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HALES-JEWETT NUMBERS REVISITED
E117

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Abstract. Hales-Jewett theorem is well known. We prove a primitive recursive bound for the density version and more.

In particular we deal with what was the main induction step in the old proof of Hales-Jewett as a result in its own right.

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References like (e.g.) [Sh:950, Th.0.2$_{=\mathsf{Ly5}}$] mean that y5 is the internal label of Theorem 0.2 in the TeXfile of [Sh:950]. The reader should note that the version in my website is usually more up-to-date than the one in arXiv. This is publication number E117 (formerly 1000) in Saharon Shelah's list.

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Annotated Content

§0 Introduction p.3

§1 The density version, the minimal case (label g) p.6

We deal with what was the main induction step in [She:329].

In the present phrasing, we have a sequence of subsets of some $A \neq \emptyset$,

$$\mathbf{a} = \langle A_\eta : \eta \in {}^M\Lambda \rangle$$

each of density $\geq \varepsilon > 0$.

The exposition is self-contained. We rely on 2.12(6) on one point.

§2 More on §1 p.13

§2(A) An induction step as a result on its own right p.13

We elaborate the results of §1. By §3 for some version there is a one exponentiation one (to be fully written).

§2(B) The average version p.15

We note that the average version is not much bigger than the minimal one. Finishing the proof in §1 and the standard variant of density HJ.

§ Better bounds in preparation

We get bounds in $\mathscr{E}^4$ for $\mathsf{HJ}(\Lambda, r, C)$ and $\mathsf{HJ}_\varepsilon(\Lambda, r)$.

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§ 0. Introduction

Here we get a primitive recursive, even $\mathcal{E}^5$, bound for $\mathsf{HJ}_\varepsilon(\Lambda, r)$, Hales-Jewett with density, as a function of $\lfloor \frac{1}{\varepsilon} \rfloor + |\Lambda| + r$.

The history of this problem is well-known; for the early parts, see the book [GraRotSpe90]. (For more recent history, see [She:679] or Wikipedia.) Here in §1 we get a bound in $\mathcal{E}^5$.

The Density Hales–Jewett Theorem, stating that any dense subset of the grid $[\ell]^n$ contains a combinatorial line, was originally proven via ergodic theory by Furstenberg and Katznelson [FurKat91]. A landmark quantitative elementary combinatorial proof was discovered (considerably later) by Polymath 1 [Pol12], a massively collaborative project. Their proof gives bounds in terms of a version of the Ackermann function. Here we give a primitive recursive bound; in fact in $\mathscr{E}^5$; it is two levels above exponentiation in the Grzegorczyk hierarchy.

Recent work in [GolMirShe:1182], among other related results, connects van der Waerden's Theorem and Hales-Jewett.

Nowadays I will surely be asked how much AI helped me:

(a) in the LaTeX typing, it helped translating my handwriting to LaTeX;
(b) it helped in proofreading (pointing out errors, inconsistencies etc.);
(c) we shall discuss §3 when making it public.

Recall the Hales-Jewett Theorem [HalJew63].

Theorem 0.1 (**Hales–Jewett Theorem**). *For all natural numbers $n, p \geq 2$ there exists an integer $\mathsf{HJ}(n,p)$ such that for all linear orders M of cardinality $\geq \mathsf{HJ}(n,p)$, Λ of cardinality n, C of cardinality p, and $\mathbf{c} : {}^M\Lambda \to C$, there exists a combinatorial line (see below) on which $\mathbf{c}$ is constant.*

(Think of $\mathbf{c}$ as a coloring: Λ is our alphabet, and C the set of colors.)

Definition 0.2. 1) A *combinatorial line* in ${}^M\Lambda$ is a set of the form

$$\{\rho \in {}^M\Lambda : \rho \supseteq \eta \text{ and } \rho \restriction u \text{ is constant}\},$$

for some non-empty $u \subseteq M$ and $\eta \in {}^{M\setminus u}\Lambda$.

2) Generalizing this, we say Γ is a *subspace* of ${}^M\Lambda$ of dimension ℓ (or ℓ-subspace) when it is of the form

$$\big\{\rho \in {}^M\Lambda : \rho \supseteq \eta, \text{ and } m < \ell \Rightarrow \rho \restriction u_m \text{ is constant}\big\}$$

for some sequence $\langle u_m : m < \ell \rangle$ of pairwise disjoint non-empty subsets of M and $\eta : M \setminus \bigcup\limits_{m<\ell} u_m \to \Lambda$.

3) Let $\mathsf{HJ}(n,j,p)$ be defined as in 0.1, but for a monochromatic j-subspace (so $\mathsf{HJ}(n,1,p) := \mathsf{HJ}(n,p)$).

The following statement reduces to the van der Waerden Theorem for $m = 1$. For $m > 1$ this is the Gallai-Witt Theorem = the multidimensional van der Waerden Theorem.

Theorem 0.3. *Let $n \geq 1$ and $m, p \geq 2$. Then there exists a number k satisfying the following:*

$(*)^{k,m}_{n,p}$ *If $\langle A_\ell : \ell < p \rangle$ is a partition of ${}^nk$, then there exist $\bar{a} \in {}^nk$, $b_0, \dots b_{n-1} \in k \setminus \{0\}$, and $\ell^* < p$ such that*

$$\big\{\bar{a} + \big\langle \eta(i) b_i : i < n \big\rangle : \eta \in {}^nm\big\} \subseteq A_{\ell^*}.$$

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(This can be deduced from $\mathsf{HJ}(n,k,p)$; see [GraRotSpe90]. The proof applies to the density version as well, so our bounds follow.)

Definition 0.4. Assume $\varepsilon \in (0,1]$. Let $\mathsf{HJ}_\varepsilon(\Lambda, r)$ be the minimal natural number k satisfying the following implication.

> If $X \subseteq {}^k\Lambda$ has density $\geq \varepsilon$ (i.e. $|X| / |{}^k\Lambda| \geq \varepsilon$) then X contains an r-dimensional subspace of ${}^k\Lambda$.

Notation 0.5. 1) Here $i, j, k, \ell, m, n, p, r, s$ will denote natural numbers; equivalently, finite ordinals. (So n is identified with $\{i : i < n\}$.)

2) Λ is a finite alphabet (mostly fixed). α, β, γ will denote members of Λ, usually fixed.

3) ε, ζ, ξ and $\varkappa$, w, W will be real numbers in the interval $(0,1]_{\mathbb{R}}$.

4) $\mathbf{s}$, $\mathbf{t}$, and $\mathbf{u}$ will denote systems (see Definition 1.5). $\mathfrak{s}$, $\mathfrak{t}$, $\mathfrak{u}$ will denote pre-systems.

5) $\exp_m(n) = \exp(m,n) := m^n$ (We will use this to avoid deeply nested superscripts.)

5A) To this end, we let $\exp_m^0(n) := n$ and $\exp_m^{k+1}(n) := \exp_m(\exp_m^k(n))$.

6) ${}^AB$ is the set of functions from A to B.

7) m^+ denotes the successor of m (so it is shorthand for $m+1$, as these are natural numbers).

8) We shall use $\{k,\ell\}$ with $k < \ell$ to denote a member of $[m]^2 := \{A \subseteq m : |A| = 2\}$. We will endeavor to make sure that when these two letters are used together it is only in this context.

9) If we write (e.g.) $\mathsf{HJ}(\Lambda, C)$, we of course mean $\mathsf{HJ}\big(|\Lambda|, |C|\big)$, and if in the definition there is no natural number satisfying the demand then it means ∞.

10) M is a finite linear order and for $a \in M \cup \{\infty\}$ let $M_{<a} = \{b \in M : b <_M a\}$.

11) Let $M = M_1 \dot\cup M_2$ mean: $M_1 \cap M_2 = \emptyset$, $(a \in M \Leftrightarrow a \in M_1 \vee a \in M_2)$ and $<_M \restriction M_\iota = <_{M_\iota}$ for $\iota = 1, 2$.

§ 0(A). **Explanation.** Here, on the one hand, we give new bounds, and on the other hand we give some bounds to some relatives; usually they are inspired by the main inductive step in the proof of [She:329, §1].

We shall first try to explain the earlier proofs, and only then the present one.

First, the earlier proofs.

The proof of the Hales-Jewett Theorem [HalJew63] uses double induction: first on n (the cardinality of the alphabet Λ), and then on p (the number of colors). Eventually they get an upper bound in terms of the Ackermann function, which is not primitive recursive.

The base idea of [She:329] is that we fix the number of colors $|C|$ and perform our induction only on $n := |\Lambda|$.

Let us try to describe the inductive step in the proof (that is, from n to $n+1$ letters). So we let $r := \mathsf{HJ}(n,p)$ and fix Λ of cardinality $n+1$, C of cardinality p, and $\alpha \neq \beta \in \Lambda$. We look for a finite linear order M which is large enough for our purposes (i.e. $|M| \gg r$) and consider a coloring $\mathbf{c} : {}^M\Lambda \to C$.

Let $\overline{N} = \langle N_i : i < r\rangle$ be a partition of M into intervals; the intention is to use each N_i as a place for one letter. That is, we will try to find $\xi = \big\langle (k_i, \ell_i, N_i) : i < r\big\rangle$

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with $k_i < \ell_i$ both from[1] N_i, and restricting ourselves to functions $\eta \in {}^{M}\Lambda$ of the form

$$i < r \wedge j \in N_i \ \Rightarrow\ \eta(j) := \begin{cases} \alpha & \text{if } j < k_i \\ \gamma_i & \text{if } j \in [k_i, \ell_i) \\ \beta & \text{if } j \geq \ell_i. \end{cases}$$

This was done such that the coloring does not change if we replace some of the α-s by β-s (or vice versa): we reduce this case of the problem to one with one letter less, hence can use the induction hypothesis, that is, the choice of r gives the desired conclusion.

So k_i and ℓ_i were chosen by downward induction on $i < r$, and naturally we required $|N_i| > |C|^{\sum_{j<i}|N_j|}$, so for our lemma we had a tetration function as a bound. Hence for HJ we had gotten a bound in $\mathcal{E}^5$.

Here we have three aims concerning $\mathsf{HJ}(\Lambda, C)$ (not disjoint):

(A) improve the bound,

(B) deal with the density versions,

(C) get good bounds for relatives, mainly for the inductive step of [She:329].

Sometimes the bounds we get for a claim are superseded if we consider only applying them to $\mathsf{HJ}_\varepsilon(\Lambda, r)$, but not so if we consider the proof and the specific results it gives on the way.

Second, toward the new.

Here we intend to improve the bound in the main inductive step, treating it as a partition theorem in its own right. A difference with [She:329] is that we choose the middle road — not fixing the number of colors, that is, in the density, but carefully controlling how quickly they are allowed to proliferate.

We choose the cardinality of N_i and ε_i by downward induction on $i < r$.

The problem is how the adding of one place (for one letter) behaves nicely. Originally we were sure that during the induction the density ε decreases, but this is not the case.

[1] Actually, we allow ℓ_i to be above all elements of N_i.

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§ 1. The density version, the minimal case

In this section we show that the main induction step in the proof of the bound on $\mathsf{HJ}(\Lambda, r)$ works for a density version, but to carry the induction we need 2.12(6), which relies on Definition 2.7 and 2.12(2).

Notation 1.1. 1) For A a finite non-empty set, let $\mu_A(-)$ be the A-*density* function. I.e. $\mu_A(B) := |B|/|A|$ for $B \subseteq A$.

(We may write μ instead of μ_A when A is clear from the context.)

2) Let $A^{[\geq \varepsilon]} := \{B \subseteq A : \mu_A(B) \geq \varepsilon\}$.

Recalling the Corrádi lemma:

Fact 1.2. 1) We have '(A) $\Rightarrow$ (B),' where

(A) (a) $n \geq 2$ and $\varepsilon \in \left[\frac{1}{n}, 1\right]_{\mathbb{R}}$.

(b) A is a finite non-empty set and $\mu = \mu_A$ is as in 1.1.

(c) For $i < n$ we have $A_i \subseteq A$ with $\mu(A_i) \geq \varepsilon$.

