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No Limit Model in Inaccessibles

Saharon Shelah

Dedicated to Michael Makkai

Abstract. Our aim is to improve the negative results, i.e., non-existence of limit models, and the failure of the generic pair property from [6] to inaccessible λ as promised there. In [6], the negative results were obtained only for non-strong limit cardinals.

0. Introduction

Let $\lambda = \lambda^{<\lambda} > \kappa$ be regular cardinals. A complete first order theory T may have (some variant of) (λ, κ)-limit model, which, if exists, is unique, see history in [6] and Definition 0.9. There we prove existence for the theory of linear order and non-existence for first order theories which are strongly independent and then just independent and even the parallel for $\kappa = 2$ (one direction of the so-called generic pair conjecture). Those non-existence results in [6] were for $\lambda = 2^\kappa$, here we deal with strongly inaccessible λ. In [7] there are existence results but for λ measurable, and we promise there the non-existence results for λ strongly inaccessible as complimentary results.

Let λ be strongly inaccessible ($> |T|$) such that $\lambda^+ = 2^\lambda$; this for transparency only.

Here in Section 1 we prove that for strongly independent T (see Definition 0.2), a strong version of the generic pair conjecture (see Definition 0.7(2)) holds. We also prove the non-existence of (λ, κ)-limit models, a related property (for all versions of "limit model").

In Section 2, we prove this even for independent T. The use of $\lambda^+ = 2^\lambda$ is just to have a more transparent formulation of the conjecture. See more on the generic pair conjecture for dependent T in [8].

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Notation 0.1. (1) $\mathscr{D}_\lambda$ is the club filter on λ for λ regular uncountable.

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(2) $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.
(3) For a limit ordinal δ let $\mathscr{P}^{\mathrm{ub}}(\delta) = \{\mathscr{U} : \mathscr{U}$ is an unbounded subset of $\delta\}$.
(4) T denotes a complete first order theory.
(5) For a model M, $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M)$ and $\bar{d} \in {}^{lg(\bar{y})}M$, let $\varphi(M, \bar{d}) = \{\bar{c} \in {}^{lg(\bar{x})}M : M \models \varphi[\bar{c}, \bar{d}]\}$.
(6) $\mathbf{S}^n(A, M) = \{\mathrm{tp}(\bar{b}, A, N) : M \prec N$ and $\bar{b} \in {}^nN\}$ where $\mathrm{tp}(\bar{c}, A, N) = \{\varphi(\bar{x}, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M), \bar{a} \in {}^{lg(\bar{y})}A$ and $M \models \varphi[\bar{c}, \bar{a}]\}$.
(7) $\mathbf{S}^n(M) = \mathbf{S}^n(M, M)$ and $\mathbf{S}^{<\omega}(M) = \bigcup\{\mathbf{S}^n(M) : n \in \omega\}$.
(8) $\beth_\alpha(\lambda) = \lambda + \Sigma\{2^{\beth_\beta(\lambda)} : \beta < \alpha\}$ and $\beth_\alpha = \beth_\alpha(\aleph_0)$.

Recall (as in [6, 2.3])

Definition 0.2. (1) T has the strong independence property (or is strongly independent) *when*: some $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ has it, where:

(2) $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ has the strong independence property for T *when* for every model M of T and pairwise disjoint finite $\mathbf{I}_1, \mathbf{I}_2 \subseteq {}^{lg(\bar{y})}(M)$ for some $\bar{a} \in {}^{lg(\bar{x})}M$ we have $l \in \{1, 2\} \wedge \bar{b} \in \mathbf{I}_l \Rightarrow M \models \varphi[\bar{a}, \bar{b}_l]^{\mathrm{if}(l=2)}$.

Remark 0.3. (1) Elsewhere we use $\varphi(x, y)$, i.e., the x and y are singletons, but the proofs are not affected.

(2) Also we may restrict ourselves to $\mathbf{I}_1, \mathbf{I}_2 \subseteq \psi(M, \bar{d})$ where $\psi \in \mathbb{L}(\tau_T)$ such that $\psi(M, \bar{d})$ is infinite, and we may restrict ourselves to $\mathbf{I}_1$, $\mathbf{I}_2$ such that every $\bar{b} \in \mathbf{I}_1 \cup \mathbf{I}_1$ realizes a fixed nonalgebraic type $p \in \mathbf{S}^m(A, M)$ with M being $(|A|^+ + \aleph_0)$-saturated. The results are not really affected.

Question 0.4. (1) Assume $\lambda_2 = \lambda_2^{<\lambda_1} \geq \lambda_1 > |T|$, T a complete first order dependent theory. Is the theory $T^*_{\lambda_1,\lambda_2}$ a dependent theory or at least when is $T^*_{\lambda_1,\lambda_2}$ a dependent theory? where

(a) $T^*_{\lambda_1,\lambda_2} = \mathrm{Th}(K^+_{\lambda_1,\lambda_2})$ where
(b) $K^+_{\lambda_1,\lambda_2} = \{(N, M) : M$ is a λ_1-saturated model of T of cardinality λ_2, N a λ_2^+-saturated elementary extension of $M\}$.

(2) Similarly for other properties of $T^*_{\lambda_1,\lambda_2}$; note[1] that this theory is complete if $\lambda_1 = \lambda_2$.

(3) When can we prove that $T^*_{\lambda_1,\lambda_2}$ does not depend on the cardinals at least for many pairs?

Remark 0.5. (1) Concerning failure of Question 0.4(1) see Kaplan – Shelah [3].

(2) Any solution of the generic pair conjecture answers positively Question 0.4(3) for dependent T in the relevant cases.

(3) It is known that in Question 0.4(1) if T extends PA or ZFC then in $T^* = \mathrm{Th}(N, M)$ we can interpret the second order theory of λ_2.

