

$\aleph_1$ -FREE ABELIAN NON-ARCHIMEDEAN POLISH GROUPS

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ABSTRACT. An uncountable $\aleph_1$ -free group cannot admit a Polish group topology but an uncountable $\aleph_1$ -free abelian group can, as witnessed, for example, by the Baer-Specker group $\mathbb{Z}^\omega$; more strongly, $\mathbb{Z}^\omega$ is separable. In this paper we investigate $\aleph_1$ -free abelian non-Archimedean Polish groups. We prove two main results. The first is that there are continuum many separable (and so torsionless, and so $\aleph_1$ -free) abelian non-Archimedean Polish groups which are pairwise not abstractly isomorphic. The second is that the following four properties are complete co-analytic subsets of the space of closed abelian subgroups of S_∞ : separability, torsionlessness, $\aleph_1$ -freeness and $\mathbb{Z}$ -homogeneity.

1. INTRODUCTION

Recall that a topological group G is said to be Polish if its topology is separable and completely metrizable, and it is said to be non-Archimedean Polish if, in addition, it admits a countable neighborhood basis at the identity consisting of open subgroups. At a meeting in Durham in 1997, D. Evans asked whether a non-Archimedean Polish group can be an uncountable free group. Around the same time, H. Becker and A. Kechris asked the same question without the non-Archimedean hypothesis (cf. [1]). In [16], the second author settled the first question, and in the subsequent paper [17] he also settled the second. It was later observed that some of the impossibility results from [17] already followed from an earlier important result of Dudley [6]. Both authors of the present paper later generalized these results substantially in [13], in the context of graph products of groups. In another direction, Khelif proved in [11] that no uncountable $\aleph_1$ -free group can be Polish. Recall that a group G is said to be $\aleph_1$ -free (or $\aleph_1$ -free abelian) if every countable subgroup of G is free (or free abelian, respectively). Furthermore, an uncountable Polish group cannot be free abelian either, by [17]. In contrast, as is well known, the Baer-Specker group $\mathbb{Z}^\omega$, equipped with the product topology, is non-Archimedean Polish and $\aleph_1$ -free abelian (but not free abelian). This state of affairs motivated us to investigate the class of $\aleph_1$ -free abelian non-Archimedean Polish groups and, in particular, to ask how complicated this class is.

Recently, there has been significant interest in abelian (non-Archimedean) Polish groups: on the one hand, in the study of the Borel complexity of their actions (see,

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e.g., [5]) and, on the other hand, in the context of homological algebra (see, e.g., [2, 3]). Our algebraic study complements these lines of inquiry by providing new structural insights into these groups.

We recall the following definitions as they will play a crucial role in our study:

Definition 1.1. Let A be a torsion-free abelian group.

- (1) We say that A is *separable* if every finite subset of A is contained in a free direct summand of A .
- (2) We say that A is *torsionless* if for every $0 \neq a \in A$ there is $f \in \text{Hom}(A, \mathbb{Z})$ such that $f(a) \neq 0$.
- (3) We say that A is $\mathbb{Z}$ -*homogeneous* if every element has type $\mathbf{0}$ (cf. 3.6).

It is well-known, see e.g. [7, Corollary 2.9], that we have the following:

$$\text{separable} \Rightarrow \text{torsionless} \Rightarrow \aleph_1\text{-free} \Rightarrow \mathbb{Z}\text{-homogeneous}.$$

Now, it so happens that $\mathbb{Z}^\omega$ is not only $\aleph_1$ -free but actually separable. Thus, the motivating question of the paper mentioned above (on the existence of $\aleph_1$ -free abelian non-Archimedean Polish groups) can be strengthened to torsionless, or even separable, abelian groups. We will deal with all these properties in our paper.

The first result concerns the number of non-isomorphic groups realizing the four properties above, and it is of the strongest possible form. We recall that a completely decomposable abelian group is a group that is a direct sum of groups of rank 1.

Theorem 1.2. *There are continuum many separable (hence torsionless, hence $\aleph_1$ -free) abelian non-Archimedean Polish groups which are pairwise not abstractly isomorphic. Furthermore, all these groups can be taken to be inverse limits of countable torsion-free completely decomposable groups.*

The second result that we present is formulated in the framework of invariant descriptive set theory (see [9] for an excellent introduction to this subject). In [10], Kechris, Nies and Tent proposed the following program: as any non-Archimedean Polish group is topologically isomorphic to a closed subgroup of the topological group S_∞ of bijections of $\mathbb{N}$ onto $\mathbb{N}$, we can study classification problems on non-Archimedean Polish groups with respect to the Effros structure on the set of closed subgroups of S_∞ , which is a standard Borel space (see [10] for details).

At the heart of this program lies the following twofold task:

- for natural classes of closed subgroups of S_∞ , determine whether they are Borel;
- if a class $\mathcal{C}$ is Borel, study the Borel complexity of topological isomorphism on $\mathcal{C}$.

In our second theorem, we prove that the four properties mentioned above are not Borel and, more strongly, that they are complete co-analytic.

Theorem 1.3. *Determining whether a non-Archimedean Polish group belongs to $\mathcal{C}$ is a complete co-analytic problem in the space of closed subgroups of S_∞ for each of the following classes $\mathcal{C}$ of abelian groups:*

- (1) $\mathbb{Z}$ -homogeneous;
- (2) $\aleph_1$ -free;
- (3) torsionless;

(4) *separable*.

The structure of the paper is as follows. In Section 2 we collect the necessary preliminaries. In Section 3 we introduce the main group-theoretic construction at the heart of the paper. In Section 4 we prove Theorems 1.2 and 1.3.

2. THE DESCRIPTIVE SET THEORY SETTING

From here on, as is customary in set theory, we denote the set of natural numbers by ω . The aim of this section is to fix the coding, i.e., to specify the spaces used in our results, and to introduce the necessary background.

Fact 2.1 ([5, Proposition 2.1]). Let G be a topological group. Then the following are equivalent:

- (1) G is a non-Archimedean Polish abelian group;
- (2) G is isomorphic (as a topological group) to a closed subgroup of $\prod_{n < \omega} H_n$ (with the product topology), with each H_n abelian, countable and discrete;
- (3) G is the inverse limit of an inverse system $(G_n, f_{(n,m)} : m \leq n < \omega)$, with each G_n abelian, countable and discrete, and G is endowed with the subspace topology inherited from the product.

Notation 2.2. Let X be a Polish space. The Effros structure on X is the Borel space consisting of the family $\mathcal{F}(X)$ of closed subsets of X together with the σ -algebra generated by the following sets $\mathcal{C}_U$, where, for $U \subseteq X$ open, we let:

$$\mathcal{C}_U = \{D \in \mathcal{F}(X) : D \cap U \neq \emptyset\}.$$

Notation 2.3. We denote by S_∞ the topological group of bijections of ω onto ω , where the topology is the one induced by the following metric:

- (1) if $x = y$, then $d(x, y) = 0$;
- (2) if $x \neq y$, then $d(x, y) = 2^{-n}$, where n is least such that $x(n) \neq y(n)$.

