. If $\mathbb{M} = (\Omega, \{R_i\}_{i \in I})$ is inductively flexible, and $\mathcal{F}$ witnesses this, then we say that $(\mathbb{M}, \mathcal{F})$ is a *flexible pair*.

We now give a technical lemma.

Lemma 4. *Suppose that*

(1) $(\mathbb{M}, \mathcal{F})$ is a flexible pair;
(2) $h, f_1, \ldots, f_n \in \mathrm{Sym}(\Omega)$;
(3) words $W_0(x_0, \ldots, x_n), \ldots, W_s(x_0, \ldots, x_n)$ each utilize x_0; and
(4) $g \in \mathcal{F}$.

Then each of the following hold.

(i) If $a \in \Omega \setminus \mathrm{dom}(g)$ then there exists $b \in \Omega$ such that $g' := g \cup \{(a, b)\} \in \mathcal{F}$, and for any $0 \le p \le s$ and $\overline{t'} \in \mathrm{It}(W_p, h \circ g', f_1, \ldots, f_n)$, if $t'_i = t'_j \in \Omega$ for some $0 \le i < j \le L'(W_p)$ then there exist $0 \le i_0 < j_0 \le L'(W_p)$ with $\bar{t} = \mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, i_0, t'_{i_0})$ satisfying $t_{i_0} = t_{j_0} \in \Omega$.
(ii) If $b \in \Omega \setminus \mathrm{ran}(g)$ then there exists $a \in \Omega$ such that $g' := g \cup \{(a, b)\} \in \mathcal{F}$, and for any $0 \le p \le s$ and $\overline{t'} \in \mathrm{It}(W_p, h \circ g', f_1, \ldots, f_n)$, if $t'_i = t'_j \in \Omega$ for some $0 \le i < j \le L'(W_p)$ then there exist $0 \le i_0 < j_0 \le L'(W_p)$ with $\bar{t} = \mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, i_0, t'_{i_0})$ satisfying $t_{i_0} = t_{j_0} \in \Omega$.

Proof. Assume the hypotheses (1)–(4). For $0 \le p \le s$ we let

$$W_p \equiv V_{p, L'(W_p)-1} V_{p, L'(W_p)-2} \cdots V_{p,0}$$

be the (††) decomposition. For each $0 \leq p \leq s$ we let $X_p^+ = \{0 \leq i \leq L'(W) - 1 : V_i \equiv x_0\}$ and $X_p^- = \{0 \leq i \leq L'(W) - 1 : V_i \equiv x_0^{-1}\}$. We shall first prove conclusion (i). Let $a \in \Omega \setminus \mathrm{dom}(g)$ be given. For each $i \in X_p^+$ we define $Z_{p,i}$ to be the set of elements $b \in \Omega$ such that

$$\mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, i+1, h(b))) \cap \mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, i, a)) = \emptyset.$$

It is straightforward to see that $\Omega \setminus Z_{p,i}$ has cardinality at most $(i+1)(L'(W_p)+1)$. For $i \in X_p^-$ define $Z_{p,i}$ to be the set of those $b \in \Omega$ such that

$$\mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, i+1, a)) \cap \mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, i, h(b))) = \emptyset.$$

By similar reasoning we can see that $\Omega \setminus Z_{p,i}$ is finite.

Defining $Z \subseteq \Omega$ by $Z = (\bigcap_{0 \leq p \leq s} \bigcap_{i \in X_p^+ \cup X_p^-} Z_{p,i}) \setminus \{h^{-1}(a)\}$ we have that $\Omega \setminus Z$ is finite. So, we may select $b \in Z$ such that $g' = g \cup \{(a, b)\} \in \mathcal{F}$. We check the conclusion of (i). Let $0 \leq p \leq s$ and suppose that $\overline{t'} \in \mathrm{It}(W_p, h \circ g', f_1, \ldots, f_n)$ is such that $t_i' = t_j' \in \Omega$ for some $0 \leq i < j \leq L'(W_p)$. Let $\overline{t''} = \mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, i, t_i')$. If $t_j' = t_j''$, then we may conclude by letting $j_0 = j$ and $i_0 = i$. Otherwise, there exists k with $j > k \geq i$ such that

- $t_i' = t_i'', \ldots, t_k' = t_k''$;
- $V_{p,k} \equiv x_0$ or $V_{p,k} \equiv x_0^{-1}$; and
- $V_{p,k}(h \circ g)t_k''$ is undefined.

