DIVIDE AND CONQUER: DIVIDING LINES AND UNIVERSALITY

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ABSTRACT. We discuss dividing lines (in model theory) and some test questions, mainly the universality spectrum. So there is much on conjectures, problems and old results, mainly of the author and also on some recent results.

§ 0. INTRODUCTION

The main theme of our work in model theory aims at the identification and exploitation of robust, successful, and fruitful dividing lines. In this introductory Part I we will explain in general terms what this means, and how this is done. In particular, we will discuss the use of test problems to find dividing lines. The bulk of the paper, Parts II and III, is devoted mainly to the discussion of a particular test problem, concerning the universality spectrum of a first order theory. What makes this test problem particularly interesting to us now is the fact that it for long appeared unpromising as a source of good dividing lines, and now appears very promising.

The way we look for dividing lines is via test problems, and these generally involve set theoretic notions which are in some sense foreign to algebraic first order model theory, but are in the heart of general model theory. For example, we ask how to determine the number of models of a first order theory in cardinal λ. That is we consider the function $I(\lambda, T)$ which associates to a first order theory T and a cardinal λ, the number of models of T of cardinality λ, counted up to isomorphism. It is more useful to think of this as a binary relation $\tilde{I}(f, T)$ where f is a class function on cardinals and T is a first order theory: the relation holds if

$$f(\cdot) = I(\cdot, T).$$

One side of the question is which functions may occur. The other side of the question is what the function tells us about the theory. We call such a property an external property, in this case, as usual, connected to set theory, but e.g. computation (e.g. recursion) theory can suffice. We are interested in cases in which the external property has an internal characterization, typically in terms of the behavior of individual formulas relative to the theory T. By the Compactness Theorem, such characterizations are likely to have a finitary character. This necessarily involves set theoretic concepts but the thesis is that it will direct us to more syntactic properties, of interest also to those to whom the original question was not appealing.

One may suggest an elementary analogy. There is a non-standard construction of the real line as the non-standard finite part of the rational line, modulo infinitesimals. This

gives, potentially, many real lines, and many notions of convergence. So the notion is momentarily "external," until the Cauchy criterion is given as an internal characterization of convergence—with a more combinatorial flavor.

Our test problems, and the associated dividing lines, tend to measure the relative complexity of theories, as expressed by some partial order (actually, a quasi-order at the level of theories). The most direct comparison of theories is by interpretability: $T_1 \leq_i T_2$ if T_1 can be interpreted in T_2. This is much too fine. A dividing line aims to break the theories into two parts, the more simple and the less simple, in a useful way. More generally we may seek to impose some coarse quasi-order on the theories, hopefully with finitely many classes, and understand something about the structure, notably the height, of the partial order, and the minimal and maximal elements.

In the case of the test problem on the number of models, a natural quasi- order would be derived from the corresponding relation on functions

$$f \leq^* g \Leftrightarrow \text{For sufficiently large } \lambda, f(\lambda) \leq g(\lambda)$$

which allows for some "sporadic" behavior (notably, for $\lambda = \aleph_0$). In particular, the minimum would then correspond to theories categorical in sufficiently high power, a case treated in Morley's seminal work. The so-called "Main Gap" refers to the behaviour between the maximum and the minimum, which is crucial for the general case of this test problem. (However, as we will see below, the way in which the quasi-order on theories is derived from the quasi-order on functions is not the most straightforward one.)

The theory developed out of the test problem of the determination of the number of models of given cardinality has been the most successful and productive to date, in most respects. We have considered a number of other test problems which we will refer to in the body of the paper. One which deserves additional mention here, before we focus on the problem of the universality spectrum, is Keisler's order. This is defined directly in terms of a quasi-order on theories, rather than a single test problem and the associated test problems are structural: is the associated partial order linear, what are the minimum and maximum elements, are there infinitely many classes? We will go into more detail below. Here we speak broadly on a few points of history.

Initially the study of the low end of Keisler's order produced a new dividing line within stability and a complete understanding of the stable case. Then the subject moved out of focus and languished for several decades with little progress, and with no concrete ideas as to how to proceed further. New developments have brought the subject back to life, showing in particular that the number of classes is infinite and that in particular, infinitely many robust classes appear; serendipitously, new applications to set theory arose, which actually was not a goal of this particular program.

Set theoretic applications aside, we wish to suggest that the study of the universality spectrum may offer some parallels. While the subject was never near death, the associated test problems appeared both a priori and empirically as a not so promising source of fundamental dividing lines. Now as we progress the associated test problems begin to look quite promising, and this is the main point we wish to discuss. The question at this stage is not, specifically, as to whether we come to new dividing lines, but rather whether we come to any important dividing line at all. (We are not averse to encountering old friends in new places.)

We have one last methodological point to address before entering into details. Namely, the use of set theory, or rather, the malleability of set theory, introduces a further complication. Cardinal arithmetic tends to be a prominent ingredient of our test problems, particularly at the "high end" in terms of complexity of theories, and as we learned starting

with Paul Cohen, cardinal arithmetic can be made to behave in almost all ways that can be imagined. In some such universes the class of (A, E), E an equivalence relation, will be maximal. As a result, we do not always want to take the answer to our question in the "real world" but in all possible worlds, or at least all which can be easily reached from the real world (by nice forcing). An alternative is to change the question somewhat to "the maximal number of models of T of cardinality λ pairwise non-embeddable to each other." Then we have a chance to discover useful "intrinsic" properties of our theories. So we build our quasi-orders in two steps, first asking set theoretic questions about first order theories, and then repeating the questions in suitable forcing extensions of the set theoretic universe—or in other ways; if our dividing lines are robust, as we hope, it ought not to matter too much how we deal with the incompleteness of set theory. But we must choose a definite way of proceeding, and whatever way we choose will then require us to make some set theoretic constructions.

One noteworthy point is that within the mass of undecidability results associated with cardinal arithmetic, there is a "hidden kernel" of absoluteness, associated with pcf theory (the theory of possible cofinalities). This theory will also make a useful appearance in the body of the present paper, as in much else of the author's work.

The *universality spectrum* of a class K of structures is the class $\mathrm{Univ}(K)$ of (infinite) cardinals in which K has a universal member, i.e. a model of that cardinality such that any other member from this class of this cardinality can be embedded into it; in some cases "cardinality" is replaced by "topological density". This is a natural notion which arose independently in several contexts (see also [Bal20]).

Below we shall write on test problems concerning the universality spectrum, but also on test problems of other types, some of a very similar character, others less so.

We focus on the family of elementary classes, i.e. the classes of models of a first order theory T and we may write T instead of the class of models of T. This is quite a wide context and is the classical one in model theory and covers many natural examples. We think that maybe we should consider trying to get a complete (or partial) characterization of universality spectra and the associated classes of first order theories. This is certainly a hard enough challenge.

This topic sits at a crossroads:

(A)   We need model theory to analyze the properties of the theories, using known properties of theories, or (which is more exciting) finding new dividing lines among them and investigating them, thus advancing a major theme of model theory which was also very successful in applications. In particular there are classical subjects related to this, like pseudo finite fields.

(B)   We need infinite combinatorics to prove relevant properties of the class that suffice for the non-existence of a universal model of cardinality λ.

(C)   We are interested also in specific classes, e.g. the classes of linear orders, groups, Abelian groups, graphs and even Banach spaces. Sometimes it suffices to prove that such a class fits a general theorem but we have to prove that the class fits it. Sometimes the quoted results need to be adapted and sometimes, even more challenging, a really new method and new properties of the class have to be found. So the results should be of interest even to researchers with marginal interest in set theory and model theory.

(D)   Last but not least, we need forcing from set theory to prove consistency results, mainly to prove that cardinal arithmetic conditions (such as $\lambda = \mu^{+k}, \mu^{\aleph_0} = \mu$) are not sufficient to give an answer.

We feel the time is ripe for advances in finding a general criterion which is necessary and sufficient for proving that any reasonable class K satisfying it has a universal member in λ, or at least for clarifying the picture.

The article is divided to three parts according to the intended audience such that each can be read independently.

Part I is meant to be accessible both to philosophers and to mathematicians not working in mathematical logic. Part II requires some familiarity with model theory and set theory. Part I is recapitulated very briefly in Part II, and mathematical logicians may prefer to start there. We note that Part II does not require familiarity with classification theory, though such familiarity with that subject, or its applications, casts a great deal of light on the significance of the dividing lines we consider below. The question as to what further theory may develop out of dividing lines associated with the test problems considered here is a separate issue, not addressed in the present paper. Parts I and II describe goals and results, without proof. In Part III we give some new results with proofs.

A drawback of this organization is that some repetition is unavoidable.

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Part I

Contents

In the following we mention some well-established dividing lines (Section 1), propose a new test question (Section 2) and describe how this new notion fits into and interacts with the established properties and the Keisler order (Sections 3-5), and mention additional goals (Section 6).

In each section, we describe the state of the art and some known theorems; as well as the new terrain, our plans for the proposed projects, in the form of problems and conjectures. We also postulate several theses (informal statements, often expressing the author's personal opinion).

To make this paper converge we present the knowledge as it is in September 2018. On recent information on the problems, see [Shef] on T with no universal model in a singular cardinality, [Sheh] on sufficient conditions for $\lambda \notin \text{univ}(T_{\text{rg}})$, [Shei] on $\lambda \in [\mu, \mu^{\text{cf}(\mu)})$ for μ singular, mainly strong limit, [KRS]. Now [Shei] will deal also with some questions left open on Abelian groups (see §10B, in particular the table there). Lastly, a work in preparation with P. Komjath shall deal with the the universality spectrum (under embeddability, in uncountable cardinals) for K_H, the class of graphs into which there is no weak embedding of the finite graphs H, (see §10D on the countable case).

§ 1. THE AIM: THE CLASSIFICATION QUESTION

Thesis 1.1. 1) It is worthwhile to classify theories, i.e. find good and meaningful dividing lines, or dichotomies.

2) Good test problems help us to find the right dividing lines.

3) In more detail: We look for properties P of a theory T such that we have relevant information on both sides: about those theories which have property P and about those which do not have it. Usually one side (having P, say) tells us that T is analyzable, or not too complicated in a certain way, and that we can develop a "positive" theory, i.e. a structure theory for the class of its models. In the other case, i.e. theories not having P, we can prove it has complicated models in a suitable sense.

Let us give names to some aspects of this intuition:

Informal Definition 1.2. We call a candidate for being a dividing line (a pre-dividing line in short):

- <u>robust</u>, if it has an internal definition, say a property on relations definable (by first order formulas with parameters) in $M \in K$, and an equivalent external definition, like having few models up to isomorphism, or that the ultra-powers of any $M \in K$ are "easily λ-saturated", etc.
- (internally) <u>successful</u>, if there is a serious structure theory on the positive side. E.g. we have a general definition of non-forking, or of dimension;
- externally <u>successful</u>, when it helps proving complicated models exist for T on the up side but we shall deal here more with the internal side.
- <u>fruitful</u>, when the positive theory has applications in parts of mathematics outside model theory.
- <u>versatile</u>, if also for contexts not falling in our framework the machinary developed is helpful.

Thesis 1.3. A dividing line (not just a candidate) is one which is robust and successful, and hopefully more. While there are worthwhile properties which do not satisfy these requirements, being such a dividing line is a great bonus:

- it directs us to the right structure theorems developed under the right conditions;
- sometimes it helps us to prove a theorem on all theories by dividing the proof into cases;
- it tells us that some theorems are best possible, i.e. applying only to those with the property;
- we expect that for some classes of interest in other parts of mathematics, the general theorems can help. However, this has not been part of our motivation.

See Figure 1 for some of the known dividing lines and candidates, and the implications between them (see http://forkinganddividing.com) for an interactive map with examples). Note that some properties look to me excellent candidates to be dividing lines, but the evidence does not exist or is slim, this is why in Figure 1 we have "and candidates."

Choice 1.4. Here we concentrate on NSOP$_4$, in particular on simple theories.

The properties stable/superstable have a very strong support (i.e. being robust and successful on the one hand, and attracting researchers and being fruitful, i.e. having applications on the other hand, see e.g. [Hru96], [HL16]). Also "simple" has strong support, as demonstrated below, as well as being dependent (earlier called NIP) [Sim15] and NTP$_2$ [Che14].
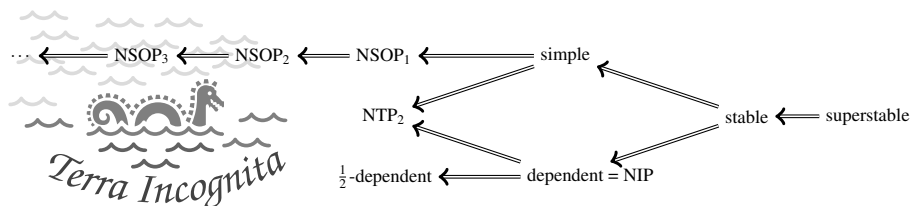
FIGURE 1. A map of dividing lines and candidates.

These are good examples of how successful dividing lines became fruitful. Many technical tools were originally developed for classification theory (etc) purposes, such as analyzing and counting models as test problems. But careful geometric analysis by Hrushovski and many others has led to powerful applications [Hru96], [Hru12] that created new bridges between model theory, algebraic and Diophantine geometry, and combinatorics, see [Bou98], [Sca01], [BGT11]. It is natural to believe that developing new dividing lines will also eventually have a similar impact.

A natural way to try to find dividing lines is considering quasi-orders, like $\leq_{\mathrm{univ}}$ discussed below and Keisler's order (see [Kei17]). Now a naturally defined order may be trivial (having only one equivalence class) or may be too fine, having too many equivalence classes, making too fine a distinction. A possible remedy is to ask about the minimal or maximal such theories, or to change the order to overcome this; we shall discuss some cases.

§ 2. THE TEST QUESTION: UNIVERSALITY AND THE UNIVERSALITY ORDER

We begin by defining the universality spectrum of a class K of models. The reader may suppose that $K = K_T$ is the class of all models of the first order theory T, as will very often be the case in the present article; alternatively, it may be some particular class of mathematical interest, such as the class of locally finite groups or Banach spaces. (We aim at a general theory applying to a wide range of classes K, but may sometimes have to settle for ad hoc adaptations of the main theory to special cases a little outside it.) We write K_λ for the class of models in K of size λ.

The Main Definition 2.1. 1) $M \in K$ is *universal* in λ (for K) if for any $N \in K_\lambda$, N can be embedded (see Section 10 for various alternatives for the meaning of 'embedded') into M.
2) $M \in K$ is *universal*, if it is universal in the cardinality $\|M\|$ of M.
3) The *universality spectrum* of K, $\mathrm{univ}(K)$ ($= \mathrm{univ}_K$), is the class of cardinals $\lambda \geq \aleph_0$ such that there is a universal $M \in K$ of cardinality λ.

We consider the following test problem, which we call the *universality spectrum problem*.

Problem 2.2. Classify first order theories T according to the universality spectrum of K_T.

This problem subsumes a number of smaller problems of considerable interest in their own right. If we temporarily ignore the essential point that issues of cardinal arithmetic will lead us (if not force us) to consider forcing extensions of the set theoretic universe, then these would include the following:

Problem 2.3. (Over-simplified) (1) Which theories T are simplest, in the sense that they have the largest universality spectrum?

(2) Which theories T are the most complex, in the sense that they have the smallest universality spectrum?

(3) Repeat questions (1,2) but working with the universality spectra modulo initial segments,

(4) What is the least cardinal λ such that two universality spectra which agree above some cardinal λ' must agree above λ?

In the foregoing we allude to a common phenomenon in model theory: first order logic has difficulty making fine distinctions in cardinality greater than the cardinality κ of the language, and even more difficulty in making distinctions above 2^κ. For simplicity we take $\kappa = \aleph_0$ so that this threshold should be something like $\aleph_1$ or $2^{\aleph_0}$.

Set theory has no such difficulty, so we will still have to consider points like the cofinality of λ or cardinal arithmetic, but when we do things properly our readiness to change the universe will remove much of this.

However for very small values of λ we anticipate that our test problems may have unusual "sporadic" solutions. The most classical instance of this is the complement to Morley's theorem: on the one hand categoricity in one uncountable cardinal is equivalent to categoricity in all uncountable cardinals; but on the other hand, categoricity in $\aleph_0$ is something else entirely.

Similarly, there is a very interesting theory of countable universality, which we discuss in some detail (§ 10(E)), but we cannot expect it to be closely related to the "generic" theory above—or well above—the cardinality of the theory. And the generic theory is most likely to give dividing lines of the type we seek. Most notably, we will comment on the case of universals in $\aleph_0$ for classes K_H of graphs omitting a given finite graph H, a topic which received considerable attention from combinatorialists as well as logicians [CS96], [CS01], [CST97], [GK96], [KMP88], [KP84], [Lat94], [Lat03], [FK97], [CSS99], [Che11], [CS07], [CS16].

A major problem in this line of research is the following decidability problem: is the collection of finite graphs H such that K_H admits a universal in $\aleph_0$ decidable [KMP88]? In a series of papers with Cherlin [CSS99], [CS07], [CS16] we devise a general strategy toward a positive answer to the decidability problem just posed. This naturally requires several involved technical investigations in finite graph theory.

But here we concentrate on uncountable (or "sufficiently large") cardinals λ. Some mathematicians are a bit skeptical towards the Cantor's paradise, i.e. they are not fond of uncountable cardinals; however, they can still enjoy its fruits, that is, the positive finitary theory for the whole family of K_T's with an "understandable" theory.

We now define a quasi-order on theories in which the smaller theories are the ones for which it is easier to find universal models, and thus the theories are presumably simpler in some respect. Our first version (with superscript 0) sets aside all set theoretic issues.

Definition 2.4. Let $\leq^0_{\mathrm{univ}}$ be the following partial (quasi) order on the collection of first-order classes: $K_1 \leq^0_{\mathrm{univ}} K_2$ if $\lambda \in \mathrm{univ}(K_2) \Rightarrow \lambda \in \mathrm{univ}(K_1)$, for every sufficiently large cardinal λ.

Problem 2.5. Can we characterize the $\leq^0_{\mathrm{univ}}$ equivalence classes? (T_1 and T_2 are equivalent if $T_1 \leq^0_{\mathrm{univ}} T_2 \leq^0_{\mathrm{univ}} T_1$.) What about the $\leq^0_{\mathrm{univ}}$ maximal theories?

The notion of universality is a natural and interesting notion, which has come up in various contexts. The existence of universal spaces allows for the consideration of mathematical objects as sub-objects of a more concrete one, and thus endows them with a greater wealth of "intrinsic" properties. It also emphasizes the relations of "parts of a

whole" . Probably the oldest instance of this notion is the observation that the complex field is universal for fields of characteristic 0 and cardinality at most continuum. (This fact was generalized by André Weil [Wei62] to a "universal domain" in algebraic geometry; which is a precursor of the inspirations of the model theoretic "monster model".) Another classical example is the complete Urysohn space [Ury27], which is the unique complete homogeneous separable metric space which embeds every finite metric space. The classical Banach-Mazur Lemma indicates that $C([0,1])$ is a universal separable Banach space. Another major source of examples is the well-known method of Fraïssé, which subsumes many examples of universal objects, such as the countable random graph [Rad64] and Hall's universal group [Hal59]. Another context in which universal objects have arisen is the study of Polish groups and, more generally, Polish spaces, see e.g. [Usp86], [BY14]. There has been a considerable interest in the question of existence of universal objects among Banach spaces, and, more generally, metric and topological spaces of a particular form, e.g. [Bou80], [Ke88], [Bel00], [Dz06], [Dz14], [BK12], [BK13]. Finally, universal objects appear also in the context of sofic groups, see e.g. [Tho10], [Pes08].

A priori, $\leq^0_{\mathrm{univ}}$ seems like an interesting order and one would expect that K's (or T's) which are "small" (or even "not maximal") under this partial order should be nice, i.e. "understandable". In short, $\leq^0_{\mathrm{univ}}$ should provide good dividing lines. However, recall the old works of Cantor and Hausdorff on linear orders. The existence of universal models was central in the early stages of model theory, see the classical works [Jón56], [Jón60], [MV62]. See also [Bal20]. In fact, if $\lambda = \mu^+ = 2^\mu$ or just $\lambda = 2^{<\lambda} > \aleph_0$ then for any reasonable T, $\lambda \in \mathrm{univ}_T$. Actually, if $\lambda = 2^{<\lambda} > \aleph_0$ there are even "homogeneous universal models" and "saturated models", where each implies universality (which was the original motivation for these investigations). So the problem of the existence of universal models leads to these new notions which became central in model theory. In particular, they motivated the introduction of what is now known as non-forking, an essential tool of classification theory.

Pedantically, as Baldwin has pointed out, Jonsson and Morely-Vaught have spoken about a model of cardinality λ being universal for models of cardinality $< \lambda$. But for the case they were dealing with, $(< \lambda)$-homogeneity, it makes no difference. Moreover, for characterizing the unique universal homogeneous model of cardinality λ, it is clearly preferable to use a weaker version.

Now generally the natural notion of universality, and the one used, is as here.

In contrast, understanding the existence of universal models directly seems harder and was not studied much by model theorists. Until recently, I too have considered it to be a poor candidate for leading us to a dividing line. The only known results were the stable classes (see below) and scattered sufficient conditions for non-existence. There are set theoretic cases on independence results in specific cases, e.g. see (and references there) [KS92a], [Mir05], [Dz11], [She96b], [She16], but now I feel that we should reconsider the general theory.

Now we must deal with the fact that $\leq^0_{\mathrm{univ}}$ commits us to whatever version of set theory holds in "the real world" (or in some arbitrary model of set theory, perhaps). Indeed, Gödel told us that possibly $\lambda = 2^{<\lambda}$ for every λ (as it is equivalent to GCH); and in this case, univ_T is maximal, i.e. $\mathrm{univ}_T = \{\lambda : \lambda > \aleph_0\}$.

But Cohen told us that GCH may fail, and, by results of Solovay and Easton [Eas70], cardinal arithmetic can be essentially anything, i.e. there are no non-trivial restriction on $2^{\aleph_0}$ and 2^{λ^+}. This motivates us to refine the order $\leq^0_{\mathrm{univ}}$ by demanding the implication not only to hold "incidentally" (due to, e.g., GCH) in the "present universe", but also in "all"

others (meaning "provably") or just in any forcing extension of the present universe. Also, some quite trivial theories have no universal models in $\aleph_1$ when $\aleph_1 < 2^{\aleph_0}$, so in order to get a clean general theory we should restrict to "sufficiently large" λ, and $\lambda \geq 2^{\aleph_0}$, or even $\lambda = \lambda^{\aleph_0}$, is a prudent restriction. The theory for small values of λ may also be of interest (this is certainly the case if $\lambda = \aleph_0$) but with a different character.

Hence, GCH could "sabotage" our test question, but we have an excellent way to avoid GCH: forcing. In fact, forcing allows us to violate GCH at any successor cardinal, while leaving the set of real numbers unchanged.

This leads to the following definition:

Definition 2.6. $T_1 \leq_{\text{univ}} T_2$ if "$\lambda \in \text{univ}(T_2) \Rightarrow \lambda \in \text{univ}(T_1)$" holds also in every larger "universe of sets" (i.e. in every forcing extension where $\lambda \geq 2^{\aleph_0}$, because of 2.5, holds).

The universality spectrum problem is the structural understanding of this quasi-order, or in other words, determination of the classes of the associated equivalence relation and the structure of the corresponding partial order on the classes. Various test problems arise naturally in this case, as well as similar problems involving small cardinals. We concern ourselves initially, as in the case of the Keisler order (see below), with the classes lying at an extreme, that is, very low or very high. *Note that small or low theories have large universality spectra, and large or high theories have small universality spectra.* At the high end, we make the following definition:

Definition 2.7. K is almost $\leq_{\text{univ}}$-maximal if in every forcing extension, K has no universal model in λ for any $\lambda = \lambda^{\aleph_0}$ with $\mu^{++} = \lambda < 2^\mu$ for some μ.

While forcing is a very successful and well-established method in set theory, one might wonder why we expect to get a natural and useful notion if we commit ourselves to this specific technique in the definition. The answer is that this detail should not be important: forcing is "good enough", or, said otherwise, potential counterexamples are "robust enough":

Thesis 2.8. 1) The order $\leq_{\text{univ}}$ is robust: We should get basically the same results if instead of forcing we talk, e.g., about provability in ZFC or other natural notions of "necessary".
2) But talking about forcing extensions is much more convenient and well defined.