Fact 2.4. The closed subgroups of S_∞ form a Borel subset of $\mathcal{F}(S_\infty)$ (see [10]). We denote this Borel space by $\text{Sgp}(S_\infty)$, with the Borel structure inherited from $\mathcal{F}(S_\infty)$.

Definition 2.5. By a tree on ω we mean a non-empty subset of $\omega^{<\omega}$ closed under initial segments (so in particular the empty sequence $\emptyset$ is the root of the tree).

Notation 2.6. Given a tree $T \subseteq \omega^{<\omega}$ we denote by $[T]$ the set of $\eta \in \omega^\omega$ such that $\eta \upharpoonright n \in T$ for all $n < \omega$, i.e., the set of infinite branches of T .

Fact 2.7. Let $\mathcal{T}(S_\infty)$ denote the set of trees on ω with no leaves, topologized by defining the distance between distinct trees T_1 and T_2 to be 2^{-n} , where n is the least integer such that $T_1 \cap \omega^n \neq T_2 \cap \omega^n$. For $C \in \text{Sgp}(S_\infty)$, define $T_C \in \mathcal{T}(S_\infty)$ as follows:

$$\mathbf{B} : C \mapsto T_C = \{g \upharpoonright n : g \in C \text{ and } n < \omega\} \cup \{g^{-1} \upharpoonright n : g \in C \text{ and } n < \omega\}.$$

Then $\mathbf{B} : \text{Sgp}(S_\infty) \rightarrow \mathcal{T}(S_\infty)$ is Borel, it is injective and it has closed range; furthermore, when restricted to its range, the map $\mathbf{B}$ is a Borel isomorphism.

By Fact 2.7, we can work with $\mathbf{B}(\text{Sgp}(S_\infty)) \subseteq \mathcal{T}(S_\infty)$ instead of $\text{Sgp}(S_\infty)$.

Proviso 2.8. *From now on, we consider only inverse systems with surjective bonding maps.*

Notation 2.9. We can look at an inverse system $A = (A_n, f_{(n,m)} : m \leq n < \omega)$ of groups as a first-order structure in a language $L_{\text{inv}} = \{P_n, \cdot_n, f_{(n,m)} : m \leq n < \omega\}$, where:

- (1) the P_n 's are disjoint predicates denoting the groups A_n 's;
- (2) $\cdot_n$ is a binary function symbol, interpreted as the group operation on A_n ;
- (3) the $f_{(n,m)}$'s are function symbols interpreted as morphisms from A_n to A_m .

Notation 2.10. With respect to the language considered in 2.9, the assertion that a structure of the form $A = (A_n, f_{(n,m)} : m \leq n < \omega)$ is an inverse system is axiomatizable in first-order logic. Thus, we can consider the Borel space of countable Pro-Groups with domain contained in ω . We also assume that our inverse systems $A = (A_n, f_{(n,m)} : m \leq n < \omega)$ satisfy $\text{dom}(A_n) \cap \text{dom}(A_m) = \emptyset$ whenever $n \neq m$. When the groups A_n are abelian, we use additive notation, writing $+_n$ instead of $\cdot_n$ and 0_{A_n} instead of e_{A_n} .

Definition 2.11. Let $A = (A_n, f_{(n,m)} : m \leq n < \omega)$ be an inverse system as in 2.10. For every $\bar{a} = (a_n : n < \omega) \in \varprojlim(A)$ we define a permutation $\pi_{\bar{a}}$ of ω as:

$$\pi_{\bar{a}}(k) = \begin{cases} k +_{A_n} a_n, & \text{if there is } n < \omega \text{ such that } k \in A_n, \\ k, & \text{if } k \notin \bigcup_{n < \omega} A_n. \end{cases}$$

Observation 2.12. The map $\bar{a} \mapsto \pi_{\bar{a}}$ from 2.11 is an embedding of $\varprojlim(A)$ into S_∞ .

We omit the details of the next proposition, as its proof is standard (see, e.g., 2.7).

Proposition 2.13. *The map $A \mapsto \varprojlim(A)$ from Pro-Groups (recalling 2.10) into $\text{Sgp}(S_\infty)$ (recalling 2.4 and the embedding described in 2.11) is Borel.*

3. A FEW CRUCIAL CLAIMS ABOUT TFAB'S

3.1. The engine

The following construction will be used to prove Theorems 1.2 and 1.3.

Notation 3.1. $\mathbb{P}$ denotes the set of prime numbers.

Definition 3.2. Let $T \subseteq \omega^{<\omega}$ be a tree with no leaves and let $L : T \rightarrow \mathcal{P}(\mathbb{P}) \setminus \{\emptyset\}$ be such that:

- (a) if $\eta \trianglelefteq \nu$, then $L(\nu) \subseteq L(\eta)$;
- (b) if $\eta \in T$, then $L(\eta) = \bigcup \{L(\nu) : \nu \in T \text{ and } \nu \text{ is a successor of } \eta\}$.

We define an inverse system $\text{inv}(T, L) = (G_n, f_{(n,m)} : m \leq n < \omega)$ as follows:

- (i) $Z_n := \bigoplus \{\mathbb{Z}x_\eta : \eta \in T \cap \omega^n\} \leq G_n \leq Q_n := \bigoplus \{\mathbb{Q}x_\eta : \eta \in T \cap \omega^n\}$;
- (ii) $G_n = \langle \frac{1}{p}x_\eta : \eta \in T \cap \omega^n \text{ and } p \in L(\eta) \rangle_{Q_n}$;
- (iii) for $m \leq n < \omega$, $f_{(n,m)}$ is the unique homomorphism from G_n into G_m such that

$$\eta \in T \cap \omega^n \Rightarrow x_\eta \mapsto x_{\eta \upharpoonright m}.$$

Notice that the inverse system $\text{inv}(T, L) = (G_n, f_{(n,m)} : m \leq n < \omega)$ is well-defined because of conditions (a) and (b) above. We then set

$$(T, L) \mapsto \text{inv}(T, L) \mapsto \varprojlim(\text{inv}(T, L)) := G(T, L).$$

Equipped with the topology inherited from the product topology on $\prod_{n < \omega} G_n$, the group $G(T, L)$ is a well-defined non-Archimedean Polish abelian group.

Remark 3.3. Notice that the inverse systems $\text{inv}(T, L) = (G_n, f_{(n,m)} : m \leq n < \omega)$ defined in 3.2 have bonding maps $f_{(n,m)}$ which are onto; this is because we ask that the tree T has no leaves and because we ask that condition (b) from there holds. Recall that in 3.2 we only consider trees with no leaves and so, for T 's as in 3.2, the set $[T]$ is always non-empty (cf. 2.6 for a definition of $[T]$).

Remark 3.4. Notice also that for $\text{inv}(T, L) = (G_n, f_{(n,m)} : m \leq n < \omega)$ as in 3.2 we have that the groups G_n are completely decomposable, i.e., they are direct sums of groups of rank 1, in fact it follows immediately from the definitions that:

$$G_n = \sum_{\eta \in \omega^n \cap T} \langle x_\eta \rangle_{G_n}^*,$$

where $\langle X \rangle_{G_n}^*$ denotes pure closure (cf. [8, pg. 151]).