So either $t_k' = a$ and $t_{k+1}' = h(b)$ (in case $V_{p,k} \equiv x_0$), or $t_k' = h(b)$ and $t_{k+1}' = a$ (in case $V_{p,k} \equiv x_0^{-1}$). Let $\overline{t^{(3)}} = \mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, k+1, t_{k+1}')$. If $t_j^{(3)} = t_j'$ then

$$t_j' \in \mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, k+1, h(b))) \cap \mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, k, a))$$

in case $V_{p,k} \equiv x_0$ (since t_j' is in the first set of this intersection and $t_i' = t_j'$ is in the second) and similarly

$$t_j' \in \mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, k+1, a)) \cap \mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, k, h(b)))$$

in case $V_{p,k} \equiv x_0^{-1}$. Either way we have contradicted our choice of $b \in Z$. Thus it is the case that $t_j^{(3)} = c$, and taking $m \geq k+1$ maximal such that $t_m^{(3)} \neq c$ we have

- $t_{k+1}' = t_{k+1}^{(3)}, \ldots, t_m' = t_m^{(3)}$;
- $V_{p,m} \equiv x_0$ or $V_{p,m} \equiv x_0^{-1}$; and
- $V_{p,m}(h \circ g)t_m^{(3)}$ is undefined.

In particular $j > k+1$ since $t_j^{(3)} = c$. We claim that $m \neq k+1$. For contradiction we assume $m = k+1$, so that $V_{p,k+1} \equiv V_{p,k}$ (since the word W_p is reduced). If $V_{p,k} \equiv x_0$ then since $V_{p,k+1}(h \circ g)t_{k+1}^{(3)}$ is not defined and $V_{p,k+1}(h \circ g')t_{k+1}^{(3)}$ is defined, we see that $t_{k+1}^{(3)} = a$. By the same reasoning we have $t_k'' = a$, and so $t_{k+1}^{(3)} = h(b)$ and $h(b) = a$, contradicting our choice of b. On the other hand, if $V_{p,k} \equiv x_0^{-1}$ then $t_{k+1}^{(3)} = h(b)$ (since $V_{p,m}(h \circ g)$ is undefined at $t_{k+1}^{(3)}$ but $V_{p,m}(h \circ g')$ is defined at $t_{k+1}^{(3)}$). By the same reasoning $t_k'' = h(b)$ and therefore $t_{k+1}^{(3)} = a = h(b)$ contradicting the choice of $b \in Z$. Thus indeed $m > k+1$.

If $t_m^{(3)} = t_{k+1}^{(3)}$ then we may conclude the argument by letting $j_0 = m$ and $i_0 = k+1$. Assuming the inequality $t_m^{(3)} \neq t_{k+1}^{(3)}$, we claim that $V_{p,m} \equiv V_{p,k}$. This is true since we have already seen so far that $V_{p,k}, V_{p,m} \in \{x_0, x_0^{-1}\}$ and

- $V_{p,k} \equiv x_0$ if and only if $t_{k+1}' = h(b)$;
- $V_{p,k} \equiv x_0^{-1}$ if and only if $t_{k+1}' = a$;
- $V_{p,m} \equiv x_0$ if and only if $t_{k+1}' = h(b)$;
- $V_{p,m} \equiv x_0$ if and only if $t_{k+1}' = a$

and since $h(b) \neq a$ and we are assuming $t^{(3)}_m \neq t^{(3)}_{k+1}$ it follows that $V_{p,m} \equiv V_{p,k}$. If additionally $V_{p,k} \equiv x_0$ then $k \in X^+_p$ and $t''_k = a = t^{(3)}_m$ and we get

$$a \in \mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, k+1, h(b))) \cap \mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, k, a))$$

which is contrary to having $b \in Z$. If $V_{p,k} \equiv x_0^{-1}$ then $k \in X^-_p$ and

$$h(b) \in \mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, k+1, a)) \cap \mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, k, h(b)))$$

which also contradicts the choice of $b \in Z$. Thus we have checked that (i) holds.