But may well be that as in Baldwin – Shelah [1]

Question 0.6. Assume $|T| < \kappa \leq \lambda_1 \leq \lambda_2 = \lambda_2^{<\lambda_1}$, T a complete first order theory. For which T's can we interpret in $M \in K^+_{\lambda_1,\lambda_2}$ a model of PA of

[1]let $(N_l, M_l) \in K^+_{\lambda_1,\lambda_2}$ for $l = 1, 2$ and let f_* be an isomorphism from M_1 onto M_2 and let $\mathscr{F} = \{f : f$ is a (N_1, N_2)-elementary mapping extending f_* of cardinality $\leq \lambda_1\}$. Now we can prove that any $f \in \mathscr{F}$ preserve satisfaction for first order formulas.

cardinality $\geq \lambda_1$ by first order formula *or* just an $\mathbb{L}_{\infty,\kappa}(\tau_T)$-formulas with parameters, the intention is that we assume λ_2 is enough larger than λ_1 which is large enough than $|T|$; if $2^\kappa \geq \lambda_1$ this is trivial.

Recall (from [6, 0.2])

Definition 0.7. (0) Let $\mathrm{EC}_\lambda(T)$ be the class of models M of (the first order) T of cardinality λ. Let $\mathrm{EC}_{\lambda,\kappa}(T)$ be the class of κ-saturated models $M \in \mathrm{EC}_\lambda(T)$.

(1) Assume $\lambda > |T|$, (we usually assume $\lambda = \lambda^{<\lambda}$) and $2^\lambda = \lambda^+$, $M_\alpha \in \mathrm{EC}_\lambda(T)$ is $\prec$-increasing continuous for $\alpha < \lambda^+$ with $M = \bigcup\{M_\alpha : \alpha < \lambda^+\} \in \mathrm{EC}_{\lambda^+}(T)$, and M is saturated. The generic pair property (for T, λ) says that for some club E of λ^+ for all pairs $\alpha < \beta$ of ordinals from E of cofinality λ, (M_β, M_α) has the same isomorphism type (we denote this property of T by $\mathrm{Pr}^2_{\lambda,\lambda}(T)$).

(2) The generic pair conjecture for $\lambda > \aleph_0$, (usually $\lambda = \lambda^{<\lambda}$) such that $2^\lambda = \lambda^+$ says that for any complete first order T of cardinality $< \lambda$, T is dependent iff it has the generic pair property for λ.

(3) Let $\mathbf{n}_{\lambda,\kappa}(T)$ be $\min\{|\{M_\delta/\cong : \delta \in E \text{ has cofinality } \kappa\}| : E \text{ a club of } \lambda^+\}$ for λ and $\overline{M} = \langle M_\alpha : \alpha < \lambda^+\rangle$ as above and $\kappa = \mathrm{cf}(\kappa) \leq \lambda$; clearly the choice of $\overline{M}$ is immaterial.

Remark 0.8. (1) Note that to say $\mathbf{n}_{\lambda,\kappa}(T) = 1$ is a way to say that T has (some variant of) a (λ, κ)-limit model, see Definition 0.9 below. There are other variants of the definition of limit.

(2) Recall that we conjecture that for $\lambda = \lambda^{<\lambda} > \kappa = \mathrm{cf}(\kappa) > |T|$, $2^\lambda = \lambda^+$ we have $\mathbf{n}_{\lambda,\kappa}(T) = 1 \Leftrightarrow \mathbf{n}_{\lambda,\kappa}(T) < 2^\lambda \Leftrightarrow T$ is dependent. The use of "$\lambda^+ = 2^\lambda$" is just for clarity. See more in [6–8].

(3) Recall that if $\lambda = \kappa = \lambda^{<\lambda}$, then $\mathbf{n}_{\lambda,\lambda}(T) = 1$ means T has a unique saturated model; (and parallely if $\lambda > \mathrm{cf}(\lambda) = \kappa$, λ strong limit). So we concentrate on the case $\mathbf{n}_{\lambda,\kappa}(T) = 1$ where $\kappa < \lambda$.

Definition 0.9. We define when M_* is a (λ, κ)-limit model of T where $\lambda \geq \kappa = \mathrm{cf}(\kappa)$ and $\lambda \geq |T|$. In general it means that: letting $K_\lambda = \{M : M \text{ is a model of } T \text{ with universe an ordinal} \in [\lambda, \lambda^+)\}$, for some function $\mathbf{F}$ with domain K and satisfying $M \prec \mathbf{F}(M) \in K$ we have:

$\oplus$ if $M_\alpha \in K$ for $\alpha < \lambda^+$ is $\prec$-increasing continuous and $\alpha < \lambda \Rightarrow \mathbf{F}(M_{\alpha+1}) \prec M_{\alpha+2}$ then for some club E of λ^+ we have:

$$\delta \in E \wedge \mathrm{cf}(\delta) = \kappa \implies M_\delta \cong M_*.$$

Remark 0.10. If $2^\lambda = \lambda^+$ we have: $\mathbf{n}_{\lambda,\kappa}(T) = 1$ iff T has a (λ, κ)-limit model.

1. Strongly independent T

Context 1.1. (1) T is a fixed first order complete theory and $\mathfrak{C} = \mathfrak{C}_T$ a monster for it; for notational simplicity τ_T is relational.

(2) We let λ be a regular uncountable cardinal $> |T|$; we deal mainly with strongly inaccessible λ.

Here for λ strongly inaccessible and (complete first order) T with the strong independence property (of cardinality $< \lambda$) we prove the non-existence of (λ, κ)-limit models for $\kappa = \mathrm{cf}(\kappa) < \lambda$ (in Theorem 1.9) and the generic pair conjecture for

λ and T, in Theorem 1.10 (which shows nonisomorphism). Recall that the generic pair property speaks on the isomorphism type of pairs of models.