(d) $0 < \zeta \leq \dfrac{\varepsilon(\varepsilon n - 1)}{n-1}$

(B) There exist $k < \ell < n$ such that $\mu(A_k \cap A_\ell) \geq \zeta$.

2) In 1.2(1) we can weaken clause (c) to $\dfrac{1}{n}\Sigma_{k<n}\mu(A_k) \geq \varepsilon$

Proof. 1) This is the Corrádi Lemma [Cor69a], [Cor69b].

2) A variant of Corrádi lemma, see [Juk11, Chapter 2, Lemma 2.2]. $\square_{1.2}$

Observation 1.3. *1) If $n \geq 1 + \lceil \frac{1}{\varepsilon} \rceil$ and non-zero $\zeta \leq \varepsilon^3$ then n, ε and ζ will satisfy the conditions of* 1.2(1)(A)(a),(d).

2) If $k \geq 1$, $n \geq 1 + \left\lceil \dfrac{1}{\varepsilon^{k+1}} \right\rceil$ and $0 < \zeta \leq \varepsilon^2\big(1 - \varepsilon^k\big)$, then the same conclusion holds.

Proof. 1) Obviously clause 1.2(1)(A)(a) holds. For clause (A)(d), recall that

$$\frac{\varepsilon(\varepsilon n - 1)}{n-1} = \varepsilon \cdot \frac{\varepsilon(n-1) - (1-\varepsilon)}{n-1} = \varepsilon^2 - \frac{(1-\varepsilon)\varepsilon}{n-1}.$$

The last expression is increasing in n (even for real-valued n), hence it is

$$\geq \varepsilon^2 - \frac{(1-\varepsilon)\varepsilon}{(1+\frac{1}{\varepsilon})-1} = \varepsilon^2 - (1-\varepsilon)\varepsilon^2 = \varepsilon^3,$$

so '$\zeta \leq \varepsilon^3$' does indeed suffice.

2) Similarly. $\square_{1.3}$

* * *

Hypothesis 1.4. Λ will denote a finite non-empty alphabet, of cardinality $\mathbf{n}$ if not stated otherwise.

Definition 1.5. 1) We say $\mathfrak{s}$ is a *pre-system* when it consists of

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(a) $M = M_{\mathfrak{s}}$, a finite non-empty linear order.

(b) $E = E_{\mathfrak{s}}$, an equivalence relation on $V_{\mathfrak{s}} := {}^{M}\Lambda$.

1A) In this paper, the default case is that

$$E_{\mathfrak{s}} = E^{\alpha,\beta}_{\mathcal{U}} := \big\{(\eta,\nu) \in V_{\mathfrak{s}} \times V_{\mathfrak{s}} : \big[a \in M \setminus \mathcal{U} \Rightarrow \eta(a) = \nu(a)\big] \text{ and } \big[\eta(a) \neq \nu(a) \Rightarrow \{\eta(a),\nu(a)\} \subseteq \{\alpha,\beta\}\big]\big\}$$

for some $\mathcal{U} \subseteq M_{\mathfrak{s}}$ and $\alpha \neq \beta \in \Lambda$; $\alpha = \alpha_{\mathfrak{s}}$, $\beta = \beta_{\mathfrak{s}}$.

We shall treat α and β as fixed, if not clear otherwise; and when we do not fix them we write $\mathfrak{s} \in \mathfrak{S}_{\alpha,\beta}$, $\mathbf{s} \in \mathbf{S}_{\alpha,\beta}$, whereas $\mathfrak{s} \in \mathfrak{S}$, $\mathbf{s} \in \mathbf{S}$, and in particular we do not fix them in 1.13(3) and some parts of §2B and of §3. Consequently, choosing $\mathcal{U} = \mathcal{U}_{\mathfrak{s}}$ will be equivalent to defining $E_{\mathfrak{s}}$, and we may write $\alpha_{\mathfrak{s}}, \beta_{\mathfrak{s}}$ in this case.

1B) Let $\mathfrak{s} = (M, =)$ mean $\mathfrak{s} = (M, E_{\mathfrak{s}})$, $E_{\mathfrak{s}}$ the equality on $V_{\mathfrak{s}}$, and $\mathfrak{s} = (M, \text{all})$ means $E_{\mathfrak{s}} = V_{\mathfrak{s}} \times V_{\mathfrak{s}}$, and $\mathfrak{s} = (M, M)$ means $\mathcal{U}_{\mathfrak{s}} = M$.

1C) If we vary Λ, we shall write $\mathfrak{s} = (M_{\mathfrak{s}}, E_{\mathfrak{s}}, \Lambda_{\mathfrak{s}})$.

2) Let $\mathfrak{S}$ denote the class of pre-systems, and

$$\mathfrak{S}_{\alpha,\beta} := \big\{\mathfrak{s} \in \mathfrak{S} : (\exists \mathcal{U} \subseteq M_{\mathfrak{s}})[E_{\mathfrak{s}} = E^{\alpha,\beta}_{\mathcal{U}}]\big\}.$$

3) We say $\mathbf{s}$ is a *system* when it consists of

(a) A pre-system $\mathfrak{s}$ (so $M_{\mathbf{s}} := M_{\mathfrak{s}}$, $E_{\mathbf{s}} := E_{\mathfrak{s}}$, *etc.*); similarly in later definitions.

(b) A finite non-empty set A and a sequence $\bar{A} = \langle A_\eta : \eta \in V_{\mathbf{s}}\rangle \subseteq \mathcal{P}(A)$ satisfying

$$\eta \; E_{\mathbf{s}} \; \nu \Rightarrow A_\eta = A_\nu.$$

3A) Let $\mathbf{S}$ denote the class of systems, and $\mathrm{pre}(\mathbf{s})$ denote the pre-system contained in $\mathbf{s}$. (We may also say that $\mathbf{s}$ *expands* $\mathrm{pre}(\mathbf{s})$.) As above, let

$$\mathbf{S}_{\alpha,\beta} := \big\{\mathbf{s} \in \mathbf{S} : \mathrm{pre}(\mathbf{s}) \in \mathfrak{S}_{\alpha,\beta}\big\}.$$

3B) For $B \subseteq A_{\mathbf{s}}$, $\mu_{\mathbf{s}}(B) = \dfrac{|B|}{|A_{\mathbf{s}}|}$, so omitting $\mathbf{s}$ when clear, e.g. in $\mu(A^{\mathbf{s}}_\eta)$.

3C) Let $\mathrm{supp}(\mathbf{s}) = \{\eta \in V_{\mathbf{s}} : A_\eta \neq \emptyset\}$.

4) Let $\min(\mathbf{s}) = \min(\bar{A}_{\mathbf{s}}) := \min\limits_{\eta \in V_{\mathbf{s}}} \mu(A_\eta)$.

For X a non-empty subset of $V_{\mathbf{s}}$, let $\min(\mathbf{s} \restriction X) = \min(\bar{A} \restriction X) := \min\limits_{\eta \in X} \mu(A_\eta)$.

5) Let $\mathrm{av}(\mathbf{s}) = \mathrm{av}(\bar{A}_{\mathbf{s}}) := \dfrac{1}{|V_{\mathbf{s}}|} \sum\limits_{\eta \in V_{\mathbf{s}}} \mu(A_\eta)$; clearly av stands for average.

For X a non-empty subset of $V_{\mathbf{s}}$, let $\mathrm{av}(\mathbf{s} \restriction X) := \dfrac{1}{|X|} \sum\limits_{\eta \in X} \mu(A_\eta)$.

6) For $\mathfrak{s}$ a pre-system, let $\mathrm{ob}(\mathfrak{s})$ be the set of pairs $\mathbf{o} = (\rho, \mathscr{E})$ such that

(A) ρ is a (proper) partial function from $M_{\mathfrak{s}}$ to Λ.

(B) $\mathscr{E}$ is an equivalence relation on $N_{\mathbf{o}} := M_{\mathfrak{s}} \setminus \mathrm{dom}(\rho)$ $(\neq \varnothing)$.

(If we write $\mathrm{ob}(\mathbf{s})$ we mean $\mathrm{ob}(\mathrm{pre}(\mathbf{s}))$.)

6A) Let $\dim(\mathbf{o}) = |N_{\mathbf{o}}/\mathscr{E}_{\mathbf{o}}|$.

6B) Let $\mathrm{ob}^+(\mathfrak{s})$ be defined similarly, allowing $N_{\mathbf{o}}$ to be empty, in this case $\dim(\mathbf{o}) = 0$.

6C) If $\mathbf{o} \in \mathrm{ob}^+(\mathfrak{s})$, $N \subseteq M_{\mathfrak{s}}$, then we let $\mathbf{o}_1 = \mathbf{o} \restriction N$ be such that $N_{\mathbf{o}_1} = N_{\mathbf{o}} \cap N$, $\mathscr{E}_{\mathbf{o}_1} = \mathscr{E}_{\mathbf{o}} \restriction N$ and $\rho_{\mathbf{o}_1} = \rho_{\mathbf{o}} \restriction N$.

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7) For $\mathbf{s}$ a system and $\mathbf{o} \in \mathrm{ob}(\mathrm{pre}(\mathbf{s}))$, let $\mathrm{sub}_{\mathbf{o}}(\mathbf{s})$ be the set of systems $\mathbf{t}$ satisfying the following.

(A) $M_{\mathbf{t}} := N_{\mathbf{o}}/\mathscr{E}$ (so $A <_{\mathbf{t}} B$ iff $\min(A) < \min(B)$).

(B) $E_{\mathbf{t}} \supseteq E^*_{\mathbf{s},\mathbf{t}} := \big\{(\eta,\nu) \in V_{\mathbf{t}} \times V_{\mathbf{t}} : h_{\mathbf{o}}(\eta)\ E_{\mathbf{s}}\ h_{\mathbf{o}}(\nu)\big\}$, where $h_{\mathbf{o}} : {}^{N/\mathscr{E}}\Lambda \to V_{\mathbf{s}}$ is defined by

$$h_{\mathbf{o}}(\nu)(a) := \begin{cases} \rho(a) & \text{if } a \notin N_{\mathbf{o}} \\ \nu(a/E) & \text{if } a \in N_{\mathbf{o}} \end{cases}$$

for all $a \in M_{\mathbf{s}}$ and $\nu \in {}^{N/\mathscr{E}}\Lambda$.[2]

(C) $A^{\mathbf{t}}_{\eta} \subseteq A^{\mathbf{s}}_{h_{\mathbf{o}}(\eta)}$ for all $\eta \in V_{\mathbf{t}}$.

(D) $A_{\mathbf{t}} = A_{\mathbf{s}}$.

8) Let $\mathrm{sub}(\mathbf{s}) := \bigcup\limits_{\mathbf{o} \in \mathrm{ob}(\mathbf{s})} \mathrm{sub}_{\mathbf{o}}(\mathbf{s})$.

9) $\mathrm{sub}_{\mathbf{o}}(\mathfrak{s})$ and $\mathrm{sub}(\mathfrak{s})$ are defined similarly, omitting clauses (7)(C),(D).

10) Let $\mathfrak{u}_{\mathbf{o}}(\mathfrak{s}) := \big(N_{\mathbf{o}}/\mathscr{E}, E^{\mathbf{o}}_{\mathfrak{s}}\big)$, where $\eta\, E^{\mathbf{o}}_{\mathfrak{s}}\, \nu$ iff $h_{\mathbf{o}}(\eta)\, E_{\mathfrak{s}}\, h_{\mathbf{o}}(\nu)$, so necessarily $\eta,\nu \in V_{\mathfrak{u}_{\mathbf{o}}(\mathfrak{s})}$. (Again, if we write $\mathfrak{u}_{\mathbf{o}}(\mathbf{s})$ we mean $\mathfrak{u}_{\mathbf{o}}(\mathrm{pre}(\mathbf{s}))$.) Let $\mathbf{u} = \mathbf{u}_{\mathbf{o}}(\mathbf{s})$ be the $\mathbf{u} \in \mathrm{sub}_{\mathbf{o}}(\mathbf{s})$ such that $\mathrm{pre}(\mathbf{u}) = \mathfrak{u}_{\mathbf{o}}(\mathbf{s})$ and in clauses (7)(C),(D) we have equality.