This notion may shed light on the terra incognita of Figure 1. There are other relevant test problems for this area. Most prominent and oldest is the Keisler order $\lhd$ and saturation of ultra-powers and its relative $\lhd^*$ and $\leq_{\text{SP}}$, see Section 5.

Problem 2.9. 1) Is there a theory T such that (provably in ZFC), $\text{univ}_T = \{\lambda : \lambda = 2^{<\lambda} > \aleph_0\}$? (I.e., univ_T contains only the cardinals guaranteed by the remark above. So such a theory is $\leq_{\text{univ}}$-maximal.)
2) In particular, is the theory of linear orders $\leq_{\text{univ}}$-maximal, ignoring $\aleph_0$ and $\aleph_1$?
3) Characterize the $<_{\text{univ}}$-maximal theories. Is it a good dividing line or is almost $\leq_{\text{univ}}$-maximality better?
4) Can we at least characterize the almost $\leq_{\text{univ}}$-maximal T's?

The test problems we consider measure notions of complexity which are likely to have something in common, since they follow a common pattern of coarsening interpretability using external set theoretic notions. At the low end, there are well established notions of simplicity flowing out more or less directly from the investigation of the number of models up to isomorphism and the existence of saturated models, notably the concept of stability and the more general concept of simplicity which actually goes under the name of

simplicity. In the next section, we present these notions in some detail together with our knowledge and expectations for the behavior of the universality spectrum in these contexts.

In §5 we shall go into some detail regarding recent results on the Keisler order, which currently serves as a rough model for the kind of results we would like to have in the case of the universality spectrum. There is still much that is not understood about the Keisler order but enough is understood to see that it is a good source of test problems which produce robust dividing lines, many of which are new.

§ 3. STABILITY

At least in this central case we give the full definition:

Definition 3.1. T is *not stable*, if some formula $\varphi(\bar{x}_n, \bar{y}_n)$ defines on ${}^n M$, for some model M of T, a directed graph which is an infinite linear order on some subset of ${}^n M$.

A priori it is not clear that this is a worthwhile dividing line, but a posteriori stable theories are understandable: They have an internal notion of free amalgamation (called non-forking) and dimension.

We expect stable theories to have a large univ_T, see [She90a, Ch.III].

There is also a stronger relative called superstable, cf. [She90a, Ch. III] (and we call strictly stable the theories which are stable but not superstable):

Theorem 3.2. *1) If T is superstable then: T is $\leq_{\mathrm{univ}}$-minimal; and if $\lambda \geq 2^{\aleph_0}$, then T has a universal model of cardinality λ, in fact, a saturated one.*
2) If T is strictly stable (i.e. stable but not superstable), then T is almost $\leq_{\mathrm{univ}}$-minimal, in fact:

   (a) *if $\lambda = \lambda^{\aleph_0}$, then T has a universal model of cardinality λ, in fact, a saturated one;*

   (b) *if $\lambda = \sum_n \lambda_n^{\aleph_0}$, equivalently if $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$, then T has a universal model in λ, in fact, a so-called special model;*

   (c) *if $2^{\aleph_0} < \mu < \lambda < \mu^{\aleph_0}$ then there are some non-existence results (by [KS95]). See also §10(B) and §10(C).*

This leads to the following:

Problem 3.3. 1) Are all strictly stable T's $\leq_{\mathrm{univ}}$-equivalent?
2) Or does $\leq_{\mathrm{univ}}$ reveal dividing lines among them?

Problem 3.4. For superstable T's, so-called regular types lead to a well-behaved dimension theory. Is there a dividing line among strictly stable T's which characterize having a parallel situation (probably for a weak relative of regular types)? See [PS18] and references there.

Of course, while the unstable theories are less "analyzable" than stable ones, not all of them are equally complicated. There are two separate "reasons" for being unstable, leading usually to two separate lines of research, that is:

Theorem 3.5. [She90a, Ch.II] *If T is unstable, then at least one of the following occurs:*

   (A) *T has the independence property, i.e. some formula $\varphi(\bar{x}_n, \bar{y}_n)$ defines in some $M \in K_T$ a graph which has a random graph as an induced subgraph;*

   (B) *T has the strict order property, i.e. some formula $\varphi(\bar{x}_n, \bar{y}_n)$ defines in some $M \in K_T$ a partial order with infinite chains.*

Now if T falls under (B), then it is almost $\leq_{\text{univ}}$-maximal, by [KS92a], and so we turn to "lacking the strict order property" which seemed to me an excellent candidate for a dividing line and I have considered several suitable test problems (to show the property to be robust); but alas, no evidence so far. However, there are weaker properties for which there is evidence: simple and to some extent NSOP_n for $n \leq 4$. In fact, we know that SOP_4 is enough for almost $\leq_{\text{univ}}$-maximalility [She96b], and so, with respect to our test question, it seems reasonable not to look beyond NSOP_4 theories. Given the many scattered results on the "positive" side for particular NSOP_4 theories, we think that now the time is ripe for relooking at the full classification problem. We first turn to simple theories.

Related directions on which we shall not say more (in part II) are:

(A) classification on a predicate - [PS85], [SUc, SUd]

(B) Game equivalence; [She08], [PS18], [She08], [PS18], [HS07].

§4. ON SIMPLE THEORIES

Simple theories were originally introduced in [She80b] as a natural candidate for a dividing line for which some results on stable theories can be generalized. In the nineties, due to the natural examples found (see below) and [KP98, Kim96], the study of simple theories become a very active area. Many properties of non-forking independence generalize from stable theories to this context (see also [GIL02], [Kim14]). Furthermore, simplicity is a robust dividing line (see 1.2), though not in an ideal way. First a definition:

Definition 4.1. We say a class $K \subseteq K_T$ is *dense in* λ if whenever $M \in K_T$ has cardinality λ, then for some $N \in K$ of cardinality λ we have $M \subseteq N$.

Theorem 4.2. T *is simple iff necessarily, if* $\aleph_0 < \mu = \mu^{<\mu} < \lambda = \lambda^{\aleph_0} < 2^\mu$ *and some generalization of Martin's axiom for* μ *holds, then the set of* μ-*saturated* $M \in K_T$ *is dense in* λ.

There are important natural examples for simple theories: the random graph, pseudo-finite fields (i.e. theories of ultraproducts of finite fields), ACFA (algebraically closed fields with a generic automorphism), as well as smoothly-approximable structures (a class of well-behaved ω-categorical countable structures, see [CH03a]).

In fact, ACFA (the algebraically closed fields with an automorphism) is a great example of a simple unstable theory whose analysis has part in various achievements outside of logic, and, in particular, to a new proof of the Manin-Mumford conjecture [Hru01].

Another highly non-trivial result states that ACFA is the limit theory of the Frobenius automorphism. In other words, for every sentence φ in the language of fields with a distinguished symbol for an automorphism, ACFA implies φ iff φ is true for almost all structures of the form (K_q, σ_q) where q is a prime power, K_q is the algebraic closure of the finite field with q elements and σ_q is the map $x \mapsto x^q$ (see [Hru04]).

How does simplicity relate to $\leq_{\text{univ}}$?

Conjecture 4.3. 1) The order $\leq_{\text{univ}}$ separates simple from non-simple theories.
2) There is a relative of Theorem 4.2 for $\leq_{\text{univ}}$.
3) There is a reasonable characterization of $\leq_{\text{univ}}$ among simple theories.

A weak support for this is: there are forcing results on the existence of models in $\lambda^+ < 2^\lambda$ universal for λ, for every simple T. Presently some non-simple T have similar results (see [She96b, DS04a, She93b]). However, for the theory T_{rg} of random graphs and relatives there are strong forcing results for the nnon-existence of universal models, cf. [She90b, Mek90].

On recent results see [She16].

We believe that this line of research will shed more light on simple theories. There are several long-standing questions (such as the "stable forking" conjecture), and it is clear that among simple theories there are several subclasses that exhibit different behavior. Uncovering new dividing lines may help us to understand this structure, and discover new interesting phenomena.

For example, T_{rg} is $\leq_{\mathrm{univ}}$-minimal among the unstable T's; so we are lead to the following problem:

Problem 4.4. 1) Characterize the theories which are $\leq_{\mathrm{univ}}$-equivalent to T_{rg}.
2) Is this a reasonable dividing line ?
3) Prove that the simple theories of fields mentioned above (pseudo-finite fields, ACFA), and more generally all the simple theories of fields, are $\leq_{\mathrm{univ}}$-equivalent to T_{rg}.

As is well-known, superstable fields are algebraically closed [Mac71], [CS80]. However, although stable fields have long been conjectured to be separably closed, this is still open. For unstable classes the situation is even less clear (while there are precise conjectures, see the beginning of [KSW11]). One hopes that the tools developed in our context will help to advance the situation at least in the class of fields with simple theories.

§ 5. The Keisler Order

Many times various kinds of limit models shed light on the original one. For example, the real and the p-adic field shed light on the rational (field). The universal sofic groups on finite permutation groups. Being universal is a weak version; being saturated (or κ-saturated) says "whatever may happen already happened", and it is central in model theory. A general way to construct such countable structures is using Fraïssé limits, a construction well-known to logicians and combinatorialists alike. Another one is considering ultra-powers of models; this motivates Keisler's order [Kei67].

Definition 5.1. $T_1 \trianglelefteq T_2$ <u>iff</u> for any set I and so-called regular ultrafilter D on I if $M_\ell \in K_{T_\ell}$ for $\ell = 1, 2$ and M_2^I/D is $|I|^+$-saturated then so is M_1^I/D.

In an initial bloom in the sixties and seventies, the stable case was sorted out (exactly two equivalence classes!); but afterwards little progress has been made until the topic was revived in the last decade. Recently:

Theorem 5.2. *[[MS16b]] The property* SOP$_2$ *implies* $\triangleleft$*-maximality.*

Note that proofs on $\triangleleft$ involve constructing ultrafilters with precise properties (separating one theory from another). While those problems are of great interest per se, there is a relative $\triangleleft^*$ of $\triangleleft$ that does not lead to such problems, but it is suspected to be too fine. Using the order $\triangleleft^*$, we can show the following:

Theorem 5.3. *(Assuming an instance of* GCH*) (One direction* [DS04a]*,* [SU08] *and the other* [MS17]*)* T *is* $\triangleleft^*$*-maximal* <u>iff</u> T *is* SOP$_2$ *(so* SOP$_2$ *is a robust dividing line!).*

Thesis 5.4. 1) It would be ideal if the order $\leq_{\mathrm{univ}}$ (as well as $\triangleleft, \triangleleft^*$) had finitely many equivalence classes (or at least countable) or just in some way nicely describable.
2) If this fails, we may try to find a relative for which this holds.

Unfortunately, this hope is shattered, at least for $\triangleleft$, in (1) above:

Theorem 5.5. [MS18] *There is an infinite* $\triangleleft$*-decreasing sequence of simple theories. Similarly for* $\leq_{\mathrm{SP}}$ *by* [SUa]*.*

This seems to indicate that this line will help to understand simple theories. This motivates:

Problem 5.6. 1) Prove that any $\lhd$-maximal theory has the SOP$_2$.
2) Characterize the NSOP$_2$ theories T such that: For every large enough μ, for every $\lambda > \mu$ there is a regular ultrafilter D on λ such that: if M a model of T, then M^λ/D is not μ^+-saturated.

So NSOP$_2$ is robust, but is it successful? (Recall 1.2.) If Problem 5.6 is answered positively, then we have:

Problem 5.7. Develop a positive/structure theory for NSOP$_2$ (but see [MS16a]).

Lately there were some advances in this direction for NSOP$_1$, related to non-forking and Kim-non-forking, see [KRS18], [KR17a], [KR17b], [CR16]. Note that it is open whether NSOP$_1$, NSOP$_2$ and NSOP$_3$ are in fact distinct for complete first order theories.

Conjecture 5.8. The order $\unlhd$ has continuum many equivalence classes.

If so, this may indicate that even $\unlhd$ is too fine for our purposes.

§ 6. OTHER DIRECTIONS

Considered a breakthrough, a triumph for model theory is [MS16b] which received a lot of attention [Har, Whi, Juh07]. It includes a very surprising result: $\mathfrak{p} = \mathfrak{t}$, recalling $\mathfrak{p}$ and $\mathfrak{t}$ are "cardinal characteristics of the continuum". These invariants measure the size of the continuum from various perspectives. These cardinals are defined combinatorially or by algebraic properties, an area which until then had absolutely no relation to model theory; moreover, the question whether $\mathfrak{p} = \mathfrak{t}$ had been open since the forties and was generally expected to be (consistently) false.
In light of this development, we propose:

Problem 6.1. Understand the connection to model theory for other cardinal characteristics of the continuum.

We expect this direction will have further connections with finite (and infinite) combinatorics, including graph theory (which already has proven to have close connections to model theory in the past). In particular generalizations of Ramsey theory, Vapnik-Chervonenkis dimension, graph regularity cf. [KPT05], [Sze75], [Gow07], [Sau72], [MS14]. It is natural to look at other test problems which we hope we lead to worthwhile dividing lines.
Let us just list a few:

- λ-exact saturation and λ-PC-exact saturation (see [KSS17]). The conjecture is that any simple theory has λ-PC-exact saturation for all singular λ, or at least for all strong limit cardinals λ. There is evidence (by work in progress and by [MS17]) that "singular PC-exact saturation" may turn out to be a good test problem for a dividing line between simple and NSOP$_2$.
- Quite old question about the order $\leq_{SP}$, related to Theorem 4.2, see [SUa].

We shall not elaborate on:
- classification over a predicate
- game equivalence, see [She08], [PS18]
- $\leq_{SP}$ see [She80b], [SUa]
- Borel reducability, Borel completeness [LS15]

- Recent advances on universality.

Thesis 6.2. When dealing with dividing lines there are good reasons to expect that answers to the proposed questions will have applications in several fields of mathematics.

Part II

Contents

§ 7. INTRODUCTION TO PART II

The universality spectrum of a class K of structures is the class of infinite cardinals in which K has a universal member; i.e. cardinals λ such that there is a model M of size λ such that any other member of K of cardinality can be embedded into M. In some cases "cardinality" is replaced by "density". This is a natural notion which arose independently in several contexts. We focus here on the family of elementary classes, i.e. the class of models of a first order theory T. This rather wide context is the classical one in model theory and covers many natural examples.

Our main test questions is: which theories have universal models in certain cardinalities? Another concerns the classical Keisler order (which just recently moved to the front line, receiving considerable attention from the mathematical community). Our expectation is that these questions will lead us to find *new dividing lines* between "understandable" and "chaotic" theories. The classes we are mainly interested in are unstable but without

the strict order property (i.e. no formula $\varphi(\bar{x}_n, \bar{y}_n)$ defines partial orders with an infinite chain), so they are "far" from e.g. the class of real-closed fields, and even generally so-called dependent T's; and they include the class of simple theories. Such dividing lines have been a major theme in model theory, with many successful applications to other areas of mathematics (such as pseudo-finite fields). However, the question itself also connects with other areas of mathematics.

The subject is at a crossroads. We need finite combinatorics to analyze the local behaviour of finite patterns of definable relations. We need infinite combinatorics to prove that certain properties of classes imply that there cannot be universal models. We are interested in specific classes, e.g. the classes of linear orders, groups, Abelian groups, graphs and Banach spaces. While the basic question for such a specific class may be solved by a general criterion, it may also be necessary to find specific arguments or even *develop new methods*. So the results should be of interest also to researchers outside of model theory and set theory. We will also employ the set-theoretic method of forcing, which will prove consistency results that can show that the obtained results are best possible (and thus really constitute a dividing line among theories).

We feel the time is ripe for trying to find a general criterion which is necessary and sufficient for proving that any reasonable class satisfying it has a universal member in a given cardinal. But even "just" developing the dividing line for T's with minimal universality spectrum, or even "just" for the so-called simple theories, would certainly be major breakthroughs and hard challenges.

§ 7(A). **Preliminaries.**

We assume the reader has some familiarity with model theory (but not necessarily classification theory) and set theory (at least some forcing). Note that there are two expected kinds of interaction with mathematics outside mathematical logic:

(i) dealing with the existence of universal models for specific classes (e.g. groups);

(ii) finding new dividing lines and developing their positive/structure theory, and consider the place of such classes in the map of model theoretic properties (cf. Figure 1). Furthermore, we expect interaction with finite as well as infinite combinatorics.

How are classification theory and the universality spectrum connected? We consider the universality spectrum as an excellent test problem, important and natural per se, and we hope it will lead us to find new dividing lines and develop new perspectives in classification theory and applications.

The main test problem for classical classification theory [She90a] was the *main gap conjecture* on the number of non-isomorphic models, but for unstable theories we expect other kinds of structure theory relevant to other problems. We expect our test problems to drive us to discover the relevant positive/structure theory.

We shall also consider other test questions — the well-established Keisler order (see [MS16b, MS16c]), on saturation of ultrapowers, but also $\leq_{SP}$ and the exact saturation problem. As a prototypical algebraic example we shall consider *ACFA*, the algebraically closed fields with a distinguished automorphism, PAC fields (= every variety which is indecomposable (even in bigger fields) has a solution).

Note that being saturated is central in model theory and problems about the existence of saturated models drove us to introduce forking, which has been central also for model theorists without interest in set theory. The existence of universal models is a fundamental problem, considered in various contexts, which is model theoretic in nature and stood high in the beginning of model theory, see the classical [Jón56], [Jón60], [MV62], but

received less attention in later developments. We believe that now is the time for making real advances.

§8 presents the main test problem. In §9 we present classifications related to the universality spectrum, most notably Keisler's order with the relevant recent breakthroughs, and to other dividing lines, also on other test problems. In §10 we turn to countable universality ($\lambda = \aleph_0$), notably in the case of graphs. §11 talks about connections with combinatorics (both finite and infinite). In §12 we turn to results on the positive side: the consistency of existence (under a strong negation of GCH) of universal models for certain theories and families of theories.

In each section, we describe the state of the art and some known theorems; as well as the new terrain, our plans for the future, in the form of problems and conjectures. We also postulate several theses (including informal statements, often expressing my personal opinion).

§ 8. A New Test Problem: Universality

Our main test problem is the universality spectrum of a class of models, but first we fix some notation:

Convention 8.1. 1) Fix a vocabulary τ (countable for simplicity); K will denote a class of τ-models.
2) If not said otherwise, an embedding of $M_1 \in K$ into $M_2 \in K$ is an isomorphism from M_1 onto a submodel of M_2.
3) K_λ is the class of $M \in K$ of cardinality λ.
4) If not said otherwise, our class is elementary, i.e. it is the class K_T of models of a first order theory T in the vocabulary τ, and for transparency we assume that it has the joint embedding property (JEP) and the amalgamation property (AP) and (see below) if T is complete then it has elimination of quantifiers.
5) We will generally not distinguish between the class of T-models and T itself.
6) Let Th(M) be the theory of M, i.e., the set of first order sentences that M satisfies.

A few comments to this convention: Elementary embeddings are central in model theory. But with respect to our problem, we can expand the vocabulary by adding predicates for every first order formula; then the notions of embedding and elementary embedding coincide. So elementary universality (universality with respect to elementary embeddings) can be viewed as a special case of universality.

In some cases we should consider other notions of embeddability and/or replace cardinality e.g. by density (most notably in Banach spaces).

Many interesting classes are not elementary (e.g. locally finite groups, Banach spaces), and we shall consider some of them later. But we believe that it will help to first focus on the first order context and already understanding the first order T's forms a hard enough challenge.

Convention 8.2. For transparency:

- In the general definitions and results (but not the specific examples) we assume that the theory T is complete.

- Formulas if not said otherwise are quantifier free and denoted by φ, ψ, ϑ.

Now there are many yardsticks to measure the complexity of a theory T, and many possible test questions. We concentrate on the following:

The Main Definition 8.3. (For the test problem)

1) We say that $M \in K$ is universal in λ (for the class K) if for any $N \in K_\lambda$, N can be embedded into M.

2) We say that $M \in K$ is universal when it is universal in the cardinality $\|M\|$ of M.

3) Let $\mathrm{univ}_K = \mathrm{univ}(K)$, the universality spectrum of K, be the class of cardinals $\lambda > \aleph_0$ such that there is a universal $M \in K$ of cardinality λ (we shall/may speak about $\lambda = \aleph_0$ separately).

4) Set $K_1 \leq_{\mathrm{univ}}^0 K_2$ if $\mathrm{univ}(K_1) \supseteq \mathrm{univ}(K_2)$.

5) Let $\mathrm{univ}_K(\cdot)$ be the class function $\mathrm{univ}_K(\lambda) = \mathrm{Min}\{\mu$: there is a sequence $\bar{M} = \langle M_\alpha : \alpha < \mu \rangle, M_\alpha \in K_\lambda$ and $\bar{M}$ is universal$\}$, i.e. every $N \in K_\lambda$ can be embedded into some M_α.

We may have chosen to use $\mathrm{univ}_K'(\lambda)$ instead of $\mathrm{univ}_K(\lambda)$ where $\mathrm{univ}_K'(\lambda) = \mathrm{Min}\{\mu$: there is $M \in K_\mu$ universal for $K_\lambda\}$ and even $\mathrm{univ}_K(<\lambda) = \mathrm{Min}\{\mu$: there is $M \in K_\mu$ universal for models from $K_{<\lambda}\}$. Note that for $\mu < \lambda^{+\omega}$, we have $\mathrm{univ}_K(\lambda) = \mu \Leftrightarrow \mathrm{univ}_K'(\lambda) = \mu$ so for the most interesting case here, $\mu = \lambda^+$, there is no difference.

We will also consider (see Problem 10.21) the case of universal models in $\aleph_0$ for the class of graphs omitting a given one, but in general we concentrate on uncountable cardinals λ. Some mathematicians are a bit skeptical toward the *Cantor paradise*, i.e. they are not fond of uncountable cardinals; however, they can still enjoy its fruits, that is, the positive finitary theory for the whole family of K_T's with a "low, understandable" place in the map of all theories.

Until recently I have not considered universality spectrum to be a good candidate for a dividing line. There were scattered results: sufficient conditions for non-existence and various independence results, but now we feel the time is ripe for dealing with a general theory.

We shall discuss below also other related dividing lines.

The notion of universality is a natural and interesting notion, which has come up in various contexts, see comments above after Problem 2.5.

Now Gödel told us that maybe $\lambda = 2^{<\lambda}$ for every λ (as it follows from GCH); so in this case, univ_T is always maximal, i.e. $\{\lambda : \lambda > \aleph_0\}$. But Cohen told us GCH may fail. This leads to refine the order $\leq_{\mathrm{univ}}$ by demanding it not only in the present universe but in others. The obvious choice is in forcing extensions. Also some quite trivial T have no universal in $\aleph_1$ when $\aleph_1 < 2^{\aleph_0}$, so we will usually require $\lambda \geq 2^{\aleph_0}$, though we may consider specific small values of λ separately.

Definition 8.4. 1) For Θ a class of cardinals we define: Let $\leq_{\mathrm{univ},\Theta}$ be the following quasi order: $T_1 \leq_{\mathrm{univ},\Theta} T_2$ <u>iff</u> in every forcing extension $\mathbf{V}_0$ of our universe and for every cardinal $\lambda \in \Theta$ (even $\mathbf{V}_0 \models \lambda \in \Theta$) we have that $\lambda \in \mathrm{univ}(T_2) \Rightarrow \lambda \in \mathrm{univ}(T_1)$. We may omit Θ if it is $\{\lambda : \lambda \geq 2^{\aleph_0}\}$.

2) K is almost $\leq_{\mathrm{univ}}$-maximal if for any $\mathbf{V}_0$ as above it has no universal model in λ for every λ, μ with $\mu^{++} = \lambda = \lambda^{\aleph_0} < 2^\mu$.

3) K is pseudo $\leq_{\mathrm{univ}}$-maximal when in any $\mathbf{V}_0$ as above, if $\mu^{++} = \lambda = \lambda^{\aleph_0}, \lambda^+ < 2^\mu$ then no $M \in K_T$ of cardinality λ^+ is universal for λ, T.

What is the rational of choosing this definition of almost $\leq_{\mathrm{univ}}$-maximal/minimal? Why $\lambda = \mu^{+2}$ and not $\lambda = \mu^+$ or $\lambda = \mu^{+3}$? Restricting ourselves to such λ's we can prove many cases of $\leq_{\mathrm{univ}}$-maximality (or -minimality) although we are far from a complete answer, see later. E.g. demanding $\lambda = \lambda^{\aleph_0}$ makes all stable T form one equivalence class. But for $\lambda < \lambda^{\aleph_0}$ there are some independence results and it seems that we get very fine

distinctions. It is of course still desirable to find what occurs at the other cardinals. Maybe advances will make us change the restrictions on λ, or even eliminate them.