For background on the notions that we now introduce, see, e.g., [8, Chapter 13]. We will often abbreviate “torsion-free abelian group” as TFAB.

Definition 3.5. Let $A \in \text{TFAB}$ and let $(p_i : i < \omega)$ be the list of the prime numbers in increasing order. For $a \in A$, we define the characteristic of a in A , denoted as $\chi_A(a) = \chi(a)$, as follows: $\chi(a) = (h_{p_i}(a) : i < \omega)$, where $h_{p_i}(a)$ is the supremum of the $k < \omega$ such that $p_i^k \mid a$, where the supremum is taken in $\omega \cup \{\infty\}$ (so the value ∞ is allowed).

Definition 3.6. (1) Two characteristics $(k_i : i < \omega)$ and $(\ell_i : i < \omega)$ are said to be equivalent if $k_i = \ell_i$ for almost all $i < \omega$ and both k_i and ℓ_i are finite whenever $k_i \neq \ell_i$. The equivalence classes of characteristics are called *types*.
 (2) Given $A \in \text{TFAB}$ we denote by $\mathbf{t}_A(a)$ the type of a in A ; we may simply write $\mathbf{t}(a)$ when A is clear from the context.
 (3) We denote by $\mathbf{0}$ the type of the characteristic $(0, 0, \dots)$.
 (4) For $A \in \text{TFAB}$ and $a, b \in A$ we say that $\mathbf{t}(a) \leq \mathbf{t}(b)$ if there are characteristics $\chi = (k_i : i < \omega) \in \mathbf{t}(a)$ and $\nu = (\ell_i : i < \omega) \in \mathbf{t}(b)$ such that $(k_i : i < \omega) \leq (\ell_i : i < \omega)$, where the order is the pointwise order, i.e., $k_i \leq \ell_i$, for all $i < \omega$.

We will need the following well-known facts.

Fact 3.7 ([8, (C) on p. 411]). Let G and H be torsion-free abelian groups, and let $f : G \rightarrow H$ be a homomorphism. Then, for every $g \in G$, $\mathbf{t}(g) \leq \mathbf{t}(f(g))$.

Fact 3.8. [7, Theorem 2.3] Let $A \in \text{TFAB}$. Then A is $\aleph_1$ -free if and only if, for every finite $S \subseteq A$, the pure closure of S in A (denoted $\langle S \rangle_A^*$) is free abelian.

3.2. Nice pairs

We now present a technical definition needed for our approach.

Definition 3.9. We call (T, L) a nice pair when for some S and $\bar{p}$ we have that:

- (a) S is a subtree of T and T is a subtree of $\omega^{<\omega}$ with no leaves;
- (b) $\eta \in S$ implies that $0 \notin \text{ran}(\eta)$;
- (c) $\bar{p} = (p_\eta : \eta \in S)$ is a sequence of pairwise distinct primes;
- (d) $S^+ = \{\eta \smallfrown \nu : \eta \in S, \nu \in \{0\}^{<\omega}\} \subseteq T$;
- (e) $p_{\eta \smallfrown \nu} = p_\eta$, when $\eta \in S$ and $\nu \in \{0\}^{<\omega}$;
- (f) L is the function with domain T defined as follows:

$$L(\eta) = \{p_\nu : (\nu \triangleleft \eta \text{ or } \eta \leq \nu) \text{ and } \nu \in S^+\}.$$

Observation 3.10. (1) If (T, L) is a nice pair, then (T, L) is as in 3.2.

(2) If $\eta \preceq \nu \in T$ and $L(\eta)$ is finite, then $L(\nu)$ is finite (as $L(\nu) \subseteq L(\eta)$).

Convention 3.11. For the rest of this subsection, we fix (T, L) , S , S^+ and $\bar{p}$ as in 3.9 and we let $G = G(T, L)$. For $\eta \in [T]$, we let $L(\eta) = \bigcap \{L(\eta \upharpoonright n) : n < \omega\}$. Also, for $n < \omega$, we let $f_{(\omega, n)}$ be the obvious projection of G onto G_n . For $\nu \in [T]$, we let $x_\nu = (x_{\nu \upharpoonright n} : n < \omega) \in G$. Recall also that $f_{(n, m)}$ denotes the bonding map from G_n onto G_m .

Claim 3.12. If S has an ω -branch (i.e., there is $\eta \in [T]$ such that, for all $n < \omega$, $\eta \upharpoonright n \in S$), then G is not $\mathbb{Z}$ -homogeneous, and so G is not $\aleph_1$ -free.

Proof. In this case $L(\eta) = \bigcap \{L(\eta \upharpoonright n) : n < \omega\} = \{p_{\eta \upharpoonright n} : n < \omega\}$ is infinite, and so in G we have that x_η is divisible by p for every $p \in L(\eta)$. ■

Definition 3.13. For $\eta \in T \cup [T]$, let K_η be the additive subgroup of $\mathbb{Q}$ generated by $\{\frac{1}{p} : p \in L(\eta)\} \subseteq \mathbb{Q}$; explicitly, K_η is the subgroup of $\mathbb{Q}$ consisting of:

$$\left\{ \frac{b}{\prod_{\ell < k} p_\ell} : b \in \mathbb{Z}, k < \omega, p_\ell \in L(\eta), p_{\ell_1} \neq p_{\ell_2}, \text{ for } \ell_1 < \ell_2 < k \right\}.$$

Main Claim 3.14. If S has no ω -branch, then $G = G(T, L)$ is separable (and so G is torsionless, and so G is $\aleph_1$ -free).

Proof. We show that G is separable. To this end, let $X \subseteq G$ be finite.

- (*) We claim that it suffices to prove that there are $n < \omega$ and finite $Y \subseteq \omega^n \cap T$ such that, writing $\pi_{(n, Y)}$ for the natural projection of G_n onto $G_{(n, Y)} := \langle Y \rangle_{G_n}^*$, there are a direct summand B of $G_{(n, Y)}$ and a retraction $r : G_{(n, Y)} \rightarrow B$ satisfying:
- (a) for every $\nu \in Y$, $L(\nu)$ is finite;
 - (b) define $g = r \circ \pi_{(n, Y)} \circ f_{(\omega, n)}$;
 - (c) $g \in \text{Hom}(G, B)$;
 - (d) $g_0 := g \upharpoonright \langle X \rangle_G^*$ is an isomorphism from $\langle X \rangle_G^*$ onto B .

We first show that (*) holds, i.e., that the conditions listed there suffice to conclude that G is separable. The group $G_{(n, Y)}$ is free abelian of finite rank, because $G_{(n, Y)} = \bigoplus \{K_\nu x_\nu : \nu \in Y\}$ and each $L(\nu)$ is finite, so $K_\nu \cong \mathbb{Z}$. Hence B is free, and by (d) so is $\langle X \rangle_G^*$. By (c) and (d), $g_0^{-1} \circ g$ is a homomorphism from G into $\langle X \rangle_G^*$, and for $a \in \langle X \rangle_G^*$ we have

$$g_0^{-1} \circ g(a) = g_0^{-1}(g_0(a)) = a.$$

Thus $g_0^{-1} \circ g$ is a retraction of G onto the free subgroup $\langle X \rangle_G^*$, so X is contained in a free direct summand of G , as desired.