11) For $\mathfrak{s}$ a pre-system, not necessarily in $\mathfrak{S}_{\alpha,\beta}$, and $k \geq 1$, let $k \times \mathfrak{s}$ denote the pre-system $\mathfrak{t}$ defined below.

- $M_{\mathfrak{t}} := k \times M_{\mathfrak{s}}$
- $E_{\mathfrak{t}} := \big\{\big((i,\eta),(i,\nu)\big) : i < k,\ \eta\ E_{\mathfrak{s}}\ \nu\big\}$.

12) If $\mathfrak{s}_1, \mathfrak{s}_2$ are pre-systems (in $\mathfrak{S}_{\alpha,\beta}$) then $\mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_2$ mean $M_{\mathfrak{s}} = M_{\mathfrak{s}_1} + M_{\mathfrak{s}_2}$ and $\mathcal{U}_{\mathfrak{s}} = \mathcal{U}_{\mathfrak{s}_1} \cup \mathcal{U}_{\mathfrak{s}_2}$ (so if $\{\mathfrak{s}_1,\mathfrak{s}_2\} \subseteq \mathfrak{S}_{\alpha,\beta}$ then $E_{\mathfrak{s}} = \{(\eta,\nu) \in V_{\mathfrak{s}} \times V_{\mathfrak{s}} : (\eta \restriction M_{\mathfrak{s}_\iota})\, E_{\mathfrak{s}_\iota}\, (\nu \restriction M_{\mathfrak{s}_\iota})$ for $\iota = 1,2\}$).

A crucial definition is

Definition 1.6. 1) For $\mathbf{o} \in \mathrm{ob}^+(\mathbf{s})$ let $\mathrm{cnd}(\mathbf{o}) = \{\Omega : \Omega \subseteq \mathrm{Rang}(h_{\mathbf{o}})$ and $(\forall \eta,\nu \in \Omega)(\forall a \in N_{\mathbf{o}})[\eta(a) \neq \nu(a) \Rightarrow \{\eta(a),\nu(a)\} = \{\alpha,\beta\}]\}$ (cnd for candidates, ob for objects).

1A) (a) For $\Omega \subseteq V_{\mathbf{s}}$ let $A^{\mathbf{s}}_{\Omega} = \cap\{A^{\mathbf{s}}_{\eta} : \eta \in \Omega\}$.

(b) For $\Omega \subseteq V_{\mathfrak{s}}$, $N \subseteq M_{\mathfrak{s}}$ let $\Omega \restriction N = \{\eta \restriction N : \eta \in \Omega\}$.

2) For $\mathbf{o} \in \mathrm{ob}^+(\mathbf{s})$, $a \in M_{\mathbf{s}} \setminus N_{\mathbf{o}}$ let $\mathbf{o} + a = \mathbf{o} +_{\mathbf{s}} a$ be the $\mathbf{o}_1 \in \mathrm{ob}(\mathbf{s})$ such that $N_{\mathbf{o}_1} = N_{\mathbf{o}} \cup \{a\}$, $\mathscr{E}_{\mathbf{o}_1} \restriction N_{\mathbf{o}} = \mathscr{E}_{\mathbf{o}}$ and $\{a\}$ is an $\mathscr{E}_{\mathbf{o}_1}$-equivalence class, and $\rho_{\mathbf{o}_1} = \rho_{\mathbf{o}} \restriction (\mathrm{dom}(\rho_{\mathbf{o}}) \setminus \{a\})$.

3) (a) Let $\mathbf{W}_{\mathfrak{s},<n} = \{(\mathbf{o},\Omega) : \mathbf{o} \in \mathrm{ob}^+(\mathbf{s})$, $\dim(\mathbf{o}) < n$ and Ω a maximal member of $\mathrm{cnd}(\mathbf{o})\}$ (of course we may write $\mathfrak{s}$ instead of $\mathrm{pre}(\mathbf{s})$). We may write n instead of $< n+1$; similarly later.

(b) Let $\mathbf{W}_{\mathbf{s},<n,\varepsilon} = \{(\mathbf{o},\Omega) \in \mathbf{W}_{\mathfrak{s},<n} : \mu_{\mathbf{s}}(A^{\mathbf{s}}_{\Omega}) \geq \varepsilon^{3^{\dim(\mathbf{o})}}\}$.

4) We say $\mathbf{s} \in \mathbf{S}_{\alpha,\beta}$ is $(<n,\varepsilon)$-good when: (A) $\Rightarrow$ (B), where

(A) for some N we have:

(a) $(\mathbf{o},\Omega) \in \mathbf{W}_{\mathfrak{s},<n}$,

(b) $N \subseteq \mathcal{U}_{\mathbf{s}} \cap N_{\mathbf{o}}$,

(c) if $a \in N$ then $\{a\}$ is an $\mathscr{E}_{\mathbf{o}}$-equivalence class,

(d)for $\rho \in {}^{N}\{\alpha,\beta\}$, $\Omega[\rho] = \{\eta \in V_{\mathbf{s}} :$ for some $\nu \in \Omega$ we have $\eta \restriction (M_{\mathbf{s}} \setminus N) = \nu \restriction (M_{\mathbf{s}} \setminus N)$ and $(\forall a \in N)(\eta(a) = \rho(a))\}$,

(e) $(\forall \rho \in {}^{N}\{\alpha,\beta\})\big(\mu(A^{\mathbf{s}}_{\Omega[\rho]}) \geq \varepsilon^{3^{\dim(\mathbf{o})-|N|}}\big)$

(f) let $\Omega[N] = \bigcup\{\Omega[\rho] : \rho \in {}^{N}\{\alpha,\beta\}\}$;

[2] That is, it sends $\nu \mapsto (\nu \circ \pi) \cup \rho$, with $\pi : N \to N/\mathscr{E}$ the quotient map.

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(B) $(\mathbf{o}, \Omega[N]) \in \mathbf{W}_{\mathbf{s},<n,\varepsilon}$.

(We may replace ε^{3^k} by ε_k when using $\bar{\varepsilon} = \langle \varepsilon_\ell : \ell < n \rangle \in {}^n(0,1]_{\mathbb{R}}$.)

Remark 1.7. The 3 in 1.6(3) comes from 1.3(1); so increasing n allows us to use a number near 2, approach 2, using 1.3(2)).

Definition 1.8. 1) For $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$ let $\mathfrak{s} \to_{\min} (\mathfrak{t})_{\varepsilon,\zeta}$ mean

(a) $\mathfrak{s}, \mathfrak{t}$ are pre-systems.

(b) $0 < \zeta < \varepsilon \leq 1$

(c) If $\mathbf{s}$ is a system expanding $\mathfrak{s}$ with $\min(\mathbf{s}) \geq \varepsilon$ then there exists a system $\mathbf{t}$ with $\mathrm{pre}(\mathbf{t}) = \mathfrak{t}$ and $\min(\mathbf{t}) \geq \zeta$ which is isomorphic to a member $\mathbf{u}$ of $\mathrm{sub}(\mathbf{s})$.

1A) For $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}_{\alpha,\beta}$ let $\mathfrak{s} \mapsto_{\min} (\mathfrak{t})_{\varepsilon,n}$ mean (here and later we can replace $< n+1$ by n)

(a), (b) as above,

(c) if $\mathbf{s}$ is a system expanding $\mathfrak{s}$ with $\min(\mathbf{s}) \geq \varepsilon$ which is (n,ε)-good then there exists a system $\mathbf{t}$ with $\mathrm{pre}(\mathbf{t}) = \mathfrak{t}$ and[3] $\min(\mathbf{t}) \geq \varepsilon$ which is isomorphic to a member $\mathbf{u}$ of $\mathrm{sub}(\mathbf{s})$ and is (n,ε)-good.

1B) For $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$ let $\mathfrak{s} \to_{\mathrm{av}} (\mathfrak{t})_{\varepsilon,\zeta}$ be defined as in part (1) replacing min by av in clause (c), twice.

1C) For $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$ let $\mathfrak{s} \to_{\mathrm{avm}} (\mathfrak{t})_{\varepsilon,\zeta}$ be defined as in part (1) replacing min by av in clause (c), only in the first place.

1D) For $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$ let $\mathfrak{s} \mapsto_{\mathrm{avm}} (\mathfrak{t})_{\varepsilon,\zeta,n}$ be as in (1A) but in clause (c) we use $\mathrm{av}(\mathbf{s}) \geq \varepsilon$ (instead of $\min(\mathbf{s}) \geq \varepsilon$) and $\min(\mathbf{t}) \geq \zeta$, and "which is (n,ε)-good" is omitted for $\mathbf{s}$.

2) Assume $0 < \zeta < \varepsilon \leq 1$. Let $\mathsf{HJ}^{\min}_{\varepsilon,\zeta}(\Lambda, r)$ be the minimal k such that for every system $\mathbf{s} \in \mathbf{S}$ we have :

(*) If $M_{\mathbf{s}}$ have k elements and $\min(\mathbf{s}) \geq \varepsilon$ then there is $\mathbf{o} \in \mathrm{ob}(\mathbf{s})$ such that $N_{\mathbf{o}}/\mathscr{E}_{\mathbf{o}}$ has r elements and $\mu\big(\cap\{A_\eta : \eta \in \mathrm{Rang}(h_{\mathbf{o}})\}\big) \geq \zeta$.

2A) We define $\mathsf{HJ}^{\mathrm{av}}_{\varepsilon,\zeta}(\Lambda, r)$ similarly, replacing $\min(\mathbf{s}) \geq \varepsilon$ by $\mathrm{av}(\mathbf{s}) \geq \varepsilon$.

3) Letting $\zeta = 0$ mean for some $\zeta > 0$, and then we can omit ζ; note that $E_{\mathbf{s}}, \alpha_{\mathbf{s}}, \beta_{\mathbf{s}}$ are irrelevant.

A crucial step is

Claim 1.9. *Suppose*

(a) $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}_{\alpha,\beta}$

(b) $M_{\mathfrak{s}} = Q + R$ *or just* $M_{\mathfrak{s}}$ *is the disjoint union of* Q *and* R, *and* $\mathcal{U}_{\mathfrak{s}} \subseteq Q$, *and* $\mathfrak{s}_1 = (Q, \mathcal{U}_{\mathfrak{s}})$.

(c) $M_{\mathfrak{t}} = Q + \{x_*\}$ *(for some* $x_* \notin Q$*) and* $\mathcal{U}_{\mathfrak{t}} := \mathcal{U}_{\mathfrak{s}} \cup \{x_*\}$.

(d) $0 < \zeta < \varepsilon < 1$

(e) $m := |R|$ *and* $m+1 \to (s)^2_k$ *where* $s = \left\lceil \frac{1}{\varepsilon^{3^{n-1}}} \right\rceil + 1$ *and* $k = \exp_2(|\mathbf{W}_{\mathfrak{s}_1,<n}|)$; n *(the bound on the dimensions) is given.*

Then $\mathfrak{s} \mapsto_{\min} (\mathfrak{t})_{\varepsilon,n}$.

[3] For this proof we may strengthen $\mathbf{u} \in \mathrm{sub}(\mathbf{s})$ by: $\nu \in V_{\mathbf{t}} \Rightarrow A^{\mathbf{t}}_\nu = A^{\mathbf{s}}_{h_{\mathbf{o}}(\nu)}$.

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Proof. So let $\mathbf{s} \in \mathbf{S}_{\alpha,\beta}$ expand $\mathfrak{s}$, with $\min(\mathbf{s}) \geq \varepsilon$, and moreover $\mathbf{s}$ is (n,ε)-good. We shall find $\mathbf{o} \in \mathrm{ob}(\mathbf{s})$ and $\mathbf{u} \in \mathrm{sub}_{\mathbf{o}}(\mathbf{s})$ and $\mathbf{t}$ expanding $\mathfrak{t}$ isomorphic to $\mathbf{u}$ as required in Definition 1.8, which is (n,ε)-good.