Note that $\leq_{\mathrm{univ},\Theta}$ is a notation which helps us express what we know, e.g.

- T $\aleph_0$-stable $\Rightarrow T$ is $\leq_{\mathrm{univ},\{\lambda:\lambda\geq\aleph_0\}}$-minimal.
- T superstable $\Rightarrow T$ is $\leq_{\mathrm{univ},\{\lambda:\lambda\geq 2^{|T|}\}}$-minimal.
- T stable $\Rightarrow T$ is $\leq_{\mathrm{univ},\{\lambda:\lambda=\lambda^{|T|}\}}$-minimal.

While forcing is a very successful and well-established method in set theory, one might wonder why we expect to get a natural and useful notion if we commit ourselves to this specific technique in the definition. The answer is that this choice should not be important: forcing is "good enough", or, said otherwise, potential counterexamples are "robust enough":

Thesis 8.5. The order $\leq_{\mathrm{univ}}$ is robust: We should get basically the same results if instead of forcing we talk, e.g., about provability in ZFC or other natural notions of "necessary".

(But talking about forcing extensions is much more convenient and well defined.)

§ 9. OUR AIM: THE CLASSIFICATION QUESTION AND UNIVERSALITY SPECTRUM

§ 9(A). **Dividing Lines.** Underlying this aim is the following thesis:

Thesis 9.1. 1) It is worthwhile to classify theories, i.e. finding dividing lines or dichotomies.
2) Good test problems help us to find the right dividing lines.

In more detail: We look for properties P of a theory T such that we have relevant information on both sides: about those theories which have property P and about those which do not have it. Usually one side (having P, say) tells us that T is analyzable, or not too complicated in a certain way, and that we can develop a "positive" theory, i.e. a structure theory for these theories. Theories that fall on the other side will usually fail the above structure theory in a strong way, and in general exhibit various non-structure properties.

We have given some names to this intuition in Definition 1.2.

Examples illustrating those notions (others are covered later):

- $\bullet_1$ Robustness:
  - (a) T is classifiable, (that is, T is superstable, NDOP, NOTOP and shallow) iff $\dot{I}(\aleph_\alpha, T) < \beth_{\omega_1}(|\alpha|)$ in every forcing extension (see [She90a, Ch.XII]).

  If you do not like the forcing extension

  - (b) small IE: recalling $\mathrm{IE}(\lambda, T) = \sup\{|\mathscr{M}| : \mathscr{M} \subseteq K_\lambda^T$ is a set of pairwise non-embeddable models$\}$, we get a trichotomy;
    - (α) T is superstable, NDOP, NOTOP and shallow iff $\mathrm{IE}(\aleph_\alpha, T) < \beth_{\omega_1}(|\alpha|)$ for every α (see [She90a, Ch.XII])
    - (β) T is unsuperstable or with DOP or with OTOP then $\mathrm{IE}(\aleph_\alpha, T) = 2^{\aleph_\alpha}$ for every $\alpha = 0$
    - (γ) if T is superstable NDOP, NOTOP and deep (not shallow) then $\mathrm{IE}(\aleph_\alpha, T)$ is the first beautiful cardinal if $\aleph_\alpha$ is at least this cardinal and is $2^{\aleph_\alpha}$ before it, see more [Bal20].
  - (c) T is stable iff every so-called $\aleph_1$-resplendid model is saturated (see [Sheg])
  - (d) T is SOP_2 iff T is $\lhd^*$-maximal (see [MS16a]).

- $\bullet_2$ (Internal) successfulness; we know a lot on superstable and also on stable, simple, dependent and NTP$_2$.

- $\bullet_3$ External successfulness, non-structure results: mainly many non-isomorphic models as above; and constructing complicated models for infinitary logics; see [HT91], [HST93], [HS94], [HS95], [HS99], [LS01], [LS03], [She08].

- $\bullet_4$ Fruitfulness: Hrushovski's work using stability of separably closed fields.

- $\bullet_5$ Versatility: There is much on it in the so-called neostability works. A nice example is of versatility is the class of K^{lf} of locally finite groups (and K_{exlf} the existentially closed sub-class). On the one hand [MS76] use [She90a, Ch.VIII] to prove $I(\lambda, K_{\mathrm{exlf}}) = 2^{\lambda}$ for uncountable λ, though K_{exlf} is not first order; so a point for externally successful. On the other hand, by [Shec] there is a dense set of definable types, so rudimentary non-forking frame holds, hence we can construct canonical existentially closed extension of $G \in K_{\mathrm{lf}}$ without changing the cardinality.
  See more [EM02] on black boxed which start in [She90a, Ch.VIII] and [GM90].

The general program of the search for dividing lines and investigation of the structural consequences and non-structural consequences is for me "classification theory", a two faced Janus; but many model theorists refer only to the structure side.

How do we envision the positive/structure theory? A thesis underlying our approach is:

Thesis 9.2. Trying to crack good test problems will drive us to the right inside theory.

We consider the stable/unstable dividing line a success (discussed in more detail in §9(C) below). Many technical tools (such as non-forking) were originally developed for addressing classification theory questions, such as existence of saturated models and counting models up to isomorphism and defining regular types, dimension and p-weights, see [She90a, V, §4]. However, continuing this and works of Zilber, careful geometric analysis has led to powerful applications (e.g., [Hru96, Hru12]) and building bridges with other areas of mathematics, such as algebraic and Diophantine geometry, and combinatorics. Attempts to generalize these tools to wider (unstable) contexts has led to major progress and breakthroughs in e.g. simple theories (also discussed in section II) and more applications in other areas of mathematics.

It is natural to try to generalize stability, but we feel that there is something missing from the emerging picture, even in the case of simple theories where this was highly successful. Perhaps investigating new test questions and examining the tools that come out of this investigation will shed more light on it. For example, the free amalgamation (given by non-forking) loses its strong uniqueness property outside of the stable context. It is natural to conjecture that uniqueness and existence of amalgamation (of finite diagrams, cf. 14.1) are the keys to a better understanding of different classes of theories. These properties are at heart of the hopeful investigation of simple theories with respect to our test question (see § 12).

It is not a priori clear that there should be robust, even successful dividing lines at all. Some appealing (at least to me) candidate test problems have not led to very good dividing lines, at least so far. For instance, the properties stable/superstable (see Definition 9.10) have a very strong support (i.e. being robust and successful on the one hand, and versatile and fruitful; i.e. attracting researchers and applications on the other hand). Also the property of being simple has strong support, see below, as well as dependent ($=$ NIP) [Sim15] and NTP$_2$ [Che14].

There are some properties which we believe are dividing lines but remain candidates and unfortunately we have only weak evidence so far, and some have none, such as $NSOP_n$ for $n > 4$. Maybe they are just approximations to the right properties, which are yet to be found. To clarify and show some of the most important dividing lines currently known consider:

See Figure 1 for some of the known dividing lines and candidates, and the implications between them (see http://forkinganddividing.com for an interactive map with examples).

Thesis 9.3. Generalizing structure theory which works for specific cases is a fruitful approach. But it may miss some phenomena which are not apparent in the usual specific cases (not all group theory is generalizing some theorems on abelian groups). In particular in model theory, looking at the known examples may be misleading. Serious test problem is a good way to discover them.

§ 9(B). **Universality Order and Dividing Lines.**

Considering the universality spectrum leads us to a specific area in the diagram of dividing lines, which is currently a part of the "terra incognita". There are other relevant (and related) test problems, the most prominent and old of them is Keisler's order $\lhd$ on saturation of ultrapowers and its relative $\lhd^*$.

Let us elaborate on what is currently known on these connections.

By [She80a], consistently $\aleph_1 < 2^{\aleph_0}$ and there is a universal linear order in $\aleph_1$, but by [KS92a] we have:

Theorem 9.4. *1) The class of linear orders (and thus many other T's) is almost $\leq_{\mathrm{univ}}$- maximal, (even $\lambda = \mu^{++} < 2^\mu$ suffices and less), that is, it is $\leq_{\mathrm{univ},\{\lambda:(\exists\mu)(\lambda=\mu^{++}=\lambda^{\aleph_0})\}}$- maximal.*
2) Moreover, the StOP *(= the strict order property) suffices (*StOP *means that some formula $\varphi(\bar{x}_n, \bar{y}_n)$ defines a partial order in every model of T, and it has an infinite chain in some model of T).*

This is annoying! We know the answer for many cardinals but not for all. This leads to:

Problem 9.5. 1) Is there a T such that $\mathrm{univ}_T = \{\lambda : \lambda = 2^{<\lambda} > \aleph_0\}$, provably in ZFC, that is, are there $\leq_{\mathrm{univ}}$-maximal T's, and for them univ_T is the minimal possible class (by the classical results).
2) Can we characterize the $<_{\mathrm{univ}}$-maximal T's (cf. Definition 8.4)? Is this a good dividing line?
3) Can we at least characterize the almost $\leq_{\mathrm{univ}}$-maximal T's? Does this give a good dividing line?
4) Is there a better notion of "almost $\leq_{\mathrm{univ}}$-maximal" which leads to good dividing lines?

The StOP seems a good candidate for being a dividing line and we had believed it may match with being almost $\leq_{\mathrm{univ}}$-maximal. However (see [She96b, §2]):

Theorem 9.6. *The* SOP_4 *(see definition below) suffices in Theorem 9.4.*

Definition 9.7. 1) For $n \geq 3$, T has SOP_n if there is a formula $\varphi(\bar{x}, \bar{y})$ that defines on some model of T a directed graph with an infinite chain and no cycles of length $\leq n$.
2) We write $NSOP_n$ for the negation of SOP_n.

A prototypical class for SOP_n (for $n \geq 3$) is the class of directed graphs with no $(\leq n)$- cycle.

The SOP$_n$ strengthen instability (the order property, Definition 9.10) and are natural approximations to the strict order property. They look like candidates for being good dividing lines, but so far we have little evidence, unlike stability (see [She90a], [Bal88]), [She96b, §2]). So is the SOP$_4$ the right answer for universality? Here we have some encouraging signs, and weak consistency results for some SOP$_3$ theories, but, quite recently, it was proved:

Theorem 9.8. *The results of Theorem 9.4 hold for any T with the so-called olive property (see* [She16]).

Out of compassion for the reader, we heed advice to omit the rather involved definition of the olive property ([She16]).

Is the olive property really a good candidate for a dividing line? Formally, it is of the same form as unstable or SOP$_4$ but it looks cumbersome and there are many variants with the same properties (with respect to universality). Hence, it is not a reasonable candidate; so why consider it in the first place? First, it is a property with serious consequences on the universality spectrum. Second, examples show that there are (not complete) T's which satisfy the olive property, are NSOP$_4$ but SOP$_3$. So NSOP$_4$ is not the relevant dividing line for us.

In fact, one such example is the class of all groups known to have NSOP$_4$ and SOP$_3$ [SU06] and previously believed to have more universal objects. But alas, this class has the olive property [She16] (exemplified by quantifier free formulas), hence, surprisingly, there are "very few" universal groups. More precisely, Theorem 9.4 applies to the class of all groups.

Problem 9.9. Are there consistency results showing all NSOP$_3$ theories T are strictly $<_{\mathrm{univ}}$-below T_{dlo}?

We still believe there is a nice dividing line lurking here. Certainly we believe that for "nicer" T's we can get more universals.

§ 9(C). **Stable Theories.**

So we are led by Theorems 9.4, 9.6, 9.8 to classes which in a sense have no partial order, so we first consider:

Definition 9.10. 1) T is not stable <u>iff</u> T has the order property which means: some formula $\varphi(\bar{x}, \bar{y})$ defines on some model of T a directed graph with an infinite chain (e.g. for the class of random graph the formula $\varphi((x_0, y_0), (x_1, y_1)) \equiv x_0 R y_0$).
2) T is not superstable <u>iff</u> there are $\varphi_n(x, \bar{y}_{k(n)})$ for $n \in \mathbb{N}$ and $M \in K_T$ and $b_\eta \in M, \bar{a}_\eta \in {}^{k(n)}M$ for $\eta \in \bigcup_n {}^n\mathbb{N}$ such that $M \models \varphi_n[b_\eta, \bar{a}_\rho]$ <u>iff</u> $\rho \triangleleft \eta$.
3) T is called strictly stable when it is stable and not superstable.

Some examples:
1) Prototypical superstable T's are: algebraically closed fields of fixed characteristic, T_{eq} (the theory of one equivalence relation), the theory of planar graphs, and $T_{\mathrm{eq}(I)}$, for I a countable well ordering, where:

   (∗) for a countable linear order I, $T_{\mathrm{eq}(I)}$ says E_s is an equivalence relation for $s \in I$ and E_t refines E_s for $s <_I t$.

2) One prototypical example of a strictly stable theory is the theory of separably closed fields of positive characteristic.

3) Another prototypical strictly stable T is the theory of free groups. This is a deep result of Sela [Sel13] . This theory is still under investigation and several advanced model theoretic questions about it remain unsolved, see [Pil09] and [Pil08].

Now stable K_T are understandable; they have an internal notion of independence or non-forking, (see below) and dimension. When applied to other parts of mathematics, e.g. replacing, say, the family of varieties (i.e. sets definable by equations) by a bigger collection of definable sets (definable by first order formulas), with better closure proper-ties, enabling e.g. results on the Mordell-Lang conjecture ([Hru96], [Bou98], [Sca01]) (building on [HZ96]) and on approximate groups [Hru12] continued by [BGT12] and non-archimedean geometry [HL16]. For simple K_T's we have similar but weaker results ([GIL02], [Cas11a]) and applications (e.g. [CH03b]).

Why do we say stable T's are analyzable? Stable theories admit a notion of free amalgamation (also called "independence") arising from non-forking, i.e. we say that $\{A_s : s \in S\}$ is independent over A inside M when they are subsets of M and the type of A_s over $\cup\{A_t : t \neq s\} \cup A$ does not fork over A. This notion is particularly well-behaved:

Theorem 9.11.
- *Preservation: Independence is preserved by elementary exten-sions and elementary submodels (of the ambient model).*
- *Basis Free: $\{A_s : s \in I\}$ is independent over A iff $\{A_s \cup A : s \in I\}$ is independent over A.*
- *Localness: $\{A_s : s \in I\}$ is independent over A iff $A_s, \bigcup_{t \in u} A_t$ is independent over A for every $s \in I$ and finite $u \subseteq I \setminus \{s\}$.*
- *Symmetry: B, C are independent over A in M iff C, B are independent over A in M.*
- *Transitivity: If $A_0 \subseteq A_1 \subseteq A_2$ and $B, A_{\ell+1}$ are independent over A_ℓ in M for $\ell = 0, 1$ then B, A_2 are independent over A_0 in M.*
- *Finitary Character: B, C are independent over A in M iff B', C' are independent over A for every finite $B' \subseteq B, C' \subseteq C$.*
- *Countable Base: If B is finite and $C \subseteq M$ then for some countable $A \subseteq C$ the sets B, C are independent over A.*
- *Monotonicity: If $A'_s \subseteq A_s$ and $\bigvee_{s \in I} A \subseteq A' \subseteq A'_s \subseteq A_s$ and $\{A_s : s \in I\}$ is independent over A then $\{A'_s : s \in I\}$ is independent over A'. (Follows by finite character and localness when $A = A'$.)*

Given such well behaved notion of amalgamation, one expects that for stable classes the universality spectrum is nice. Indeed ([She90a, Ch.II], [KS93]):

Theorem 9.12. *1) If T is superstable, then:*
(a) *T is $\leq_{\mathrm{univ},\{\lambda:\lambda \geq 2^{\aleph_0}\}}$-minimal;*
(b) *if $\lambda \geq 2^{\aleph_0}$ then T has a universal model of cardinality λ, in fact, a saturated one.*

2) If T is strictly stable (i.e. stable not superstable), then T is almost $\leq_{\mathrm{univ}}$-minimal, in fact:
(a) *if $\lambda = \lambda^{\aleph_0}$, then T has a universal model of cardinality λ, in fact, a saturated one;*
(b) *if $\lambda = \sum_n \lambda_n^{\aleph_0}$, equivalently if $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$, then T has a universal model $\leq \lambda$, in fact, a so-called special one.*

3) If T is strictly stable (as in Definition 9.7(2)) and $2^{\aleph_0} < \mu < \lambda < \mu^{\aleph_0}$ and e.g. $\lambda = \mu^{++}$ then $\lambda \notin \mathrm{univ}_T$.

Also the so-called oak property has similar consequences, see [DS06], [She17].

Problem 9.13. 1) Are all strictly stable T's pairwise $\leq_{\mathrm{univ}}$-equivalent?
2) Or does $\leq_{\mathrm{univ}}$ reveal dividing lines among them?
3) Can we in 9.12(3) allow $\lambda = \mu^+$? Can we just require $\lambda \in (\mu, \mu^{\aleph_0})$? Maybe it is more a forcing problem.

We need on the one hand to prove more on the relevant forcing; and on the other hand, further investigate strictly stable T's (which is of independent interest, cf. e.g. the theory of free groups).

Problem 9.14. 1) Find sufficient conditions on T_1, T_2 for $\lambda \in \mathrm{univ}(T_1) \Leftrightarrow \lambda \in \mathrm{univ}(T_2)$ at least when "λ is reasonable", meaning:

(∗) $\aleph_0 < \mu = \mu^{<\mu} < \lambda = \lambda^{<\mu} < 2^\mu$ for some μ.

2) In particular for $T_1 = $ the theory of random graphs.

Why do we call those cardinals "reasonable"? Because for those cardinals we have a way to force complementary consistency results, not involving large cardinals.

§ 9(D). **Simple Theories.**
    Simple theories were originally introduced in [She80b] as a natural candidate for a dividing line having a reasonable theory of non-forking. In the nineties, due to the natural examples found (e.g., smoothly approximable structures [CH03a], ACFA: algebraically closed fields with a generic automorphism [Hru04]) and the discovery that non-forking behaves well in simple theories [KP98], [Kim96], the study of simple theories was a very active area. Many properties of non-forking independence generalize from stable theories to this context. For more, see [GIL02], [Kim14], [Cas11b]. In recent years the study of dependent = NIP theories has been more popular than the study of simple ones (although there is still research on simple theories e.g. [Sha15]), but we believe that there is much more to investigate in simple theories.

Definition 9.15. 1) T is non-simple when there is a (quantifier free for our T) formula $\varphi(\bar{x}, \bar{y})$ and model M of T and $\bar{b}_\eta \in {}^{\ell g(\bar{x})}M, \bar{a}_\eta \in {}^{\ell g(\bar{y})}M$ for $\eta \in \bigcup_n {}^n\mathbb{N}$ such that:

    (a) if $\eta \in {}^{\omega >}\omega$ then $\varphi(b_\eta, \bar{a}_{\eta\restriction n})$: for $n < \ell g(\eta)$;
    (b) if $\eta \in {}^{\omega >}\omega$ then $\{\varphi(\bar{x}, \bar{a}_{\eta^\smallfrown\langle n\rangle}) : n < \omega\}$ are pairwise contradictory.

2) T is dependent iff it is not independent which means that some formula $\varphi(\bar{x}_n, \bar{y}_n)$, in some $M \in K_T$ defines a graph on ${}^nM$ such that any finite graph is embedded into it.
3) A prototypical simple T is the theory T_{rg} of random graphs.
4) T_{ceq} seem to me a minimal non-simple theory; fully: a model M is a model of T_{ceq}^0 iff P^M, Q^M is a partition of the model, E^M is an equivalence relation on Q^M and F^M : $Q^M \times P^M \to P^M$ satisfies for each $b \in P^M$ the function $x \mapsto F^M(x, b)$ is a choice function for E^M. (Earlier a relative T_{feq} was used, for our purpose there is no difference.)
5) T_{dlo}, the theory of dense linear order, is prototypical for dependent. Other examples include the theory of the real field (and each p-adic field).

    There has been considerable amount of work using non-forking (as said, a kind of free amalgamation). In fact, amalgamation has been (naturally) of crucial importance in proofs of (consistent) existence of universal objects.

Let us give some examples of positive results of this kind, which signify the role of simple theories (and also some of the other dividing lined mentioned above) and various kinds of amalgamation in the picture.

Assume (see below) $\mu = \mu^{<\mu}$, first for $\lambda = \mu^+$. By [She90b] we can force that there is a universal graph in λ while $\lambda < 2^\mu$ not collapsing cardinals, nor adding sequences of length $< \mu$. Moreover, if $\mu = \mu^{<\mu} < \lambda < \chi = \chi^\lambda$ (and GCH holds in $[\mu, \chi)$) then we can force a universal graph in λ, not collapsing cardinals and preserving $^{\mu>}$Ord. Subsequently, Mekler [Mek90] proves that for universal T closed under submodels without algebraicity and with $\mathscr{P}^-(3)$-amalgamation (see Definition 14.1) there is a universal model of cardinality λ for $\lambda = \mu^+$, and that there is one for every λ assuming both $\mathscr{P}^-(3)$ and $\mathscr{P}^-(4)$ amalgamation. This is what the proof gives, it is not clear for which T's we cannot force this.

Can simple theories be $\leq_{\text{univ}}$-maximal? This is unsettled, but [She96b, §1] gives the consistency of "weak universal". The consistency of the existence of "weak universal" where

Definition 9.16. We say K has a weak universal in $\lambda < 2^{<\lambda}$ when there is $M \in K_{\lambda^+}$ into which every $N \in K_\lambda$ can be embedded (equivalently, in our context of $\text{univ}_K(\lambda) \leq \lambda^+$).

This does not register in the universality spectrum but comes fairly close, in terms of the function $\text{univ}_T(\cdot)$ taking λ to the minimal cardinality of a λ-universal model. So $\text{univ}_T(\lambda) = \lambda$ corresponds to the universality spectrum, and $\text{univ}_T(\lambda) = \lambda^+$ is a close approximation of interest when $\lambda^+ < 2^\lambda$. Moreover, the same result applies also to T_{tfr}, (the class of triangle free graphs) and T_{ceq} see Definition 9.15(4). Both are not simple and are NSOP$_4$. But T_{ceq} is NSOP$_2$ while T_{tfr} has even SOP$_3$. This is a weaker form of "there is a universal", and although it is not what we really want, it still indicates that there is a difference. So certainly this tells us that being simple is relevant, but not fully satisfying (from this point of view).

Problem 9.17. 1) Can we force $\lambda < 2^{<\lambda} + $ "$\lambda \in \text{univ}_T$ for all simple T"?
2) Is there a simple T such that: if $\aleph_0 < \mu = \mu^{<\mu} < \lambda = \text{cf}(\lambda) = \lambda^{\aleph_0} = \chi^{++} < 2^\mu$ then $\lambda \notin \text{univ}_T$? This may require adding $\mu^+ < \lambda$.
3) What are the $\leq_{\text{univ}}$-minimal unstable T's? Should they be similar enough to T_{rg}?
4) What are the $\leq_{\text{univ}}$-maximal among simple T's? This should throw new light on the family of simple T's.
5) What about other dividing lines for non-simple T's? Recently, there has been work on NSOP$_2$ theories being a robust dividing line (discussed in detail in the next subsection), and on NSOP$_1$, see § 9(F).
6) Are all non-simple T almost $\leq_{\text{univ}}$-maximal? Or at least pseudo $\leq_{\text{univ}}$-maximal? Seems too optimistic, still.

§ 9(E). **Keisler's Order and Saturation of Ultrapowers.**

Definition 9.18. We define SOP$_n$ for $n = 1, 2$.
1) T has SOP$_2$ iff there is $\varphi(x, \bar{y})$, $M \in K_T$ and $\bar{a}_\eta \in {}^{\ell g(\bar{y})}M, b_\eta \in M$ for η a finite sequence of zero's and one's such that:

   (a) $\varphi(b_\eta, \bar{a}_{\eta \restriction \ell})$ for $\ell < \ell g(\eta)$;
   (b) $\varphi(x, \bar{a}_{\eta_0}), \varphi(x, \bar{a}_{\eta_1})$ are contradictory when $\eta^\smallfrown\langle \ell \rangle \trianglelefteq \eta_\ell$ for $\ell = 0, 1$.

2) T has SOP$_1$ <u>iff</u> as above but in (b) we add $\eta^\smallfrown\langle 1 \rangle = \eta_1$.

We define the partial order $\triangleleft$ and its relative $\triangleleft^*$ as follows.