So now that we have shown that (*) suffices, for the rest of the proof we will focus on showing that the conditions from (*) can be fulfilled (cf. the final (*₁₉)).

- (*) For every $n < \omega$, $f_{(n+1, n)}$ maps $f_{(\omega, n+1)}(\langle X \rangle_G^*)$ onto $f_{(\omega, n)}(\langle X \rangle_G^*)$, hence:
- (·₁) $\text{rk}(f_{(\omega, n+1)}(\langle X \rangle_G^*)) \geq \text{rk}(f_{(\omega, n)}(\langle X \rangle_G^*))$;
 - (·₂) those ranks are $\leq |X|$ which is finite.
- (*) For some $n_* < \omega$, $(\text{rk}(f_{(\omega, m)}(\langle X \rangle_G^*)) : m \geq n_*)$ is constant, call it k_* , and for every $n_2 > n_1 \geq n_*$, $f_{(n_2, n_1)} \upharpoonright f_{(\omega, n_2)}(\langle X \rangle_G^*)$ is an isomorphism from $f_{(\omega, n_2)}(\langle X \rangle_G^*)$ onto $f_{(\omega, n_1)}(\langle X \rangle_G^*)$.

[Why? As $n_2 > n_1 \geq n_*$ and so $\text{rk}(f_{(\omega, n_2)}(\langle X \rangle_G^*)) = \text{rk}(f_{(\omega, n_1)}(\langle X \rangle_G^*))$, recalling that $f_{(n_2, n_1)}$ maps $f_{(\omega, n_2)}(\langle X \rangle_G^*)$ onto $f_{(\omega, n_1)}(\langle X \rangle_G^*)$, we have that the map $f_{(n_2, n_1)} \upharpoonright f_{(\omega, n_2)}(\langle X \rangle_G^*)$ has trivial kernel and so it is 1-to-1, and trivially it is also onto.]

(*₄) If $n \geq n_*$, then $f_{(\omega, n)} \upharpoonright \langle X \rangle_G^*$ is an isomorphism from $\langle X \rangle_G^*$ onto $f_{(\omega, n)}(\langle X \rangle_G^*)$.

Why (*₄) holds? Let $n_* \leq n < \omega$ and suppose the negation of the thesis. Then there is $y \in \langle X \rangle_G^* \setminus \{0\}$ such that $f_{(\omega, n)}(y) = 0$. Let $y = (y_m : m < \omega)$. Then we can find $k \geq n$ such that $f_{(\omega, k)}(y) = y_k \neq 0$. But then $y_k \in f_{(\omega, k)}(\langle X \rangle_G^*) \setminus \{0\}$ and $f_{(k, n)}(y_k) = 0$, and so $f_{(k, n)}$ has non-trivial kernel, contradicting the choice of n_* (recall that by hypothesis $n \geq n_*$ and that $f_{(k, n)} : G_k \rightarrow G_n$ is onto G_n). So (*₄) holds indeed.

(*₅) W.l.o.g. $|X| = k_*$, equivalently, X is independent.

(*₆) For $n < \omega$.

(a) Let $Y_n \subseteq T \cap \omega^n$ be finite and such that we have:

$$f_{(\omega, n)}(\langle X \rangle_G^*) \subseteq G_{(n, Y_n)} = \bigoplus \{K_\nu x_\nu : \nu \in Y_n\},$$

see 3.13 for the definition of K_ν .

(b) For $y \in \langle X \rangle_{G_\omega}^*$, let:

$$(\cdot_1) \quad f_{(\omega, n)}(y) = \sum \{a_{(y, n, \nu)} x_\nu : \nu \in Y_n\}, \text{ where } a_{(y, n, \nu)} \in K_\nu;$$

$$(\cdot_2) \quad Z_n(y) = \{\nu \in Y_n : a_{(y, n, \nu)} \neq 0\}.$$

(c) W.l.o.g. for every $n \geq n_*$ and $\nu \in Y_n$, there is $y \in X$ such that $a_{(y, n, \nu)} \neq 0$.

[Why? Obvious.]

(*₇) Let $Y_n^* = \{\nu \in Y_n : L(\nu) \text{ is finite}\}$.

(*₈) Recalling the definition of $\pi_{(n, Y_n^*)}$ from (*₁), we have that the sequence

$$(\text{rk}(\pi_{(n, Y_n^*)} \circ f_{(\omega, n)}(\langle X \rangle_G^*)) : n \geq n_*)$$

is eventually constant, say for $n \geq n_*$, and let k_* be this constant value.

Why (*₈)? Put $A = \langle X \rangle_G^*$ and $H_n = \pi_{(n, Y_n^*)} \circ f_{(\omega, n)}(A)$. We show that the sequence $(\text{rk}(H_n) : n \geq n_*)$ is non-decreasing. Fix $m > n \geq n_*$ and let

$$Y_{(m, n)}^* = \{\rho \in Y_m : \rho \upharpoonright n \in Y_n^*\}.$$

Notice that $Y_{(m, n)}^* \subseteq Y_m^*$: indeed, if $\rho \upharpoonright n \in Y_n^*$, then $L(\rho) \subseteq L(\rho \upharpoonright n)$ is finite. For $y \in A$, decompose

$$f_{(\omega, m)}(y) = z_{(m, n, y)}^1 + z_{(m, n, y)}^2 + z_{(m, n, y)}^3,$$

where the three summands are the parts supported, respectively, on $Y_{(m, n)}^*$, on $Y_m^* \setminus Y_{(m, n)}^*$, and on $Y_m \setminus Y_m^*$. The last two summands are annihilated by $\pi_{(n, Y_n^*)} \circ f_{(m, n)}$. Consequently,

$$\pi_{(n, Y_n^*)} \circ f_{(m, n)}(\pi_{(m, Y_m^*)} \circ f_{(\omega, m)}(y)) = \pi_{(n, Y_n^*)} \circ f_{(\omega, n)}(y).$$

It follows that $\pi_{(n, Y_n^*)} \circ f_{(m, n)} \upharpoonright H_m$ maps H_m onto H_n , and hence $\text{rk}(H_n) \leq \text{rk}(H_m)$. Finally, these ranks are bounded by $\text{rk}(A) = k_* < \omega$, so they are eventually constant, as claimed.

(*₉) Assume $n_* \leq n_1 \leq n_2 < \omega$, then:

($\cdot_1$) if $\nu \in Y_{n_1}$, then there is $\rho \in Y_{n_2}$ such that $\nu \triangleleft \rho$;

($\cdot_2$) if $\nu \in Y_{n_1}^*$ and $\nu \triangleleft \rho \in Y_{n_2}$, then $\rho \in Y_{n_2}^*$ ($Y_{n_1}^*$ was defined in (*₇)).

[Why (*₉)($\cdot_1$)? By (*₆)(c).]

(*₁₀) Let $(y_\ell : \ell < k_*)$ be a list of X (recall (*₅)).

(*₁₁) For $\ell < k_*$, let $w_{(n,\ell)} = \pi_{(n,Y_n^*)} \circ f_{(\omega,n)}(y_\ell)$ and $W_n = \{w_{(n,\ell)} : \ell < k_*\}$.