$(*)_0$ For $(\mathbf{o},\Omega) \in \mathbf{W}_{\mathfrak{s}_1,<n}$ and $\varrho \in {}^R\Lambda$ we let $(\mathbf{o},\Omega)+\varrho$ be the $(\mathbf{o}_1,\Omega_1) \in \mathbf{W}_{\mathfrak{s},<n}$ such that $(N_{\mathbf{o}_1},\mathscr{E}_{\mathbf{o}_1}) = (N_{\mathbf{o}},\mathscr{E}_{\mathbf{o}})$ and $\rho_{\mathbf{o}_1} = \rho_{\mathbf{o}} \cup \varrho$ and $\Omega_1 = \{\nu \cup \varrho : \nu \in \Omega\}$.

We define a relation $\mathbf{E}$ on ${}^R\Lambda$ as follows.

$(*)_1$ For $\eta_1,\eta_2 \in {}^R\Lambda$, $\eta_1 \mathbf{E} \eta_2$ iff: $\mu(A^{\mathbf{s}}_{\Omega_1} \cap A^{\mathbf{s}}_{\Omega_2}) \geq \varepsilon^{3^{\dim(\mathbf{o}_1)\cdot 3}}$ when for some $(\mathbf{o},\Omega) \in \mathbf{W}_{\mathfrak{s}_1,<n}$ we have $(\mathbf{o}_\iota,\Omega_\iota) = (\mathbf{o},\Omega)+\eta_\iota$ and $(\mathbf{o}_\iota,\Omega_\iota) \in \mathbf{W}_{\mathbf{s},n,\varepsilon}$ for $\iota = 1,2$ (note that "$(\mathbf{o}_\iota,\Omega_\iota) \in \mathbf{W}_{\mathbf{s},n,\varepsilon}$" does not follow from "$\mathbf{s}$ is (n,ε)-good").

Now,

$(*)_2$ (a) For transparency, we let $R := \{0,\dots,m-1\}$.

(b) We define $\varrho_\ell \in {}^R\Lambda$ (for $\ell \leq m$) by

$$\varrho_\ell(i) := \begin{cases} \alpha & \text{if } i < \ell \\ \beta & \text{if } i \in [\ell,m). \end{cases}$$

$(*)_3$ There are $k_* < \ell_* \leq m$ such that $\varrho_{k_*} \mathbf{E} \varrho_{\ell_*}$.

[Why? We have

$(*)_{3.1}$ if $(\mathbf{o},\Omega) \in \mathbf{W}_{\mathfrak{s}_1,<n}$, $d = \dim(\mathbf{o})$, $S \subseteq m+1$, $|S| = s$, and $(\mathbf{o},\Omega)+\varrho_k \in \mathbf{W}_{\mathbf{s},n,\varepsilon}$ for every $k \in S$, then there are $k < \ell$ from S with $\mu(A^{\mathbf{s}}_{\Omega_1} \cap A^{\mathbf{s}}_{\Omega_2}) \geq \varepsilon^{3^{d+1}}$, where $(\mathbf{o}_1,\Omega_1) = (\mathbf{o},\Omega)+\varrho_k$, $(\mathbf{o}_2,\Omega_2) = (\mathbf{o},\Omega)+\varrho_\ell$.

[Why? By 1.2(1) and 1.3(1) for s sets of measure $\geq \delta := \varepsilon^{3^{\dim(\mathbf{o})}}$, as $s \geq 1+\lceil 1/\delta \rceil$, recalling $\dim(\mathbf{o}) < n$.]

Next

$(*)_{3.2}$ There are $k_* < \ell_* \leq m$ such that $\varrho_{k_*} \mathbf{E} \varrho_{\ell_*}$

[Why? Define a function $\mathbf{c}$ with domain $\{(k,\ell) : k < \ell \leq m\}$ such that $\mathbf{c}((k,\ell)) = \{(\mathbf{o},\Omega) \in \mathbf{W}_{\mathfrak{s}_1,<n} :$ letting $(\mathbf{o}_1,\Omega_1) = (\mathbf{o},\Omega)+\varrho_k$, $(\mathbf{o}_2,\Omega_2) = (\mathbf{o},\Omega)+\varrho_\ell$ we have $\{(\mathbf{o}_1,\Omega_1),(\mathbf{o}_2,\Omega_2)\} \subseteq \mathbf{W}_{\mathbf{s},<n,\varepsilon}$ and $\mu(A^{\mathbf{s}}_{\Omega_1} \cap A^{\mathbf{s}}_{\Omega_2}) \geq \varepsilon^{3^{\dim(\mathbf{o})+1}}\}$. Clearly the range of the colouring $\mathbf{c}$ has[4] cardinality $\leq \exp_2(|\mathbf{W}_{\mathfrak{s}_1,<n}|)$ so by clause (e) of the claim assumption there is a set $S \subseteq R \cup \{m\}$ of cardinality s such that $\mathbf{c} \restriction \{(k,\ell) : k < \ell$ are from $S\}$ is constant. By $(*)_{3.1}$ for each $(\mathbf{o},\Omega) \in \mathbf{W}_{\mathfrak{s}_1,<n}$ necessarily we have $(\mathbf{o},\Omega) \in \mathbf{c}(k,\ell)$ for some hence for every pair (k,ℓ) from S. So we are done.] (Recall that as $\mathbf{s}$ is (n,ε)-good, for every $(\mathbf{o},\Omega) \in \mathbf{W}_{\mathfrak{s}_1,<n}$ and $\eta \in {}^R\Lambda$ we have $(\mathbf{o},\Omega)+\eta \in \mathbf{W}_{\mathbf{s},n,\varepsilon}$.)

Towards defining $\mathbf{t}$, let us construct an object $\mathbf{o} = (\rho,\mathscr{E}) \in \mathrm{ob}(\mathfrak{s})$ as follows.

$(*)_4$ (a) $\mathrm{dom}(\rho) := [0,k_*) \cup [\ell_*,m)$.

(b) $\rho(i) := \begin{cases} \alpha & \text{if } i < k_* \\ \beta & \text{if } i \in [\ell_*,m). \end{cases}$

(c) Let $\mathscr{E}$ be $\big\{(x,x) : x \in Q\big\} \cup \Big([k_*,\ell_*) \times [k_*,\ell_*)\Big)$; this is an equivalence relation on $M_{\mathfrak{s}} \setminus \mathrm{dom}(\rho)$.

Now we define the system $\mathbf{u}$ as follows.

$(*)_5$ (a) $\mathfrak{u} = \mathrm{pre}(\mathbf{u})$ is the pre-system with $M_{\mathfrak{u}} := \big(Q \cup [k_*,\ell_*)\big)/\mathscr{E}_{\mathbf{o}}$ and $\mathcal{U}_{\mathfrak{u}} := \mathcal{U}_{\mathfrak{s}} \cup \{y_*\}$, where $y_* := [k_*,\ell_*)/\mathscr{E}_{\mathbf{o}}$.

(b) For $\nu \in V_{\mathbf{u}}$, we define $A^{\mathbf{u}}_\nu := A^{\mathbf{s}}_{h_{\mathbf{o}}(\nu)}$.

[4] Of course we can be careful and save, but for 1.13 not seriously in this section.

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$(*)_6$ $\mathbf{u}$ is (n,ε)-good.

[Why? Easy, recalling ob^+ was used in the definition of (n,ε)-good.] Note: considering $(\mathbf{o},\Omega) \in \mathbf{W}_{\mathfrak{s}_1,n}$, if $\dim(\mathbf{o}) = n$ the assumption "$\mathbf{s}$ is (n,ε)-good" suffices; while if $\dim(\mathbf{o}) < n$ we use the choice of $k_* < \ell_*$.

This is the main point we need to prove

$$\mathfrak{s} \mapsto_{\min} (\mathfrak{u})_{\varepsilon,n},$$

Clearly $\mathfrak{u}$ is isomorphic to $\mathfrak{t}$. (Just consider the map which is the identity on Q and sends $y_* \mapsto x_*$.)

Let $\mathbf{t}$ be the system which expands $\mathfrak{t}$ and is isomorphic to $\mathbf{u}$: clearly it is as required, and so we are done. $\square_{1.9}$

Observation 1.10. *1) So in 1.9, $k = \exp_2((|\Lambda|+n+1)^{|Q|})$ suffices, and essentially $m = k^{ks}$ suffices.*

2) If $\mathbf{s} \in \mathbf{S}_{=,\alpha,\beta}$ and $\min(\mathbf{s}) \geq \varepsilon$ and ($\mathcal{U}_{\mathbf{s}} = \emptyset$ or $n = 0$) then $\mathbf{s}$ is (n,ε)-good.

Proof. 1) A rough estimate on the number of $(\mathbf{o},\Omega) \in \mathbf{W}_{\mathfrak{s}_1,<n}$, being very generous, is $\leq (|\Lambda| + n + 1)^{|Q|}$.

[Why? We have to choose $N_{\mathbf{o}}$, $\rho_{\mathbf{o}}$ and $\mathscr{E}_{\mathbf{o}}$, so we can code all by $f : Q \to \Lambda \cup (n+1)$ with $a \in N \Rightarrow f(a) \leq n$, and $f(a) = \gamma \in \Lambda$ if $(\forall \nu \in \Omega)(\nu(a) = \gamma)$, and $f(a) = \ell$ if a belongs to the ℓ-th equivalence class of $\mathscr{E}_{\mathbf{o}}$ (the question if $a \in \mathrm{dom}(\rho_{\mathbf{o}})$ or not is immaterial). Pedantically we count the number of Ω, not the number of pairs $(\mathbf{o},\Omega)$. If you like to avoid this then you get $(|\Lambda| + n + |\Lambda| + 1)^{|Q|}$.]

Hence $|\mathbf{W}_{\mathfrak{s}_1,<n}| \leq (|\Lambda| + n + 1)^{|Q|}$, hence $k = \exp_2((|\Lambda| + n + 1)^{|Q|})$ suffices.

2) If $n = 0$ then being (n,ε)-good is an empty demand. If $\mathcal{U}_{\mathbf{s}} = \emptyset$ then the N in clause (A) of Definition 1.6(4) is $\subseteq \mathcal{U}_{\mathbf{s}} = \emptyset$ (by (A)(b) there), necessarily the empty set, hence we have $M_{\mathbf{s}} \setminus N = M_{\mathbf{s}}$, so (by (A)(d) there) $(\mathbf{o},\Omega) \in \mathbf{W}_{\mathbf{s},<n,\varepsilon}$, which is (B) there. $\square_{1.10}$

Claim 1.11. *Suppose*

(a)
- $\bullet_1$ $\mathfrak{s},\mathfrak{t} \in \mathfrak{S}_{\alpha,\beta}$
- $\bullet_2$ $M_{\mathfrak{t}} := \{i : i < r\}$, *and* $M_{\mathfrak{s}}$ *is of the form* $\sum\limits_{i<r} M_i$.
- $\bullet_3$ $\mathcal{U}_{\mathfrak{s}} := \varnothing$ *(for transparency).*
- $\bullet_4$ $\mathcal{U}_{\mathfrak{t}} := M_{\mathfrak{t}}$

(b) $s_i \geq 1$ *for* $i < r$

(c) $\varepsilon \in (0,1)_{\mathbb{R}}$

(d) *Letting* $M_{<i} := \sum\limits_{j<i} M_j$, *such that (for* $i < r$*)*
- $\bullet_1$ $s_i = \left\lceil \frac{1}{\varepsilon^{3^{n-1}}} \right\rceil + 1$
- $\bullet_3$ $|M_i| \to (s_i)^2_{k(i)}$ *where* $k(i) = \exp_2(|\mathbf{W}_{\mathfrak{s}_i,<n}|)$, *where* $\mathfrak{s}_i = (M_{<i} + \{(i,r)\}, \{x_j : i \leq j < r\})$*; the* $x_j \notin M_{<i}$ *are with no repetition*

Then $\mathfrak{s} \mapsto_{\min} (\mathfrak{t})_{\varepsilon,n}$.

Proof. Let $\mathbf{s} \in \mathbf{S}_{\alpha,\beta}$ expand $\mathfrak{s}$, with $\min(\mathbf{s}) \geq \varepsilon$, and $\mathbf{s}$ is (n,ε)-good. Without loss of generality, $M_i := \{i\} \times m_i$ (with the natural order).