Definition 9.19. Let D be an ultrafilter on a set I.
1) We define $N = M^I/D$ as the model with the same vocabulary (= kind) with set of elements $\{f/D : f \in {}^I M\}$ where $f_1/D = f_2/D$ means that $\{s \in I : f_1(s) = f_2(s)\} \in D$, and e.g. $N \models f/D = F(f_1/D, \ldots, f_n/D)$ <u>iff</u> $\{s \in I : f(s) = F^M(f_1(s), \ldots, f_n(s))\} \in D$.
2) We say D is regular when, if $f_\alpha/D \in M^I/D$ for $\alpha < |I|$ then for some finite $w_s \subseteq M$ for $s \in I$, we have $\alpha < |I| \Rightarrow \{s \in I : f_\alpha(s) \in w_s\} \in D$.
3) $T_1 \trianglelefteq T_2$ <u>iff</u> for any set I and regular ultrafilter D on I if $M_\ell \in K_{T_\ell}$ and M_2^I/D is $|I|^+$-saturated then so is M_1^I/D (this order is known as the *Keisler order*).
4) $T_1 \trianglelefteq^* T_2$ <u>iff</u> there is a model M in which T_1, T_2 are interpreted such that if $N \models \mathrm{Th}(M)$ and N_1, N_2 are the models of T_1, T_2 interpreted in N correspondingly and N_2 is λ^+-saturated then so is N_1.

In an initial bloom in the sixties and seventies, the stable case was sorted out (exactly two equivalence classes!); but afterwards little progress had been made on the Keisler order, until recently.

Note that $\triangleleft^*$ is more model theoretic, as for a proof of $\ntrianglelefteq$ we need to build suitable ultrafilters on the relevant infinite sets; however, we suspect that $\triangleleft^*$ is making too fine a distinction. Ideally, we would have finitely many equivalence classes; however, even for $\triangleleft$ any such hopes were shattered by [MS18], where it was proved that there is an infinite decreasing $\triangleleft$-chain of T's. Still:

Problem 9.20. Are there continuum many pairwise $\triangleleft$-incomparable T's?

Like in the case of $\leq_{\mathrm{univ}}$, the StOP seemed a candidate for $\triangleleft$-maximality, but alas, SOP$_4$ suffices at least for almost $\leq_{\mathrm{univ}}$-maximal [She96b, SU06]. However by [DS04b], [SU08] if T is NSOP$_2$ then T is not $\triangleleft^*$-maximal. (Well, assuming an instance of GCH.) The proof seems like a beginning of a positive theory.

Indeed, more recently, [MS17, 7.14] proves the other direction: if T is SOP$_2$, then it is $\triangleleft$-maximal and even $\triangleleft^*$-maximal, a further suggestion of a positive theory. Together, we conclude that SOP$_2$ is a *robust* dividing line.

Interestingly enough, the results above were preceded by a breakthrough [MS16b] telling us a somewhat weaker result: SOP$_3 \Rightarrow \triangleleft$-maximal, and also, out of the blue, that $\mathfrak{p} = \mathfrak{t}$, i.e. that two classical cardinal characteristics of the continuum are equal. This is another example of how investigating "good" dividing lines may lead to interesting and unexpected discoveries in other areas of mathematics.

Problem 9.21. 1) Prove that any NSOP$_2$-theory is not $\triangleleft$-maximal.
2) Characterize the NSOP$_2$ theories T such that for every large enough μ, for every $\lambda > \mu$ there is a regular ultrafilter D on λ such that: if M a model of T, then M^λ/D is not μ^+-saturated.

Problem 9.22. Develop a structure theory for NSOP$_2$. That is, show that NSOP$_2$ is a *successful* dividing line.

§ 9(F). NSOP$_1$ **and Exact Saturation.**
Lately, there have been serious advances on NSOP$_1$, making NSOP$_1$ a successful dividing line. By [CR16], any theory satisfying a certain amalgamation criterion is NSOP$_1$, but it was not clear that the converse holds (i.e., that this amalgamation criterion holds for NSOP$_1$ theories). In [KR17a], an independence relation called Kim-non-forking was introduced. This relation is defined in a similar way to non-forking, with the difference that

dividing is now required to be witnessed by generic sequences. In particular, Kim-non-forking satisfies the so-called *independence theorem*. By [KRS18] Kim-non-forking satisfies local character in $NSOP_1$ (and also a dual phenomenon). [KR17b] studies expansions of $NSOP_1$ theories; e.g., under mild assumptions, any $NSOP_1$ theory can be expanded to one with built-in Skolem functions. Examples of $NSOP_1$ theories include T_{feq} ([She93b]), T_{ceq} (see Definition 9.15 (2)) the generic binary function [KR17b], certain fields (e.g., ω-free PAC fields [Cha02], [CR16]), vector spaces with a generic bi-linear form [CR16], and the generic projective plane [CK].
Another serious candidate for being a good test problem is:

Definition 9.23. 1) T has the λ-exact saturation property when T has a λ-saturated model which is not λ^+-saturated.
2) T has the λ-PC-exact[1] saturation property when for every $T_1 \supseteq T$ there is a model M of T which can be expanded to a model of T_1, it is λ-saturated, and it is not λ^+-saturated.

Model theoretically these give rise to reasonable test problems. Now, exact saturation was first dealt with in [She15a], where it was shown that for unstable T's it holds for every regular $\lambda (> \aleph_0)$, and that for stable T's it holds for every λ; so we may as well assume that T is unstable and λ singular. For dependent theories T, the relevant dividing line turns out to be "T is distal" [KSS17]. Again, this result was reached by stages: [She15a] gives a sufficient condition for existence of universal models for dependent theories, and a sufficient condition for nonexistence for really complicated theories, e.g., Peano arithmetic.

However, for the PC-exact version the situation is different. On the one hand, SOP_2 implies that T does not have PC-exact saturation (for λ singular) by [MS17]; on the other hand for T simple, exact saturation holds for cardinals for which Jensen's square holds [KSS17]; in [KSS] a parallel result for exact PC-saturation is proved.

Conjecture 9.24. Any simple T has singular PC-exact saturation, at least for λ strong limit.

But more exciting is:

Problem 9.25. Is "singular PC-exact saturation" a good test problem, pointing to a dividing line between simple and $NSOP_2$? Is the right characterization $NSOP_2$ or $NSOP_1$?

This may involve continuing the work on $NSOP_1$, as well as developing it.
Back to $\leq_{univ}$ we think that T_{ceq} is the prototypical and simplest among non-simple theories; it is $NSOP_1$, but is it actually minimal? Now (see [She93b] and [DS04a]) we can force for it "there is a universal in μ^+ for μ of cardinality $\mu^{++} < 2^\mu$", similar to the case of simple theories. Again, still this gives only an indication (and not a proof) that it is not maximal. On the other hand, it is prototypical for the oak property, which gives sufficient conditions for non-existence near a singular cardinal. Like strictly stable T's; "the oak property" was introduced in [DS06]; see more in [She17]. The oak property, though of the right form, like $SOP_n, n \geq 3$ and StOP is as of now not supported by robust evidence for being a robust dividing line.

Problem 9.26. 1) Can we force fully $\mu^+ \in univ(T_{ceq}), \mu = \mu^{<\mu}, \mu^+ < 2^\mu$.
2) Is T_{ceq} really prototypical for TP_2 theories? Is it $\leq_{univ}$-minimal among them?
3) Can we prove for T_{ceq} non-existence results like in the case of SOP_4 theories?

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[1] PC refers to projective class.

Note that the family NTP_2 is too big for our program, as it includes theories that are clearly too complicated (by the present yardstick), like T_{dlo}, the theory of dense linear orders. So we will need to restrict ourselves to a subfamily.

§ 10. SPECIFIC CLASSES (NOT NECESSARILY ELEMENTARY)

§ 10(A). **Examples.**

We consider some classical classes. For which of those specific classes can we prove non-existence of universals?

- (A) linear orders: recall that $K_{\text{linear order}}$ is almost $\leq_{\text{univ}}$-maximal but consistently $\aleph_1 < 2^{\aleph_0}$ and there is a universal linear order in $\aleph_1$; we will later discuss the remaining cardinals;

- (B) Boolean algebras and ordered fields (which are like linear orders);

- (C) the theory of the random graph is $\leq_{\text{univ}}$-minimal among the unstable theories; so for reasonable cardinals (cf. Problem 9.14) they may have universals. But, e.g. for $\lambda = \beth_\omega^+ < 2^{\beth_\omega}$ we have only weak consistency results: some graph of cardinality $\lambda^+ < 2^{\beth_\omega}$ is universal for λ (see [DS03], [CDM$^+$]);

- (D) T_{trf}, the theory of triangle free graphs, we know only of cases (of consistency) of weak universality (as in [DS04a, She93b]) covered by the results on simple T's); [18.11.11 - check]

- (E) K_{groups}, the class of groups has the non-existence results of $K_{\text{linear order}}$ (for universality), though it has more amalgamation properties (in our framework, is NSOP_4). We now denote (see [SUb])

- (F) Abelian groups, there are several natural versions, a major one is the class of torsion free Abelian groups under pure embeddings (i.e. non-divisibility by a natural number is preserved), it seems a prototypical case for strictly stable T's in general and specifically to trees with ω- levels; on this we have much to say (see [Fuc70, Fuc73b, KS95, She97] and §10(B));

We shall consider more examples below.

Note that many relevant results in ZFC have roots in "difficulties" in extending independence results (see e.g. §10(B) below) thus providing a case for:

Thesis 10.1. Even if you do not like independence results you better look at them, as you will not even consider your desirable ZFC results when they are camouflaged by the litany of many independence results. Once forcing gets the rubbish out of the way, you can try to find diamonds.

Of course, independence has interest per se; still in a given problem in general a solution in ZFC is for me preferable on an independence result. But if it gives a method of forcing (so relevant to a series of problems) the independence result is preferable (of course, we assume there are no other major differences; the depth of the proof would be of first importance).

As occurs often in the author's papers lately, references to pcf theory appear.

This paper is also a case of the following thesis:

Thesis 10.2. Assumption of cases of the negation of GCH at singular (more generally $pp\lambda > \lambda^+$) are "good", "helpful" assumptions; i.e. traditionally uses of GCH proliferate mainly not from conviction but as you can prove many theorems assuming $2^{\aleph_0} = \aleph_1$ but very few from $2^{\aleph_0} > \aleph_1$, but assuming $2^{\beth_\omega} > \beth_\omega^+$ is helpful in some proofs, and see § 10(B) on $2^{\beth_\omega} > \beth_\omega^{++}$.

§ 10(B). **Abelian Groups.**

The class of Abelian groups (for usual embeddings) has universal models in every cardinality λ, they are suitable divisible ones, so we better restrict ourselves to so-called reduced ones, i.e. Abelian group G with no divisible non-trivial subgroup. Another natural notion of embedding is pure embedding which are embedding preserving non-divisibility by natural numbers, again we better consider only reduced Abelian group as any such G is the direct sum of divisible(G) and reduced(G), where divisible(G) = ∪{$H : H \subseteq G$ is divisible} and reduced(G) is not unique but is isomorphic to G/divisible(G).

Now if π is a pure embedding of G_1 into G_2 then it induces a pure embedding of divisible(G_1) into divisible (G_2) and of reduced(G_1) into reduced (G_2).

There are many interesting classes of Abelian groups but we shall consider here only (all with $\subseteq$ the used embedding or with $\subseteq_{\mathrm{pr}}$, the pure one):

- K_{rtf}, reduced torsion free
- $K_{\mathrm{rs}(p)}$, reduced separable p-group, the separable means that every element has finite height, equivalently belongs to a finite direct summand.

In particular, we shall not deal here with

- $K^{\mathrm{r}(p)}$, the class of reduced Abelian p-groups, because in every cardinal λ, there is no universal member, see [KS95, 3.1] and Fact 10.6, (though we could fix the height to some $\alpha < \lambda^+$, but have not dealt with this)
- classes depending on a sequence $\bar{t}$ of natural number (see [She97], [She01])
- $K_{\aleph_1\text{-free}}$, the class of $\aleph_1$-free Abelian groups (see [She97]).

Note that this class fails amalgamation "badly", hence $\lambda = \lambda^{\aleph_0} \Rightarrow \lambda \notin \mathrm{univ}(K_{\aleph_1\text{-free}})$; see [She97]. Note also $(K_{\mathrm{rtf}}, \subseteq)$ fail amalgamation but not so badly - that is, for enough existentially closed ones it has amalgamation, see [She01, §1] or Claim 10.9, but we shall not elaborate on $K_{\aleph_1\text{-free}}$.

We also shall barely mention the class (see [She96c])

- $(< \lambda)$-stable one.

There are closely related "combinatorial classes".

Thesis 10.3. General abelian groups and trees with $\omega + 1$ levels behave in universality theorems like stable non-superstable theories.

The simplest example of such a class is the class $K^{\mathrm{tr}} :=$ trees $\mathscr{T}$ with $(\omega + 1)$-levels, i.e. (up to isomorphism) $\mathscr{T} \subseteq {}^{\omega \geq}\alpha$ for some α, with the relations $\eta E_n^0 \nu := \eta \restriction n = \nu \restriction n$ and $<_T$ being an initial segment. For K^{tr} we know that $\mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$ implies there is no universal for K^{tr}_λ (by [KS92b]). Classes as K^{rtf} or $K^{\mathrm{rs}(p)}$ (reduced separable abelian p-groups) are similar (though they are not elementary classes) were first considered with pure embeddings (by [KS95]). But it is not less natural to consider usual embeddings (remembering they, the (abelian) groups under consideration, are reduced). Though the problem is the more natural it is harder to treat. The problem is that the invariant has been defined using divisibility, and so under non-pure embedding those seemed to be erased.

Unfortunately, most results in [She97] are only almost in ZFC as they use extremely weak assumptions from pcf, assumptions whose independence is not known. So practically it is not so tempting to try to remove them as they may be true, and it is unreasonable to try to prove independence results before independence results on pcf have advanced.

Then in [She96c] the non-existence of universals is proved restricting ourselves to $\lambda > 2^{\aleph_0}$ and $(< \lambda)$-stable groups (see there). These restrictions hurt the generality of the theorem; because of the first requirement we lose some cardinals. The second requirement changes the class to one which is not established among abelian group theorists (though to me it looks natural). The aim in [She97] was to eliminate those requirements, or show that they are necessary.

Next [She97] gives an explanation of some of the earlier difficulties: considering K_λ^x (for $x \in \{\mathrm{rtf}, \mathrm{rs}(p)\}$, we let $\mu = \min\{\mu : \mu^{\aleph_0} \geq \lambda\}$ and let $K_{\lambda,\mu}^x$ be the class of $G \in K_\lambda^x$ with density μ by the metric $d(x,y) = \inf\{\frac{1}{2^n} : n! \text{ divide } x-y\}$. The natural combinatorial problem is about $K_{\lambda,\mu}^{\mathrm{tr}} = \{\mathscr{T} \in K_\lambda^{\mathrm{tr}}: \text{the level } n \text{ of } \mathscr{T}, P_n^{\mathscr{T}} \text{ has cardinality } \leq \mu \text{ for } n < \omega\}$.

Parallely we define:

- $K_{\lambda,\mu}^{\mathrm{rs}(p)} = \{G \in K_\lambda^{\mathrm{rs}(p)} : |G/p^n G| \leq \mu \text{ for every } n\}$

- $K_{\langle \lambda_\alpha : \alpha \leq \omega \rangle}^{\mathrm{rs}(p)} = \{G \in K_{\lambda_\omega}^{\mathrm{rs}(p)} : p^n G/p^{n+1} G \text{ has dimension } \leq \lambda_n \text{ for every } n\}$, (the proof in [She97, 1.1] gives a reduction to this case).

In [She97, 1.1=pg.271], or see Proposition 10.7, it is proved[2] that $(K_\lambda^{\mathrm{rs}(p)}, \subseteq)$ can be replaced by $K_{\lambda,\mu}^{\mathrm{rp}(p)}$, moreover can be replaced by $K_{\langle \lambda_n : n < \omega \rangle ^\frown \langle \lambda \rangle}^{\mathrm{rs}(p)}, \bigwedge_n \lambda_n = \lambda_n^{\aleph_0} < \mu$.

The problem (of the existence of universals for $K^{\mathrm{rs}(p)}$) is not like looking for K_λ^{tr} but for $K_{\lambda,\mu}^{\mathrm{tr}}$ or even $K_{\langle \lambda_\alpha : \alpha \leq \omega \rangle}^{\mathrm{tr}}$ (where $K_{\lambda,\mu}^{\mathrm{tr}} = K_{\langle \mu : n < \omega \rangle ^\frown \langle \lambda \rangle}^{\mathrm{tr}}$) when

- $\lambda_n \leq \lambda_{n+1} < \mu = \Sigma_n \lambda_n$ and $\mu < \lambda = \lambda_\omega = \mathrm{cf}(\lambda) \leq \mu^{\aleph_0}$;

where

⊕ for $\bar{\lambda} = \langle \lambda_n : n \leq \omega \rangle$ non-decreasing, $\lambda \leq \Pi_n \lambda_n$, $K_{\bar{\lambda}}^{\mathrm{tr}}$ is $\{T : T$ a tree with $\omega + 1$ levels, in level $n < \omega$ there are λ_n elements$\}$.

For $K^{\mathrm{rs}(p)}$ this, or 10.7, reduction is proved fully (see [She17]), for K^{rtf} this is proved for the natural examples.

In [She97, §2] we define in addition to K_λ^{tr}, also $K_{\langle \lambda_\alpha : \alpha \leq \omega \rangle}^{\mathrm{fc}}$. In the second we have $\omega + 1$ kinds of elements and for each n a function from the ω-th-kind to the n-th kind. We can interpret a tree $\mathscr{T}$ as a member of the second example: $P_\alpha^{\mathscr{T}} = \{x : x \text{ is of level } \alpha\}$ and

$$F_n(x) = y \Leftrightarrow x \in P_\omega^{\mathscr{T}} \ \& \ y \in P_n^{\mathscr{T}} \ \& \ y <_{\mathscr{T}} x.$$

Now $K_{\lambda,\mu}^{\mathrm{fc}} = K_{\langle \mu : n < \omega \rangle ^\frown \langle \lambda \rangle}^{\mathrm{fc}}$ is defined as it is easier to get the non-existence theorems. But this is not one of the classes we considered originally.

In [She97, §3] we return to K^{rtf} (reduced torsion free Abelian groups with the usual embedding) and prove the non-existence of universal ones in λ if $2^{\aleph_0} < \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$ and an additional very weak set theoretic assumption (the consistency of its failure is not known). In [She01, 1.6] the non-existence result for K_λ^{rtf} for λ as above is fully proved, i.e. no extra assumption. For $K_\lambda^{\mathrm{rs}(p)}$, the parallel result is proved only almost in ZFC: we need $\lambda \geq \beth_\omega$, as we quote the RGCH, see [She00]. But recall that for the class of $\aleph_1$-free Abelian groups we have non-existence for $\lambda = \lambda^{\aleph_0}$.

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[2]Note that [She97, 1.1] the statement is referred to and proved rightly but stated wrongly.

We have noted above that for K_λ^{rtf} requiring $\lambda \geq 2^{\aleph_0}$ is reasonable because: we can prove (i.e. in ZFC) that there is no universal member. What about $K_\lambda^{\mathrm{rs}(p)}$? By [She97, §1] we should look at $K_{\langle \lambda_i : i \leq \omega \rangle}^{\mathrm{tr}}, \lambda_\omega = \lambda < 2^{\aleph_0}, \lambda_n < \aleph_0$.

In [She97, §4] we prove the consistency of the existence of universals for $K_{\langle \lambda_i : i \leq \omega \rangle}^{\mathrm{tr}}$ when $\lambda_n \leq \aleph_0, \lambda_\omega = \lambda < 2^{\aleph_0}$ but weakly, i.e. of cardinality λ^+; this is not the original problem but it seems to be a reasonable variant, and more seriously, it shoots down the hope to use the present methods of proving non-existence of universals. Anyhow this is $K_{\langle \lambda_i : i \leq \omega \rangle}^{\mathrm{tr}}$ not $K_{\lambda_\omega}^{\mathrm{rs}(p)}$, so we proceed to reduce this problem to the previous one under a mild variant of MA.

The reader should remember that the consistency of e.g. the other side, i.e. of $(*)$ below is much easier to obtain, even in a wider context (just add many Cohen reals) where

$(*)$ $2^{\aleph_0} > \lambda > \aleph_0$ and there is no M such that $M \in K^{\mathrm{rs}(p)}$ is of cardinality $< 2^{\aleph_0}$ and universal for $K_\lambda^{\mathrm{rs}(p)}$.

As in [She97, §4] the problem for $K_\lambda^{\mathrm{rs}(p)}$ was reasonably resolved for $\lambda < 2^{\aleph_0}$ (and for $\lambda = \lambda^{\aleph_0}$, see [KS95]), we then, in [She97, §5] turn to $\lambda > 2^{\aleph_0}$ (and μ, λ_n) as in $(\oplus)$ above.

How above do we overcome the problem of "divisibiity is not preserved? and how does pcf help"? As in an earlier proof we use $\langle C_\delta : \delta \in S \rangle$ guessing clubs for λ (see references or §11(B)), so C_δ is a subset of δ (hence the invariant depends on the representation of G but this disappears when we divide by suitable ideal on λ). What we do is: rather than trying to code a subset of C_δ (for $\bar{G} = \langle G_i : i < \lambda \rangle$ a representation or filtration of the structure G as the union of an increasing continuous sequence of structures of smaller cardinality) by an element of G, we do it, say, by some sequence $\bar{x} = \langle x_t : t \in \mathrm{Dom}(I) \rangle$, I an ideal on $\mathrm{Dom}(I)$ (so really we use $\bar{x}/I$). At first glance if $\mathrm{Dom}(I)$ is infinite we cannot list *a priori* all possible such sequences for a candidate H for being a universal member, as their number is $\geq \lambda^{\aleph_0} = \mu^{\aleph_0}$. But we can find a family

$$\mathscr{F} \subseteq \{ \langle x_t : t \in A \rangle : A \subseteq \mathrm{Dom}(I), A \notin I, x_t \in \lambda \}$$

of cardinality $< \mu^{\aleph_0}$ such that for any $\bar{x} = \langle x_t : t \in \mathrm{Dom}(I) \rangle$, for some $\bar{y} \in \mathscr{F}$ we have $\bar{y} = \bar{x} \restriction \mathrm{Dom}(\bar{y})$.

As in [She97, §3] there is such $\mathscr{F}$ except when some set theoretic statement related to pcf holds. This statement is extremely strong, also in the sense that we do not know how to prove its consistency at present. But again, it seemed unreasonable to try to prove its consistency before the pcf problem was dealt with. Of course, we may try to improve the combinatorics to avoid the use of this statement, but were naturally discouraged by the possibility that the pcf statement can be proved in ZFC; thus we would retroactively get the non-existence of universals in ZFC. However, in [She01], this pcf assumption for K_λ^{rtf} is eliminated but for $K_\lambda^{\mathrm{rs}(p)}$ is only replaced by $\lambda > \beth_\omega$, using the RGCH.

Further results are in [She97, §6-§9].

An interesting aspect of the above and of [She01], is the use of pcf for ideals defined using algebra. From another aspect it is probably better replace K^{fc} (fc stands for function) by K^{fd} defined by:

$\boxplus$ (A) $M \in K^{\mathrm{fd}}$ <u>iff</u>:

   (a) $\langle P_\alpha^M : \alpha \le \omega \rangle$ is a partition

   (b) $F_{n,\ell}^M$ is a function from P_ω^M into P_n^M for $n, \ell < \omega$

   (c) $\langle F_{n,\ell}^M(a) : \ell < \omega \rangle$ is eventually constant for every $n < \omega$

(B) $K_{\lambda,\mu}^{\text{fd}}, K_{\langle \lambda_\alpha : \alpha \le \omega \rangle}^{\text{fd}}$ are defined naturally.

The point is:

Claim 10.4. *1)* $\text{univ}(K_\lambda^{\text{tr}}) \le \text{univ}(K_\lambda^{\text{rtf}})$.
2) $\text{univ}(K_\lambda^{\text{tr}}) \le \text{univ}(K_\lambda^{\text{rs}(p)}) \le \text{univ}(K_\lambda^{\text{fd}})$.
3) $\text{univ}(K_\lambda^{\text{tr}}) \le \text{univ}(K_\lambda^{\text{fc}}) \le \text{univ}(K_\lambda^{\text{fd}})$.
   See [She97].