(*₁₂) For every $n < \omega$ and $\ell < k_*$, let:

$$y_{(\ell,n)} = \pi_{(n,Y_n^*)} \circ f_{(\omega,n)}(y_\ell),$$

where $\pi_{(n,Y_n^*)} : G_n \rightarrow \langle Y_n^* \rangle_{G_n}^*$ is the obvious projection onto $\langle Y_n^* \rangle_{G_n}^*$ (as in (*₁)).

(*₁₃) Hence, we have:

$$y_{(\ell,n)} = \sum \{a_{(y_{\ell,n},\nu)} x_\nu : \nu \in Y_n^*\}.$$

(*₁₄) For $n \geq m \geq n_*$, let:

(a) $Y_{(n,m)}^* = \{\nu \in Y_n^* : \nu \restriction m \in Y_m^*\};$

(b) $y_{(\ell,n,m)} = \sum \{a_{(y_{\ell,n},\nu)} x_\nu : \nu \in Y_{(n,m)}^*\}.$

(*₁₅) For $n \geq m \geq n_*$, $y_{(\ell,n,m)} = \pi_{(n,Y_{(n,m)}^*)}(f_{(\omega,n)}(y_\ell)).$

(*₁₆) $k_* = k_*(= |X|)$ (recall (*₅) and (*₈)).

We prove (*₁₆). The proof of (*₁₆) will be rather long and will contain several sub-claims (*_{16,m}). Toward contradiction, assume that $k_* < k_*$ (notice that $k_* \leq k_*$). First of all, notice that:

(*_{16.1}) if $u \subseteq k_*$, $\ell \in k_* \setminus u$ and $n_1 < n_2$, then (a) implies (b), where (recall that $y_{(\ell,m)}$ was defined in (*₁₃)):

(a) $y_{(\ell,n_2)}$ depends on $\{y_{(i,n_2)} : i \in u\};$

(b) $y_{(\ell,n_1)}$ depends on $\{y_{(i,n_1)} : i \in u\}.$

(*_{16.2}) For some $n_\bullet \in (n_*, \omega)$ we have that for all $n_2 \geq n_1 \geq n_\bullet$ and $u \subseteq k_*$, $\ell \in k_* \setminus u$ we have (a) iff (b), where:

(a) $y_{(\ell,n_2)}$ depends on $\{y_{(i,n_2)} : i \in u\};$

(b) $y_{(\ell,n_1)}$ depends on $\{y_{(i,n_1)} : i \in u\}.$

Now, relying on (*_{16.2}), by the assumption toward contradiction, modulo renaming the elements y_ℓ , we have:

(*_{16.3}) for some $k \in [k_*, k_*)$ we have that for every $n \geq n_\bullet$ we have:

($\cdot$) the $\{y_{(\ell,n)} : \ell < k\}$ are independent in G_n ;

($\cdot$) $y_{(k,n)}$ depends on $\{y_{(\ell,n)} : \ell < k\}$ in G_n .

(*_{16.4}) For $n \geq n_\bullet$, let $\sum \{b_\ell^n y_{(\ell,n)} : \ell \leq k\} = 0$, where, for $\ell \leq k$, $b_\ell^n \in \mathbb{Z}$ and $b_k^n \neq 0$ (where k is of course the k fixed in (*_{16.3})).

(*_{16.5}) W.l.o.g. for every $n_2 \geq n_1 \geq n_\bullet$ we have that, for every $\ell \leq k$, $b_\ell^{n_2} = b_\ell^{n_1}$, call this number b_ℓ , so that in particular $b_k \neq 0$.

Why (*_{16.5})? For $n \geq n_\bullet$, $(b_\ell^n : \ell \leq k) \in \mathbb{Z}^{k+1}$ is not unique. However we can strengthen (*_{16.4}) demanding in addition the following:

($\cdot$) $b_k^n > 0$;

($\cdot$) there is no $a \in \omega \setminus \{0, 1\}$ such that a divides b_ℓ^n for all $\ell < k$;

($\cdot$) the greatest common divisor of $(b_\ell^n : \ell \leq k)$ is 1.

Now, under these restrictions $(b_\ell^n : \ell < k)$ is unique, since otherwise we contradict the fact that $(y_{(\ell,n)} : \ell < k)$ is independent. Finally, notice that if $n_\bullet \leq n_1 \leq n_2$, then $(b_\ell^{n_2} : \ell \leq k)$ can serve as $(b_\ell^{n_1} : \ell \leq k)$. Together we are done.

(*_{16.6}) In G we have that $z := \sum \{b_\ell y_\ell : \ell \leq k\} \neq 0$, so there is $n_+ > n_\bullet$ such that $f_{(\omega,n_+)}(z) \neq 0$, and thus we can find $\nu \in Y_{n_+}$ such that $a_{(z,n_+, \nu)} \neq 0$.

(*_{16.7}) $\nu \notin Y_{n_+}^*$.

[Why? By (*_{16.4}) and (*_{16.5}), $\pi_{(n_+, Y_{n_+}^*)} \circ f_{(\omega,n_+)}(z) = \sum \{b_\ell y_{(\ell,n_+)} : \ell \leq k\} = 0.]$

(*16.8) There are $n_{\dagger}$ and ρ such that:

- (a) $n_{\dagger} \geq n_+ \geq n_{\bullet} \geq n_*$;
- (b) $\rho \in Y_{n_{\dagger}}$ and $\nu \triangleleft \rho$ (where ν is as in (*16.6), (*16.7));
- (c) $\rho \in Y_{n_{\dagger}}^*$;
- (d) $a_{(z, n_{\dagger}, \rho)} \neq 0$.

Why does (*16.8) hold? We choose ν_n by induction on $n \geq n_+$ such that:

- (a) $\nu_{n_+} = \nu \in Y_{n_+}$;
- (b) $\nu_n \in Y_n$;
- (c) if $n = m + 1 > n_+$, then $\nu_m \triangleleft \nu_n$;
- (d) $a_{(z, n, \nu_n)} \neq 0$.

As in the beginning of the proof we can carry the induction. Now, as we are assuming that S has no ω -branch, clearly for some $n_{\dagger} > n_+$ we have that $\nu_{n_{\dagger}} \notin S$, but then necessarily $L(\nu_{n_{\dagger}})$ has to be finite, i.e., $\nu_{n_{\dagger}} \in Y_{n_{\dagger}}^*$, since by assumption S is a subtree of T and we defined, for $\eta \in T$, $L(\eta) = \{p_{\nu} : (\nu \triangleleft \eta \text{ or } \eta \trianglelefteq \nu) \text{ and } \nu \in S^+\}$.

$$(*16.9) \quad \pi_{(n_{\dagger}, Y_{n_{\dagger}}^*)} \circ f_{(\omega, n_{\dagger})}(z) \neq 0.$$

[Why? By (*16.8)(d).]

$$(*16.10) \quad \pi_{(n_{\dagger}, Y_{n_{\dagger}}^*)} \circ f_{(\omega, n_{\dagger})}(z) = 0.$$

[Why? Notice that:

$$\begin{aligned} \pi_{(n_{\dagger}, Y_{n_{\dagger}}^*)} \circ f_{(\omega, n_{\dagger})}(z) &= \pi_{(n_{\dagger}, Y_{n_{\dagger}}^*)} \circ f_{(\omega, n_{\dagger})}(\sum \{b_{\ell} y_{\ell} : \ell \leq k\}) \\ &= \sum \{b_{\ell} y_{(\ell, n_{\dagger})} : \ell \leq k\} \\ &= \sum \{b_{\ell}^{n_{\dagger}} y_{(\ell, n_{\dagger})} : \ell \leq k\} \\ &= 0 \end{aligned}$$

where the first equality is by the definition of z preceding (*16.6); the second equality is by (*12), the third equality is by (*16.5) and the fourth equality is by (*16.4).]