Let $\mathbf{u}_r := \mathbf{s}$, and we will choose k_i, ℓ_i, $\mathbf{o}_i$, and $\mathbf{u}_i$ by downward induction on $i < r$ as follows.

$(*)^i_1$ (a) $\mathbf{o}_i \in \mathrm{ob}(\mathbf{s})$, $\mathbf{u}_i \in \mathrm{sub}_{\mathbf{o}_i}(\mathbf{s})$, and $\mathfrak{u}_i := \mathrm{pre}(\mathbf{u}_i)$ for $i < r$.

(b) $\bullet_1$ $k_i < \ell_i \leq m_i$

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$\bullet_2$ $\mathscr{E}_{\mathbf{o}_i}$ is the equivalence relation on $N_{\mathbf{o}_i}$ whose classes are the singletons $\{x\}$ for $x \in M_{<i}$ and the sets $\{j\} \times [k_j, \ell_j)$ for $j \in [i, r)$
(This means $N_{\mathbf{o}_i} := M_{<i} \cup \bigcup\limits_{j=i}^{r-1} \Big(\{j\} \times [k_j, \ell_j)\Big)$.)

$\bullet_3$ $\rho_{\mathbf{o}_i} : M \setminus N_{\mathbf{o}_i} \to \Lambda$ is defined by

$$\rho_{\mathbf{o}_i}(j,a) := \begin{cases} \alpha & \text{if } a \in [0, k_j) \\ \beta & \text{if } a \in [\ell_j, m_j) \end{cases}$$

for $j \geq i$.

(c) $\mathcal{U}_{\mathbf{u}_i} := \bigcup\limits_{j=i}^{r-1} \Big(\{j\} \times [k_j, \ell_j)\Big)$

(d) $\min(\mathbf{u}_i) \geq \varepsilon$,

(e) $\mathbf{u}_i$ is (n, ε)-good.

For $i < r$, we can carry the induction by applying Claim 1.9, with $m_i, \mathrm{pre}(\mathbf{u}_{i+1})$, $\mathrm{pre}(\mathbf{u}_i)$, ε, s_i here standing for $m, \mathfrak{s}, \mathfrak{t}, \varepsilon, n$ there.

In details, we apply 1.9 with

- $\mathbf{u}_{i+1}$, $M_{\mathbf{u}_{i+1}} \setminus M_i$, M_i, ε, s_i and $\mathfrak{u}_i$ here standing for
- $\mathbf{s}$, Q, R, ε, n and $\mathfrak{t}$ there.

We have also $\mathbf{u}_{i+1}$ as in 1.9(a). So by 1.9, $\mathfrak{u}_{i+1} \mapsto_{\min} (\mathfrak{u}_i)_{\varepsilon,n}$. By the proof of 1.9 we can find $k_i < \ell_i \leq |M_i|$ (like k_*, ℓ_* in $(*)_{3.2}$ in that proof) and using them define $\mathbf{u}_i$ (like $\mathbf{u}$ in $(*)_5$ in that proof). Clearly the induction step is done.

Arriving at $i = 0$ we are done. $\square_{1.11}$

Remark 1.12. In 1.11, for any r and ε from $(0,1]$ we can find an M whose cardinality is bounded by a power tower of height $r + \lceil \frac{1}{\varepsilon} \rceil$.

Conclusion 1.13. *1) If $\mathfrak{s}, \mathfrak{t}$, $\langle s_i, k(i), M_i : i < r \rangle$ are as in 1.11, $\varepsilon < \varepsilon_* < 1$, and $\mathfrak{s}_* = k_* \times \mathfrak{s}$ where $k_* = \dfrac{|V_{\mathfrak{s}}| - 1}{\varepsilon_* - \varepsilon}$, then $\mathfrak{s}_* \mapsto_{\mathrm{avm}} (\mathfrak{t})_{\varepsilon_*,\varepsilon,n}$.*

2) So the function giving $|M_{\mathfrak{s}_*}|$ as a function of $\lceil \frac{1}{\varepsilon} \rceil + \lceil \frac{1}{\varepsilon_* - \varepsilon} \rceil + n$ is iteration of exponentiation of length $2r+1$ (so is in $\mathcal{E}^4$).

3) $\mathsf{HJ}_\varepsilon(\Lambda, r)$ is bounded by a function in $\mathcal{E}^5$.

Proof. 1) By 2.12(6) we have $\mathfrak{s}_* \to_{\mathrm{avm}} (\mathfrak{s})_{\varepsilon_*,\varepsilon,n}$. Hence there are $\mathbf{o} \in \mathrm{ob}(\mathfrak{s}_*)$ and $\mathbf{u} \in \mathrm{Sub}_{\mathbf{o}}(\mathbf{s})$ isomorphic to $\mathfrak{s}$ such that $\min(\mathbf{u}) \geq \varepsilon$ (recalling Definition 2.7(3)). Now apply 1.11 and 1.10(2). Note that ε does not change after applying 2.12(6).

2) Clear from (1).

3) By part (1), used $|\Lambda| - 1$ times. In fact only in the first we should use part (1); in the rest we can use 1.11, recalling again 1.10(2). $\square_{1.13}$

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§ 2. More on §1

We deal with two additions to §1. In §2A we point to what 1.11 tell us, not just on the way to HJ. In §2B we note that the version of the density HJ proved in §1 is essentially equivalent to a more standard one.

§ 2(A). **An induction step as a result on its own right.** What is the difference between 2.1(1) below and 1.11? The difference is the use of $\nu \in {}^{M_{<i}}\Lambda$ in $(*)$(c); moreover it follows from the proof of 1.11.

Theorem 2.1. 1) *In* 1.11 *we can find* $\langle (k_i, \ell_i) : i < r\rangle$ *such that*

$(*)$ *(a)* $k_i < \ell_i \le m_i$

(b) for $\rho \in {}^r\Lambda$ *let* $\varrho_\rho \in {}^{M_s}\Lambda$ *be such that: if* $i < r$, $a \in M_i$ *and* $|(M_i)_{<a}| = n$, *then* $\varrho_\rho(a)$ *is* α *if* $n < k_i$, *is* β *if* $n \ge \ell_i$, *and is* $\rho(i)$ *if* $n \in [k_i, \ell_i)$

(c) if $i < r$, $\nu \in {}^{M_{<i}}\Lambda$, $\rho_\gamma \in {}^r\{\gamma\}$ *for* $\gamma \in \{\alpha, \beta\}$, *then*

$$\zeta \le \mu\big(A^{\mathbf{s}}_{\nu \cup (\varrho_{\rho_\alpha} \restriction M_{\ge i})} \cap A^{\mathbf{s}}_{\nu \cup (\varrho_{\rho_\beta} \restriction M_{\ge i})}\big).$$

2) *We can strengthen* $(*)$ *of part (1) to*

$(c)_2$ *if* $i < r$, $\nu \in {}^{M_{<i}}\Lambda$, $\rho \in {}^{(i,r)}\Lambda$ *and* $\varrho_\gamma \in {}^{\{i\}}\{\gamma\}$ *for* $\gamma \in \{\alpha, \beta\}$ *then*

- $\bullet_1$ $\zeta \le \mu\big(A_{\nu \cup (\varrho_{\rho \cup \varrho_\alpha} \restriction M_{\ge i})} \cap A_{\nu \cup (\varrho_{\rho \cup \varrho_\beta} \restriction M_{\ge i})}\big)$, *with* $\varrho_{\rho'} \restriction M_{\ge i}$ *for* $\rho' \in {}^{[i,r)}\Lambda$ *defined as in (1)(b)*
- $\bullet_2$ *moreover we can replace* ζ *by* ε_0.

3) *In* 1.11 *we have*

- $\bullet_1$ $\zeta \le \mu\big(\cap \{A_{\nu \cup (\varrho_\rho \restriction M_{\ge i})} : \rho \in {}^r\{\alpha, \beta\}\}\big)$
- $\bullet_2$ *moreover we can replace* ζ *by* ε_0.

4) *For all three parts, the colouring version (i.e.* $\mathbf{c} : {}^M\Lambda \to C$, *with* $|C|$ *parallel to* $\lfloor \frac{1}{\varepsilon} \rfloor + 1$*) holds.*

Proof. 1), 2), 3) Same proof. 4) Easy. $\square_{2.1}$

Remark 2.2. Also see 1.9, 1.11.§1. The three statements in 2.1 have reasonable lower bound, quite close to the upper bound in 2.1.

Claim 2.3. *Assume* $M = \sum\{M_i : i < r\}$ *and* C *finite non-empty.*

1) If the colouring version parallel to 2.1(1) holds, then necessarily for every $i < r$ *we have*

- $|M_i| > \exp_{|C|}\big(\exp_{|\Lambda|}(|M_{<i}| + 1)\big)$.

2) Similarly for the density version.

3) Similarly for the parallel to 2.1(3) but now

- $|M_i| > \exp_{|C|}\big(\exp_{|\Lambda|}(|M_{<i}| + r - i - 1)\big)$.

Proof. Easy. E.g. 1) How do we arrive to "$|M_i| > \exp_{|C|}\big(\exp_{|\Lambda|}(|M_{<i}| + 1)\big)$"? In the proof of 1.9 (we may use $n = 0$, i.e. Ω a singleton $\subseteq {}^Q\Lambda$) we use "$k = \exp_2(|{}^Q\Lambda|)$", this from considering ${}^Q\Lambda$, i.e. for each $\ell \le |R|$, i.e. $\varrho_\ell[R]$, we consider the set $\langle A^{\mathbf{s}}_{\nu \cup \varrho_\ell[R]} : \nu \in {}^Q\Lambda\rangle$. Here however we have, except for $i = r - 1$, for each $\ell \le |M_i|$, i.e. $\varrho_\ell[M_i]$, the set $\langle A^{\mathbf{s}}_{\nu \cup \varrho_\ell[M_i] \cup \varrho_\gamma} : \nu \in {}^{M_{<i}}\Lambda$ and $\gamma \in \{\alpha, \beta\}\rangle$. $\square_{2.3}$

Discussion 2.4. We may in 2.1 have some extra parameter with stronger conclusion, mainly replacing "there are $k_i < \ell_i \le m_i$" by "there are $w_i \in [m_i + 1]^{k_i}$". For variety we state a colouring version, the simplest one.[5]

[5] In fact slightly less suffices.

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$\boxplus$ We have $(A)_1 + (A)_2 \Rightarrow (B)$ <u>when</u>

$(A)_1$ $C, M, \overline{M}, \bar{m}, \bar{s}, \bar{k}$ satisfy:

(a) $C \neq \emptyset$ finite, $r \geq 2$,

(b) $M = \sum\{M_i : i < r\}$,

(c) $\bar{m} = \langle m_i : i < r\rangle$, $m_i = |M_i|$,

(d) $\bar{s} = \langle s_i : i < r\rangle$, $2 \leq s_i \leq m_i$,

(e) $\bar{k} = \langle k_i : i < r\rangle$, $k_i \geq 1$,

(f) $\alpha \neq \beta \in \Lambda$.

$(A)_2$ $|M_i| \to (s_i)^{k_i}_{n(i)}$ where $n(i)$ is $\exp_{|C|}\left(\prod_{j<r, j\neq i} |\Lambda|^{k_j}\right)$.