Question 10.5. 1) Can we prove the consistency of: μ strong limit of cofinality $\aleph_0, \lambda = \mu^+ < 2^\mu$ and $\text{univ}(K_\lambda^{\text{tr}}) = 1$ and even $\text{univ}(K_\lambda^{\text{fd}}) = 1$?
2) Prove that $\lambda \le \text{univ}_{\text{group}}$ <u>iff</u> $\lambda = 2^{<\lambda} > \aleph_0$.

   Maybe it will help to formally divide to cases and give a table clarifying the situation: See Table 1 at the end of this article.

10(B).1. *Fulfilling some points.*

Fact 10.6. [KS95, Th.3.1] For every $\lambda, \lambda \notin \text{univ}(K^{r(p)})$ recalling $K^{r(p)} = \{G : G$ a reduced Abelian p-group$\}$.

Proof. Recall

   (A) For $G \in K^{r(p)}$ and $x \in G'$ defined $\text{rk}, \text{ht}(x, G) = \sup\{\text{rk}(y, G) + 1 : y \in G, G \models py = x\} \in \{\text{Ord}\}$. So $\text{ht}(x, G) = \infty \Leftrightarrow x = 0_G \Leftrightarrow \text{ht}(x, G) < |G|^+$. Let $\text{ht}(G) = \sup\{\text{ht}(x, G) + 1 : x \in G \backslash \{0_G\}\}$

   (B) For every $\alpha < \lambda^+, \alpha \ge 2$ there is $G \in K_\lambda^{r(p)}$ with $\text{rk}(G) = \alpha$

   (C) If $g \in \text{Hom}(G_1, G_2)$ then $\text{ht}(x, G_2) \le \text{ht}(g(x), G_2)$

   (D) Together $\text{univ}(K_\lambda^{r(p)}) \ge \lambda^+$.

$\square_{10.6}$

Proposition 10.7. [She97, 1.1] *Assume* $\mu = \Sigma_{n<\omega} \lambda_n = \limsup_n \lambda_n$, $\mu \le \lambda \le \mu^{\aleph_0}, G$ *is a reduced separable (Abelian) p-group,* $|G| = \lambda$ *and* $\lambda_n(G) =: \dim((p^n G)[p]/(p^{n+1}G)[p]) \le \mu$ *(this is a vector space over $\mathbb{Z}/p\mathbb{Z}$ hence the dimension is well defined).*
   <u>Then</u> *there is H, a reduced separable p-group such that $|H| = \lambda, H$ extends G (not necessarily purely) and $(p^n H)[p]/(p^{n+1}H)[p]$ is a group of cardinality λ_n.*

Remark 10.8. So for H the invariants from [KS95] are trivial.

Proof. (See Fuchs [Fuc73a]). We can find z_i^n ($n < \omega, i < \lambda_n(G) \le \mu$) such that:

   (a) $z_i^n \in G$ has order p^n,

   (b) $B = \underset{n,i}{\Sigma} \langle z_i^n \rangle_G$ is a direct sum,

   (c) B is dense in G (in the topology induced by the norm
$$\|x\| = \min\{2^{-n} : p^n \text{ divides } x \text{ in } G\}).$$

For each $n < \omega$ and $i < \lambda_n(G) (\le \mu)$ choose $\eta_i^n \in \prod_{m<\omega} \lambda_m$, pairwise distinct such that: for $(n^1, i^1) \ne (n^2, i^2)$ for some $n(*)$ we have:

$$\lambda_n \ge \lambda_{n(*)} \Rightarrow \eta_{i^1}^{n^1}(n) \ne \eta_{i^2}^{n^2}(n).$$

Let H be generated by G, x_i^m (for $i < \lambda_m, m < \omega$), $y_i^{n,k}$ (for $i < \lambda_n, n < \omega, n \le k < \omega$) freely except for:

(α) the equations of G

(β) $y_i^{n,n} = z_i^n$

(γ) $py_i^{n,k+1} - y_i^{n,k} = x_{\eta_i^n(k)}^k$

(δ) $p^{n+1} x_i^n = 0$,

(ε) $p^{k+1} y_i^{n,k} = 0$.

Check. $\square_{10.7}$

Implicit in [She01].

Claim 10.9. *Letting $K_{\mathrm{ftf}} = \{G \in K_{\mathrm{rtf}} : G$ is full (see [She01, Definition 1.3(3)])\}.*
1) K_λ^{ftf} is dense in λ and it satisfies:

(a) *every member of K_λ^{rtf} can be extended to a full one*

(b) *if $M_1, M_2 \in K_{\mathrm{rtf}}$ and M_1 is full and h embeds M_1 into M_2, then h is a pure embedding.*

2) $(K_\lambda^{\mathrm{ftf}}, \subseteq) = (K_\lambda^{\mathrm{ftf}}, \subseteq_{\mathrm{pr}})$ and it has amalgamation.
*3) $\mathrm{univ}(K_\lambda^{\mathrm{ftf}}, \subseteq) = \mathrm{univ}(K_\lambda^{\mathrm{rtf}}, \subseteq) = \mathrm{univ}(K_\lambda^{\mathrm{ftf}}, \subseteq) = \mathrm{univ}(K_\lambda^{\mathrm{rtf}}, \subseteq).*$

Claim 10.10. *1) If (A) then (B) where:*

(A) (a) $\mu = \mu^{<\mu} < \theta = \mathrm{cf}(\theta) < \chi = \chi^\mu$ *and θ has weak club guessing for μ or for some at least for some $\sigma \ge \mu$, see §11(B)*

(b) $\mathbb{Q}$ *is the forcing of adding χ many μ-Cohens*

(c) $\mathbb{Q} \lessdot \mathbb{P}$ *and $\mathbb{P}$ satisfies the μ^{++}-c.c., and $\mathbf{V}^{\mathbb{P}}$ and $\mathbf{V}$ have the same cardinals*

(d) $\mathbf{V}^{\mathbb{P}} \models$ *"$\mathrm{cf}(\mu) = \aleph_0$" so if $|\mathbb{P}| = \chi$ then $\Vdash_{\mathbb{P}}$ "$\mu^{\aleph_0} = \chi$"*

(B) (a) $\mathrm{univ}(K_\lambda^{\mathrm{tr}}) \ge \chi$

(b) $\mathrm{univ}(K_{\lambda,\mu}^{\mathrm{tr}}) \ge \chi.$

2) Assume $\mu < \theta = \mathrm{cf}(\theta) \le \lambda \le \mu^{\aleph_0}$ and θ has club guessing for μ (even weakly, see §11(B)):

(A) *if $\mathrm{cf}([\lambda]^\theta, \subseteq) < \mu^{\aleph_0}$ then*

⊞   (a) $\mathrm{univ}(K_\lambda^{\mathrm{tr}}), \mathrm{univ}(K_\lambda^{\mathrm{fc}}) \ge \mu^{\aleph_0}$

(b) $\mathrm{univ}(K_{\lambda,\mu}^{\mathrm{tr}}), \mathrm{univ}(K_{\lambda,\mu}^{\mathrm{fc}}) \ge \mu^{\aleph_0}$

(c) $\mathrm{univ}(K_\lambda^{\mathrm{rtf}}) \ge \mu^{\aleph_0}$

(d) $\mathrm{univ}(K_\lambda^{\mathrm{rs}(p)}) \ge \mu^{\aleph_0}$

(B) *if $\mathrm{cov}(\lambda, \theta^+, \theta^+, \theta) < \mu^{\aleph_0}$, then clause (a),(b),(c) of ⊞ holds.*

3) All the standard models for violating the SCH *fit under part (1).*

Proof. 1) By clause (A) of (2).
2) See [She01] for K_λ^{rtf} use 10.10.
3) Clear. $\square_{10.10}$

Claim 10.11.   *(1) Assume* $\lambda = \sum_n \lambda_n$ *where* $\lambda_n = (\lambda_n)^{\aleph_0}$. *Then* $\mathrm{univ}(K_\lambda^{\mathrm{tr}}) = \mathrm{univ}(K_\lambda^{\mathrm{fc}}) =$
$\mathrm{univ}(K_\lambda^{\mathrm{fd}}) = 1$.
(2) K_λ^{fc} *has no universal element when* $\lambda < 2^{\aleph_0}$, *moreover, this holds for* $K_{\bar\lambda}^{\mathrm{fc}}$, *where*
$\bar\lambda = \langle \lambda_n : n \le \omega \rangle$, $\aleph_0 \le \lambda_\omega < 2^{\aleph_0}$, $\lambda_n \le \lambda_\omega$ *for all* $n < \omega$, *and* $\lambda_n \ge 2$ *for infinitely*
many $n < \omega$.

Proof. Easy. E.g.
2) Let $u_* = \{n < \omega : \lambda_n \ge 2\}$, an infinite set. For every $u \subseteq u_*$ we define the model M_u
as follows:

(a) The universe of M_u is the set $\{(n,i) : n < \omega, i \in \{0,1\}$ and $n \in u \Rightarrow i = 0\} \cup$
$\{(\omega,0),(\omega,1)\}$
(b) $P_n^{M_u} = \{(n,i) : i < 2\}$
(c) $F_n^{M_u}((\omega,i))$ is $(n,1)$ if $n \in u$, and $(n,0)$ if $n \notin u$.

Clearly,

- $M_u \in K_{\aleph_0}^{\mathrm{fc}}$
- If $N \in K^{\mathrm{fc}}$, then the set $\{u \subseteq u_* : M_u$ embeddable in $N\}$ is of cardinality $\le \|N\|$.

We are done. $\square_{10.11}$

§ 10(C). **Continuity.**
In [She97, §9] we turn to a closely related class: the class of metric spaces with (one to
one) continuous embeddings, similar results hold for it. We also phrase a natural criterion
for deducing the non-existence of universals from one class to another.

(G) A very active and successful area concerns complete metric spaces and Banach
spaces; this area is known as continuous model theory. The analogous problems
there are usually considerably more difficult.

The basic idea is to consider a Banach space or just model M with metric $\mathbf{d}_\mu$ which is
a complete metric space under the metric $\mathbf{d}_\mu$; so relations $R_{\mathbf{d},\le r} = \{(x,y) : \mathbf{d}_M(x,y) \le r\}$
$r\}, R_{\mathbf{d},\ge r} = \{(x,y) : \mathbf{d}_M(x,y) \ge r\}$ for $r \in \mathbb{R}_{\ge 0}$; and individual constants 0_M, and countably
many other closed relations. Taking ultra-products, "throwing" away elements of infinite
distance from 0 and dividing by the equivalence relation of being of distance $\le 1/(n+1)$
for every n things are fine, but we preserve only some formulas and sentences.

For model theorist cardinality is replaced by topological density and "M embeddable
into N" means M is isometric to a submodel of N; Banach space theorists prefer isomorphic
which means if $f(a) = b$ and $\varepsilon \in (0,1)_\mathbb{R}$, then for some $\xi \in (0,1)_\mathbb{R}$ we have

$$(b \in M \wedge \mathbf{d}_M(b,a) < \xi) \Rightarrow \mathbf{d}_N(f(b),f(a)) < \varepsilon,$$

or ξ-isomorphism for $\xi \in (0,1)_\mathbb{R}$, where ξ-isomorphism means that if $a \ne b \in M$, then

$$\mathbf{d}_N(g(a),g(b))/\mathbf{d}_M(a,b) \in [\xi, 1/\xi].$$

On categoricity the first step, see [BY05], [SU11], the first uses isometry, the second an
almost isometric $((1+\varepsilon)$-isomorphic for every $\varepsilon \in \mathbb{R}_{>0})$. But for our purpose central is
[SU06] showing that the class of Banach spaces has a version of SOP_4 (even SOP_n, for all
n), and deduce it is almost $\le_{\mathrm{univ}}$-maximal under isometric embeddings.

Central problems are:

Problem 10.12. Prove "categoricity spectrum" is nice under isomorphisms.

Problem 10.13. Develop the parallel of superstable in this context.

See more in [BYU10], [SUb], [SU06].

In particular, [SU06] shows that the class of Banach spaces is almost $\leq_{\text{univ}}$-maximal for isometries; more recently, [BK12] proved no universal Banach space of density the continuum (assuming the continuum is far from GCH). More general negative results on isomorphic embeddings are obtained in [Dz14]. Recently there has been interest from the operator algebras community, and some similar preliminary results have been obtained, see [FV].

(H) Some general results were applied to Eberlein spaces, see e.g. [Mir05], [Dz06].

Now, [SS] addresses some other classes and notions of universality, universality of functions, and also distinctions of existence of universals for closely related classes.

Definition 10.14. (1) For $\mathbf{k} \in [2, \aleph_0]$ let $\text{Un}^1_{\mathbf{k}}$ be the statement: there is $\mathbf{c} : [\omega_1]^2 \to \mathbf{k}$ such that for any $\mathbf{d} : [\omega_1]^2 \to \mathbf{k}$ there is a one-to-one function $\pi : \omega_1 \to \omega_1$ such that $\alpha < \beta < \omega_1 \Rightarrow \mathbf{d}\{\alpha,\beta\} = \mathbf{c}\{\pi(\alpha),\pi(\beta)\}$.
(2) Let $\text{Un}^2_{\mathbf{k}}$ be defined similarly but there is also a permutation $\varkappa$ of $\mathbf{k}$ such that $\alpha < \beta < \omega_1 \Rightarrow \mathbf{d}\{\alpha,\beta\} = \varkappa(\mathbf{c}\{\pi(\alpha),\pi(\beta)\})$.

Can we prove:

Problem 10.15. Can we distinguish $\text{Un}^1_{\aleph_0}$ and $\text{Un}^2_{\mathbf{k}}$?

§ 10(D). **Locally Finite Groups.**

The class K_{lfg} of locally finite groups is the class of groups G such that every finitely generated subgroup of G is finite, see on it the books [KW73] and [GS84]. Characteristic of the area is mixing finite group theory and finite combinatorics. By [GS83] we have:

Theorem 10.16. *If $\lambda = \lambda^{\aleph_0}$, then K_{lfg} has no universal in λ.*

This result is very different from the results cited above, the reason is that K_{lfg} is not a first order class; moreover, amalgamation fails very badly in this class. The first cardinal which is not covered by Theorem 10.16 under GCH is $\beth_\omega$.

Problem 10.17. 1) Does K_{lfg} have a universal member in $\beth_\omega$?
2) Assume μ is strong limit (i.e. $\theta < \mu \Rightarrow 2^\theta < \mu$) uncountable and of cofinality $\aleph_0$, i.e. $\mu = \sum_n \mu_n$, $\aleph_0 < \mu_n < \mu_{n+1}$. Does $\mu \in \text{Univ}(K_{\text{lfg}})$?

Note that if $\lambda = \mu^{++}$ then no group of cardinality λ is universal even for locally finite groups or cardinality λ, [She16]. Concerning Problem 10.17, by [GS83] we have:

Theorem 10.18. *If κ is a very large cardinal, specifically a so-called compact cardinal and $\mu > \theta$ is as in Problem 10.17(2), then $\mu \in \text{Univ}(K_{\text{lfg}})$.*

A very special cardinal is $\aleph_1$.

Problem 10.19. Can we prove $\aleph_1 \notin \text{Univ}(K_{\text{lfg}})$?

This, in particular, would require use of finite group theory. As in earlier cases, if $\beth_\omega < \lambda < 2^{\beth_\omega}$, then in many cases we can prove $\lambda \notin \text{Univ}(K_{\text{lfg}})$, but note:

Problem 10.20. 1) Assume $\mu = \Sigma \lambda_n$, $\lambda_n = \lambda_n^{\aleph_0} < \mu_{n+1}$ and $\mu < \lambda < \mu^{\aleph_0}$. Prove $\lambda \notin$ Univ(K_{lfg}).

2) What about singular not strong limit cardinals? By [Sheb] we know that having $G \in K_{\mathrm{lfg}}$ of cardinality $\mu = \Sigma_{n<\omega} 2^{\lambda_n} = \Sigma_{n<\omega} \lambda_n$ is equivalent to having something like a special model.

§ 10(E). Countable Graphs and Countable Density.

Universality in $\aleph_0$ has its mysteries, which attracted the attention of logicians (see Part I and e.g. [CSS99], [CT07]), combinatorialists (see e.g. [FK97], [KMP88]) and algebraists alike (see e.g. [Hal59] and [Don06]). One of the most studied cases of universality in $\aleph_0$ is the case of graphs, and specifically of classes of countable graphs defined by forbidding a finite subgraph (see [Che11] for a general discussion). A characteristic well-known case is the case of triangle free graphs [Hen71], where a universal object is known to exist by the usual method of Fraïssé. Corresponding to the usual graph theoretic distinction between subgraphs (weak embeddings) and induced subgraphs (strong embeddings), we have here two notions of universality for a class K of countable graphs, one with respect to weak embeddings (i.e. every $A \in K$ embeds weakly into the proposed universal object) and one with respect to strong embeddings (i.e. every $A \in K$ embeds strongly into the proposed universal object).

In this context, one usually prefers to prove the existence of universal objects in the strong form, and the nonexistence in the weak form, taking special note of the rare instances where a weakly universal object exists but a strongly universal one does not. The main problem in this line of inquiry is the following:

Problem 10.21 ([Che11])**.** 1) Given a finite graph H, can we characterize the class K_H of countable graphs omitting H as a subgraph in the weak sense, does K_H admits a universal object in the strong sense?

2) Is the class of finite graphs H such that K_H admits a universal object (in the strong sense) *decidable*?

This problem has been studied by many scholars, from different backgrounds, see e.g. [CS96], [CS01], [CST97], [GK96], [KMP88], [KP84], [Lat94], [Lat03]. One of the most general results in the area was due to Komjáth [Kom99], who proved that for any incomplete 2-connected H the class K_H does not admit a universal graph.

Now in [CSS99, CS07, CS16] was a general strategy which lead to fundamental progress toward a positive answer to the decidability problem above. Crucial to this strategy is the discovery of a fundamental connection with the model theoretic algebraic closure operator, where we say that $a \in G$ is algebraic over a finite subset $B \subseteq G$ if the orbit of a under $Aut_A(G)$ (automorphisms of G fixing A pointwise) is finite. Now, fixing a connected graph H, obviously K_H has the joint embedding property; let G be an existentially closed $\aleph_0$-homogeneous graph in K_H. It turns out that:

⊞ The following three conditions are intimately related at both a theoretical and an empirical level:

  (A) the algebraic closure operation associated to G is (uniformly) locally finite, in the sense that the algebraic closure of a set of size n is bounded in size by a function of n;

  (B) there is a strongly universal H-free graph;

  (C) there is a weakly universal H-free graph.

(These three conditions are successively weaker and not much different in practice.)

This breakthrough allowed for serious progress in Problem 10.21, for example [CS07] gave a *complete characterization* of the finite trees (in the graph theoretic sense) T such that K_T admits a universal object. In the case of trees, all blocks have at most two vertices. Also, the opposite case was considered, in which all blocks have at least three vertices. We now have an explicit list of the finite graphs of this type: exactly the ones for which K_H admits a universal object. The general case should be a mix (rather than a union) of the case of trees and this opposite case. Of course this would be the best possible result.

In order to settle the conjecture we need several crucial technical results, which naturally involves investigating deeply the structure of finite graphs. Our advancement has led to the following specific conjecture, which in many respects is truly the missing piece of our venue:

Conjecture 10.22. The Solidity Graph conjecture: if K_H has a universal member in $\aleph_0$ then every block of H (i.e. a maximal 2-connected subgraph) is complete.

Apart from its inherent theoretical interest, the question of universality in $\aleph_0$ and the study of relative classes K_H's as above connects crucially to our main line of investigation. In fact, we conjecture:

Conjecture 10.23. Investigating K_H's as above (and relative generalizations to hyper-graphs, and combinatorial structures in general) will be a source of interesting examples of unstable theories, as in the case of the involved discovery of infinite chains in Keisler's order (see [MS18]).

Finally, the study of graphs forbidding a given substructure hopefully will help us in building new examples of T's. Another source of such examples is $0 - 1$ laws (see e.g. [KPR87], [SS88], [BS97], [BS98], [She15b], [She96a], [She02]).

§ 11. COMBINATORICS

§ 11(A). **Finite Combinatorics.**

In this section we discuss the combinatorial ingredients of the theory of universality and connections with established themes in combinatorics, both finite and infinite. It is no surprise that infinite combinatorics will play a crucial role, including club guessing and pcf theory which are particularly relevant, but there are also strong connections with finite combinatorics, which we wish to stress, such as structural Ramsey theory, the Szemeredi Regularity Lemma, and probabilisitic methods, see e.g. [She80a], [KS92a], [She93a], [She13].

Why is this? In principle because the dividing lines we mainly are looking for use finitary properties of the family of subsets defined by mainly one formula $\varphi(\bar{x}_n, \bar{y}_n)$, in a model of T, varying the parameter $\bar{y}_n$. Let us look at some examples of connections.

1) Ramsey's theorem

Ramsey's theorem stands behind Ehrenfeucht-Mostowski models [EM56]; their aim was to find models with many automorphisms and this is where indiscernible sequences originate; this was used by the seminal work of Morley. In fact indiscernible sequences are one of the most important tools in model theory.

Generalizations of Ramsey are also very important. The proof that theories with the independence property have complicated models in terms of Karp complexity in [LS03] uses the Ramsey theorem for ordered graphs; see more generally [Shee].

2) Stability theory and the number of types

At the beginning of stability theory a crucial point was that the order property (one of the definitions of being unstable) is equivalent to some rank being large (and having

"many" φ-types over a set). This was done in [She71] quoting a combinatorial result from [EM66]; now [She72] gives stronger infinitary results solving problems from the list of Erdös-Hajnal, and [Hod81] gives the finitary theorem. This is also connected to counting the number of (local) types.

Another connection to counting types is the proof that: if for some model M of T, $\lambda = |\mathbf{S}(M)|$ (i.e., the number of complete 1-types over M) is too large (essentially no linear order of cardinality $\|M\|$ has $\geq \lambda$ cuts) then T has the independence property for some φ. This leads me to interest in the finitary question: if $\mathscr{P} \subseteq \mathscr{P}(n)$ and $|\mathscr{P}|$ big enough, then there are k-independent sets from $\mathscr{P}$, see [VvC71], [Sau72], and VC-dimension via the infinite theorem we knew ([She71]) that: if some $\varphi(\bar{x}, \bar{y})$ has the independence property in (a complete T) then also some $\varphi(x, \bar{y})$ does. Now this needs forcing (as if GCH holds the infinitary theorem is vacuous), considerably later a finitary proof was found by [Las92].

3) Graph regularity

Szemerédi's lemma is suspected to be relevant as well as results like [Sze75], [Gow07], [Gow13]. So far we have an application in the other direction (i.e., improving the general combinatorial results under model theoretic assumptions.) For example, in graph with no complete k-half graphs the bounds in the Szemerédi Lemma can be much improved (see [LS10], [MS14]). Other examples include better regularity lemmas assuming distality or dependence=NIP, see e.g., [CSb], [CSa].

4) Uniform definability over finite sets

In [CS15] a long standing open problem was solved: if T is dependent, then for every $\varphi(x, \bar{y})$ there is $\psi(\bar{y}, \bar{z})$ such that: if A is a finite subset of $M \in K_T$ and $a \in M$ then $\mathrm{tp}_{\varphi}(a, A, M)$ is definable by $\psi(\bar{y}, \bar{c})$ for some $\bar{c} \in {}^{\ell g(\bar{z})}A$. They have essentially used tools from both model theory and finite combinatorics (in the form of the (p, k)-theorem, see [Mat04]).

§ 11(B). **Infinite Combinatorics.**

We would like to give a taste of the infinite combinatorics involved. After [She80a], surprisingly [KS92a] prove that if e.g. $\lambda = \mu^{++} < 2^{\mu}$ then $\lambda \notin \mathrm{univ}(T_{\mathrm{dlo}})$ and more. How is this done?

A major point is:

Definition 11.1. 1) We say λ has club guessing for S when some $\bar{C}$ witnesses it, which means:

    (a) S is a stationary subset of λ;

    (b) $\bar{C} = \langle C_{\delta} : \delta \in S \rangle$, where C_{δ} is a club of δ of order type $\mathrm{cf}(\delta)$, the cofinality of δ;

    (c) if E is a club of λ, then for stationarily many $\delta \in S$ we have $C_{\delta} \subseteq E$.

2) Replacing S by κ means κ is a cardinal $< \lambda$ and S is a stationary subset of $\{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\kappa)\}$ and demanding $\mathrm{otp}(C_{\delta}) = \kappa$.