Definition 1.2 gives us a more constructive invariant of $(N, M)/\cong$. Unfortunately it seemed opaque how to manipulate it so we shall use a related but different version, the one from Definition 1.4. Naturally it concentrates on types in one formula $\varphi(\bar{y}, \bar{x})$ witnessing the strong independence property. But mainly gives the pair (N, M) an invariant $\langle \mathscr{P}_\delta : \delta < \lambda \rangle / \mathscr{D}_\lambda$ where $\mathscr{P}_\delta \subseteq \mathscr{P}(\mathscr{P}(\delta))$. Now always $|\mathscr{P}_\delta| \leq 2^{|\delta|}$ and it is easily computable from one $\mathscr{P} \subseteq \mathscr{P}(\delta)$, in fact from the invariant $\mathrm{inv}_4(M, N)$ from Definition 1.2, but in our proofs its use is more transparent. It has monotonicity property and we can increase it.

We need different but similar version for the proof of non-existence of (λ, κ)-limit models.

Definition 1.2. (1) Let $\mathscr{E}_T^*$ be the following two-place relation on $\{(M, \mathbf{P}) : M \models T \text{ and } \mathbf{P} \subseteq \mathbf{S}^{<\omega}(M)\}$; let $(M_1, \mathbf{P}_1)\mathscr{E}_T^*(M_2, \mathbf{P}_2)$ iff there is an isomorphism h from M_1 onto M_2 mapping $\mathbf{P}_1$ onto $\mathbf{P}_2$.

(2) For models $M \subseteq N$ we define (the important case is $M \prec N \models T$):

(a) $\mathrm{inv}_1(M, N) = \{p \in \mathbf{S}^{<\omega}(M) : p \text{ is realized in } N\}$

(b) $\mathrm{inv}_2(M, N) = (M, \mathrm{inv}_1(M, N))/\mathscr{E}_T^*$.

(3) If $M \prec N$ are models of T such that the universe of N is $\subseteq \lambda$, recalling $\mathscr{D}_\lambda$ is the club filter on λ, let:

(a) for any ordinal $\delta < \lambda$

$$\begin{aligned} &\mathrm{inv}_3(\delta, M, N) \\ &\quad = (M \restriction \delta, \{p \in \mathbf{S}^{<\omega}(M \restriction \delta) : p \text{ is realized by some sequence from } N \restriction \delta\}/\mathscr{E}_T^*) \end{aligned}$$

(b) $\mathrm{inv}_4(M, N) = \langle \mathrm{inv}_3(\delta, M, N) : \delta < \lambda \rangle / \mathscr{D}_\lambda$.

(4) If $M \prec N$ are models of T of cardinality λ then $\mathrm{inv}_4(M, N)$ is equal to $\mathrm{inv}_4\big(f(M), f(N)\big)$ for every one-to-one function f from N into λ (equivalently some f, see Observation 1.3(1), (2) below).

Observation 1.3. (0) *In Definition* 1.2(3) *for a club of δ's below λ we have $M \restriction \delta \prec M$ and $N \restriction \delta \prec N$ and so $M \restriction \delta \prec N \restriction \delta \models T$.*

(1) *Concerning Definition* 1.2(3), *if $M \prec N$ are models of T of cardinality λ and f_1, f_2 are one-to-one functions from N into λ then* $\mathrm{inv}_4\big(f_1(M), f_1(N)\big) = \mathrm{inv}_4\big(f_2(M), f_2(N)\big)$ *using Definition* 1.2(3)(b).

(2) *Definitions* 1.2(3), 1.2(4) *are compatible and in Definition* 1.2(4), "*some f such that f is a one-to-one function from N to λ*" *is equivalent to* "*every f such that. . . .*"

Proof (Observation 1.3). Straight, e.g., (this argument will be used several times).

(1) Let $E = \{\delta < \lambda : \delta$ is a limit ordinal such that $M \restriction \delta \prec M$, $N \restriction \delta \prec N$ and $\mathrm{Rang}(f_l \restriction \delta) = \mathrm{Rang}(f_l) \cap \delta$ for $l = 1, 2\}$. So E is a club of λ and $\delta \in E \Rightarrow f_2 \circ f_1^{-1}$ is an isomorphism from $f_1(N \restriction \delta)$ onto $f_2(N \restriction \delta)$, mapping $f_1(M \restriction \delta)$ onto $f_2(M \restriction \delta)$. $\square$

Definition 1.4. Assume $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ and $N_1 \prec N_2$ are models of T of cardinality λ.

(1) For one-to-one mapping f from N_2 to λ and $\delta < \lambda$ we define

$$\mathrm{inv}_5^\varphi(\delta, f, N_1, N_2) = \{\mathscr{P} \subseteq \mathscr{P}(\delta) : \text{there are } \bar{a}_\gamma \in {}^{lg(\bar{x})}N_2 \text{ satisfying } f(\bar{a}_\gamma) \in {}^{lg(\bar{x})}\delta \text{ for } \gamma < \delta \text{ such that for every } \mathscr{U} \subseteq \delta \text{ the following are equivalent: (i) } \mathscr{U} \in \mathscr{P} \text{ (ii) for some } \bar{b} \in {}^{lg(\bar{y})}N_1 \text{ we have } \gamma < \delta \Rightarrow N_2 \models \varphi[\bar{a}_\gamma, \bar{b}]^{\mathrm{if}(\gamma \in \mathscr{U})}\}.$$

(2) We let $\mathrm{inv}_6^\varphi(N_1, N_2)$ be $\langle \mathrm{inv}_5^\varphi(\delta, f, N_1, N_2) : \delta < \lambda\rangle/\mathscr{D}_\lambda$ for some (equivalently every) f as above.