(B) if $\mathbf{c} : {}^M\Lambda \to C$ <u>then</u> there is $\bar{w}$ such that

(a) $\bar{w} = \langle w_i : i < r\rangle$,

(b) $w_i \in [M_i \cup \{\infty\}]^{s_i}$ for $i < r$,

(c) let $\Gamma_{\bar{w}}$ be the set of η such that for each i for a unique $u_i = u_{\eta,i} \subseteq w_i$, $1 \leq |u_i| \leq k_i$, $\min(w_i) \geq \min(M_i)$ for technical reasons, letting $\langle b_j : j < |u_i|\rangle$ list u_i in $<_{M_i}$-increasing order and letting $b_{-1} = \min(M_i)$, $b_{|u_i|} = \infty$ we have

$\bullet_1$ if $j \in [-1, |u_i|)$ then $\eta \restriction \{a \in M_i : a \in [b_j, b_{j+1})_{M_i}\}$ is constant,

$\bullet_2$ $\eta \restriction [b_{-1}, b_0)_{M_i}$ is constantly α (or empty),

$\bullet_3$ u_i is minimal (under those restrictions),

(d) $\mathbf{c}(\eta_1) = \mathbf{c}(\eta_2)$ <u>when</u> $\eta_1, \eta_2 \in \Gamma_{\bar{w}}$ and if $\eta_1 \restriction M_i \neq \eta_2 \restriction M_i$ and $i < r$ then

- $\mathrm{Rang}(\eta_1 \restriction M_i), \mathrm{Rang}(\eta_2 \restriction M_i) \subseteq \{\alpha, \beta\}$,
- $u_{\eta_1,i} = u_{\eta_2,i}$.

Another relative

Definition 2.5. 1) [The colouring version] Let $\bar{m} \to (\bar{s})^{\bar{k}}_C$ mean (A) and (B) <u>when</u>

(A) (a) C finite non-empty, $r \geq 2$,

(b) $\bar{m} = \langle m_i : i < r\rangle$ where $m_i \geq 2$ for $i < r$,

(c) $\bar{s} = \langle s_i : i < r\rangle$, $2 \leq s_i < m_i$,

(d) $\bar{k} = \langle k_i : i < r\rangle$, $1 \leq k_i < s_i$.

(B) We have (a) $\Rightarrow$ (b) <u>when</u>

(a) For $i < r$:

- $\mathbf{c}_i$ is a function with range $\subseteq C$,
- $\mathrm{dom}(\mathbf{c}_i)$ is the set of $\bar{u}$ such that $\bar{u} = \langle u_j : j < r\rangle$ where
  - $j \neq i \Rightarrow u_j \in [m_j]^{k_j}$,
  - $j = i \Rightarrow u_j \in [m_j]^{k_i}$.

(b) there is $\bar{u} = \langle u_j : j < r\rangle \in \prod_j [m_j]^{s_j}$ such that

- if $i < r$, and $\nu_1, \nu_2 \in [u_i]^{k_i}$ and $\eta_1, \eta_2 \in \mathrm{dom}(\mathbf{c}_i)$ and $j \in r \setminus \{i\} \Rightarrow \eta_1(j) = \eta_2(j) \subseteq u_j$ and $\eta_\iota(i) = \nu_\iota$ for $\iota = 1, 2$, <u>then</u> $\mathbf{c}_i(\eta_1) = \mathbf{c}_i(\eta_2)$.

2) The density version (and $\to_{\min}$, $\to_{\mathrm{av}}$) are phrased similarly.

Claim 2.6. *1) Given r, C, $\bar{s}$ and $\bar{k}$ with $k_i \leq s_i$, <u>then</u> $\min\{m :$ for $\bar{m} = \langle m : i < r\rangle$ we have $\bar{m} \to (\bar{s})^{\bar{k}}_C\}$ is bounded by iteration of length $2r$ of exponentiation.*

2) Similarly for the density version.

Proof. 1) By 1.9, 1.11 and their proof.

2) Use the proof of 1.9, 1.11. $\square_{2.6}$

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§ 2(B). **The average version.** Here we note that $\mathsf{HJ}^{\rm av}_{\varepsilon}(\Lambda,r)$ is not much larger than $\mathsf{HJ}^{\min}_{\varepsilon}(\Lambda,r)$ (see below), but (recall 2.10(3)) $\mathsf{HJ}_{\varepsilon}(\Lambda,r)\le\mathsf{HJ}^{\rm av}_{\varepsilon}(\Lambda,r)$, hence by §1 also $\mathsf{HJ}_{\varepsilon}(\Lambda,r)$ has an $\mathcal{E}^5$ bound. We use a more general relative of 1.8. Also in 2.12(6) we pay a debt from §1.

Definition 2.7. 1) If $B\subseteq A_{\mathbf{s}}$ is not empty then $\mathbf{t}=\mathbf{s}\restriction B$ is $(M_{\mathbf{s}},\mathcal{U}_{\mathbf{s}},A_{\mathbf{t}},\bar{A}_{\mathbf{t}})$ where $A_{\mathbf{t}}=B$ and $A^{\mathbf{t}}_{\eta}=A^{\mathbf{s}}_{\eta}\cap B$.

1A) Let $\mathbf{t}\in\mathrm{Sub}_0(\mathbf{s})$ if for some non-empty $B\subseteq A_{\mathbf{s}}$ we have $\mathbf{t}=\mathbf{s}\restriction B$.

2) For $\mathbf{s}\in\mathbf{S}_{=}$ (that is, $E_{\mathbf{s}}$ is equality), let $\mathrm{Sub}_1(\mathbf{s})$ be the set of $\mathbf{u}$ such that for some $\bar{N}$ we have:

(a) $\bar{N}=(N_0,N_1)$ is a partition of $M_{\mathbf{s}}$,
(b) $M_{\mathbf{u}}=N_0$,
(c) $\mathcal{U}_{\mathbf{u}}=\mathcal{U}_{\mathbf{s}}\cap N_0$ (usually $\mathcal{U}_{\mathbf{s}}\subseteq N_0$),
(d) $A_{\mathbf{u}}=A_{\mathbf{s}}\times{}^{(N_1)}\Lambda$,
(e) for $\nu\in V_{\mathbf{u}}$, i.e. $\nu\in{}^{(N_0)}\Lambda$, we have $A^{\mathbf{u}}_{\nu}=\{(a,\varrho):a\in A_{\mathbf{s}},\ \varrho\in{}^{(N_1)}\Lambda$ and $a\in A^{\mathbf{s}}_{\nu\cup\varrho}\}$.

3) Let $\mathrm{Sub}_2(\mathbf{s})$ be the set of $\mathbf{u}$ such that for some $\mathbf{o}\in\mathrm{ob}(\mathbf{s})$: (a) $\mathrm{pre}(\mathbf{u})=\mathfrak{u}_{\mathbf{o}}(\mathrm{pre}(\mathbf{s}))$, (b) $A_{\mathbf{u}}=A_{\mathbf{s}}\times X$ for some $X\neq\emptyset$, (c) $A^{\mathbf{u}}_{\varrho}\subseteq A^{\mathbf{s}}_{h_{\mathbf{o}}(\varrho)}\times X$.

4) Let $\mathbf{t}\in\mathrm{Sub}_3(\mathbf{s})$ mean that $\mathbf{t}\in\mathrm{sub}_{\mathbf{o}}(\mathbf{s})$ for some $\mathbf{o}\in\mathrm{ob}(\mathbf{s})$. Actually this is a special case of part (3) using X a singleton.

5) Let $\mathrm{Sub}(\mathbf{s})$ be the set of $\mathbf{t}$ such that for some n and $\langle\mathbf{t}_\ell:\ell\le n\rangle$ we have $\mathbf{t}_0=\mathbf{t}$, $\mathbf{t}_n=\mathbf{s}$, and for each $\ell<n$, $\mathbf{t}_\ell\in\mathrm{Sub}_\iota(\mathbf{t}_{\ell+1})$ for some $\iota<4$.

Definition 2.8. 0) Below $\iota\in\{0,1,2,3,\infty\}$, and $\mathrm{Sub}_\infty=\mathrm{Sub}$.

1) For $\mathfrak{s},\mathfrak{t}\in\mathfrak{S}$ let $(\mathfrak{s},\Omega_1)\to_{\iota,\min}(\mathfrak{t},\Omega_2)_{\varepsilon,\zeta}$ mean

(a) $\mathfrak{s},\mathfrak{t}$ are pre-systems, $\Omega_1\subseteq V_{\mathfrak{s}}$, $\Omega_2\subseteq V_{\mathfrak{t}}$, both non-empty.
(b) $0<\zeta<\varepsilon\le1$
(c) If $\mathbf{s}$ is a system expanding $\mathfrak{s}$ with $\min(\mathbf{s},\Omega_1)\ge\varepsilon$ then there exists a system $\mathbf{t}$ with $\mathrm{pre}(\mathbf{t})=\mathfrak{t}$ and $\min(\mathbf{t},\Omega_2)\ge\zeta$ which is isomorphic to a member $\mathbf{u}$ of $\mathrm{Sub}_\iota(\mathbf{s})$.

2) Let $(\mathfrak{s},\Omega_1)\to_{\iota,\mathrm{av}}(\mathfrak{t},\Omega_2)_{\varepsilon,\zeta}$ mean[6]

(a) $\mathfrak{s},\mathfrak{t}$ are pre-systems, $\Omega_1\subseteq V_{\mathfrak{s}}$, $\Omega_2\subseteq V_{\mathfrak{t}}$, and both also non-empty.
(b) $0<\zeta<\varepsilon\le1$
(c) If $\mathbf{s}$ is a system expanding $\mathfrak{s}$ with $\mathrm{av}(\mathbf{s}\restriction\Omega_1)\ge\varepsilon$ then there exists a system $\mathbf{t}$ with $\mathrm{pre}(\mathbf{t})=\mathfrak{t}$ and $\mathrm{av}(\mathbf{t}\restriction\Omega_2)\ge\zeta$ which is isomorphic to a member $\mathbf{u}$ of $\mathrm{Sub}_\iota(\mathbf{s})$.

3) We define $(\mathfrak{s},\Omega_1)\to_{\iota,\mathrm{avm}}(\mathfrak{t},\Omega_2)_{\varepsilon,\zeta}$ to mean:

(a) $\mathfrak{s},\mathfrak{t}$ are pre-systems, $\Omega_1\subseteq V_{\mathfrak{s}}$, $\Omega_2\subseteq V_{\mathfrak{t}}$, both non-empty.
(b) $0<\zeta<\varepsilon\le1$
(c) If $\mathbf{s}$ is a system expanding $\mathfrak{s}$ with $\mathrm{av}(\mathbf{s},\Omega_1)\ge\varepsilon$ then there exists a system $\mathbf{t}$ with $\mathrm{pre}(\mathbf{t})=\mathfrak{t}$ and $\min(\mathbf{t},\Omega_2)\ge\zeta$ which is isomorphic to a member $\mathbf{u}$ of $\mathrm{Sub}_\iota(\mathbf{s})$.

We define $(\mathfrak{s},\Omega_1)\to_{\mathrm{mav}}(\mathfrak{t},\Omega_2)_{\varepsilon,\zeta}$ similarly but in clause (c), $\min(\mathbf{s})\ge\varepsilon$, $\mathrm{av}(\mathbf{t})\ge\zeta$.

4) If $\Omega_1=V_{\mathfrak{s}}$ we may write $\mathfrak{s}$; similarly for Ω_2.

5) Assume $0<\zeta<\varepsilon\le1$. Let $\mathsf{HJ}^{\rm av}_{\varepsilon,\zeta}(\Lambda,r)$ be the minimal k such that for every $\mathbf{s}$ we have:

[6] Of course, if Ω_1 contains no sub-space of dimension r this will fail.

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(∗) If $M_{\mathbf{s}}$ have k elements and $\mathrm{av}(\mathbf{s}) \geq \varepsilon$ then there is $\mathbf{o} \in \mathrm{ob}(\mathbf{s})$ we have $N_{\mathbf{o}}/\mathscr{E}_{\mathbf{o}}$ has r elements and $\mu\big(\cap\{A_\eta : \eta \in \mathrm{Rang}(h_{\mathbf{o}})\}\big) \geq \zeta$.

6) Letting $\zeta = 0$ mean for some $\zeta > 0$, and then we can omit ζ in part (5); note that $E_{\mathbf{s}}, \alpha_{\mathbf{s}}, \beta_{\mathbf{s}}$ are irrelevant.

7) For $\mathbf{s} \in \mathbf{S}$ let $\mathrm{sob}(\mathbf{s})$ be the set of $\mathbf{o} \in \mathrm{ob}(\mathbf{s})$ such that $\mathscr{E}_{\mathbf{o}}$ is the equality on $N_{\mathbf{o}}$; so for $\mathbf{u} \in \mathrm{sub}_{\mathbf{o}}(\mathbf{s})$ we may identify $M_{\mathbf{u}}$ with $M_{\mathbf{s}} \restriction N_{\mathbf{o}}$.