The proof of conclusion (ii) is entirely analogous and we give the sketch. We have X^+_p and X^-_p defined as before, for each $0 \leq p \leq s$. Let $b \in \Omega \setminus \mathrm{ran}(g)$ be given. For each $i \in X^+_p$ define $Y_{p,i}$ to be the set of elements $a \in \Omega$ such that

$$\mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, i+1, h(b))) \cap \mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, i, a)) = \emptyset.$$

For $i \in X^-_p$ we let $Y_{p,i}$ be the set of $a \in \Omega$ such that

$$\mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, i+1, a)) \cap \mathrm{set}(\mathrm{It}(W_p, h \circ g, f_1, \ldots, f_n, i, h(b))) = \emptyset.$$

Let $Y = (\bigcap_{0 \leq p \leq s} \bigcap_{i \in X^+_p \cup X^-_p} Y_{p,i}) \setminus \{h(b)\}$. It is easy to see that $\Omega \setminus Y$ is finite, so select $a \in Y$ such that $g' := g \cup \{(a, b)\} \in \mathcal{F}$. Now follow the proof as in (i), with the obvious slight modifications, to check (ii). □

Definition 4. We say a subset $\mathcal{G} \subseteq \mathrm{Sym}(\Omega)$ is *strongly independent* if for every nontrivial $W(x_0, \ldots, x_\ell) \in F$ and pairwise distinct elements $f_0, \ldots, f_\ell \in \mathcal{G}$, the element $W(f_0, \ldots, f_\ell) \in \mathrm{Sym}(\Omega)$ has only finitely many fixed points in Ω. We will use $\mathrm{fix}(f)$ to denote the set of fixed points of $f \in \mathrm{Sym}(\Omega)$.

Lemma 5. *There exists a countably infinite, strongly independent set $\mathcal{J}$ which is dense in* $\mathrm{Sym}(\Omega)$.

Proof. Let $\mathbb{M} = (\Omega, \emptyset)$ be the relational model having no relations. Then $\mathrm{Aut}(\mathbb{M}) = \mathrm{Sym}(\Omega)$ and letting $\mathcal{F}$ be the set of all finite partial bijections on Ω, the ordered pair $(\mathbb{M}, \mathcal{F})$ is clearly a flexible pair. Let $\mathcal{J}_n = \{f_1, \ldots, f_n\}$ be a (possibly empty) finite strongly independent set with $f_1, \ldots, f_n$ pairwise distinct. We show that if $q \in \mathcal{F}$ then there exists $\overline{q} \in \mathrm{Sym}(\Omega) \setminus \mathcal{J}_n$ such that $\overline{q} \restriction \mathrm{dom}(q) = q$ and $\mathcal{J}_n \cup \{\overline{q}\}$ is strongly independent. This immediately implies the lemma since we can then enumerate $\mathcal{F} = \{q_\ell\}_{\ell \in \omega \setminus \{0\}}$ and inductively define strongly independent sets $\mathcal{J}_n = \{\overline{q_n}\}_{1 \leq \ell \leq n}$ so that the $\overline{q_1}, \ldots, \overline{q_n}$ are pairwise distinct and $\overline{q_\ell} \restriction \mathrm{dom}(q_\ell) = q_\ell$, so $\mathcal{J} = \bigcup_{n \in \omega \setminus \{0\}} \mathcal{J}_n$ is as required.