Claim 1.5. (1) *In Definition* 1.4(2) *we have* $\mathrm{inv}_6^\varphi(N_1, N_2)$ *is well defined.*

(2) *In Definition* 1.4*, for* δ, λ, N_1, N_2, $\varphi(\bar{x}, \bar{y})$ *as there*

(a) *the set* $\mathrm{inv}_5^\varphi(\delta, f, N_1, N_2)$ *has cardinality at most* $2^{|\delta|}$

(b) *if* π *is a one-to-one function from* $f(N_2)$ *into* λ *mapping* $f(N_2) \cap \delta$ *onto* $\pi(f(N_2)) \cap \delta$ *then* $\mathrm{inv}_5^\varphi(\delta, \pi \circ f, N_1, N_2) = \mathrm{inv}_5^\varphi(\delta, f, N_1, N_2)$.

Proof (Claim 1.5). Easy. $\square$

Definition 1.6. (1) For $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$, a model N of T with universe λ, δ a limit ordinal $< \lambda$ and $\kappa < \lambda$ let

$$\mathrm{inv}_{7,\kappa}^\varphi(\delta, N) = \{\mathscr{P} \subseteq \mathscr{P}(\delta) : \text{we can find } \bar{a}_\gamma^i \in {}^{lg(\bar{x})}\delta \text{ for } \gamma < \delta,\ i < \kappa \text{ such that the following conditions on } \mathscr{U} \subseteq \delta \text{ are equivalent: (i) } \mathscr{U} \in \mathscr{P} \text{ (ii) for some } \bar{b} \in {}^{lg(\bar{y})}N \text{ we have: for every } i < \kappa \text{ large enough for every } \gamma < \delta \text{ we have } N \models \varphi[\bar{a}_\gamma^i, \bar{b}]^{\mathrm{if}(\gamma \in \mathscr{U})}\}.$$

(2) For $\varphi = \varphi(\bar{y}, \bar{x}) \in \mathbb{L}(\tau_T)$ and a model N of T of cardinality λ let $\mathrm{inv}_{8,\kappa}^\varphi(N) = \langle \mathrm{inv}_7^\varphi(\delta, N') : \delta < \lambda\rangle/\mathscr{D}_\lambda$ for every, equivalently some model N' isomorphic to N with universe λ.

Observation 1.7. (1) $\mathrm{inv}_{8,\kappa}^\varphi(N)$ *is well defined for* $N \in \mathrm{EC}_\lambda(T)$ *when* $|T| + \kappa < \lambda$.

(2) *In Definition* 1.6(1) *we have* $|\mathrm{inv}_{7,\kappa}^\varphi(\delta, N)| \le 2^{|\delta|+\kappa}$.

Proof (Observation 1.7). Easy. $\square$

Claim 1.8. *Assume* $\lambda > |T|$ *is regular,* $S \subseteq \lambda$ *is stationary,* $\varphi = \varphi(\bar{x}, \bar{y})$ *and*

(a) $\langle N_i : i < \kappa\rangle$ *is a* $\prec$*-increasing sequence*

(b) $N_i \in \mathrm{EC}_\lambda(T)$

(c) $N = \bigcup\{N_i : i < \kappa\}$

(d) $\overline{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda\rangle$ *where* $\mathscr{P}_\alpha \subseteq \mathscr{P}(\alpha)$

(e) f *is a one-to-one function from* N *onto* λ

(f) *for a club of* δ*'s below* λ *there are* $\bar{a}_\gamma^j \in N_{j+1} \cap f^{-1}(\delta)$ *for* $\gamma < \delta$, $j < \kappa$ *satisfying*

(α) *for every* $\bar{c} \in {}^{lg(\bar{x})}(N_j)$ *there is* $\mathscr{U} \in \mathscr{P}_\delta$ *such that* $\gamma < \delta \Rightarrow N \models \varphi[\bar{a}_\gamma^j, \bar{c}]^{\mathrm{if}(\gamma \in \mathscr{U})}$

(β) *for every* $\mathscr{U} \in \mathscr{P}_\delta$ *for some* $\bar{b} \in {}^{lg(\bar{y})}(N_\delta)$ *we have* $\gamma < \delta \Rightarrow N \models \varphi[\bar{a}_\gamma^j, \bar{b}]^{\mathrm{if}(\gamma \in \mathscr{U})}$.

Then $\{\delta \in S : \mathscr{P}_\delta \in \mathrm{inv}^{\varphi}_{7,\kappa}(\delta, f(N))\} \in \mathscr{D}_\lambda + S$.

PROOF (CLAIM 1.8). Straight. $\square$

Now we come to the main two results of this section.

Theorem 1.9. *For some club E of λ^+, if $\delta_1 \neq \delta_2$ belong to $E \cap S^{\lambda^+}_\kappa$* then *$M_{\delta_1}, M_{\delta_2}$ are not isomorphic, moreover $\mathrm{inv}^{\varphi}_{8,\kappa}(M_{\delta_1}) \neq \mathrm{inv}^{\varphi}_{8,\kappa}(M_{\delta_2})$* when :

$\boxtimes$ (a) *T has the strong independence property* (*see Definition* 0.2)
(b) *$\lambda = \lambda^{<\lambda}$ is regular uncountable, $\lambda > |T|$, $\lambda > \kappa = \mathrm{cf}(\kappa)$ and $\lambda^+ = 2^\lambda$*
(c) *M is a saturated model of T of cardinality λ^+*
(d) *$\langle M_\alpha : \alpha < \lambda^+\rangle$ is $\prec$-increasing continuous sequence with union M, each of cardinality λ.*

Theorem 1.10. *Assume $\boxtimes$ of Theorem* 1.9.