Remark 2.9. How are 1.8(1), 2.8, 0.4 connected? For $\mathfrak{s} = (m, =)$ and $\mathfrak{t} = (r, \mathrm{all})$: $\mathsf{HJ}^{\min}_{\varepsilon,\zeta}(\Lambda, r) \leq m$ iff $\mathfrak{s} \to_{\min} (\mathfrak{t})_{\varepsilon,\zeta}$ in the sense of 1.8(1), and $\mathsf{HJ}^{\mathrm{av}}_{\varepsilon,\zeta}(\Lambda, r) \leq m$ iff $\mathfrak{s} \to_{\mathrm{av}} (\mathfrak{t})_{\varepsilon,\zeta}$ in the sense of 2.8(2) (this is 2.10(5),(6)). The HJ_ε of 0.4 is the case $|A_{\mathbf{s}}| = 1$ (so $\mu(A_\eta) \in \{0, 1\}$ and $X = \{\eta : A_\eta \neq \emptyset\}$); hence $\mathsf{HJ}_\varepsilon(\Lambda, r) \leq \mathsf{HJ}^{\mathrm{av}}_{\varepsilon,\zeta}(\Lambda, r)$ for any $\zeta > 0$ (2.10(3)), and $\mathsf{HJ}^{\min} \leq \mathsf{HJ}^{\mathrm{av}}$ (2.10(2)). So 0.4 is the special case, 1.8(1) the min-form and 2.8 the av-form of the same arrow.

Claim 2.10. 1) *Let* $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$.

- $\mathfrak{s} \to_{\mathrm{av}} (\mathfrak{t})_{\varepsilon,\zeta}$ *implies* $\mathfrak{s} \to_{\min} (\mathfrak{t})_{\varepsilon,\zeta}$ *when* $\mathfrak{t} = (M_{\mathfrak{t}}, \mathrm{all})$.
- $(\mathfrak{s}, \Omega_1) \to_{\mathrm{av}} (\mathfrak{t}, \Omega_2)_{\varepsilon,\zeta}$ *implies* $(\mathfrak{s}, \Omega_1) \to_{\min} (\mathfrak{t}, \Omega_2)_{\varepsilon,\zeta}$ *when* $\mathfrak{t} = (M_{\mathfrak{t}}, \mathrm{all})$.
- $(\mathfrak{s}, \Omega_1) \to_{\min} (\mathfrak{s}, \Omega_2)_{\varepsilon,\zeta}$ *when* $\Omega_2 \subseteq \Omega_1$.

2) $\mathsf{HJ}^{\min}_{\varepsilon,\zeta}(\Lambda, r) \leq \mathsf{HJ}^{\mathrm{av}}_{\varepsilon,\zeta}(\Lambda, r)$.

3) $\mathsf{HJ}_\varepsilon(\Lambda, r) = \mathsf{HJ}^{\mathrm{av}}_\varepsilon(\Lambda, r)$.

4) $\mathsf{HJ}^{\min}_{\varepsilon,\zeta}(\Lambda, r) \leq m$ *when we have* $(\mathfrak{s}, \Omega) \to_{\min} (\mathfrak{t}, V_{\mathfrak{t}})_{\varepsilon,\zeta}$ *for some non-empty* $\Omega \subseteq V_{\mathfrak{s}}$, *where* $\mathfrak{s} = (m, =)$ *and* $\mathfrak{t} = (r, \mathrm{all})$.

5) $\mathsf{HJ}^{\min}_{\varepsilon,\zeta}(\Lambda, r) \leq m$ *iff* $\mathfrak{s} \to_{\min} (\mathfrak{t})_{\varepsilon,\zeta}$ *when* $\mathfrak{s} = (m, =)$, $\mathfrak{t} = (r, \mathrm{all})$.

6) $\mathsf{HJ}^{\mathrm{av}}_{\varepsilon,\zeta}(\Lambda, r) \leq m$ *iff* $\mathfrak{s} \to_{\mathrm{av}} (\mathfrak{t})_{\varepsilon,\zeta}$ *when* $\mathfrak{s} = (m, =)$, $\mathfrak{t} = (r, \mathrm{all})$.

7) (a) If $\mathfrak{s} \to_{\infty,x} (\mathfrak{t})_{\varepsilon,\zeta}$ *then* $\mathfrak{s} \to_x (\mathfrak{t})_{\varepsilon,\zeta}$, *for* $x \in \{\min, \mathrm{av}, \mathrm{mav}, \mathrm{avm}\}$.
(b) If $\mathfrak{s}_1 \to (\mathfrak{s}_2)_{\varepsilon_1,\varepsilon_2}$ *and* $\mathfrak{s}_2 \to (\mathfrak{s}_3)_{\varepsilon_2,\varepsilon_3}$ *then* $\mathfrak{s}_1 \to (\mathfrak{s}_3)_{\varepsilon_1,\varepsilon_3}$.
(c) Similarly for $\to_\infty$.

8) (a) If $\mathfrak{s}_1 \to_{\mathrm{mav}} (\mathfrak{s}_2)_{\varepsilon_1,\varepsilon_2}$ *and* $\mathfrak{s}_2 \to_{\mathrm{avm}} (\mathfrak{s}_3)_{\varepsilon_2,\varepsilon_3}$ *then* $\mathfrak{s}_1 \to_{\min} (\mathfrak{s}_3)_{\varepsilon_1,\varepsilon_3}$.
(b) If $\mathfrak{s}_1 \to_{\mathrm{avm}} (\mathfrak{s}_2)_{\varepsilon_1,\varepsilon_2}$ *and* $\mathfrak{s}_2 \to_{\mathrm{mav}} (\mathfrak{s}_3)_{\varepsilon_2,\varepsilon_3}$ *then* $\mathfrak{s}_1 \to_{\mathrm{av}} (\mathfrak{s}_3)_{\varepsilon_1,\varepsilon_3}$.
(c) In both cases we can replace $\to$ *by* $\to_\infty$.

Proof. 1), 2) Very easy.

3) For "$\mathsf{HJ}_\varepsilon(\Lambda, r) \leq \mathsf{HJ}^{\mathrm{av}}_\varepsilon(\Lambda, r)$": Let $m = \mathsf{HJ}^{\mathrm{av}}_\varepsilon(\Lambda, r)$, and let $\mathfrak{s}$ be $(m, =)$, i.e. defined by (m, E) with E being the equality on $V_{\mathfrak{s}}$. Let $X \subseteq V_{\mathfrak{s}}$ have density $\geq \varepsilon$. Define $\mathbf{s}$ by

- A any fixed non-empty set
- $\mathrm{pre}(\mathbf{s}) := \mathfrak{s}$
- $\mathbf{a} = \langle A_\eta : \eta \in V_{\mathfrak{s}} \rangle$ is defined by

$$A_\eta := \begin{cases} A & \text{if } \eta \in X \\ \emptyset & \text{if } \eta \notin X. \end{cases}$$

Clearly

$$(*)\quad \mathrm{av}(\mathbf{s}) = \frac{1}{|V_{\mathfrak{s}}|} \Sigma\{|A_\eta|/|A| : \eta \in V_{\mathfrak{s}}\} = \frac{1}{|{}^m\Lambda|}\,|X| \geq \varepsilon.$$

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Now we can apply $m = \mathsf{HJ}^{\rm av}_{\varepsilon}(\Lambda, r)$ to get what $m \geq \mathsf{HJ}_{\varepsilon}(\Lambda, r)$ promises.

For the other inequality, $\mathsf{HJ}_{\varepsilon}(\Lambda, r) \geq \mathsf{HJ}^{\rm av}_{\varepsilon}(\Lambda, r)$, assume $k = \mathsf{HJ}_{\varepsilon}(\Lambda, r)$ and consider $\mathbf{s} \in \mathbf{S}$, $|M_{\mathbf{s}}| = k$ and $\mathrm{av}(\mathbf{s}) \geq \varepsilon$. Let $f : A_{\mathbf{s}} \to [0,1]_{\mathbb{R}}$ be defined by $f(a) = |\{\eta \in V_{\mathbf{s}} : a \in A^{\mathbf{s}}_{\eta}\}|/|V_{\mathbf{s}}|$. So the average value of f, that is $\frac{1}{|A|}\Sigma\{f(a) : a \in A\}$, is equal to $\mathrm{av}(\mathbf{s})$, hence for some $a \in A$ we have $f(a) \geq \varepsilon$. Now apply the choice of k.

4) Easy.

5), 6), 7), 8) By the definitions. $\square_{2.10}$

Remark 2.11. In 2.10(3), for the inequality $\mathsf{HJ}_{\varepsilon}(\Lambda, r) \geq \mathsf{HJ}^{\rm av}_{\varepsilon}(\Lambda, r)$: the subspace found in $X_a = \{\eta : a \in A_\eta\}$ has $\mu(\cap A_\eta) \geq \frac{1}{|A_{\mathbf{s}}|}$, which depends on $\mathbf{s}$; so the equality is for $\zeta = 0$ in the sense of 2.8(6), not for a fixed ζ (for $\mathsf{HJ}^{\rm av}_{\varepsilon,\zeta}$ with fixed $\zeta > 0$ only "$\leq$" is clear).

Claim 2.12. 1) *Suppose:*

(a) $\zeta < \varepsilon_* < \varepsilon \leq 1$

(b) $m_1 \geq \mathsf{HJ}^{\min}_{\varepsilon_*,\zeta}(\Lambda, r)$

(c) $\varepsilon \leq \varepsilon_\bullet \leq \varepsilon + \dfrac{\varepsilon - \varepsilon_*}{|^{m_1}\Lambda| - 1}$

(d) $m_2 \geq \mathsf{HJ}^{\rm av}_{\varepsilon_\bullet,\zeta}(\Lambda, r)$.

Then $m_1 + m_2 \geq \mathsf{HJ}^{\rm av}_{\varepsilon,\zeta}(\Lambda, r)$.

2) Suppose we have sequences $\langle \varepsilon_i : i < k\rangle$ *with* $k \geq 1$ *such that*

(a) $\zeta < \varepsilon_* < \varepsilon_0 \leq 1$ *and*

(b) $m \geq \mathsf{HJ}^{\min}_{\varepsilon_*,\zeta}(\Lambda, r)$

(c) $\varepsilon_i \leq \varepsilon_{i+1} \leq \min\left\{1,\ \varepsilon_i + \dfrac{\varepsilon_i - \varepsilon_*}{|^m\Lambda| - 1}\right\}$ *for* $i < k-1$.

(d) $\varepsilon_{k-1} = 1$.

(e) $\mathfrak{s} = (m, =)$ *and* $\mathfrak{t} = (r, \mathrm{all})$.

Then $k \times \mathfrak{s} \to_{\infty,\rm av} (\mathfrak{t})_{\varepsilon_0,\zeta}$.

3) If $r \geq 1$, $\mathbf{s} \in \mathbf{S}$ *is a system and* $\min(\mathbf{s}) > 1 - \dfrac{1}{|\Lambda|^r} + \zeta$ *then for every* r*-dimensional subspace* V *of* $V_{\mathbf{s}}$ *we have* $\zeta \leq \mu\big(\cap\{A^{\mathbf{s}}_\eta : \eta \in V\}\big)$.

4) Assume

(a), (b) *of part (2),*

(c) $n = \left\lceil \dfrac{|^m\Lambda| - 1}{\varepsilon - \varepsilon_*} \right\rceil$.

Then $m \cdot (n+1) \geq \mathsf{HJ}^{\rm av}_{\varepsilon,\zeta}(\Lambda, r)$*; that is,* $(n+1) \times \mathsf{HJ}^{\min}_{\varepsilon_*,\zeta}(\Lambda, r) \geq \mathsf{HJ}^{\rm av}_{\varepsilon,\zeta}(\Lambda, r) \geq \mathsf{HJ}^{\min}_{\varepsilon,\zeta}(\Lambda, r)$.