But (*16.10) contradicts (*16.9). Hence, (*16) holds indeed.

- (*17) (a) Let $f_{(\omega, n)}^* = \pi_{(n, Y_n^*)} \circ f_{(\omega, n)}$;
- (b) $f_{(\omega, n)}^*$ is a homomorphism from G into $\langle Y_n^* \rangle_{G_n}^* \leq G_n$.

(*18) There is $N \geq n_*$ such that, for every $n \geq N$, the map $h_n := f_{(\omega, n)}^* \upharpoonright \langle X \rangle_G^*$ is an isomorphism from $\langle X \rangle_G^*$ onto $\langle W_n \rangle_{G_n}^*$ (recall: W_n was defined in (*11)).

Why (*18)? For $n \geq n_*$, let:

- (*18.1) (.) $A = \langle X \rangle_G^*$;
- (.) $B_n = \langle W_n \rangle_{G_n}^*$;
- (.) $H_n = h_n(A)$;
- (.) $F_n = \langle Y_n^* \rangle_{G_n}^*$.

Recall from Definition 3.13 that, for every $\nu \in T$, K_{ν} is the additive subgroup of $\mathbb{Q}$ generated by $\{1/p : p \in L(\nu)\}$. By (*7), $L(\nu)$ is finite for $\nu \in Y_n^*$, and then $K_{\nu} = d_{\nu}^{-1}\mathbb{Z}$, where $d_{\nu} = \prod_{p \in L(\nu)} p$. Thus $F_n = \bigoplus_{\nu \in Y_n^*} K_{\nu} x_{\nu}$ is free abelian of finite rank. Since $W_n \subseteq F_n$ and F_n is pure in G_n , also $B_n \leq F_n$, so B_n is free abelian of finite rank. By (*8) and (*16), $\text{rk}(H_n) = \text{rk}(A)$; hence h_n is injective, since its kernel is torsion-free of rank zero. Moreover, $W_n \subseteq H_n \leq B_n$, and since $B_n = \langle W_n \rangle_{G_n}^*$, we have $\text{rk}(H_n) = \text{rk}(B_n)$. Consequently, B_n/H_n is a finitely generated torsion group, hence finite. In particular, if H_n is pure in B_n , this quotient is also torsion-free and must be zero, giving $H_n = B_n$. Since $h_n : A \rightarrow H_n$ is already an isomorphism

for $n \geq n_*$, to establish $(*_{18})$ it remains to prove that $H_n = B_n$ for all sufficiently large n ; this is the aim of the rest of its proof.

Recall $Q_m = \bigoplus_{\nu \in T \cap \omega^m} \mathbb{Q}x_\nu$ from Definition 3.2. Recall now that we have:

$$G = \{(u_m : m < \omega) \in \prod_{m < \omega} G_m : f_{(m,n)}(u_m) = u_n \text{ for all } n \leq m < \omega\} \leq \prod_{m < \omega} Q_m.$$

Thus $A \leq G$ gives the following embedding:

$$\iota : A \longrightarrow \prod_{m < \omega} Q_m, \quad a \longmapsto (f_{(\omega,m)}(a) : m < \omega).$$

Using the inverse-limit presentation of G , we can identify A with $\iota(A)$ and put

$$V = \{\iota(a)/d : a \in A, 0 < d \in \mathbb{Z}\} = \text{span}_{\mathbb{Q}}(\iota(A)) \leq \prod_{m < \omega} Q_m.$$

This realizes the rational hull $\mathbb{Q} \otimes_{\mathbb{Z}} A$: its natural map to the displayed subspace sends $d^{-1} \otimes a$ to $\iota(a)/d$ and is injective, since $\iota(a)/d = 0$ implies $a = 0$. Under this identification, writing $v = a/d$ means that $v = (v_m : m < \omega)$ and $v_m = d^{-1}f_{(\omega,m)}(a)$.

For $n \leq m < \omega$, let $\hat{f}_{(m,n)} : Q_m \rightarrow Q_n$ be the $\mathbb{Q}$ -linear map defined by $\hat{f}_{(m,n)}(x_\nu) = x_{\nu \upharpoonright n}$ for $\nu \in T \cap \omega^m$. It extends the bonding map $f_{(m,n)}$ defining G . Hence, for $v = a/d \in V$, we have that $\hat{f}_{(m,n)}(v_m) = d^{-1}f_{(m,n)}(f_{(\omega,m)}(a)) = d^{-1}f_{(\omega,n)}(a) = v_n$. These are the compatibility relations satisfied by the components of v . For $n \geq n_*$, extend h_n by $\tilde{h}_n(a/d) = h_n(a)/d$ and put $C_n = \{v \in V : \tilde{h}_n(v) \in B_n\}$.

The map $\tilde{h}_n$ is injective and maps V onto the rational span of B_n . Indeed, since $A = \langle X \rangle_G^*$, X spans V over $\mathbb{Q}$, and $h_n(X) = W_n$ by $(*_{11})$ and $(*_{17})$. Hence $\tilde{h}_n(V) = \text{span}_{\mathbb{Q}}(W_n) = \text{span}_{\mathbb{Q}}(B_n)$, where the last equality follows from $B_n = \langle W_n \rangle_{G_n}^*$. Its restriction is therefore an isomorphism $C_n \rightarrow B_n$ taking A onto H_n , and induces an isomorphism $C_n/A \cong B_n/H_n$. We have already shown that B_n/H_n is finite, so C_n/A is finite. If $m \geq n \geq n_*$ and $v \in C_m$, the identity from $(*_8)$, extended by $\mathbb{Q}$ -linearity, gives $\tilde{h}_n(v) = \pi_{(n,Y_n^*)} \circ f_{(m,n)}(\tilde{h}_m(v)) \in G_n \cap \text{span}_{\mathbb{Q}}(W_n) = B_n$. Here the right-hand side lies in G_n because $\tilde{h}_m(v) \in B_m \leq G_m$, so $A \leq C_m \leq C_n$.

$$(*_{18.2}) \quad \bigcap_{n \geq n_*} C_n = A.$$

Why $(*_{18.2})$? Since $A \leq C_n$ for every $n \geq n_*$, it suffices to prove the reverse inclusion. Fix v in this intersection, and choose $a \in A$ and $0 < d \in \mathbb{Z}$ such that $v = a/d$. For every $m < \omega$, $(*_6)$ applied to $a \in A$ gives

$$f_{(\omega,m)}(a) \in \bigoplus_{\nu \in Y_m} K_\nu x_\nu.$$

Dividing by d shows that v_m belongs to the rational span of $\{x_\nu : \nu \in Y_m\}$. Since the x_ν 's are a basis of Q_m , there are unique coefficients

$$v_m = \sum_{\nu \in Y_m} c_{(m,\nu)} x_\nu, \quad c_{(m,\nu)} \in \mathbb{Q}.$$