(1) *For some club E of λ^+, if $\delta_1 < \delta_2 < \delta_3$ are from E and $\delta_l \in S^{\lambda^+}_\lambda$ for $l =$* 1, 2, 3 then *$(M_{\delta_2}, M_{\delta_1}) \ncong (M_{\delta_3}, M_{\delta_1})$, moreover $\mathrm{inv}^{\varphi}_6(M_{\delta_2}, M_{\delta_1}) \neq \mathrm{inv}^{\varphi}_6(M_{\delta_3}, M_{\delta_1})$ for some φ.*

(2) *If $M \prec N_0$ are models of T of cardinality λ,* then *for some elementary extension $N_1 \in \mathrm{EC}_\lambda(T)$ of N_0 we have $N_1 \prec N_2 \in \mathrm{EC}_\lambda(T) \Rightarrow (N_0, M) \ncong (N_2, M)$.*

Discussion 1.11. We shall below start with $M \in \mathrm{EC}_\lambda(T)$ and a sequence $\langle b_i : i < \lambda\rangle$ of distinct members such that $\langle \varphi(\bar b_i, \bar y) : i < \lambda\rangle$ are independent, and like to find N, $\langle \bar a_i : i < \lambda\rangle$ such that $M \prec N \in \mathrm{EC}_\lambda(T)$ and the $\langle \bar b_i : i < \lambda\rangle$ has a real affect on the relevant φ-invariant, in the case of Theorem 1.10(1) this is $\mathrm{inv}^{\varphi}_6(M, N)$: for a stationary set of δ's below λ it adds something to the δth component in a specific representation, i.e., assuming $f\colon N \to \lambda$ is a one-to-one function and we deal with $\langle \mathrm{inv}^{\varphi}_5(\delta, f, M, N) : \delta < \lambda\rangle$; we have freedom about $\varphi(\bar a_\alpha, \bar b_i)$ and we can assume $\bar b \in {}^{lg(\bar y)}M \backslash \{\bar b_i : i < \lambda\} \Rightarrow N \models \neg\varphi[\bar a_\alpha, \bar b]$.

But the relevant $\mathscr{P}_\delta$ is influenced not just by say $\langle \bar b_i : i \in [\delta, 2^{|\delta|})\rangle$ but also by later $\bar b_i$'s (and earlier $\bar b_i$). To control this we use below $\langle \bar a_\alpha : \alpha < \lambda\rangle$, S, E such that we deal with different $\delta \in S$ in an independent way to large extent; this is the reason for choosing the C^*_α's.

PROOF (THEOREM 1.9). By the proof of [6, Section 2] without loss of generality λ is strongly inaccessible. Choose $\theta \in \mathrm{Reg} \cap \lambda \backslash \{\aleph_0\}$, will be needed when we generalize the proof in Section 2.

Let $\langle \mathscr{U}_i : i < \kappa\rangle$ be a $\subseteq$-increasing sequence of subsets of λ such that the set $\mathscr{U}^-_i = \mathscr{U}_i \backslash \bigcup\{\mathscr{U}_j : j < i\}$ has cardinality λ for each $i < \kappa$ and let $\mathscr{U}_\kappa = \bigcup\{\mathscr{U}_i : i < \kappa\}$. Let $\varphi(\bar x, \bar y) \in \mathbb{L}(\tau_T)$ have the strong independence property, see Definition 0.2. We can choose $\langle C^*_\alpha : \alpha < \lambda^+\rangle$ such that $C^*_\alpha \subseteq \mathrm{nacc}(\alpha)$, $\mathrm{otp}(C^*_\alpha) \leq \kappa$, $\beta \in C^*_\alpha \Rightarrow C^*_\beta = C^*_\alpha \cap \beta$ and $\lambda | \alpha \wedge \mathrm{cf}(\alpha) = \kappa \Rightarrow \alpha = \sup(C^*_\alpha)$ and $\mathrm{cf}(\alpha) \neq \kappa \Rightarrow \mathrm{otp}(C^*_\alpha) < \kappa$.

[See [4] but for completeness we show this; by induction on $\alpha < \lambda^+$ we choose $\langle C^*_\varepsilon : \varepsilon < \lambda\alpha\rangle$ such that:

(a) the relevant demand holds
(b) if $\alpha = \beta + 1$, $C \subseteq \lambda\beta$, $(\forall i \in C)(C^*_i = C \cap i)$ and $\mathrm{otp}(C) < \kappa$ then for some $i \in (\lambda\beta, \lambda\alpha)$ we have $C^*_{i+1} = C$.

As $\lambda = \lambda^{<\kappa}$ this is easy but we elaborate. For $\alpha = 0$ trivial for α limit obvious. Assume $\alpha = \beta + 1$ let $\alpha_* = \lambda\alpha$, $\beta_* = \lambda\beta$ and $\langle C^*_i : i < \beta_*\rangle$ has been defined.

First, we choose $C^*_{\beta_i}$. If $\mathrm{cf}(\beta_*) \neq \kappa$ let $C^*_{\beta_*} = \varnothing$, so assume $\mathrm{cf}(\beta_*) = \kappa$ then necessarily $\mathrm{cf}(\beta) = \kappa$.

Let $\langle \alpha_\varepsilon : \varepsilon < \kappa \rangle$ be increasing with limit β, and choose $\beta_\varepsilon \in [\lambda\alpha_\varepsilon, \lambda\alpha_\varepsilon + \lambda)$ by induction on $\varepsilon < \kappa$ such that $C^*_{\beta_\varepsilon} = \{\beta_\zeta : \zeta < \varepsilon\}$.

Lastly, let $C^*_{\lambda\alpha} := \{\beta_\varepsilon : \varepsilon < \kappa\}$. So $C^*_{\beta_*}$ has been defined in any case.

Now let $\mathscr{C}_\alpha = \{C \subseteq \lambda\beta : \text{otp}(C) < \kappa \text{ and } \gamma \in C \Rightarrow C^*_\gamma = C \cap \gamma\}$, so $|\mathscr{C}| \leq \lambda$, also $\varnothing \in \mathscr{C}$, so let $\langle C^*_{\lambda\beta+i} : i \in (0, \lambda)\rangle$ list $\mathscr{C}_\alpha$ possibly with repetitions. So we have defined $\langle C^*_\varepsilon : \varepsilon < \lambda\alpha \rangle$, so have carried the induction.]