4A) In part (4) we also have $\mathfrak{s}' \to_{\infty,\rm av} (\mathfrak{t})_{\varepsilon,\zeta}$ *where* $\mathfrak{s}' = ((n+1)\cdot m, =)$ *and* $\mathfrak{t} = (r, \mathrm{all})$.

5) Parts (4), (4A) apply also to $\zeta = 0$.

6) If (A) then (B), where

(A) (a) $\mathfrak{s} = m \times \mathfrak{t}$,

(b) $0 < \zeta < \varepsilon < 1$,

(c) $m \cdot \dfrac{\varepsilon - \zeta}{|V_{\mathfrak{t}}| - 1} \geq 1 - \varepsilon$;

(d) $\mathcal{U}_{\mathfrak{t}} = \emptyset$, *hence* $\mathcal{U}_{\mathfrak{s}} = \emptyset$.

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(B) $\mathfrak{s} \to_{\infty,\mathrm{avm}} (\mathfrak{t})_{\varepsilon,\zeta}$.

7) Similarly to 6) with $\Omega \subseteq V_{\mathfrak{s}}$.

Proof. 1) For $\iota = 1,2$ we define the pre-system $\mathfrak{s}_\iota = (\{\iota\} \times m_\iota, =)$ where $\{\iota\} \times m_\iota$ is naturally ordered; similarly define the pre-system $\mathfrak{s} = (M_{\mathfrak{s}_1} + M_{\mathfrak{s}_2}, =)$.

Let $\mathbf{s}$ expand $\mathfrak{s}$, and we should find $\mathbf{t}$ as promised. Clearly

$(*)_0$ $\eta \in V_{\mathfrak{s}}$ iff $(\exists \rho \in V_{\mathfrak{s}_1})(\exists \nu \in V_{\mathfrak{s}_2})(\eta = \rho \cup \nu)$.

We define $\mathbf{s}_1$ expanding $\mathfrak{s}_1$ as follows.

$(*)_1$ (a) $A_{\mathbf{s}_1} = V_{\mathfrak{s}_2} \times A_{\mathbf{s}}$

(b) for $\rho \in V_{\mathbf{s}_1}$ we let $A_{\mathbf{s}_1,\rho} = \{(\nu, a) : \nu \in V_{\mathfrak{s}_2}$ and $a \in A_{\rho \cup \nu}\}$.

Now

$(*)_2$ (a) $\mathbf{s}_1$ indeed is an expansion of $\mathfrak{s}_1$

(b) $\mathrm{av}(\mathbf{s}_1) = \mathrm{av}(\mathbf{s})$.
[Why? Easy to check.]

$(*)_3$ If $(\forall \rho \in V_{\mathbf{s}_1})\big[\mu_{\mathbf{s}_1}(\rho) \geq \varepsilon_*\big]$ then the desired conclusion holds.
[Why? Because in the claim we assume $m_1 \geq \mathsf{HJ}^{\min}_{\varepsilon_*,\zeta}(\Lambda, r)$, so there is $\mathbf{t}$ expanding $\mathfrak{t}$ and $\mathbf{o}_1 \in \mathrm{ob}(\mathfrak{s}_1)$ such that $\mathbf{t}$ is isomorphic to some $\mathbf{u} \in \mathrm{sub}_{\mathbf{o}_1}(\mathbf{s}_1)$ such that $\mu_{\mathbf{s}_1}(B) \geq \zeta$ where

$$B = \cap\{A_{\mathbf{s}_1,\rho} : \rho \in \mathrm{Rang}(h_{\mathbf{o}_1})\}.$$

Now B is the disjoint union of $\langle B \cap (\{\nu\} \times A_{\mathbf{s}}) : \nu \in V_{\mathfrak{s}_2}\rangle$. Also $\langle \{\nu\} \times A_{\mathbf{s}} : \nu \in V_{\mathfrak{s}_2}\rangle$ is a partition of $A_{\mathbf{s}_1}$ to sets of equal cardinality. Together we have

$(*)_{3.1}$ $\mu_{\mathbf{s}_1}(B) = \dfrac{1}{|V_{\mathfrak{s}_2}|}\Sigma\big\{|B \cap (\{\nu\} \times A_{\mathbf{s}})| \,/\, |A_{\mathbf{s}}| : \nu \in V_{\mathfrak{s}_2}\big\}$.

Hence

$(*)_{3.2}$ for some $\nu_* \in V_{\mathfrak{s}_2}$ we have $|B \cap (\{\nu_*\} \times A_{\mathbf{s}})| \,/\, |A_{\mathbf{s}}| \geq \mu_{\mathbf{s}_1}(B) \geq \zeta$.

Now we define $\mathbf{o} \in \mathrm{ob}(\mathfrak{s})$ by

$(*)_{3.3}$ $\bullet_0$ $N_{\mathbf{o}} = N_{\mathbf{o}_1}$

$\bullet_1$ $\mathscr{E}_{\mathbf{o}} = \mathscr{E}_{\mathbf{o}_1}$

$\bullet_2$ $\rho_{\mathbf{o}}(c) = \rho_{\mathbf{o}_1}(c)$ if $c \in \mathrm{dom}(\rho_{\mathbf{o}_1})$

$\bullet_3$ $\rho_{\mathbf{o}}(c) = \nu_*(c)$ for $c \in M_{\mathfrak{s}_2}$.

Now check, so $(*)_3$ holds indeed.]

Hence without loss of generality the assumption of $(*)_3$ fails.

$(*)_4$ (a) So we can choose $\rho_1 \in V_{\mathbf{s}_1}$ such that $\mu_{\mathbf{s}_1}(\rho_1) < \varepsilon_*$.

(b) We can choose $\rho_2 \in V_{\mathbf{s}_1} \setminus \{\rho_1\}$ such that

$$\mu_{\mathbf{s}_1}(\rho_2) \geq \varepsilon + (\varepsilon - \varepsilon_*)\frac{1}{|^{m_1}\Lambda| - 1} = \varepsilon_\bullet.$$

[Why? Clause (a) as we assume $(*)_3$ does not apply; clause (b) by easy computation.]

$(*)_5$ Define $\mathbf{s}_2$, an expansion of $\mathfrak{s}_2$, by

(a) $A_{\mathbf{s}_2} = A_{\mathbf{s}}$

(b) for $\nu \in V_{\mathbf{s}_2}$ we let $A_{\mathbf{s}_2,\nu} = A^{\mathbf{s}}_{\rho_2 \cup \nu}$.

Now

$(*)_6$ (a) $\mathbf{s}_2$ indeed is an expansion of $\mathfrak{s}_2$

(b) $\mathrm{av}(\mathbf{s}_2) = \mu_{\mathbf{s}_1}(\rho_2) > \varepsilon_\bullet$.
[Why? By the choice of $\mathbf{s}_2$ and of ρ_2.]

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$(*)_7$ There is $\mathbf{t}$ as required.

[Why? In clause (d) we have assumed $\mathfrak{s}_2 \to_{\rm av} (\mathfrak{t})_{\varepsilon_\bullet,\zeta}$. Hence there is $\mathbf{o}_2 \in {\rm ob}(\mathfrak{s}_2)$ and $\mathbf{t} \in {\rm sub}_{\mathbf{o}_2}(\mathbf{s}_2)$ such that $\mu_{\mathbf{s}_2}(B) \geq \zeta$ where $B = \cap\{A_{\mathbf{t},\nu} : \nu \in V_{\mathfrak{t}}\}$.

We define $\mathbf{o} \in {\rm ob}(\mathfrak{s})$ by

$(*)_8$
- $\bullet_0$ $N_{\mathbf{o}} = N_{\mathbf{o}_2}$
- $\bullet_1$ $\mathscr{E}_{\mathbf{o}} = \mathscr{E}_{\mathbf{o}_2}$
- $\bullet_2$ $\rho_{\mathbf{o}}(c) = \rho_{\mathbf{o}_2}(c)$ if $c \in {\rm dom}(\rho_{\mathbf{o}_2})$
- $\bullet_3$ $\rho_{\mathbf{o}}(c) = \rho_2(c)$ if $c \in M_{\mathfrak{s}_1}$.

Now check that $\mathbf{u}_{\mathbf{o}}(\mathbf{s})$ is as required.]

2) We prove by downward induction on $j \leq k-1$ that $(k-j) \times \mathfrak{s} \to_{\rm av} (\mathfrak{t})_{\varepsilon_j,\zeta}$.

For $j = k-1$ clause (b) of the claim assumption gives the desired conclusion. [Why? Here $(k-j) \times \mathfrak{s} = 1 \times \mathfrak{s}$ is (isomorphic to) $\mathfrak{s} = (m,=)$. Let $\mathbf{s}$ expand it with ${\rm av}(\mathbf{s}) \geq \varepsilon_{k-1} = 1$. As ${\rm av}(\mathbf{s})$ is an average of numbers ≤ 1, necessarily $\eta \in V_{\mathbf{s}} \Rightarrow A^{\mathbf{s}}_\eta = A$, so the desired conclusion holds as $m \geq r$.]

For $j = \ell - 1 < k-1$ we again apply part (1) with $(m, (k-\ell) \times m, \varepsilon_{j+1}, \varepsilon_j, \varepsilon_*, \zeta)$ here standing for $(m_1, m_2, \varepsilon_\bullet, \varepsilon, \varepsilon_*, \zeta)$ there.

Having carried the induction, for $j = 0$ we get the desired conclusion.

3) Easy.

4) We choose ε_i by induction on $i \leq n$: $\varepsilon_0 := \varepsilon$ and $\varepsilon_{i+1} := \min\{1,\ \varepsilon_i + \frac{\varepsilon - \varepsilon_*}{|^m\Lambda|-1}\}$. This sequence is non-decreasing, and as $\varepsilon_i \geq \varepsilon$ we have $\varepsilon_{i+1} \leq \min\{1, \varepsilon_i + \frac{\varepsilon_i - \varepsilon_*}{|^m\Lambda|-1}\}$; also $\varepsilon_n = 1$, since $\varepsilon + n\frac{\varepsilon-\varepsilon_*}{|^m\Lambda|-1} \geq \varepsilon + 1 > 1$ by clause (c). Now apply part (2) with m, $n+1$, $\langle \varepsilon_i : i \leq n\rangle$ here standing for m, k, $\langle \varepsilon_i : i \leq k\rangle$ there (its clause (a) holds as $\zeta < \varepsilon_* < \varepsilon = \varepsilon_0$). We get $(n+1) \times \mathfrak{s} \to_{\rm av} (\mathfrak{t})_{\varepsilon,\zeta}$ for $\mathfrak{s} = (m,=)$, $\mathfrak{t} = (r, {\rm all})$, i.e. $m \cdot (n+1) \geq {\rm HJ}^{\rm av}_{\varepsilon,\zeta}(\Lambda, r)$.

4A) By the proof of (4), as $(n+1) \times \mathfrak{s}$ is isomorphic to $((n+1)m, =)$.

5) Follows.

6) As in part (2), using the projection of 2.7(2) $({\rm Sub}_1)$ on the first copy of $\mathfrak{t}$ and, when it fails, the slice through a heavy fibre (a member of ${\rm Sub}_3(\mathbf{s})$), m times; (A)(c) guarantees that we do not fail m times.

7) Similarly to part (6).

$\square_{2.12}$

Claim 2.13. *1) If* $\mathbf{t} \in {\rm Sub}(\mathbf{s})$ *and* $\mathbf{u} \in {\rm Sub}(\mathbf{t})$ *then there is* $\mathbf{u}' \in {\rm Sub}(\mathbf{s})$ *isomorphic to* $\mathbf{u}$.

2) If $\mathfrak{s}_1 \to_\infty (\mathfrak{s}_2)_{\varepsilon_1,\varepsilon_2}$ *and* $\mathfrak{s}_2 \to_\infty (\mathfrak{s}_3)_{\varepsilon_2,\varepsilon_3}$ *then* $\mathfrak{s}_1 \to_\infty (\mathfrak{s}_3)_{\varepsilon_1,\varepsilon_3}$.

Proof. Straightforward. $\square_{2.13}$

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