Let $q \in \mathcal{F}$ be given. Let $\{W_p(x_0, \ldots, x_n)\}_{p<\omega}$ be an enumeration of the words which use x_0 and which do not use x_i for $i > n$. Let $\Omega = \{a_t\}_{t<\omega}$ be an enumeration. By iterated use of Lemma 4 define a sequence u_s, $s < \omega$, of elements of $\mathcal{F}$ such that

- $(*)_1$ $q = u_0 \subseteq u_1 \subseteq \cdots \subseteq u_r \subseteq \cdots$;
- $(*)_2$ $a_s \in \mathrm{dom}(u_{s+1}) \cap \mathrm{ran}(u_{s+1})$;
- $(*)_3$ if $p < s$ and $\overline{t'} \in \mathrm{It}(W_p, u_s, f_1, \ldots, f_n)$ is an itinerary for which $t'_j = t'_i \in \Omega$ for some $0 \leq i < j \leq L'(W_p)$ then there exist $0 \leq i_0 < j_0 \leq L'(W_p)$ such that $\overline{t} = \mathrm{It}(W_p, u_p, f_1, \ldots, f_n, i_0, t'_{i_0})$ satisfies $t_{i_0} = t_{j_0} \in \Omega$.

More precisely, set $u_0 = q$. Assuming u_s is defined, let $u'_s = u_s$ in case $a_s \in \mathrm{dom}(u_s)$. Otherwise use Lemma 4(i), with $h \in \mathrm{Sym}(\Omega)$ being the identity map, to find $b \in \Omega$ with $u'_s = u_s \cup \{(a_s, b)\} \in \mathcal{F}$ such that if

(1) $0 \leq p \leq s$;
(2) $\overline{t'} \in \mathrm{It}(W_p, u'_s, f_1, \ldots, f_n)$; and
(3) $0 \leq i < j \leq L'(W_p)$ are such that $t'_i = t'_j \in \Omega$

then there are $0 \leq i_0 < j_0 \leq L'(W_p)$ such that $\overline{t} = \mathrm{It}(W_p, u_s, f_1, \ldots, f_n, i_0, t'_{i_0})$ satisfies $t_{i_0} = t_{j_0} \in \Omega$. Next, define $u''_s = u'_s$ in case $a_s \in \mathrm{ran}(u'_s)$. Otherwise use Lemma 4 (ii) to find $a \in \Omega$ such that

$u_s'' = u_s' \cup \{(a, a_s)\} \in \mathcal{F}$ and satisfying the analogous conditions. Set $u_{s+1} = u_s''$. Now conditions $(*)_1$ and $(*)_2$ are clear, and condition $(*)_3$ holds for a fixed p by induction on $s \geq p$.

Write $\overline{q} = \bigcup_{s<\omega} u_s$. By $(*)_1$ and $(*)_2$ we know $\overline{q} \in \mathrm{Sym}(\Omega)$. Notice that $\overline{q} \neq f_k$ for each $1 \leq k \leq n$, since the word $W \equiv x_0 x_k^{-1}$ is such that $W(\overline{q}, f_1, \ldots, f_n) = \overline{q} f_k^{-1}$ has only finitely many fixed points (at most $|\mathrm{dom}(u_p)| + |\mathrm{fix}(f_k)|$ where $W \equiv W_p$).

Note that in checking strong independence of $\mathcal{J}_n \cup \{\overline{q}\}$ it suffices to prove that if $W(x_0, \ldots, x_n)$ uses x_0 then $W(\overline{q}, f_1, \ldots, f_n)$ has finitely many fixed points (since $\mathcal{J}_n$ is already strongly independent, and by permuting the variables in a free word so that $\overline{q}$ is substituted for x_0). Let $p < \omega$. Write

$$W_p \equiv V_{L'(W_p)-1} \cdots V_0$$

for the (††) decomposition of W_p. By construction, if $\overline{t} \in \mathrm{It}(W_p, u_p, f_1, \ldots, f_n)$ and $0 \leq i_0 < j_0 \leq L'(W_p)$ are such that $t_{i_0} = t_{j_0} \in \Omega$ then there is some $0 \leq k \leq L'(W_p) - 1$ for which

- $V_k \equiv x_0$ or $V_k \equiv x_0^{-1}$, and $t_k \in \mathrm{dom}(V_k(u_p))$; or
- $V_k \not\equiv x_0, x_0^{-1}$ and $V_k(f_1, \ldots, f_n)t_k = t_{k+1} = t_k$.