Let $S_* = \{\mu : \mu = \beth_{\alpha+\omega}$ for some $\alpha < \lambda\}$. Let E_*, $\langle C_\alpha : \alpha < \lambda \rangle$ be such that:

- $\circledast_1$ (a) $C_\alpha \subseteq \alpha \cap S_*$
  - (b) $\beta \in C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta$
  - (c) $\text{otp}(C_\alpha) \leq \theta$
  - (d) E_* is the club of λ included in$\{\delta < \lambda : \theta < \delta = \beth_\delta\}$
  - (e) $\text{otp}(C_\alpha) = \theta$ iff $\alpha \in E_* \cap S^\lambda_\theta$
  - (f) if $\alpha \in S := E_* \cap S^\lambda_\theta$ then $\alpha = \sup(C_\alpha)$
  - (g) if $\alpha \in E_*$ and $i < \kappa$ then $|\alpha \cap \mathscr{U}^-_i| = |\alpha|$.

[Why can we choose? By induction on the cardinal $\chi \in [\aleph_0, \lambda)$ we choose $\langle C_\alpha : \alpha < \beth_\chi \rangle$ and $E_\chi = E_* \cap \beth_\chi$ such that the relevant demands hold and: if $\chi = 2^{\chi_1}$ and C is a subset of $S_* \cap \beth_{\chi_1}$ of order type $< \theta$ satisfying $\alpha \in C \Rightarrow C_\alpha = C \cap \alpha$ *then* for some $\alpha \in S_* \cap (\beth_{\chi_1}, \beth_\chi)$ we have $C_\alpha = C$. Why this extra induction hypothesis help? As arriving to $\alpha \in S$ so $\alpha = \beth_\alpha$ let $\langle \chi_i : i < \theta \rangle$ be an increasing sequence of cardinals with limit $\beth_\alpha = \alpha$ and we choose $\alpha_i \in (\beth_{\chi_i}, \beth_{2^{\chi_i}}) \cap S_*$ by induction on $i < \theta$ such that $C_{\alpha,i} = \{\alpha_j : j < i\}$ and the let $C_\chi = \{\alpha_i : i < \theta\}$.]

We shall prove that

- $\circledast_2$ if $\boxdot_2$ below holds, then there is a β such that $\odot_2$ holds *where*:

$\boxdot_2$
- (a) $\alpha < \lambda^+$, $i < \kappa$
- (b) f is a one-to-one function from M_α into $\mathscr{U}'_i = \bigcup\{\mathscr{U}_j : j < i\}$
- (c) $E \subseteq E_*$ is a club of λ such that $\delta \in E \Rightarrow f(M_\alpha) \restriction \delta \prec f(M_\alpha)$
- (d) $\overline{\mathscr{P}} = \langle \mathscr{P}_\delta : \delta \in S \rangle$
- (e) $\mathscr{P}_\delta \subseteq \mathscr{P}(\delta)$ and $\varnothing \in \mathscr{P}_\delta$ and $\mathscr{P}_\delta \subseteq \bigcup_{l \leq 2} \mathscr{P}^{*,l}_\delta$ where[2]
  - (α) $\mathscr{P}^{*,0}_\delta = \{A \subseteq \delta : \sup(A) = \delta \text{ and } A \subseteq \bigcup\{[\mu, 2^\mu) : \mu \in C_\delta\}\}$,
  - (β) $\mathscr{P}^{*,1}_\delta = \bigcup\{\mathscr{P}^{*,0}_{\delta_1} : \delta_1 \in S \cap \delta\}$,
  - (γ) $\mathscr{P}^{*,2}_\delta = \{A \subseteq \delta :$ for some $\delta_1 \in \lambda \backslash (\delta + 1)$ we have $A \subseteq \bigcup\{[\partial, 2^\partial) : \partial \in C_{\delta_1} \cap \delta\}\}$
- (f) if $\delta_1 < \delta_2$ are from S *then*
  - (α) $A \in \mathscr{P}_{\delta_1} \Rightarrow A \in \mathscr{P}_{\delta_2}$
  - (β) $A \in \mathscr{P}_{\delta_2} \Rightarrow A \cap \delta_1 \in \mathscr{P}_{\delta_1}$,
  - (γ) for any $\delta \in S$ the family $\mathscr{P}^{*,1}_\delta \cup \mathscr{P}^{*,2}_\delta$ is a set of bounded subsets of δ; (this follows)
- (g) $\bar{b}_{\delta,\mathscr{U}} \in M_\alpha$ for $\delta \in E \cap S$, $\mathscr{U} \in \mathscr{P}_\delta$ are such that $\bar{b}_{\delta_1,\mathscr{U}_1} = \bar{b}_{\delta_2,\mathscr{U}_2} \wedge \mathscr{U}_1 \in \mathscr{P}_{\delta_1} \wedge \mathscr{U}_2 \in \mathscr{P}_{\delta_2} \Rightarrow \delta_1 = \delta_2 \wedge \mathscr{U}_1 = \mathscr{U}_2$

$\odot_2$
- (α) $\beta \in (\alpha, \lambda^+)$
- (β) there are $\bar{a}_\gamma \subseteq M_\beta$ for $\gamma < \lambda$ such that for a club of $\delta \in E$, if $\delta \in S$ then the following conditions on $\mathscr{U} \subseteq \delta$ are equivalent:
  - (i) $\mathscr{U} \in \mathscr{P}_\delta$

---

[2]Note that $\mathscr{P}^{*,1}_{\delta,i}$, $\mathscr{P}^{*,2}_{\delta,i}$ are the families of sets we like to ignore as they are influenced by our choices for $\delta_1 \in S \backslash \{\delta\}$, so we work to have them families of bounded subsets of δ.