The point is that, unlike e.g. Jensen's diamond, cases of this principle are provable in ZFC (see [She94, Ch.III] and for part (3), see [She93a] and more in [She13, 0.11], or [Shea, 1.3(b)(β)]).

Theorem 11.2. *1) If $\lambda > \kappa$ are regular cardinals and $\kappa^+ < \lambda$ then λ has club guessing for κ.*
2) Moreover this holds for any such stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\kappa)\}$.
3) Moreover we can (for many such sets S) find a sequence $\bar{C} = \langle C_{\delta} : \delta \in S \rangle$, $\mathrm{otp}(C_{\delta}) = \kappa$, witnessing λ has a club guessing sequence for κ such that:

$\oplus$ if $\alpha < \lambda$ then $\{\delta \in S : \alpha \in \text{nacc}(C_\delta)\}$ has cardinality $< \lambda$.

4) If $\lambda = \kappa^{++}$, κ not necessarily regular, then λ has club guessing for κ. Moreover, we can take S as in 11.2(2) and demand

$$\alpha \in C_{\delta_1} \cap C_{\delta_2} \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha.$$

Now [KS92a] proves:

Theorem 11.3. *1) If λ has club guessing for κ and $2^\kappa > \lambda$ then $\lambda \notin \text{univ}(T_{\text{dlo}})$.*
2) This holds for any T with the strict order property.

How does the proof go? Let $\bar{C} = \langle C_\delta : \delta \in S \rangle$ wittness λ has club guessing for κ and be as in 11.2(3). How does club guessing help? For transparency, instead of T_{dlo} we use T_{tr}, the theory of trees. Suppose M is a "pretender" to be a universal of cardinality λ, without loss of generality it has universe λ. Given M, what are the demands for another model N to be unembeddable into it? Let f embed N into M, expand M to M' by $P^{M'} = \text{Rang}(f)$ (and possibly more) and let $E = \{\delta < \lambda : \delta$ a limit ordinal $< \lambda$ and $M' {\restriction} \delta \prec M'\}$, clearly it is a club of λ hence for some $\delta \in S, C_\delta \subseteq E$. Now

$\boxplus_1$ (a) for $\beta < \lambda$ let $\text{inv}^0(\beta, C_\delta, M)$ be the set of $\alpha \in \text{nacc}(C_\delta)$ satisfying there is $\gamma \in (\sup(\alpha \cap C_\delta), \alpha)$ such that $\gamma <^M \beta \wedge (\forall \gamma_1)(\gamma_1 < \sup(C_\delta \cap \alpha) \wedge \gamma_1 <^M \beta \Rightarrow \gamma_1 <^M \gamma)\}$.
(b) $\text{inv}(\beta, C_\delta, M) = \{\text{otp}(\alpha \cap C_\delta) : \alpha \in \text{inv}(\beta, C_\gamma, M)\}$, so $\in \mathscr{P}(\kappa)$.

$\boxplus_2$ (a) let $\text{Inv}(M)$ be the set of $u \subseteq \kappa$ such that the set

$$\{\delta \in S : \text{for no } \beta < \lambda \text{ do we have } \text{inv}(\beta, C_\delta, M) = u\}$$

belongs to $\text{id}_{\bar{C}}$, see below.
(b) Let $\text{inv}(M) = \langle \{\text{inv}(\beta, C_\delta, M) : \beta < \lambda\} : \delta \in S \rangle / \text{id}_{\bar{C}}$ where

$$\text{id}_{\bar{C}} = \{B \subseteq S : \text{ for some club } E \text{ of } \lambda \text{ we have } \delta \in B \Rightarrow C_\delta \nsubseteq E\}.$$

The point is that by $\boxplus_1$, $\text{inv}(M)$ really depends on M only up to isomorphism which can be seen as follows: If M' and M'' are trees with λ as universe, let f be an isomorphism from M' onto M'' letting E be the set of limit ordinals $< \lambda$ closed under f. Clearly, E is a club and if $\delta \in S, C_\delta \subseteq E$ and $\beta < \lambda$, then $\text{inv}(\beta, C_\delta, M') = \text{inv}(f(\beta), C_\delta, M'')$. Moreover, if N is embeddable into M and without loss of generality has universe λ then the set

$$\{\delta : \{\text{inv}(\beta, C_\delta, M) : \beta < \lambda\} \subseteq \{\text{inv}(\beta, C_\delta, N) : \beta < \lambda\}\},$$

belongs to $\text{id}_{\bar{C}}$. Why? As in the proof of $\boxplus_3$.

$\boxplus_3$ $\text{Inv}(M)$ is a subset of $\mathscr{P}(\kappa)$ of cardinality $\leq \lambda$

$\boxplus_4$ if $u \subseteq \kappa$ then there is a model M of T of cardinality λ and even universe λ such that:

$\bullet_1$ $(\forall \delta \in S)(\exists \beta < \lambda)[\text{inv}(\beta, C_\delta, M) = u]$, hence

$\bullet_2$ $u \in \text{Inv}(M)$.

For transparency $\alpha \in C_{\delta_1} \cap C_{\delta_2} \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$ and for transparency $\delta_1, \delta_2 \in S \Rightarrow \delta_1 \notin C_{\delta_2}$. Let $M = (\lambda, <_u)$ where $\alpha \leq_u \beta$ <u>iff</u> for some $\delta \in S$ we have:

$\bullet$ $\{\alpha, \beta\} \subseteq C_\delta \cup \{\delta\}$, $\alpha \leq \beta$ and $\text{otp}(C_\delta \cap \alpha), \text{otp}(C_\delta \cap \beta)$ belong to u.

Now let $\langle C^*_{\alpha,\beta} : \beta \in [\alpha, \beta^*_\alpha) \rangle$ for $\alpha < \lambda$ list $\{C_\delta \cap \alpha : \delta \in S$ satisfies $\alpha \in \mathrm{nacc}(C_\delta)\}$ without repetition, so $\beta^*_\alpha < \lambda$, let $S^\bullet = \{\delta \in S : C_\delta \subseteq E_0\}$ where $E_0 = \{\delta < \lambda : \delta$ limit such that $\alpha < \delta \Rightarrow \beta^*_\alpha < \delta\}$. Lastly, let $<_u$ be the set of pairs (β_1, β_2) such that for some $\delta \in S$ we have:

- $\beta_1 \leq \beta_2$ and β_1, β_2 belong to $\{\beta : \beta = \delta + 1$ or for some $\alpha \in C_\delta$ and $i \in u$ we have $\mathrm{otp}(C_\delta \cap \alpha) = i + 1$ and $C_\delta \cap \alpha = C^*_{\alpha,\beta}\}$.

To summarize

⊞ (a) if $M \in K^{\mathrm{tr}}_\lambda$ (a pretender) then $\mathrm{Inv}(M) \subseteq \mathscr{P}(\kappa)$ has cardinality λ

  (b) for every $u \in \mathscr{P}(\kappa)$ for some $M \in K^{\mathrm{tr}}_\alpha, u \in \mathrm{Inv}(M)$.

So we are done.

The proof for T with SOP$_4$ or with the olive property continues this proof. In the case $\lambda = \mu^{++} < 2^\mu$, μ singular we have to change the choice of $\bar{C}$ by 11.2, so the remaining case for regular λ is:

Problem 11.4. Assume $\lambda = \mu^+$ and $\kappa < \mu \Rightarrow 2^\kappa \leq \lambda$. Can we prove $\lambda \notin \mathrm{univ}(T)$ for T as above, i.e. with SOP$_4$ or with the olive property?

For singular λ, this was proved for most λ's, but the general case is still open. Maybe it is time to say what the olive property is. It means:

⊞ There are $(\varphi_0, \varphi_1, \psi)$ and a model $\mathfrak{C}$ of T such that:

  (a) for some m, $\varphi_0 = \varphi_0(\bar{x}_{[m]})$, $\varphi_1 = \varphi_1(\bar{x}_{[m]}, \bar{y}_{[m]})$, $\psi = \psi(\bar{x}_{[m]}), \bar{y}_{[m]}, \bar{z}_{[m]})$ are quantifier free formulas (and $\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{z}_{[m]}$ are m-tuples of variables, see Notation 0.10 below)

  (b) for every k and $\bar{f} = \langle f_\alpha : \alpha < k \rangle$ where f_α is a function from α to $\{0,1\}$ we can find a model M of T and $\bar{a}_\alpha \in {}^m M$ for $\alpha < k$ such that:

   (α) $\varphi_1[\bar{a}_\alpha, \bar{a}_\beta]$ for $\alpha < \beta < k$ when $1 = f_\beta(\alpha)$

   (β) $\psi[\bar{a}_\alpha, \bar{a}_\beta, \bar{a}_\gamma]$ when $\alpha < \beta < \lambda$ and $f_\gamma \restriction [\alpha, \beta]$ is constantly 0

  (c) there are no $\bar{a}_i \in {}^m M$ for $\ell = 0,1,2,3$ such that the following conditions are satisfied in M:

   (α) $\varphi_0[\bar{a}_0, \bar{a}_i]$ for $\ell = 1,2,3$, $\varphi[\bar{a}_i, \bar{a}_\ell]$ for $\ell = 1,2$ and $\varphi_0[\bar{a}_2, \bar{a}_3]$

   (β) $\psi[\bar{a}_0, \bar{a}_2, \bar{a}_3]$.

§ 12. CONSISTENCY RESULTS

§ 12(A). **Forcing.**

Naturally, as possibly $\leq^0_{\mathrm{univ}}$ is trivial (when GCH holds), we need consistency results. In particular we have to apply (and develop!) forcing methods.

The first relevant forcing is [She80a], aiming to prove that some results of [She90a, Ch VIII] were best possible. First, it introduces oracle c.c.c. forcing that $T = \mathrm{Th}({}^\omega 2, E_n)_{n<\omega}$ has a universal model in $\aleph_1$ and $\aleph_1 < 2^{\aleph_0}$; this method was inspired by omitting types arguments in model theory. Second, introducing proper forcing (in fact, with oracle) we force that there is a universal linear order in $\aleph_1$ while $\aleph_1 < 2^{\aleph_0}$. This is done proving that there is a non-meagre $A \subseteq \mathbb{R}$, $|A| = \aleph_1$, that realizes as many cuts as possible.

Problem 12.1. 1) Can we generalize those results to $\lambda = \mu^+ < 2^\mu, \mu = \mu^{<\mu}$, at least for μ inaccessible or even a suitable large cardinal.

2) Similarly for suitably defined λ^+-oracle forcing preserving $\mathrm{non}_\lambda(\lambda - \mathrm{meagre}) = \mu^+$.

3) Or in the inverse direction, if $\aleph_1 < \lambda < 2^{<\lambda}$, then there is no universal linear order.

The forcing in [She90b] started with $\mu = \mu^{<\mu} < \lambda < \chi = \chi^\lambda$, a sequence $\langle A_\alpha : \alpha < \chi \rangle$ of members of $[\lambda]^\lambda$ which is say μ-almost disjoint (i.e. $\alpha < \beta < \chi \Rightarrow |A_\alpha \cap A_\beta| < \mu$) gotten by [Bau76] and then used a $(< \mu)$-support iteration $\mathbf{q} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \chi, \beta < \chi \rangle$. Now $\mathbb{Q}_0$ forces a graph $\mathbf{G}_0$ on λ by conditions of cardinality $< \mu$, so this is just adding λ, μ-Cohens. Next for $\alpha > 0$, $\underset{\sim}{H}_\alpha$ is a $\mathbb{P}_\alpha$-name of a graph with universe (= set of nodes) λ, and $\mathbb{Q}_\alpha$ adds an embedding of $\underset{\sim}{H}_\alpha$ into $\mathbf{G}_0 \restriction A_\alpha$. The problem is to choose $\mathbb{Q}_\alpha$ such that we can prove that the μ^+-c.c. is preserved; [Mek90] continues this, proving it for many theories (universal theories with no algebraicity and $\mathscr{P}^-(3)$, $\mathscr{P}^-(4)$-amalgamation). We naturally ask (see [Shed]):

Problem 12.2. 1) For $\aleph_0 < \mu = \mu^{<\mu} \ll \lambda \ll \chi = \chi$ find a $(< \mu)$-complete $(< \mu)$-support iteration for forcing notions which are suitably μ^+-c.c. forcing: making $\lambda \in \mathrm{univ}(T)$ for a maximal set of T's; (Or at least for all simple T's.)

2) Develop a framework for distinguishing T's, e.g. the $T_{n,k}^{\mathrm{hgr}}$ (the theory of $(k+1)$-hyper-graphs not complete on any $n+1$ nodes).

This requires "hair splitting" distinction of forcing. Close to this are [She93b, DS04a], where we restrict ourselves to $\lambda = \mu^+$ and force a sufficient condition which covers T_{ceq} and triangle free graphs. However, those works prove only weak universality i.e. the M universal for $\lambda = \mu^+$ was of cardinality $\lambda^+ < 2^\mu$. It makes sense that for the case $\lambda = \mu^+$ we shall get stronger consistency results.

Problem 12.3. 1) Are there T and λ such that

(a): There is no universal model in K for λ.

(b): There is a family of λ^+ models in K of cardinality λ which together are universal.

(c): $\lambda^+ \ll 2^\lambda$.

2) Can we force the full result for $\lambda = \mu^+$ for $T = T_{\mathrm{ceq}}$? For $T = $ triangle free graphs, for simple theories?

To approach this let us describe the way this is accomplished in the forcing. We start with $\mu = \mu^{<\mu}$ and for $\mu^{++} < 2^\mu = 2^{\mu^+}$. Now we use a $(< \mu)$-support iteration of length μ^{++}, $\mathbf{q} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \mu^{++}, \beta < \mu^{++} \rangle$ where $\mathbb{Q}_\beta$ adds a model $\underset{\sim}{M}_i$ of T with set of elements λ which is universal for all "old" models of T of cardinality λ (so without loss of generality with set of elements λ), old meaning from the universe $\mathbf{V}^{\mathbb{P}_\alpha}$. Clearly this is easier than first forcing the hopeful universal and then try to embed new models into it. Also $\lambda = \mu^+$ should help us.

Problem 12.4. Find the dividing lines suitable for such forcing.

There is a different approach in [SS] for $\mu = \aleph_0$. The forcing with finite support necessarily adds many μ-Cohens. So if we like to not add Cohen or even retain $\mathfrak{d} = \aleph_1$ we have to use CS iteration. Now it is natural to start with a universe $\mathbf{V}$ satisfying CH, so we have a saturated random graph H_0, but after forcing it cannot be saturated but we retain a weak property being "random saturated" or "Cohen saturated" which follows from the forcing preserving "a set of reals is non-null/non-meagre". The forcing method used relies on iterating well known reals, such as Miller and Laver reals, and alternating this with the PID forcing which adds no new reals.

Problem 12.5. 1) Generalize those methods to other cardinals $\mu > \aleph_0$.
2) Find the maximal family of theories for which this applies.

§ 12(B). **Strong Limit Singular μ.**

Cardinals near a singular μ, in particular for μ a strong limit, raise special problems. For example, even forcing $2^\mu > \mu^+$ while still controlling the situation below μ has been a serious problem, and necessarily involves "large cardinals", cf. [Sil75], [Mag77a], [Mag77b], [FW91], [Git10], [AM10].

Can we get further consistency results for $\lambda \in (\mu, 2^\mu)$, e.g. like Martin's Axiom or like consistency results on universals? There are some results; [MS89] proves that for such λ's we can get uniformization results; by [GS98] we can get consistency of the density of base product spaces. And [DS03] proves that we can force $\lambda = \mu^+ < 2^\mu$, $\mathrm{cf}(\mu) = \aleph_0$ and there is a graph on $\mu^{++} < 2^\mu$ universal for graph in λ^+. Subsequently, [CDM$^+$] also for the case $\mathrm{cf}(\mu) > \aleph_0$ and gives a general forcing axiom.

Problem 12.6. 1) Can we get $\lambda \in \mathrm{univ}_{\mathrm{graph}}$ above?
2) For which T's does this work?
3) Generally: state and prove consistency of relevant forcing axioms.

Part III

Contents

§ 13. INTRODUCTION TO PART III

This part assumes more familiarity with model theory; we shall speculate, point some answers and give some proofs. Here we shall consider simple theories, suggest relevant properties which hopefully lead to good dividing lines and prove that all simple theories have weak consistency results on "$\mathrm{univ}_{T/K}(\lambda)$ being small" better than in [She96b, §2].

We had thought for long that maybe NSOP$_4$ is the right dividing line for universality. An important case is the class of groups (which has various amalgamation properties hence NSOP$_4$, though it has SOP$_3$, like triangle free graphs (see [SU06])) so expect it will have at least weak universality consistency results (like [She96b]), and even had some typed proofs. Alas, mathematics is a harsh mistress, and proved me wrong on both accounts; by [She16] group theory is almost $\leq_{\mathrm{univ}}$-maximal; in fact there is a property called "olive property", which implies this (and it is satisfied by the class of groups). However, this property seems ad hoc, and it is doubtful that it is a good candidate for a successful dividing line.

Still, all this tells us that hidden there is a family of theories which has a good structure theory, its models are somehow no more complicated than random graphs, and has the external definition of being almost $\leq_{\text{univ}}$-maximal (or something similar). Just as in the main gap theorem [She90a, Ch.XII] found the family of theories with models coding set theoretic information like stationary sets rather then just (generalized) dimension [She85], [She93b].

§ 14. SPECULATIONS

§ 14(A). **Hyper Amalgamation and Simple Theories.** We feel we know considerably on simple classes but much remains to be done; consider:

Definition 14.1. Let us fix a theory T.
1) Here always $\mathscr{P} \subseteq \mathscr{P}(\theta)$ is closed under intersections of two; and includes $\{\{i\} : i < \theta\} \cup \{\emptyset\}$.
2) $\bar{M}$ is a $(\theta, \mathscr{P})$-*problem* (for T) when : $\bar{M} = \langle M_u : u \in \mathscr{P} \rangle$ satisfies: M_u is a model of T , $\|M\| \leq \theta$; and $M_u \prec M_v$ for $u \subseteq v \in \mathscr{P}$; and $M_u \cap M_v = M_{u \cap v}$ for $u, v \in \mathscr{P}$; and if $u \subset v \in \mathscr{P}$ then $\text{tp}(M_u, \cup\{M_{v_1} : v_1 \subseteq v, u \nsubseteq v_1\})$ does not fork over $\cup\{M_{u_1} : u_1 \subsetneq u\}$ inside M_v, where tp means type.
3) For $\bar{M}$ as above we say $(N, \bar{f})$ is a *solution* of $\bar{M}$ when :

    (a) N is a model of T

    (b) $\bar{f} = \langle f_u : u \in \mathscr{P} \rangle$

    (c) f_u is an embedding of M_u into N

    (d) $u \subseteq v \in \mathscr{P} \Rightarrow f_u \subseteq f_v$

    (e) $\text{tp}(f_u(M_u), \cup\{f_v(M_u) : v \in \mathscr{P}, u \nsubseteq v\}, N)$ does not fork over $\cup\{f_v(M_v) : v \nsupseteq u\}$.

4) T has $(\theta, \mathscr{P})$-existence when every $(\theta, \mathscr{P})$-problem has a solution.
5) $\bar{M}$ has $(\theta, \mathscr{P})$-uniqueness when $\bar{M}$ is a $(\theta, \mathscr{P})$-problem and $(N_\iota, \bar{f}_\iota)$ is a solution of $\bar{M}$ for $\iota = 1, 2$, then there is a $N \in K_T$ and embedding g_ι of N_ι into N for $\iota = 1, 2$ such that $u \in \mathscr{P} \Rightarrow g_1 \circ (f_1 \restriction M_u) = g_2 \circ (f_2 \restriction M_u)$.
6) We say T has $(\mathscr{P}, \mu)$-uniqueness when every $(\theta, \mathscr{P})$-problem $\bar{M}$ has uniqueness.

Problem 14.2. 1) Sort out the implications between those properties for any simple theory T.
2) Prove that they determine the place of T under $\leq_{\text{univ}}$ assuming "reasonable" cardinal arithmetic.

Question 14.3. Can we have existence of decompositions such that the existence problems is local, so in Definition 14.1, θ finite is enough?

Recall that in the main gap theorem (see $\bullet_1(b)(\gamma)$ after Thesis 9.1), it is proved for superstable T with none of the three relevant order properties that every model M of T is prime over $\cup\{M_\eta : \eta \in \mathscr{T}\}$ where $\mathscr{T} \subseteq {}^{\omega >}\|M\|$ is a subtree and $\langle M_\eta : \eta \in \mathscr{T} \rangle$ is a non-forking tree of models. Hence for any T, either T has many models coding complicated objects like stationary sets, or all of its models can be described by suitable dimensions, the machinery developed along the way is not less important. Its proof [She90a, Ch.12] uses such $(\theta, \mathscr{P})$-problems.

In our case we cannot expect anything so strong; but we try to go in this direction so let us describe a scenario for analyzing models of T, assume for transparency that T is super-simple; and M is a model of T of cardinality $\lambda \geq \theta \geq |T|$. We can find $\bar{N} = \langle N_u :$

$u \in [\lambda]^{<\aleph_0}\rangle$ which is a $(\theta, [\lambda]^{<\aleph_0})$-problem and $M \prec \cup\{N_u : u \in [\lambda]^{<\aleph_0}\}$ (see more with a proof later).

Now if T has $(\theta, \mathscr{P}^-(n))$-uniqueness for every n, then irrespective of what is T, as we can inflate the N_n's (i.e. increase them but only boundedly so), we can reduce the problem of "$\lambda \in \mathrm{univ}_T$" to "there is a universal $f : [\lambda]^{<\aleph_0} \to 2^{\aleph_0}$". Restricting ourselves to "reasonable λ's" we can use only $f : [\lambda]^{<\aleph_0} \to \{0, 1\}$, which can be translated to $\lambda \in \mathrm{univ}(T_{\mathrm{un}(<\omega)})$, where $K_{T_{\mathrm{un}(<\omega)}} =$ the class of $\tau_{\mathrm{un}(<\omega)}$-models where $\tau_{\mathrm{un}(<\omega)} = \{R_n : n < \omega\}$, R_n an n-place predicate; hopefully all such T's are equivalent to $T_{\mathrm{un}(<\omega)}$.

Now T being just simple (rather than supersimple) is a burden and generally moving from a scenario to reality will have its complications. Anyhow let $\mathbf{n} = \mathbf{n}(T)$ be such that $(\theta, \mathscr{P}^-(n))$-uniqueness <u>iff</u> $n \leq \mathbf{n}$, We describe how the structure theory is used but we expect this will lead us to finding finitary, syntactical characterization too.

Relatives of such properties should be relevant to the $\leq_{\mathrm{SP}}$-problem (see [SUa]), whereas Keisler's order seems to be of a different character (see [MS]). By such arguments we can prove that any simple T is not pseudo $\leq_{\mathrm{univ}}$-maximal.

Discussion 14.4. Though the existence properties are nice and natural, they do not provide the right dividing lines for the problems we are interested in, they are just first approximations. Why?

Naturally we assume:

$(*)_1$ $\lambda < 2^{<\lambda}$.

Let $\mu = \min\{\mu : \lambda \leq 2^\mu\}$. Now if $\mu^{|T|} \geq \lambda$ this is connected to the SCH, which introduces extra set theoretic complications, so we assume:

$(*)_2$ $\mu = \mu^{|T|}$ and even $\mu = \mu^{<\mu}, \mu > 2^{|T|}$ for transparency (the point is that it is natural when we try to force).

To analyze $M \in K_{\lambda!}$ by Claim **??** we can consider an NF-diagram $\bar{M} = \langle M_u : u \in [\lambda]^{2^{|T|}}\rangle$. Now choose $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle \in {}^\lambda({}^\mu 2)$, the η_α's pairwise distinct. After mild forcing letting $\bar{\mathbf{a}}_u$ list the elements of M_u, we may assume:

$(*)_3$ $(\bar{M}, \bar{\eta})$ is tree indiscernible, i.e. $(a) \Rightarrow (b)$, where

    (a) (α) $\alpha(\eta, \iota) < \lambda$ for $n < n(*), \iota < 2$

          (β) if $n < m < n(*)$ and $\iota < 2$ then $\alpha(n, \iota) \neq \alpha(m, \iota)$

          (γ) $\varepsilon < \mu$ and

          (δ) if $\iota < 2$ then $\langle \eta_{\alpha(n,\iota)} \restriction \varepsilon : \eta < n(*)\rangle$ is without repetitions

          (ε) $\eta_{\alpha(n,0)} \restriction \varepsilon = \eta_{\alpha(n,1)} \restriction \varepsilon$ for $n < n(*)$

    (b) there is f such that:

          (α) f is an isomorphism from $M_{\{\alpha(n,0):n<n(*)\}}$ onto $M_{\{\alpha(n,1):n<n(*)\}}$

          (β) for $u \subseteq n(*)$, f maps $\mathbf{a}_{\{\alpha(n,0):n\in u\}}$ to $\mathbf{a}_{\{\alpha(n,1):n\in u\}}$.