Suppose that for $\overline{t''} \in \mathrm{It}(W_p, \overline{q}, f_1, \ldots, f_n)$ there exist some $0 \leq i < j \leq L'(W)$ for which $t_i'' = t_j''$. By $(*)_3$ we select $0 \leq i_0 < j_0 \leq L'(W)$ for which

$$\overline{t} = \mathrm{It}(W_p, u_p, f_1, \ldots, f_n, i_0, t_{i_0}'')$$

satisfies $t_{i_0} = t_{j_0} \in \Omega$. Then we can pick a k for which $V_k \equiv x_0$ or $V_k \equiv x_0^{-1}$, and $t_k \in \mathrm{dom}(V_k(u_p))$; or such that $t_{k+1} = t_k$ and $V_k \not\equiv x_0, x_0^{-1}$. Therefore the number of points of Ω which are fixed by $W_p(\overline{q}, f_1, \ldots, f_n)$ is given by the bound

$$\begin{aligned} |\mathrm{fix}(W_p(\overline{q}, f_1, \ldots, f_n))| &\leq |\{0 \leq i \leq L'(W_p) - 1 : V_i \equiv x_0 \vee V_i \equiv x_0^{-1}\}| \cdot |\mathrm{dom}(u_p)| \\ &+ \textstyle\sum_{\{0 \leq i \leq L'(W_p)-1 : V_i \not\equiv x_0, x_0^{-1}\}} |\mathrm{fix}\, V_i(f_1, \ldots, f_n)| \end{aligned}$$

since the number on the right bounds the number of itineraries which are not injective into Ω. Therefore the set of fixed points is finite.

So the set $\mathcal{J}_n \cup \{\overline{q}\}$ is strongly independent and of cardinality $n + 1$. □

3. Proof of Theorem 1 and applications

We recall some set-theoretic conventions. Every ordinal is the set of ordinals which are strictly below itself, so $4 = \{0, 1, 2, 3\}$ and $\omega+2 = \{0, 1, 2, \ldots, \omega, \omega+1\}$. Each cardinal is an ordinal (which is the least ordinal having that cardinality). We use $2^{\aleph_0}$ to denote the cardinality of the continuum.

Lemma 6. *Suppose that for any strongly independent set $\mathcal{G} \subseteq \mathrm{Sym}(\Omega)$, with $|\mathcal{G}| < 2^{\aleph_0}$, and isomorphic, inductively flexible structures $\mathbb{M}_1 = (\Omega, \{R_i\}_{i\in I})$ and $\mathbb{M}_2 = (\Omega, \{R_i'\}_{i\in I})$ there exists an isomorphism $f : \mathbb{M}_1 \to \mathbb{M}_2$ such that $\mathcal{G} \cup \{f\}$ is strongly independent. Then there exists a subgroup $G \leq \mathrm{Sym}(\Omega)$ which is*

(1) proper;
(2) dense; and
(3) for any pair of isomorphic, inductively flexible structures $\mathbb{M}$ and $\mathbb{M}'$ having universe Ω there exists $f \in G$ which is an isomorphism between $\mathbb{M}$ and $\mathbb{M}'$.

Proof. Assume the hypotheses. Throughout this proof we shall consider two relational structures $\mathbb{M}_1 = (\Omega, \{R_i\}_{i\in I})$ and $\mathbb{M}_2 = (\Omega, \{R_t\}_{t\in T})$ to be equal provided $\{R_i\}_{i\in I} = \{R_t\}_{t\in T}$ as sets. Thus, it is easy to see that the number of distinct relational structures on Ω having countably many relations is $2^{\aleph_0}$. Let $\{(\mathbb{M}_{1,\alpha}, \mathbb{M}_{2,\alpha})\}_{\alpha<2^{\aleph_0}}$ be an enumeration of all pairs of isomorphic, inductively flexible structures which both have universe Ω. By Lemma 5 let $\mathcal{J}$ be a countably infinite strongly independent set which is dense in $\mathrm{Sym}(\Omega)$.