Put $c_{(m,\nu)} = 0$ for $\nu \in (T \cap \omega^m) \setminus Y_m$. The relation $\hat{f}_{(m+1,m)}(v_{m+1}) = v_m$ reads

$$\sum_{\rho \in Y_{m+1}} c_{(m+1,\rho)} x_{\rho \upharpoonright m} = \sum_{\nu \in Y_m} c_{(m,\nu)} x_\nu.$$

Comparing coefficients of x_ν , for each $\nu \in T \cap \omega^m$, gives

$$(\dagger) \quad c_{(m,\nu)} = \sum \{c_{(m+1,\rho)} : \rho \in Y_{m+1}, \rho \restriction m = \nu\}.$$

The sum is finite because Y_{m+1} is finite. Suppose $v_m \notin G_m$ for some $m < \omega$. Since $G_m = \bigoplus_{\nu \in T \cap \omega^m} K_\nu x_\nu$, this means that $c_{(m,\nu)} \notin K_\nu$ for some $\nu \in Y_m$. If every coefficient $c_{(m+1,\rho)}$ in the sum in $(\dagger)$ belonged to K_ρ , then $K_\rho \subseteq K_\nu$ would imply $c_{(m,\nu)} \in K_\nu$, a contradiction. Hence some immediate successor $\rho \in Y_{m+1}$ satisfies $c_{(m+1,\rho)} \notin K_\rho$. Iterating gives a chain $(\nu_k : m \leq k < \omega)$ with $\nu_k \in Y_k$, $\nu_k \triangleleft \nu_{k+1}$ and $c_{(k,\nu_k)} \notin K_{\nu_k}$ at every level.

For $n_\star \leq k < \omega$, the definition in $(\ast_{17})$ and $\tilde{h}_k(a/d) = h_k(a)/d$ give

$$\tilde{h}_k(v) = d^{-1} \pi_{(k,Y_k^\star)}(f_{(\omega,k)}(a)) = \sum_{\nu \in Y_k^\star} c_{(k,\nu)} x_\nu.$$

Indeed, $f_{(\omega,k)}(a) = dv_k$, and $\pi_{(k,Y_k^\star)}$ retains exactly the terms indexed by $Y_k^\star$. Moreover, $v \in C_k$ implies $\tilde{h}_k(v) \in B_k$, and we already proved $B_k \leq F_k$. As $F_k = \bigoplus_{\nu \in Y_k^\star} K_\nu x_\nu$, uniqueness of coefficients now gives $c_{(k,\nu)} \in K_\nu$ for every $\nu \in Y_k^\star$. Consequently $\nu_k \notin Y_k^\star$, and thus $L(\nu_k)$ is infinite, whenever $k \geq \max\{m, n_\star\}$. By Definition 3.9, nodes with infinite label sets belong to S , so this chain would yield an ω -branch of S , a contradiction. It follows that $v_m \in G_m$ for every $m < \omega$. Since $\hat{f}_{(m,n)}$ restricts to $f_{(m,n)}$ on G_m , the compatibility relations above now give $f_{(m,n)}(v_m) = v_n$ for all $n \leq m < \omega$. By the displayed description of G , this proves $v \in G$. For our fixed a and d , we have $dv = a \in A$. Purity of A in G yields $b \in A$ with $db = a = dv$; torsion-freeness of G then gives $v = b \in A$. This proves $(\ast_{18.2})$.

The sequence $(C_n/A : n_\star \leq n < \omega)$ is a descending chain of subgroups of the finite group $C_{n_\star}/A$. There can be only finitely many strict inclusions, so the chain eventually stabilizes. Its intersection is zero by $(\ast_{18.2})$, hence its stable value is zero. Therefore there is $N \geq n_\star$ such that $C_n = A$, and thus $H_n = B_n$, for all $n \geq N$. This proves $(\ast_{18})$.

$(\ast_{19})$ The conditions from $(\ast_1)$ hold, and so the proof is complete.

Why $(\ast_{19})$? Let N be as in $(\ast_{18})$, fix $N \leq n < \omega$ and take $Y = Y_n^\star$ and $B = B_n = \langle W_n \rangle_{G_n}^\star$. By $(\ast_6)$ and $(\ast_7)$, Y is a finite subset of $T \cap \omega^n$ and $L(\nu)$ is finite for every $\nu \in Y$, verifying clause (a) of $(\ast_1)$. Moreover, using the notation introduced in the proof of $(\ast_{18})$, $G_{(n,Y)} = F_n$. Since $B_n = \langle W_n \rangle_{G_n}^\star$ is pure in F_n , the finitely generated group F_n/B_n is torsion-free, hence free abelian. Lifting a basis of F_n/B_n to F_n gives a complement to B_n , and therefore a retraction $r_n : F_n \rightarrow B_n$. Let

$$g = r_n \circ \pi_{(n,Y_n^\star)} \circ f_{(\omega,n)}^\star = r_n \circ f_{(\omega,n)}^\star \in \text{Hom}(G, B_n).$$

This verifies clauses (b) and (c) of $(\ast_1)$. Furthermore, for $a \in A = \langle X \rangle_G^\star$, we have

$$g(a) = r_n(h_n(a)) = h_n(a),$$

where h_n is as in $(\ast_{18})$, this is because $h_n(a) \in B_n$ and r_n is the identity on B_n . Thus $g_0 = g \restriction A = h_n$, which is an isomorphism onto B_n by $(\ast_{18})$, verifying clause (d) of $(\ast_1)$. All the conditions of $(\ast_1)$ are therefore fulfilled and the proof is complete. $\blacksquare$

4. PROOF OF MAIN THEOREMS

In this section we show how to use our machinery to prove Theorems 1.2 and 1.3.

4.1. Proof of Theorem 1.2

Definition 4.1. Let S be the following well-founded tree. For $n < \omega$, let T_n be a copy of $\omega^{<n}$ such that $T_n \cap T_m = \emptyset$ whenever $n < m < \omega$. Let S be the tree obtained by adding a minimum x_* to $\bigcup\{T_n : n < \omega\}$, so x_* is the root of S . The tree S can be realized as a subtree of $\omega^{<\omega}$ satisfying condition 3.9(b), and it is well-founded. Fix an enumeration of S of order type ω , say $(\eta_i : i < \omega)$. Let $\mathbb{P}_*$ be an infinite set of primes, enumerate it in increasing order as $(p_i^{\mathbb{P}_*} : i < \omega)$, and let $p_{\eta_i} = p_i^{\mathbb{P}_*}$. Then S and $\bar{p} = (p_{\eta_i} : i < \omega)$ are as in Definition 3.9. Without loss of generality, we assume that if η is not the root of the tree, then $\eta(0) = n$ if and only if $\eta \in T_n$. Finally, let $T = \omega^{<\omega}$ and let $L_{\mathbb{P}_*}$ be the labeling induced by S and $\bar{p}$ according to Definition 3.9.