Hopeful Theorem 14.5. *If (A) then (B) where:*

  (A) *(a)* $\kappa < \mu = \mu^{<\mu} < \lambda = \lambda^{2^\kappa} < \lambda < \chi = 2^\mu = \chi^\lambda$

        (b) $\mathscr{A} \subseteq [\lambda]^\lambda$ *is* $\mu - \mathrm{AD}$ *of cardinality* χ

  (B) *there is a forcing notion* $\mathbb{P}$ *such that:*

      (a) $\mathbb{P}$ *is* $(<\mu)$-*complete,* μ^+-*c.c. (even* $*_\mu^\omega$ *holds, see* [Shed]*) and has cardinality* 2^μ

      (b) *in* $\mathbf{V}^{\mathbb{P}}$, *for any complete simple first order theory* T *cardinality* $\leq \kappa$, *we have* $\mathrm{univ}_T(\lambda) \leq \lambda^+$.

Proof. It is enough to prove:

(∗) there is $\mathbb{P}$ such that:

(a) as in (B)(a)

(b) for every complete simple first order T of cardinality $\leq \kappa$, in $\mathbf{V}^{\mathbb{P}}$ there is $M \in K_{T,\lambda}$ into which every $M' \in K_{T,\lambda}^{\mathbf{V}}$ can be embedded.

$\square_{14.5}$

§ 14(B). Non-existence of Universals.

For $T = T_{\mathrm{dlo}}$ (or is SOP$_4$ or have the olive property) we know it is almost $\leq_{\mathrm{univ}}$-maximal, but maybe it is even $\leq_{\mathrm{univ}}$-maximal. There are two separate cases - singular cardinals and regular cardinals.

For the singular case, trying to look for an exact reference, I see (as I recall) that while the results in [KS92a] for regular $\lambda > \aleph_1$, use linear order with many cuts in the proof but not as an assumption in the theorem; however I discover, for singular λ, there is such a demand in the theorem. So it says less than what I attributed to it (so my memory improves the result, but alas, not the proof). For singular λ there are not only pcf demands but:

$\boxplus_\lambda$ there is a linear order I of cardinality $< \lambda$ with $> \lambda$ Dedekind cuts

and moreover

$\boxplus_\lambda^+$ for arbitrarily large regular $\kappa < \lambda$, there is a tree with κ nodes and $> \lambda$ κ-branches.

Why this does not occur for regular λ? If $\kappa = \min\{\kappa : 2^\kappa > \lambda\}$ and $\lambda < 2^{<\lambda}$ then $\kappa < \lambda$ hence $2^{<\kappa}$ is the sum of $< \lambda$ cardinals $< \lambda$ hence $2^{<\kappa} < \lambda$ and $(^\kappa{>}2, \lhd)$ exemplify $\boxplus_\lambda$. But for λ singular, this is not the case, so [KS92a] was correct to distinguish but does not explicate the natural problem below; and [She16] (as not said there) makes a real advance on the singular case: for T with (any variant of) the olive property we have only pcf obstacles. So it is natural to ask:

Question 14.6. Is the use of $\boxplus_\lambda^+$ for singular λ, in proving $N\mathrm{univ}_{T_{\mathrm{dlo}}}(\lambda)$ for $\lambda \notin u$, necessary?

Claim 14.7. *1) If (A) + (B) then (C) <u>when</u> :*

(A) (a) κ is a Mahlo cardinal

(b) $\lambda > \kappa$ is strong limit singular of cofinality κ

(c) $\kappa_i = \kappa_i^{<\kappa_i}$ is increasing for $i < \kappa$ with limit κ

(d) $\lambda_i = \lambda_i^{\kappa_i} \in (\kappa, \lambda)$ is increasing with limit λ

(e) $\lambda_i > \prod_{j<i} 2^{\lambda_j}$

(f) for limit $\delta < \kappa$ we have $2^{\sup\{\kappa_i : i < \delta\}} = (\sum\{\kappa_i : i < \delta\})^+$ *and* $\prod_{i<\delta} \lambda_i = (\Sigma\{\lambda_i : i < \delta\})^+$

(B) (a) let $\mathbb{Q}_i = \mathrm{Cohen}_{\kappa_i}(\lambda_i)$, *the forcing of adding* λ_i *many* κ_i-*Cohens so* $\mathbb{Q}_i$ *satisfies the* κ_i^+-*c.c. and is* κ_i-*complete of cardinality* λ_i

(b) $\mathbb{P} = \prod_{i<\kappa} \mathbb{Q}_i$ *is the product with the Easton support*

(C) *in* $\mathbf{V}^{\mathbb{P}}$ *we have:*

(a) *cardinals and cofinalities are the same as in* $\mathbf{V}$

    (b) $\kappa_i = \mathrm{cf}(\kappa_i)$ *increasing with* $i < \kappa = \mathrm{cf}(\kappa)$ *and* $\lambda_i > \kappa_i$

    (c) $2^{\kappa_i} = \lambda_i, 2^{<\kappa} = \sum_{i<\kappa} \lambda_i = \lambda$ *and* $\theta < \kappa_0 \Rightarrow 2^\theta = (2^\theta)^{\mathbf{V}} < \kappa$

    (d) λ *is singular,* $\lambda = \sum_{i<\kappa} 2^{\kappa_i}$ *hence* $\kappa = \mathrm{cf}(\lambda)$ *and* $\lambda < 2^\kappa$

    (e) *if* I *is a linear order of cardianlity* $< \lambda$, *then* I *has* $\leq \lambda$ *cuts*

    (f) *if* $\theta = \mathrm{cf}(\theta) < \lambda$ *and* $\mu \in [\theta, \lambda)$, *then* $\mathrm{trp}_\theta(\mu) \leq (\mu^\theta)^{\mathbf{V}}$.

Proof. 1) We are assuming κ is a Mahlo cardinal so let

$(*)_1$ (A) $p \in \mathbb{Q}_i$ iff

        (a) p is a function

        (b) $\mathrm{dom}(p) \in [\lambda_i]^{<\kappa_i}$

        (c)   if $\alpha \in \mathrm{dom}(p)$ then $p(\alpha) = {}^{\kappa_i>}2$

    (B)   order natural

    (C)   the generic of $\mathbb{Q}_i$ is $\bar{\eta}_i = \langle \eta_{i,\alpha} : \alpha < \lambda_i \rangle, \eta_{i,\alpha} \in {}^{(\kappa_i)}2$

$(*)_2$ (a)   for $u \subseteq \lambda_i$ let $\mathbb{Q}_{i,u} = \{p \in \mathbb{Q}_i : \mathrm{dom}(p) \subseteq u\}$

    (b)   for $v \subseteq \kappa$ and $\bar{u} = \langle u_i : i \in v \rangle$ with $u_i \subseteq \lambda_i$ let $\mathbb{P}_{\bar{u}} = \prod_{i \in v} \mathbb{Q}_{i,u_i}$ product with Easton

        support

    (c)   for u, v and $\bar{u}$ as above, $\mathbb{P}_{\bar{u}} \lessdot \mathbb{P}$.

The cardinal arithmetic is well known so should be clear. Clause (e) of (C) of the Claim follows from clause (f). To prove it, i.e. concerning $\mathrm{trp}_\theta(\mu)$ where $\theta = \mathrm{cf}(\theta) \leq \mu < \lambda$, toward contradiction assume $\mu < \lambda, \theta = \mathrm{cf}(\theta) \leq \mu$ and $p \Vdash_\mathbb{P}$ "$\mathscr{T}$ is a subtree of ${}^{\theta>}\mu$ with $\leq \mu$ nodes and $> \lambda$ θ-branches", without loss of generality $\mu > \kappa$ and as $\Vdash_\mathbb{P} \lambda = \Sigma\{2^{\kappa_i} : i < \kappa\}, \lambda_i = 2^{\kappa_i}$ increasing with i, necessarily $\theta \geq \kappa$.

As $\mathbb{P}$ satisfies the κ^+-c.c., we can find $\bar{u} = \langle u_i : i < \kappa \rangle, u_i \in [\lambda_i]^{\leq\mu}$ such that $p \in \mathbb{P}_{\bar{u}}$ and $\mathscr{T}$ is a $\mathbb{P}_{\bar{u}}$-name. Now $\mathbb{P}_{\bar{u}}$ has cardinality $\leq \mu \times \kappa$ which is $< \lambda$ because λ is a strong limit cardinal. As $\theta > \kappa$ or $\theta = \kappa$ recalling κ is Mahlo clearly $\mathbb{P}$ satisfies the θ-c.c. so the forcing notion $\mathbb{P}/\mathbb{P}_{\bar{u}}$ satisfies the θ-c.c. hence it does not add θ-branches to $\mathscr{T}$, contradiction. $\square_{14.7}$

For regular λ the problem is different. Necessarily, λ is a successor cardinal, say $\lambda = \mu^+$ so (see earlier) $2^{<\mu} \leq \lambda$. We may need better guessing of clubs (see [She94, Ch.III]), and probably division to cases will help. A very different problem but with related character appears when for $\lambda \in (\mu, \mu^{\aleph_0})$ where $\mu = \Sigma\mu_n, \mu_n^{\aleph_0} < \mu_{n+1}$ for $n < \omega$ and T is strictly stable. The most puzzling case is $\lambda = \mu^+$ unlike $\lambda = \mu^{++} < \mu^{\aleph_0}$, in the case $\lambda = \mu^+$ we do now know strong enough guessing of clubs.

The arguments above may lead to a nice complete non-existence answer, but it may well reveal a hidden dividing line among those theories.

§ 14(C). **Back to Dividing Lines.**

First, while the olive property (see Theorem 9.8) is not satisfactory, probably there is a weaker, neater, "right" property which will give necessary and sufficient condition, at least for almost $\leq_{\mathrm{univ}}$-maximality. This calls us to analyze the proof in [She16] and/or try to find a better proof; all this is trying to find the dividing line from up.

We can try to look from below. A weak version of having universal concern

Question 14.8. Given T, when does $(A) \Rightarrow (B)$?

(A) $\mu = \mu^{<\mu} > |T|, \lambda = \mu^+, \bar{M} = \langle M_\alpha : \alpha < 2^\lambda \rangle$ list the models of T with universe λ and E_α is a thin enough club of λ related to M_α (e.g. definable in $\mathfrak{B}_\alpha = (\mathscr{H}(\beth_2^+(\mu)), \in, <_*, \theta, \mu, M_\alpha)$, where $<_*$ is a well-ordering of $\mathscr{H}(\beth_2^+(\mu))$; let $E_\alpha = \{\delta < \lambda : $ the Skolem Hull $\mathfrak{B}_{\alpha,\delta}$ of $\{\iota : \iota < \delta\}$ in $\mathscr{H}(\beth_2^+(\mu))$ satisfies $\mathfrak{B}_{\alpha,\delta} \cap \lambda = \delta\})$

(B) there are $(\mathbb{Q}, \underset{\sim}{M})$ such that:

   (a) $\mathbb{Q}$ is $(< \mu)$-complete, μ^+-c.c. moreover a $*_\mu^\varepsilon$-forcing notion, ([Shed])

   (b) $\underset{\sim}{M}$ is a $\mathbb{Q}$-name of a model of T with universe λ

   (c) $\Vdash_\mathbb{Q}$ "M_α is embeddable into $\underset{\sim}{M}$", moreover, if $\delta \in E \wedge \text{otp}(\delta_1 \cap E_\alpha) = \beta$ then the embedding maps $M_\alpha \restriction \delta$ into $\underset{\sim}{M} \restriction (\lambda + \lambda\beta)$.

This is the point of [DS04a], [She93b], [She96b], so we get a positive answer for every simple T, and for some non-simple T, but those theories are the parallel of "simple T with trivial forcing notions".

Problem 14.9. Does the theory of non-forking for NSOP$_1$ theories suffice to get such a result? This is fine but $K_{\text{triangle-free}}$ is a SOP$_3$ theory for which that answer to Question 14.8 is positive.

Problem 14.10. Is there a relevant definition of non-forking describing a best positive answer for Problem 14.9?

§ 14(D). **Forcing Near Regular.**

The case we concentrate on here is $\lambda \in [\mu, \chi)$, $\mu = \mu^{<\mu} < \lambda < \chi = \chi^\lambda$ and even GCH holds between μ and λ; an important case is $\lambda = \mu^+$. The forcing of [She90b] being $(< \mu)$-support, μ^+-c.c. has some limitation. It is persuasive to remember [SS], which allows us to preserve for $\mu = \aleph_0$, to preserve $\mathfrak{d} = \aleph_1$ and even "$^\omega 2$ is non-null"; alternatively "$^\omega 2$ is non-meagre". So the universal model can have, even in $\mathbf{V}^{\mathbb{P}_\alpha}$, some distinct remnants of having been saturated (in the original $\mathbf{V}$).

Problem 14.11. 1) Develop iterated forcing for such universality problems.
2) In particular force "$\lambda = \mu^+ < 2^\mu$ and $\lambda \in \text{univ}(K_{\text{triangle-free}})$".
3) For which T's this work?
4) Can we in Question 14.8 use Q a creature forcing (at least for μ inaccessible)?

§ 14(E). **Back to $\aleph_1$.**

Clearly the situation in $\aleph_1 < 2^{\aleph_0}$ is different.

We would like to characterize the (countable complete) T's with $\mathbf{D}(T) = \{\text{tp}(\bar{a}, \emptyset, M) : M \in K_T, \bar{a} \in {}^{\omega>}M\}$ countable such that consistently T has a universal member. By [KS92a] for some such T's this fail so the problem is delicate; hence we may first concentrate on the superstable case; for one such theory this is done in [She80a].

An interesting case is K_{lfg}, locally finite groups; the results cited earlier do not cover this case, as it is not first order and fails amalgamation for countable models. A proof of non existence will require finding finite groups behaving as in the example above, so require using at least some finite groups theory. But we may try to work as in §(5D), to force existence recalling that we may use the so called creature forcing (see [RS99]; that requires a choice of so called norms; again those norms has to be related to finite group theory.

§ 14(F). **Non-simple Theories.**

As hinted earlier, we feel that T_{ceq} should be a test case. It is not simple but seem to us to be the "simplest non-simple". This may indicate trying to prove things for all

NSOP$_1$ theories, which seems reasonable but we like to include also T_{trf}, the theory of triangle free graphs which is not even NSOP$_3$. It seem reasonable that for many cardinals $\lambda = \text{cf}(\lambda) \in (\mu, 2^\mu)$ we shall succeed to prove $\lambda \notin \text{univ}(T_{\text{ceq}})$, still for this we probably need to have better club guessing and/or better ways to use them.

But maybe we better start also from the other direction. Recall we have gotten a weak consistency result the existence of a weak universal for T_{ceq}, (it is of cardinality λ^+, is universal for $\lambda = \mu^+$ and $\lambda^+ < 2^\mu$). We should try to get a real universal in λ, maybe using creature forcing for inaccessibles. Of course all this will be preliminary works for achieving this for the class of T's similar enough to T_{ceq}. This will require developing a new non-forking theory under quite hard conditions (for NSOP$_1$ there is one but not for NSOP$_3$).

§ 15. CONSISTENCY FOR SIMPLE T

Claim 15.1. *Assume T is complete simple first order theory. T has $\mathscr{P}$-existence[3] when:*

$(*)_{\mathscr{P}}$ *for some $n, \ell(*), \bar{v}$ we have:*

    (a) *$\mathscr{P} \subseteq \mathscr{P}(I)$ is downward closed where $I_\iota = [\iota n, \iota n + n)$ for $\iota = 0, 1, 2$ and $I = \bigcup_\iota I_\iota$*

    (b) *$\mathscr{P}_1 = \mathscr{P}(I_0 \cup I_1) \cap \mathscr{P}$*

    (c) *$\mathscr{P}_2 = \mathscr{P}(I_0 \cup I_2) \cap \mathscr{P}$*

    (d) *for some $\langle v_\ell : \ell < \ell(*) \rangle$ we have $\mathscr{P} = \cup \{ \mathscr{P}(v_\ell) : \ell < \ell(*) \} \cup \mathscr{P}_1 \cup \mathscr{P}_2$ and $\ell < k < \ell(*) \Rightarrow v_\ell \cap v_k \subseteq I_0$.*

Proof. By induction on $\ell(*)$.

If $\ell(*) = 0$ or $\ell(*) = 1$ by basic properties of non-forking.

If $\ell(*) > 1$ let $J = I_0 \cup (I_1 \setminus v_0) \cup (I_2 \setminus v_0)$ and $\mathscr{P}_* = \mathscr{P}(J)$. By the induction hypothesis we can find a non-forking $\bar{N} = \langle N_u : u \in \mathscr{P}_* \rangle$ such taht $u \in \mathscr{P}_* \cap \mathscr{P} \Rightarrow N_u = M_u$. By the case $\ell(*) = 1$, there is a non-forking diagram $\bar{M}' = \langle \mu'_u : u \subseteq I_0 \cup v_0 \rangle$ such that $u \in \mathscr{P}(I_0 \cup v_0) \cap \mathscr{P} \neq M'_u = M_u$. Applying twice the basic properties we are done. $\square_{15.1}$

Discussion 15.2. Can we apply 15.1 to prove the "hopeful theorem" 14.5?

Definition 15.3. 1) Assume $\theta < \mu = \mu^{<\mu} < \lambda$, T is simple, $2^{|T|} \leq \theta$ for transparency. Let $\mathbb{Q}_0 = \mathbb{Q}^0_{\lambda,\mu} = \mathbb{Q}_{\lambda,\mu,T}$ be defined as follows:

    (A) $p \in \mathbb{Q}$ iff:

        (a) $p = \langle M_{p,u} : u \in [p]^{\leq |T|} \rangle$

        (b) $v_p \in [\lambda]^{<\mu}$

        (c) $M_{p,u}$ has universe $\mathscr{P}(u) \times \theta$

        (d) p is a stable system of models of T

    (B) $\leq_{\mathbb{Q}}$ is defined naturally

    (C) the generic is $\langle \underline{N}_u : u \in [\lambda]^{\leq \theta} \rangle$.

2) Assume that $T = \langle (\lambda_i, T_i) : i < i(*) \rangle$ each (μ_i, λ_i, T_i) is as above for each i. Let $\mathbb{Q}_{\mu,T}$ be the product $\pi \{ \mathbb{Q}^0_{\lambda_0,\mu,T_i} : i < i(*) \}$ with support $< \mu$ and $\langle \underline{N}_{0,i,u} : u \in [\lambda_i]^{\leq |T_i|} \rangle$ the generic, as above.

---

[3]See Definitions 14.1 and 15.8.

Claim 15.4. *Assume* $\mu = \mu^{<\mu} < \lambda < \chi = \chi^{\lambda}$, *let* $\bar{T} = \langle(\lambda_i, T_i) : i < i(*)\rangle$ *be as in Definition 15.3 (2).*
1) $\mathbb{Q}_{\lambda_i,\mu,T_i}$ *satisfies* $*^{\omega}_{\mu}$.
2) So does $\mathbb{Q}_{\mu,\bar{T}}$ *the product* $\Pi\{\mathbb{Q}_{\lambda_i,\mu,T_i} : i < i(*)\}$ *with support* $< \mu$.

Proof. Should be clear. $\square_{15.4}$

Hypothesis 15.5. 1) $\aleph_0 < \mu = \mu^{<\mu} < \lambda_* \leq \chi = \chi^{<\lambda_*}$.
2) $\mathscr{A}_\lambda \subseteq [\lambda]^\lambda$ for $\lambda \in (\mu, \lambda_*)$ is $\mu - \mathrm{AD}$ of cardinality χ (justified by [Bau76]).
3) $\langle A^\bullet_\alpha : \alpha \in [1,\chi)\rangle$ list $\bigcup_\lambda \mathscr{A}_\lambda$ with no repetitions with $A^\bullet_\alpha \in \mathscr{A}_{\lambda_{i(\alpha)}}, i(\alpha) \leq i(*)$ and h_α is the increasing function from $\lambda_{i(\alpha)}$ onto $A^\bullet_\alpha$.
4) $\bar{T} = \langle(\lambda_i, T_i) : i < i(*) \leq \chi\rangle$ list the pairs (λ, T) such that:

- $\lambda_i \in [\mu^+, \lambda_*), T_i \in \mathscr{H}(\lambda_*)$ is simple complete first order $\theta_i = \theta(i) = 2^{|T_i|}$.

Definition 15.6. Let **Q** be the class of **q** consisting of:

(a) $\alpha_{\mathbf{q}} = \alpha(\mathbf{q}) = \ell g(\mathbf{q})$, the length
(b) $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \alpha_{\mathbf{q}}, \beta < \alpha_{\mathbf{q}}\rangle$ is $(< \mu)$-support iteration
(c) $\mathbb{Q}_0 = \mathbb{Q}_{\mu,\bar{T}}$ and $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{\alpha(\mathbf{q})}$
(d) for $\alpha \in (0, \chi)$
  (α) $i(\alpha) = i_{\mathbf{q}}(\alpha) < i(*)$
  (β) $\underset{\sim}{M} = \langle \underset{\sim}{M}_{\alpha,u} : u \in [\lambda_{i(\alpha)}]^{\leq \theta(i(\alpha))}\rangle$ is a $\mathbb{P}_\alpha$-name of a non-forking diagram of models of T_i such that $\underset{\sim}{M}_{\alpha,u}$ has universe $\mathscr{P}(u) \times \theta(i_\alpha)$
(e) for $\alpha \in [1, \chi), \mathbb{Q}_\alpha$ is defined by (the order is inclusion)
  $(*)$ $p \in \underset{\sim}{\mathbb{Q}}_\alpha$ iff
    - p is a function with domain $\in [\lambda_{i(\alpha)}]^{<\mu}$
    - if $\beta \in \mathrm{dom}(p)$ then $p(\beta) < \mu$
    - every $u \in [\mathrm{dom}(p)]^{<\theta(i(\alpha))}$ the set $f_p = \{\langle(v, \zeta), (h''_\alpha(v), \zeta)\rangle : v \subseteq u, \zeta < \mu\}$ is a function; moreover, an isomorphism from $\underset{\sim}{M}_{\alpha,u}$ onto $\underset{\sim}{N}_{0,i(\alpha),h''_\alpha(u)}$ mapping $\underset{\sim}{M}_{\alpha,v}$ onto $\underset{\sim}{N}_{0,i(\alpha),h''_\alpha(v)}$ for every $v \subseteq u$
(f) $\mathbb{P}'_\alpha$ is a dense subset of $\mathbb{P}_\alpha$ defined by induction on α by:
  $(*)$ $\mathbb{P}'_\alpha$ is the set of $p \in \mathbb{P}_\alpha$ such that for some $I = I_p, u = u_p$ we have:
    (α) $u \in [\lambda_*]^{<\mu}$ and $I \subseteq \mathbf{i}(*)$
    (β) assuming $\mathrm{dom}(p) \neq \emptyset$ we have
      $(\bullet_1)$ $0 \in \mathrm{dom}(p)$ and $\mathrm{dom}(p(0)) = I$
      $(\bullet_2)$ $i \in \mathrm{dom}(p(0)) \Rightarrow \mathrm{dom}(p(i)) = u \cap \lambda_i$
      $(\bullet_3)$ if $0 < \alpha \in \mathrm{dom}(p)$ then $\mathbf{i}(\alpha) \in I$ and $\mathrm{dom}(p(\alpha)) = u \cap \lambda_{\mathbf{i}(\alpha)}$
      $(\bullet_4)$ if $0 < \alpha \in \mathrm{dom}(p)$ and $v \in [u_p]^{\leq|T_i|}$ then the $\mathbb{P}_\alpha$-name $\underset{\sim}{M}_{\alpha,v}$ is defined by maximal antichains included in $\mathbb{P}'_\alpha$ and for transparency we fix it.

Crucial Claim 15.7. *If* $\mathbf{q} \in \mathbf{Q}$ *then the forcing notion* $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_{\mathbf{q},\ell g(\mathbf{q})}$ *is* $(< \mu)$-*complete satisfying the* μ^+-*c.c. (and even* $*^{\omega}_{\mu}$?).