By the assumptions of the lemma pick $f_0 \in \mathrm{Sym}(\Omega)$ which is an isomorphism from $\mathbb{M}_{1,0}$ to $\mathbb{M}_{2,0}$ such that $\mathcal{J} \cup \{f_0\}$ is strongly independent. Assuming we have chosen f_β for all $\beta < \alpha < 2^{\aleph_0}$, select $f_\alpha \in \mathrm{Sym}(\Omega)$ which is an isomorphism from $\mathbb{M}_{1,\alpha}$ to $\mathbb{M}_{2,\alpha}$ such that $\mathcal{J} \cup \{f_\beta\}_{\beta\leq\alpha}$ is strongly independent. Let $G = \langle \mathcal{J} \cup \{f_\alpha\}_{\alpha<2^{\aleph_0}}\rangle$. By construction we know that (3) holds. As $\mathcal{J} \subseteq G$ we know that (2) holds. We also know that each nontrivial element of G has finitely many fixed points (since G is generated by a strongly independent set). Then, letting $\Omega = \{a_z\}_{z\in\mathbb{Z}}$ be an enumeration without repetition and

$$f(a_z) = \begin{cases} a_{z+2} & \text{if } z \in 2\cdot\mathbb{Z}, \\ a_z & \text{otherwise} \end{cases}$$

we have $f \in \mathrm{Sym}(\Omega) \setminus G$ and so (1) holds. □

We review some notions regarding partial orders (see [2, Ch.14]).

Definitions 7. *Let $\mathbb{P} = (P, \leq)$ be a partially ordered set. A set $Q \subseteq P$ is*

- open *if $q \in Q$ and $q \leq p$ imply $p \in Q$;*
- dense *if for each $p \in P$ there exists $q \in Q$ with $p \leq q$;*
- centered *if every finite subset of Q has an upper bound in P (i.e. if $\{q_0, \ldots, q_k\} \subseteq Q$ is finite then there exists $p \in P$ with $q_j \leq p$ for each $0 \leq j \leq k$);*
- *a* filter *if*

  (i) $Q \neq \emptyset$;
  (ii) for any $q_0, q_1 \in Q$ there is $q \in Q$ with $q_0, q_1 \leq q$;
  (iii) if $p \leq q \in Q$ then $p \in Q$.

*We say $\mathbb{P}$ is σ-*centered *if P is a countable union of centered sets. The assertion MA(σ-centered) is the following:*

Whenever $\mathbb{P} = (P, \leq)$ is a σ-centered partially ordered set, $P \neq \emptyset$, and $\mathcal{D}$ is a collection of open dense subsets of P with $|\mathcal{D}| < 2^{\aleph_0}$ there exists a filter $Q \subseteq P$ with $Q \cap O \neq \emptyset$ for all $O \in \mathcal{D}$.

Lemma 8 (*ZFC + MA(σ-centered)*)**.** *Suppose $\mathcal{G} \subseteq \mathrm{Sym}(\Omega)$ is strongly independent, with $|\mathcal{G}| < 2^{\aleph_0}$, and $\mathbb{M}_1 = (\Omega, \{R_i\}_{i\in I})$ and $\mathbb{M}_2 = (\Omega, \{R'_i\}_{i\in I})$ are isomorphic, inductively flexible structures. Then there exists an isomorphism $f : \mathbb{M}_1 \to \mathbb{M}_2$ such that $\mathcal{G} \cup \{f\}$ is strongly independent.*