(ii) for some $\bar{b} \in {}^{lg(\bar{y})}M_\alpha$ we have: for every $\gamma < \delta$, $M_\beta \models \varphi[\bar{a}_\gamma, \bar{b}]$ iff $\gamma \in \mathscr{U}$

(iii) clause (ii) holds for $\bar{b} = \bar{b}_{\delta,\mathscr{U}}$ and $\mathscr{U} \in \mathscr{P}_\delta$

[Why? For each $\delta \in E \cap S$ let $\langle \mathscr{U}_{\delta,\varepsilon} : \varepsilon < |\mathscr{P}_\delta| \leq 2^{|\delta|}\rangle$ list $\mathscr{P}_\delta$ and let $\bar{b}_{\delta,\varepsilon} := \bar{b}_{\delta,\mathscr{U}_{\delta,\varepsilon}}$.

Let

$$\Gamma = \{\varphi(\bar{x}_\gamma, \bar{b}_{\delta,\varepsilon})^{\mathrm{if}(\gamma \in \mathscr{U}_{\delta,\varepsilon})} : \gamma < \lambda, \delta \in E \text{ and } \varepsilon < |\mathscr{P}_{\delta,i}|\}$$
$$\cup \{\neg\varphi(\bar{x}_\gamma, \bar{b}) : \gamma < \lambda,\ \bar{b} \in {}^{lg(\bar{y})}(M_\alpha) \text{ and for no } \delta \in E,\ \varepsilon < |\mathscr{P}_\delta| \text{ do we have } \bar{b} = \bar{b}_{\delta,\varepsilon}\}.$$

As $\varphi(\bar{x}, \bar{y})$ has the strong independence property, recalling that by clause (g) of $\boxdot_2$ the sequence $\langle \bar{b}_{\delta,\varepsilon} : \delta \in E \cap S$ and $\varepsilon < |\mathscr{P}_\delta|\rangle$ is with no repetitions, clearly Γ is finitely satisfiable in M_α, but M is λ^+-saturated, $M_\alpha \prec M$ and $|\Gamma| = \lambda$ hence we can find $\bar{a}_\gamma \in {}^{lg(\bar{x})}M$ for $\gamma < \lambda$ such that the assignment $\bar{x}_\gamma \mapsto \bar{a}_\gamma(\gamma < \lambda)$ satisfies Γ in M. Lastly, choose $\beta \in (\alpha, \lambda^+)$ such that $\{\bar{a}_\gamma : \gamma < \lambda\} \subseteq M_\beta$.

Now check recalling $\varnothing \in \mathscr{P}_\delta$ for $\delta \in S$.]

Note

$\odot_3$ in $\odot_2$ if h is a one-to-one mapping from M_β into $\mathscr{U}_i$ extending f *then* for some club E of λ if for every $\delta \in S \cap E$ we have $(\forall \gamma < \lambda)(\gamma < \delta \to h(\bar{a}_\gamma) \in {}^{lg(\bar{x})}\delta)$ and so for every $\mathscr{U} \subseteq \delta$ the conditions (i), (ii), (iii) from $\odot_2$ are equivalent.

Next we can choose $\bar{f}$ such that

$\circledast_3$ (a) $\bar{f} = \langle f_\alpha : \alpha < \lambda^+\rangle$

(b) f_α is a one-to-one function from M_α into $\mathscr{U}_{\mathrm{otp}(C^*_\alpha)}$

(c) if $\alpha \in C^*_\beta$ then $f_\alpha \subseteq f_\beta$.

Now

$\circledast_4$ for every $\alpha < \lambda^+$ there is $\overline{\mathscr{P}}^\alpha = \langle \mathscr{P}^\alpha_\varepsilon : \varepsilon \in S\rangle$ such that

(i) $\mathscr{P}^\alpha_\varepsilon \subseteq \mathscr{P}(\varepsilon)$ are as in $\boxdot_2$(e), (f) above

(ii) for every $\beta \leq \alpha$, for a club of δ's from S we have

$$\mathscr{P}^\alpha_\delta \notin \mathrm{inv}^\varphi_{7,\kappa}(\delta, f_\beta(M_\beta)).$$

[Why? For every $\beta \leq \alpha$ and $\delta \in (\kappa, \lambda)$ we have $\mathrm{inv}^\varphi_{7,\kappa}(\delta, f_\beta(M_\beta))$ is a subset of $\mathscr{P}(\mathscr{P}(\delta))$ of cardinality $\leq 2^{|\delta|}$. As the number of β's is $\leq \lambda$, by diagonalization we can do this: let $\alpha+1 = \bigcup_{\varepsilon<\lambda} u_\varepsilon$ and $u_\varepsilon \in [\alpha+1]^{<\lambda}$ increasing continuous for $\varepsilon < \lambda$; moreover, $\alpha < \lambda \Rightarrow u_\varepsilon = \alpha$ and $\alpha \geq \lambda \Rightarrow u_\varepsilon \cap \lambda \subseteq \varepsilon$ and $|u_\varepsilon| \leq |\varepsilon|$. By induction on $\varepsilon \in (\kappa, \lambda) \cap S$ choose $\mathscr{P}^\alpha_\varepsilon \subseteq \bigcup_{l<3} \mathscr{P}^{*,l}_{\alpha_\varepsilon}$ which includes $\bigcup\{\mathscr{P}^\alpha_\zeta : \zeta \in u_\varepsilon \cap S\} \cup \mathscr{P}^{*,2}_\varepsilon$ and satisfies $\mathscr{P}^{*,0}_\varepsilon \cap \mathscr{P}^\alpha_\varepsilon \in \mathscr{P}(\mathscr{P}^{*,0}_{\varepsilon,i})) \backslash \{\mathscr{P} \cap \mathscr{P}^{*,0}_\varepsilon : \mathscr{P} \in \mathrm{inv}^\varphi_{7,\kappa}(\varepsilon, f_\beta(M_\beta)), \beta \in u_\varepsilon\}$. Note that for each $\beta \leq \alpha$ the set $\{\varepsilon < \lambda : \beta \in u_\varepsilon\}$ contains an end-segment of λ hence a club of λ as required.]