Proof. Clearly:
  $(*)_1$ for $\alpha \leq \alpha_{\mathbf{q}}, \mathbb{P}_\alpha$ is $(< \mu)$-complete; moreover, for any increasing sequence $\langle p_\alpha : \alpha < \delta\rangle$ of members of $\mathbb{P}'_\alpha$ of length $\delta < \mu$ the union naturally defined is a lub

$(*)_2$ (a) let $\mathbf{T}$ be the set of triples $(I, \mathcal{U}, \mathcal{W})$ such that $I \subseteq i(*), \mathcal{U} \in [\lambda_*]^{\leq \mu}, \mathcal{W} \in [\alpha_{\mathbf{q}}]^{\leq \mu}$ each of cardinality $\leq \mu$ ordered naturally

   (b) for $(I, \mathcal{U}, \mathcal{W}) \in \mathbf{T}$ we define $\mathbb{P} = \mathbb{P}'_{I, \mathcal{U}, \mathcal{W}}$ by induction on $\alpha = \cup\{\beta + 1 : \beta \in \mathcal{W}\}$ as follows: $p \in \mathbb{P}$ iff :

   (α) $p \in \mathbb{P}'_\alpha$ as witnessed by I_p, u_p

   (β) $I_p \subseteq I, u_p \subseteq \mathcal{U}, \mathrm{dom}(p) \subseteq \mathcal{W}$

   (γ) if $\beta \in \mathrm{dom}(p) \setminus \{0\}$ then the maximal antichains in the definition of $p(\beta)$, see 15.6(f)(*)(β)($\bullet_4$), are included in $\mathbb{P}_{I, \mathcal{U}, \mathcal{W} \cap \beta}$

$(*)_3$ we say a triple $(I, \mathcal{U}, \mathcal{W})$ as above is $\mathbf{q}$-closed when: if $\alpha \in \mathcal{W}$ and $v \subseteq \mathcal{U}$ has cardinality $\leq |T_{\mathbf{i}(\alpha)}|$ then the maximal antichains defining $\underset{\sim}{M}_{\alpha, v}$ are $\subseteq \mathbb{P}_{I, \mathcal{U}, \mathcal{W} \cap \alpha}$

$(*)_4$ if $(I_1, \mathcal{U}_1, \mathcal{W}_1) \in \mathbf{T}$, then there is a triple $(I_2, \mathcal{U}_2, \mathcal{W}_2) \in \mathbf{T}$ above $(I_1, \mathcal{U}_1, \mathcal{W}_1)$.

[Why? Should be clear.]

$(*)_5$ if $p \in \mathbb{P}'_{\mathbf{q}}$, then $p \in \mathbb{P}_{I, \mathcal{U}, \mathcal{W}}$ for some $(I, \mathcal{U}, \mathcal{W}) \in \mathbf{T}$

$(*)_6$ if $\langle (I_\varepsilon, \mathcal{U}_\varepsilon, \mathcal{W}_\varepsilon) : \varepsilon < \delta \rangle$ is an increasing sequence of closed members of $\mathbf{T}$ and $\mathrm{cf}(\delta) = \mu$, then the union is a closed member of $\mathbf{T}$.

[Why? Think!]

Now to finish:

$(*)_7$ assume $p_\zeta \in \mathbb{P}_{\mathbf{q}}$ for $\zeta < \mu^+$ and we should prove then for some $\zeta_1 < \zeta_2 < \mu^+$, the conditions p_{ζ_1}, p_{ζ_2} are compatible

$(*)_{7.1}$ (a) without loss of generality $p_\zeta \in \mathbb{P}'_{\mathbf{q}}$

   (b) there is a sequence $\langle (I_\zeta, \mathcal{U}_\zeta, \mathcal{W}_\zeta) : \zeta < \mu^+ \rangle$ of closed members of $\mathbf{T}$ such that $p_\zeta \in \mathbb{P}_{I_{\zeta + f}, \mathcal{U}_{\zeta + f}, \mathcal{W}_{\zeta + f}}$.

$\square_{15.7}$

Definition 15.8. 1) We say $\mathrm{Pr}_1(T)$ when:

   (a) T is simple (complete first order)

   (b) T has existence for $\mathscr{P}$ (i.e. $\mathscr{P}$-existence see 14.1(4)) when
   (*) for some n letting $I_\iota = [\iota n, \iota n + n]$ for $\iota = 0, 1, 2$ and $I = \bigcup\limits_\iota I_\iota$

   (α) $\mathscr{P} \subseteq \mathscr{P}(T)$ is downward closed

   (β) $\mathscr{P}_1 = \mathscr{P}(I_0 \cup I_1) \subseteq \mathscr{P}$

   (γ) $\mathscr{P}_2 = \mathscr{P}(I_0 \cup I_2) \subseteq \mathscr{P}$

   (δ) if $u \in \mathscr{P} \setminus (\mathscr{P}_1 \cup \mathscr{P}_2)$ then for some u_0, u_1, u_2 we have:

   • $u \subseteq u_0 \cup u_1 \cup u_2 \in \mathscr{P}$

   • $u_l \subseteq I_l$ for $l = 0, 1, 2$

   • $n + k \in u_1 \Leftrightarrow 2n + k \in u_2$ for $k < n$.

   (ε) there is an isomorphism f from $M_{I_0 \cup I_1}$ onto $M_{I_0 \cup I_2}$ over M_{I_0} such that:

   • if $u_1 \subseteq I_0 \cup I_1$ and $u_2 = \{\ell : \ell \in I_0 \text{ or } \ell - n \in I_1 \text{ (so } \ell \in I_2)\}$ then f maps M_{u_1} onto M_{u_2}.

2) We say $\mathrm{NPr}_1(T)$ when:

   (a) as above

   (b) the failure of (b) of part (1).

Claim 15.9. *There are T such that*

    (a) *T is simple complete countable*

    (b) *T fails* Pr_1

    (c) $\text{univ}_T(\lambda)$

Proof. Let $k \geq 3, \tau = \{R\}, R$ is a $(2k+2)$-place relation.

Let $T^{7,0}_{n,k}$ be the following universal first order theory in $\mathbb{L}(\tau)$

    $(*)_1$ for an τ-model $M, M \models T^0_k$ iff :

        (a) R^M is symmetric irreflexive

        (b) we cannot find $a_\ell, b_\ell \in M$ for $\ell < n+1$ such that: if $u \in [n]^{k+1}$ then $(\ldots, a_\ell, \ldots)_{\ell \in u}{}^\wedge \langle a_\ell :$
           $\ell \in u \rangle^\wedge \langle b_\ell : \ell \in u \rangle \in R^M$

    $(*)_2$ $T^{7,0}_{n,k}$ is universal (first order) with finite vocabulary, with no function symbols, JEP
        and amalgamation.

[Why? Check, here $k \geq 2$ suffice.]

    $(*)_3$ $T^{7,1}_{n,k}$, the model completion of $T^{7,0}_{n,k}$ is well defined.

[Why? By $(*)_2$.]

    $(*)_4$ $T^{7,0}_{n,k}$ has $\mathscr{P}^-(3)$-amalgamation.

[Why? Check.]

    $(*)_5$ $T^{7,0}_{n,k}$ is simple.

[Why? By $(*)_4$.]

    $(*)_6$ $T^{7,0}_{n,k}$ fails Pr_1.

[Why? Compare the definitions in $(*)_1$ and 15.8.]

                                                     $\square_{15.9}$

§ 16. NON-FORKING SYSTEMS

§ 16(A). **Basics.**

Hypothesis 16.1. 1) T is simple, hence $\kappa(T) \leq |T|^+$.
2) $\mathfrak{C} = \mathfrak{C}_T$ is a monster model for T.

Claim 16.2. *Let $\theta \geq |T|, \theta^+ \geq \kappa = \text{cf}(\kappa) \geq \kappa(T)$, (e.g. $\theta = |T|, \kappa = \theta^+$) and assume $M_* \prec \mathfrak{C}_T$ has cardinality $\leq \theta$ and $\mathbf{a} = \langle \bar{a}_\alpha : \alpha < \lambda \rangle, \bar{a}_\alpha \in {}^{\omega>}(\mathfrak{C}_T)$ for $\alpha < \lambda$ with no repetitions for transparency.*
1) We can find $\mathbf{s} = (\bar{I}, \bar{u}, \bar{M}) = (\bar{I}_\mathbf{s}, \bar{u}_\mathbf{s}, \bar{M}_\mathbf{s})$ such that:

    ⊞ (a) *$i_\mathbf{s} = i(\mathbf{s}) \leq \kappa$ and $\bar{I} = \langle I_i : i < i_\mathbf{s} \rangle$ and I_i is a set of ordinals of cardinality $\leq \lambda$*
              disjoint to $I_{<i} := \bigcup_{j<i} I_j$ (even of order type $\leq \lambda$)

       (b) *$\bar{u} = \langle \bar{u}_i : i < i_\mathbf{s}$ and $\bar{u}_i = \langle u_s = u_{i,s} : s \in I_i \rangle$*

       (c) *$u_{i,s} \subseteq I_{<i}$ has cardinality $< \kappa$ and $t \in u_s \Rightarrow u_t \subseteq u_s$*

       (d) *$\bar{M} = \langle \bar{M}_i : i < i_* \rangle$ and $\bar{M}_i = \langle M_s = M_{i,s} : s \in I_i \rangle$*

       (e) *$M_t \prec \mathfrak{C}_T$ has cardinality $\leq \theta, M_* \prec M_t$ and $s \in u_t \Rightarrow M_s \prec M_t$ for $t \in I_i$*

       (f) *(non-forking) if $t \in I_i$ then $\text{tp}(M_t, \cup\{M_s : s \in I_{<i} \text{ or } s \in I_i \backslash \{t\}\})$ does not fork*
            over $\cup\{M_s : s \in u_t\}$

(g) $\bar{a}_\alpha \in \cup\{{}^{\omega>}M_s : s \in I_{<i_s}\}$

(h) let $I = I_{<i_s}$ and let $u_s^+ = u_s \cup \{s\}$.

2) If $\kappa(T) = \aleph_0$, then renaming, w.l.o.g. $\bar{M} = \langle M_u : u \in [\lambda]^{<\aleph_0}\rangle$.

Proof. By induction on $i < \kappa$ we try to choose $\mathbf{s}_i = (I_i, \bar{u}_i, \bar{M}_i)$ increasing with i (in the natural sense) satisfying the clauses (a)-(f), (h) of $\boxplus$ and in addition:

$\oplus$ (a) If $\alpha < \lambda$ and $\bar{a}_\alpha \notin \cup\{M_s : s \in I_{<i}\}$, then $\mathrm{tp}(\bar{a}_\alpha, \cup\{M_s : s \in I_{\leq i}\})$ forks over $\cup\{M_s : s \in I_{<i}\}$.

(b) If $\alpha < \lambda$, and $\bar{a}_\alpha$ is included in the algebraic closure of $\in \cup\{M_s : s \in I_{<i}\}$, then for some $t \in I$ we have $\bar{a}_\alpha \in {}^{\omega>}(M_t)$.

For $i = 0$ there is nothing to do.

For $i = 1$ let $\mathbf{s}_0$ be defined by $I_0 = \{0\}$, $M_0 \prec \mathfrak{C}_T$ of cardinality θ (and $u_0 = \emptyset$).

For i a limit ordinal ($\leq \kappa$) let $\bar{I}_{\mathbf{s}_i}$ be a sequence of length i such that $j < i$ implies $\bar{I}_{\mathbf{s}_i \restriction j} = \bar{I}_{\mathbf{s}_j}$, and similarly with the other objects.

For $i = j+1$, we try to choose $(\bar{I}_{j\restriction\varepsilon}, \bar{u}_{j\restriction\varepsilon}, \bar{M}_{j\restriction\varepsilon})$ by induction on $\varepsilon < \lambda$ such that

(∗) $M_{j\restriction\varepsilon} \prec \mathfrak{C}_T$ has cardinality $\leq \lambda$.

In step ε first choose $\alpha_{j,\varepsilon}$ as the minimal ordinal $\alpha < \lambda$ such that $s \in I_{<j}$ implies $\bar{a}_\alpha \notin {}^{\omega>}(M_s)$ and $\xi < \omega$ implies $\bar{a}_\alpha \notin {}^{\omega>}(M_{j,\xi})$; and

$$\mathrm{tp}(\bar{a}_\alpha, \bigcup\{M_s : s \in I_{<j}\} \cup \bigcup\{M_{\varepsilon,\xi} : \xi < \varepsilon\})$$

does not fork over $\bigcup\{M_s : s \in I_{<j}\}$. If there is no such α, let $\varepsilon_j = \varepsilon$ and we do not continue.

Second, choose $u_{j,\varepsilon}$ as a subset of $I_{<j}$ of cardinality $< \kappa(T)$ such that

$$\mathrm{tp}(\bar{a}_{\alpha_{j,\varepsilon}}, \bigcup\{M_s : s \in I_{<j}\})$$

does not fork over $\bigcup\{M_s : s \in u_{\varepsilon,j}\}$; possible by the definition of $\kappa(T)$. By the transitivity property of non-forking

$\bullet_0$ $\mathrm{tp}(\bar{a}_\alpha, \bigcup\{M_s : s \in I_{<j}\} \cup \{M_{\varepsilon,\xi} : \xi < \varepsilon\})$ does not fork over $\bigcup\{M_s : s \in u_{\varepsilon,j}\}$.

Third, choose $M_{j,s}$ such that

$\bullet_1$ $M_{j,\varepsilon} \prec \mathfrak{C}_T$ has cardinality $\leq \theta$.

$\bullet_2$ $\bigcup\{M_s : s \in u_{\varepsilon,j} \cup \bar{a}_\alpha\} \subseteq M_{j\restriction\varepsilon}$.

Fourth, w.l.o.g. (possible by the extension property of non-forking):

$\bullet_3$ $\mathrm{tp}(M_{j,\varepsilon} \cup \bigcup\{M_s : s \in I_{<j}\} \cup \{M_{\varepsilon,\xi} : \xi < \varepsilon\})$ does not fork over $\bigcup\{M_s : s \in u_{\varepsilon,j}\} \cup \bar{a}_\alpha$.

By the transitivity of non-forking (and $\bullet_0 - \bullet_3$) we have $\mathrm{tp}(M_{j,\varepsilon} \cup \bigcup\{M_s : s \in I_{<j}\} \cup \{M_{\varepsilon,\xi} : \xi < \varepsilon\})$ does not fork over $\bigcup\{M_s : s \in u_{\varepsilon,j}\} \cup \bar{a}_\alpha$.

Clearly, for some $\varepsilon_j \leq \lambda$, $\alpha_{j,\varepsilon}$ is well-defined iff $\varepsilon < \varepsilon_j$. Let $I_j = \{\alpha_{j,\varepsilon} : \varepsilon < \varepsilon_j\}$ and $M_{\alpha_{j,\varepsilon}} = M_{j,\varepsilon}$, $u_{\alpha_{j,s}} = u_{j,\varepsilon}$ for $\varepsilon < \varepsilon_j$. So $\mathbf{s}_i = I_{j+1}$ is well-defined.

So we have carried out the induction on $i \leq \kappa$ and so $\mathbf{s} = \mathbf{s}_\kappa$ is well-defined and satisfies clauses $(a) - (f)$ and (h) of $\boxplus$; what about clause (g)? Let $\alpha < \lambda$, so as $\kappa = \mathrm{cf}(\kappa) \geq \kappa(T)$, there is $j < \kappa$ such that $\mathrm{tp}(\bar{a}_\alpha, \bigcup\{M_s : s \in I_{<j}\})$ does not fork over $\bigcup\{M_s : s \in u_{\varepsilon,j}\}$. By monotonicity of non-forking, $\alpha \in \{\alpha_{j,\varepsilon} : \varepsilon < \varepsilon_j\}$ hence for some $\varepsilon < \varepsilon_1$, $\alpha = \alpha_{j,\varepsilon}$, hence $\bar{a}_\alpha \in {}^{\omega>}(M_{\alpha_{j,\varepsilon}})$, as promised. $\square_{16.2}$

Claim 16.3. *Assume* $\kappa \geq \kappa(T), \theta^+ \geq \kappa, \theta \geq |T|, \mu = \mu^{<\kappa} < \lambda \leq 2^\mu$ *and the triple* $(\bar{I}, \bar{u}, \bar{M})$ *is as in 16.2 and for* $s \in I_{<i_*}$ *we let* $\mathbf{b}_s = \langle b_{s,j} : j < \theta\rangle$ *list the elements of* M_s, *so* $s \in u_t \Rightarrow \mathrm{Rang}(\bar{b}_s) \subseteq \mathrm{Rang}(\bar{b}_t))$.

Then we can find a sequence $\langle (\zeta_\alpha, s_\alpha) : \alpha < \lambda\rangle$ *such that:*

(a) $\zeta_\alpha < \theta^+$

(b) $s_\alpha \in I$ and $\bar{a}_\alpha \subseteq M_{s_\alpha}$, (recall I is a set of ordinals)

(c) if $\zeta_\alpha = \zeta_\beta$, then $\mathrm{otp}(u_{s_\alpha}^+) = \mathrm{otp}(u_{s_\beta}^+)$ and letting $h_{\beta,\alpha}$ be the unique order preserving function from $u_{s_\alpha}^+$ onto $u_{s_\beta}^+$ we have:

  (α) $h_{\beta,\alpha}(s_\alpha) = s_\beta$ and $h_{\beta,\alpha}(s) = t \wedge i < i_* \Rightarrow (s \in I_i \equiv t \in I_i)$ and $\bigwedge\limits_{i=1}^{2} h_{\beta,\alpha}(s_\ell) = t_\ell \Rightarrow (s_1 \in u_{t_1} \equiv s_2 \in u_{t_2})$

  (β) $\bar{\mathbf{b}}_{s_\alpha}, \bar{\mathbf{b}}_{s_\beta}$ realize the same type over M_*

  (γ) moreover, if $h_{\beta,\alpha}(r_1) = r_2$ then $\bar{\mathbf{b}}_{s_\alpha}{}^\smallfrown\bar{\mathbf{b}}_{r_1}, \bar{\mathbf{b}}_{s_\beta}{}^\smallfrown\bar{\mathbf{b}}_{r_2}$ realize the same type over M_*

  (δ) $h_{\beta,\alpha}$ is the identity on $u_{s_\beta} \cap u_{s_\alpha}$.

Remark 16.4. Note that presently there may be $s_1 \neq s_2 \in I, u_{s_1} = u_{s_2}$. Overcoming this we get $2^{<\kappa}$ at least. For well foundations use $F : {}^\omega\lambda \to \lambda$.

Proof. Let $\langle \eta_s : s \in I \rangle$ be a sequence with no repetition of members of ${}^\mu 2$, etc. or quote [EK65]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square_{16.3}$

§ 16(B). **Discussion.**

Discussion 16.5. The super-simple case: We first deal with the super-simple case and let θ satisfy $\alpha < \theta \Rightarrow |\alpha|^{|T|} < \theta$. In this case

* Every model M of T of cardinality λ can be elementarily embedded into a $(|T|, 2^{|T|})$-system $\bar{M} = \langle M_u : u \in [\lambda]^{<\aleph_0} \rangle$; assuming $\theta = \beth_\omega(|T|)$ we can add

  (a) $\|M_u\| = \beth_{\|u\|}(\|T\|)$

  (b) If $n = \|u\|$ and $\beth_{n+1}(\|T\|) = (\beth_n(\|T\|))^+$, then M_u is saturated of cardinality $> \|\bigcup\{M_v : v \subset u\}\|$.

If we have uniqueness for every $\mathscr{P}^-(n)$ we can have uniqueness for every finite $\mathscr{P}$. Also we have compactness for any $\mathscr{P}, \theta$ fixed. This is dual to using only finite $\mathscr{P}$ and varying the cardinality. We may or may not use fullness, see 10.20. If we assume $(\theta, \mathscr{P}^-(n))$-uniqueness for every $n, \theta = |T|^+$ or $\theta = \beth_\omega(|T|)$, we get the existence of universal (first for every finite $\mathscr{P}$ then existence for such $\mathscr{P}$, etc. But if $\mathbf{n}(T)$ is minimal such that uniqueness fails, can we sort out what occurs for $\mathscr{P} \subseteq [I]^{<n}$?

Discussion 16.6. Assume just that T is simple.

We may use $\mathscr{P} = [I]^{<\theta}, \theta > 2^{|T|}$, see **??**.

We may use well-founded $\mathscr{P}$ such that $\mathrm{dp}_\mathscr{P} : \mathscr{P} \to \theta$ is depth, i.e. if $u \subsetneq v$ are from $\mathscr{P}$ then $\mathrm{dp}_\mathscr{P}(u) < \mathrm{dp}_\mathscr{P}(v)$. Using $F : {}^\omega\lambda \to \lambda$ with no decreasing ω-sequence of F-closed subsets, this is O.K.

A question is: does failure of uniqueness for $\mathscr{P} \subseteq \mathscr{P}(u_*), |u_*| < \theta$, implies one for $\mathscr{P} = \mathscr{P}^-(n)$?

Similarly for existence?

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	Cardinality	Class						
		K^{fc}_*	K^{tr}	K^{tr}_*	$(K^{\mathrm{rtf}}, \subseteq_{\mathrm{pr}})$	$(K^{\mathrm{rtf}}, \subseteq)$	$(K^{\mathrm{rs(p)}}, \subseteq_{\mathrm{pr}})$	$(K^{\mathrm{rs(p)}}, \subseteq)$
A	$\lambda = \aleph_0$	×	✓	✓	×	×	✓	✓
B	$\lambda = \lambda^{\aleph_0}$	✓ 10.11(1)	✓ 10.11(1)	✓	✓ [KS95, 3.5]	✓ [She17, 1.2(1)]	✓ [KS95, 3.2]	✓
C	$\aleph_0 < \lambda < 2^{\aleph_0}$	10.11(2)	⊞ for λ singular $> \aleph_1$, by 10.10(2A)(a). × for $\lambda = \mathrm{cf}(\lambda) > \aleph_1$, by 10.10(2A)(a). Con(✓) for $\aleph_1$ 10.11(4) Con(×) see 10.11(3)	Con(✓*), by [She97, 4.7]. Con(×), by 10.11(3). Con(✓) for $\aleph_1$, by 10.11(4), [She80a]	× [KS95, 3.7]	× [She01, 1.2]	⊞ 10.10(2A)(d)	Con(×), 10.11(3). Con(✓*), [She97, §4].
D	$2^{\aleph_0} + \mu^+ < \lambda < \mu^{\aleph_0}$ and λ regular	?	× 10.10(2A)(a)	?	× [KS95, 3.8]	× [She01, 1.6]	× [KS95, 3.3]	⊞ 10.11(b)
E	$2^{\aleph_0} + \mu^+ < \lambda < \mu^{\aleph_0}$ and λ singular	?	Con(×) 10.10(2A)(a)	?	⊠	⊠	⊠	⊠
F	$\mu^+ = \lambda < \mu^{\aleph_0}$ and $(\forall\chi<\mu)(\chi^{\aleph_0}<\mu)$	Con(×) e.g. $\lambda = \beth_\omega^+$	Con(×) 10.10(2)	Con(×)	Con(×)	Con(×)	Con(×)	Con(×)
G	$\mathrm{cf}(\lambda) = \aleph_0$ and $(\forall\chi<\lambda)(\chi^{\aleph_0}<\lambda)$	✓ 10.11(1)	✓ 10.11(1)	✓ 10.11(2)	✓ ([She17, 1.2(2)])	✓ [She17, 1.2(2)]	✓	✓ [She17]

TABLE 1. Table

$\times$	There is no universal model
$\boxtimes$	There is no universal model except possibly when some extreme pcf condition holds
$\checkmark$	There is a universal model
$\checkmark^*$	There is a weak universal model for λ of cardinality $\lambda^+ < 2^\lambda$
$\rightarrow$	A solution on the left serves also on the right
$\leftarrow$	A solution on the right serves also on the left
pcf	Under mild pcf conditions
K_*	Stands for $K_{\bar\lambda}$, $\bar\lambda = \langle \lambda_n : n < \omega \rangle$ such that (a) or (b), where (a) $\Sigma_n \lambda_n = \mu < \lambda = \lambda_\omega < \mu^{\aleph_0}$ for some μ, (b) $\lambda_n = \lambda_\omega = \lambda$.

TABLE 2. Legend for Table 1.