Proof. Assume the hypotheses and let $h : \mathbb{M}_1 \to \mathbb{M}_2$ be an isomorphism. Let $(\mathbb{M}_1, \mathcal{F})$ be a flexible pair. For a reduced word W in a free group $F(\{x_n\}_{n<\omega})$ let $L(W)$ denote the standard length of the word W (for example $L(x_1x_3^{-2}x_1^{-1}) = 4$). We define a partial order $\mathbb{P} = (P, \leq)$ as follows. An element $p \in P$ has form $p = (g_p, H_p)$ where $g_p \in \mathcal{F}$ and H_p is a finite subset of $\mathcal{G}$. For the order $\leq$ on P, we write $p \leq p'$ if and only if

(a) $g_p \subseteq g_{p'}$;
(b) $H_p \subseteq H_{p'}$; and
(c) if $f_1, \ldots, f_n$ lists the elements of H_p in a pairwise distinct fashion, $W(x_0, \ldots, x_n)$ is a reduced word which uses x_0, and if $L(W) \leq n$ then

  (A) $\overline{t'} \in \mathrm{It}(W, h \circ g_{p'}, f_1, \ldots, f_n)$; and
  (B) $t'_i = t'_j \in \Omega$ for some $0 \leq i < j \leq L'(W)$
  imply that there exist $0 \leq i_0 < j_0 \leq L'(W)$ such that

$$\overline{t} = \mathrm{It}(W, h \circ g_p, f_1, \ldots, f_n, i_0, t'_{i_0})$$

  has $t_{i_0} = t_{j_0} \in \Omega$.

Notice first that $\mathbb{P}$ is σ-centered. To see this, let $\mathbb{E} = \{(p, q) : p, q \in P \wedge g_p = g_q\}$. This is clearly an equivalence relation, having precisely $\aleph_0$ equivalence classes. If J is an equivalence class of $\mathbb{E}$, say $J = \{p : g_p = g\}$, and $\{(g_1, H_1), \ldots, (g_k, H_k)\} \subseteq J$ is finite then $(g, \bigcup_{i=1}^k H_i)$ is an upper bound for the set.

Notice as well that given $a \in \Omega$, the set $\mathbb{I}^1_a = \{p \in P : a \in \mathrm{dom}(g_p)\}$ is open and dense. That $\mathbb{I}^1_a$ is open is obvious, and that it is dense is immediate from Lemma 4(i). That the set $\mathbb{I}^2_a = \{p \in P : a \in \mathrm{ran}(g_p)\}$ is open and dense is clear using Lemma 4 (ii). For a function $f \in \mathcal{G}$ the set $\mathbb{I}_f = \{p \in P : f \in H_p\}$ is open and dense. That it is open is clear. To see that it is dense, let $p \in P$ be given and let $p' = (g_p, H_p \cup \{f\})$, so that indeed $p \le p' \in \mathbb{I}_f$.

The collection $\mathcal{D} = \{\mathbb{I}^1_a\}_{a\in\Omega} \cup \{\mathbb{I}^2_a\}_{a\in\Omega} \cup \{\mathbb{I}_f\}_{f\in\mathcal{G}}$ has cardinality less than $2^{\aleph_0}$, so by MA(σ-centered) we select a filter Q which intersects each element of $\mathcal{D}$ nontrivially. Letting $g = \bigcup\{g_p : p \in Q\}$ it is clear that g is an automorphism of $\mathbb{M}_1$, so the function $f = h \circ g$ is an isomorphism of $\mathbb{M}_1$ with $\mathbb{M}_2$.

It remains to check that $\mathcal{G} \cup \{f\}$ is strongly independent. Let $\Omega = \{a_s\}_{s<\omega}$ be an enumeration. Let $f_1 \in \mathcal{G}$ be given and consider the word $W(x_0, x_1) = x_0x_1^{-1}$. Pick $p_0 \in Q$ such that $|H_{p_0}| \ge 2$ and $f_1 \in H_{p_0}$. Let $p_1, p_2, \ldots$ be a sequence in Q such that

- $p_0 \le p_1 \le p_2 \le \cdots$; and
- $a_s \in \mathrm{dom}(g_{p_s}) \cap \mathrm{ran}(g_{p_s})$;

Then, reasoning as in the proof of Lemma 5, the number of fixed points of $W(f, f_1)$ is at most $|\mathrm{dom}(g_{p_0})| + |\mathrm{fix}(f_1)|$ and in particular $f \neq f_1$.