Observation 4.2. By 3.14, for $(\omega^{<\omega}, L_{\mathbb{P}_*})$ as in 4.1, $G(\omega^{<\omega}, L_{\mathbb{P}_*})$ is $\aleph_1$ -free.

Lemma 4.3. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be infinite sets of primes such that $\mathbb{P}_1 \cap \mathbb{P}_2$ is finite, and let $(\omega^{<\omega}, L_{\mathbb{P}_\ell})$ be as in 4.1 and $G_\ell = G(\omega^{<\omega}, L_{\mathbb{P}_\ell})$ (recall 3.2). Then there is no topological isomorphism from G_1 onto G_2 .

Proof. Let $A = G_1 = \varprojlim (G_{(1,n)})$ and $B = G_2 = \varprojlim (G_{(2,n)})$, where the $G_{(\ell,n)}$'s belong to the inverse system $\text{inv}(\omega^{<\omega}, L_{\mathbb{P}_\ell})$. Let $G_{(1,n)} = A_n$ and $G_{(2,n)} = B_n$. Also, for $n \leq m < \omega$, denote the bonding map from A_n onto A_m by $t_{(m,n)}$, and the bonding map from B_n onto B_m by $s_{(m,n)}$. Suppose that there is a topological isomorphism from A onto B . Then there are ϕ, ψ, f_n and g_n , for $n < \omega$, such that:

- (a) $\phi : \omega \rightarrow \omega$ is increasing;
- (b) $\psi : \omega \rightarrow \omega$ is increasing;
- (c) $f_n : A_{\phi(n)} \rightarrow B_n$;
- (d) $g_n : B_{\psi(n)} \rightarrow A_n$;
- (e) for every $n \leq k < \omega$, the following diagram is commutative:

$$\begin{array}{ccc} A_{\phi(k)} & \xrightarrow{f_k} & B_k \\ \downarrow t_{(\phi(k), \phi(n))} & & \downarrow s_{(k,n)} \\ A_{\phi(n)} & \xrightarrow{f_n} & B_n \end{array}$$

- (f) for every $n \leq k < \omega$, the following diagram is commutative:

$$\begin{array}{ccc} B_{\psi(k)} & \xrightarrow{g_k} & A_k \\ \downarrow s_{(\psi(k), \psi(n))} & & \downarrow t_{(k,n)} \\ B_{\psi(n)} & \xrightarrow{g_n} & A_n \end{array}$$

- (g) for every $n < \omega$, $f_n \circ g_{\phi(n)} : B_{\psi(\phi(n))} \rightarrow B_n$ is equal to $s_{(\psi(\phi(n)), n)}$;
- (h) for every $n < \omega$, $g_n \circ f_{\psi(n)} : A_{\phi(\psi(n))} \rightarrow A_n$ is equal to $t_{(\phi(\psi(n)), n)}$.
- (*) If $\eta \in \omega^{\phi(k)+1} \cap S$ is not maximal in S , then $f_k(x_\eta) = 0_{B_k}$.

[Why (*)? By our choice η is not maximal in S , hence we have that:

$$A_{\phi(k)} \models "x_\eta \text{ is divisible by infinitely many primes in } \mathbb{P}_1",$$

but since $\mathbb{P}_1 \cap \mathbb{P}_2$ is finite, necessarily $f_k(x_\eta) = 0_{B_k}$, by 3.7.]

Let $n = \phi(\psi(0))$ and let $\eta \in \omega^{n+1}$ be such that η is not maximal in the tree S . Then by (*) we have that $f_{\psi(n)}(x_\eta) = 0_{B_{\psi(n)}}$ and so necessarily $g_0 \circ f_{\psi(0)}(x_\eta) = 0_{A_0}$, but

this contradicts item (h) above, in fact, since $g_n \circ f_{\psi(n)} : A_{\phi(\psi(0))} \rightarrow A_0$ is equal to $t_{(\phi(\psi(0)),0)}$ we also have that $g_0 \circ f_{\psi(0)}(x_\eta) = x_\emptyset$ and $x_\emptyset \neq 0_{A_0}$, a contradiction. ■

Recall that an abelian group A is said to be slender if every homomorphism $\phi : \mathbb{Z}^\omega \rightarrow A$ satisfies $\phi(e_n) = 0$ for all but finitely many n , where e_n is the n -th canonical unit vector (cf. [8, p. 489]).

Lemma 4.4. *Suppose $\{\phi_{(n,m)} : A_n \rightarrow A_m : m < n < \omega\}$ is an inverse system in which each A_n is a countable slender abelian group, and let $\psi : G \rightarrow A = \varprojlim A_n$ be an abstract homomorphism, where G is a completely metrizable topological group. Then ψ is continuous.*

Proof. Letting $\phi_m : A \rightarrow A_m$ denote the natural projection, we have that $\phi_m \circ \psi : G \rightarrow A_m$ is a homomorphism from a completely metrizable group to a countable slender abelian group, and so by Theorem C of [4] we know that the kernel of $\phi_m \circ \psi$ is open. Since the topology of A is determined by the kernels $\ker(\phi_m)$, we see that ψ is continuous. ■

Proof of 1.2. Take a collection $\{\mathbb{P}_\alpha : \alpha < 2^{\aleph_0}\}$ of almost disjoint infinite subsets of the set of prime numbers, where A and B are almost disjoint if $A \cap B$ is finite. The result then follows from 4.3 and 4.4: each group used in the inverse system defining $G(\omega^{<\omega}, L_{\mathbb{P}_\alpha})$ is countable, abelian, torsion-free and reduced, and hence slender by [15]. ■

4.2. Proof of Theorem 1.3

Definition 4.5. Let $A \subseteq \omega^{<\omega}$ be an infinite tree (in this case, crucially, possibly with leaves). For $\eta \in A$, let $\eta^{+1} = (\eta(n) + 1 : n < \omega)$ and let $A^{+1} = S = \{\eta^{+1} : \eta \in A\}$. Let $\bar{p} = \{p_\eta : \eta \in S\}$ and $T = \omega^{<\omega}$. Let $G(\omega^{<\omega}, L_A)$ be defined as in 3.9 for this choice of S and $\bar{p}$.

Lemma 4.6. *For $(\omega^{<\omega}, L_A)$ as in 4.5, letting $G_A = G(\omega^{<\omega}, L_A)$ we have:*

- (1) *if A is well-founded, then G_A is separable, and hence torsionless, $\aleph_1$ -free and $\mathbb{Z}$ -homogeneous;*
- (2) *if A is not well-founded, then G_A is not $\mathbb{Z}$ -homogeneous, and hence is neither $\aleph_1$ -free, nor torsionless, nor separable.*

Proof. This is by 3.12 and 3.14. ■

Proof of 1.3. Recall first that, as is well known, well-founded trees form a complete co-analytic set. The result follows from 2.13 and 4.6, together with the observation that the map $A \mapsto \text{inv}(\omega^{<\omega}, L_A)$ is Borel when considering models whose domain is contained in ω . ■

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