Let $W(x_0, \ldots, x_n)$ be a nontrivial word, which without loss of generality uses x_0, and $f_1, \ldots, f_n$ be a list of pairwise distinct elements of $\mathcal{G}$. Pick $p_0 \in Q$ such that $|H_{p_0}| \ge L(W)$ and $\{f_1, \ldots, f_n\} \subseteq H_{p_0}$. Picking a sequence $p_0, p_1, \ldots$ in Q with

- $p_0 \le p_1 \le p_2 \le \cdots$; and
- $a_s \in \mathrm{dom}(g_{p_s}) \cap \mathrm{ran}(g_{p_s})$; we can argue as before that $\mathrm{fix}(W(f, f_1, \ldots, f_n))$ is finite. $\square$

Theorem 1 is now immediate since Lemma 8 states that under the assumption ZFC + MA(σ-centered) the hypotheses of Lemma 6 are satisfied, and the conclusion of Lemma 6 gives precisely the group required. We point out some consequences. It is straightforward to check that if $<$ is a total linear order on Ω of order type $\mathbb{Q}$, then $\mathbb{M} = (\Omega, \{<\})$ is inductively flexible.

Corollary 9 (*ZFC + MA(σ-centered)*)**.** *There exists a subgroup G of* $\mathrm{Sym}(\Omega)$ *(e.g. G from Theorem 1) which is a dense proper subgroup acting transitively on $\overline{\mathbb{M} = (\Omega, \{<\})}$, where $<$ is of order type $\mathbb{Q}$.*

Define an $\aleph_0$-section on Ω to be a partition of Ω into $\aleph_0$ many subsets of cardinality $\aleph_0$. One can consider an $\aleph_0$-section on Ω as a collection $\{R_\ell\}_{\ell<\omega}$ of unary relations satisfying the following three requirements

$(s)_1$ $(\forall \ell < \omega)|\{a \in \Omega : R_\ell a\}| = \aleph_0$;
$(s)_2$ $(\forall \ell_0 < \ell_1 < \omega)(\forall a \in \Omega)\neg R_{\ell_0} a \vee \neg R_{\ell_1} a$; and
$(s)_3$ $(\forall a \in \Omega)(\exists \ell < \omega) R_\ell a$

but with the caveat that two such collections of relations $\{R_\ell\}_{\ell<\omega}$, $\{R'_\ell\}_{\ell<\omega}$ are considered identical provided there is some $\sigma \in \mathrm{Sym}(\omega)$ with $R_\ell a \iff R'_{\sigma(\ell)} a$. It is easy to see that a model $\mathbb{M} = (\Omega, \{R_\ell\}_{\ell<\omega})$ with the set of relations satisfying $(s)_1$ - $(s)_3$ is inductively flexible. Thus G from Theorem 1 will act transitively on $\overline{\mathbb{M}}$ and *a fortiori* G will act transitively on $\aleph_0$-sections. So we may record the following.

Corollary 10 (*ZFC + MA(σ-centered)*)**.** *There exists a subgroup $G <$ $\mathrm{Sym}(\Omega)$ (e.g. G from Theorem 1) which is a dense proper subgroup acting transitively on $\aleph_0$-sections.*

Peter M. Neumann asked whether there is a proper subgroup $G < \mathrm{Sym}(\Omega)$ which acts transitively on Ω and transitively on $\mathbb{Q}$-type orderings on Ω [3, Problem 9.42]. Since our group G is dense, it is k-transitive for each $k < \omega$, and therefore transitive, and so a consistent positive answer to his question follows from Corollary 9. Similarly he asks for a proper transitive subgroup of $\mathrm{Sym}(\Omega)$ which is transitive on $\aleph_0$-sections [3, Problem 9.41 (c)], and Corollary 10 supplies a consistent positive answer.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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