Now choose pairwise distinct $\bar{b}_{\delta,\mathscr{U}} \in {}^{lg(\bar{y})}(M_0)$ for $\delta \in E_*$, $\mathscr{U} \in \mathscr{P}^{*,0}_\delta$

$\circledast_5$ for every $\alpha_* \leq \alpha < \lambda^+$ for some $\beta \in (\alpha, \lambda^+)$ and $\bar{a}_\gamma \in {}^{lg(\bar{x})}M_\beta$ for $\gamma < \lambda$ the condition in clause (γ) of $\odot_2$ holds with $\overline{\mathscr{P}}^{\alpha_*}$ here standing for $\overline{\mathscr{P}}$ there and the $\bar{b}_{\delta,\mathscr{U}}$ chosen above.

[Why? By $\circledast_2$.]

$\circledast_6$ let $E = \{\delta < \lambda^+ : \delta$ is a limit ordinal such that for every $\alpha_* \le \alpha < \delta$ there is $\beta < \delta$ as in $\circledast_5\}$.

Clearly E is a club of λ^+.

$\circledast_7$ if $\delta_1 < \delta_2$ are from $E \cap S^{\lambda^+}_\kappa$ then $M_{\delta_1}, M_{\delta_2}$ are not isomorphic.

[Why? Let $\alpha_* = \min(C^*_{\delta_2} \backslash \delta_1)$. We consider $\overline{\mathscr{P}}^{\alpha_*}$ which is from $\circledast_4$. On the one hand $\{\varepsilon < \lambda : \mathscr{P}^{\alpha_*}_\varepsilon \notin \mathrm{inv}^\varphi_{7,\kappa}(\varepsilon, f_{\delta_1}(M_{\delta_1}))\}$ contains a club by $\circledast_4$(ii). Note that $\langle f_\alpha : \alpha \in C^*_{\delta_2} \backslash \delta_1\rangle$ is $\subseteq$-increasing sequence of functions with union f_{δ_2}.

On the other hand choose an increasing $\langle \alpha_i : i < \kappa\rangle$ with limit δ_2 satisfying $\alpha_0 = 0$, $\alpha_1 = \delta_1$ such that $(\alpha_*, \alpha_{1+i}, \alpha_{1+i+1})$ are like $(\alpha_*, \alpha, \beta)$ in $\circledast_5$ for each $i < \kappa$ and $i \in (1, \kappa) \Rightarrow \alpha_i \in C^*_{\delta_2}$. Now by Claim 1.8, $\{\varepsilon < \lambda : \mathscr{P}^{\alpha_*}_\varepsilon \in \mathrm{inv}^\varphi_{7,\kappa}(\varepsilon, f_{\delta_2}(M_{\delta_2}))\}$ contains a club. Hence by the last sentence and the end of the previous paragraph $M_{\delta_1} \not\cong M_{\delta_2}$ as required.]

So we are done. $\square$

Remark 1.12. We can avoid using C^*_δ and also C_δ (e.g., using $A \in \mathscr{P}^{*,0}_\delta \Rightarrow \mathrm{otp}(A) = \delta$) but this seems less transparent.

Proof (Theorem 1.10). Similar but easier (for λ regular not strong limit (but $2^\lambda > 2^{<\lambda}$) also easy), or see the proof of Theorem 2.9. $\square$

2. Independent T

We would like to do something similar to Section 1, but our control on the relevant family of subsets of M is less tight. We control it to some extent by using the completion of a free Boolean algebra.

Context 2.1. T a complete first order theory, $\varphi(x, \bar{y})$ has the independence property (of course the existence of such φ follows from the strong independence property but is weaker).

We continue [6, 2.1 – 2.12], but we do not rely on it.

Definition 2.2. For a set I let

(a) $\mathbb{B} = \mathbb{B}_I$ be the Boolean algebra generated by $\langle e_t : t \in I\rangle$ freely,
(b) $\mathbb{B}^c_I$ is the completion of $\mathbb{B}$
(c) for $J \subseteq I$ let $\mathbb{B}^c_{I,J}$ be the complete subalgebra of $\mathbb{B}^c_I$ generated by $\{e_s : s \in J\}$
(d) let $\mathrm{uf}(\mathbb{B}^c_I)$ be the set of ultrafilters on I.

Claim 2.3. *Assume*

$\circledast$ (a) $M \models T$
(b) $\bar{b}_t \in {}^{lg(\bar{y})}M$ *for* $t \in I$
(c) $\langle \varphi(x, \bar{b}_t) : t \in I\rangle$ *is an independent sequence of formulas.*

Then *there is a function* F *from* ${}^{lg(\bar{y})}M$ *to* $\mathbb{B} = \mathbb{B}^c_I$ *such that*

(α) $F(\bar{b}_t) = e_t$
(β) *for every ultrafilter* D *of* $\mathbb{B}$ *there is* $p = p_D = p_{F,D} \in \mathbf{S}_\varphi(M)$, *in fact, a unique one, such that for every* $\bar{b} \in {}^{lg(\bar{y})}M$ *we have* $\varphi(x, \bar{b}) \in p \Leftrightarrow F(\bar{b}) \in D$.

Remark 2.4. (1) Note that the mapping $D \mapsto p_D$ is not necessarily one to one, but $D_1 \cap \{e_t : t \in I\} \neq D_2 \cap \{e_t : t \in I\} \Rightarrow p_{D_1} \neq p_{D_2}$.

(2) If $I = I_1 \cup I_2, I_1 \cap I_2 = \varnothing$ and $|I_2| = |I_1|^{\aleph_0}$ *then* we can find a mapping F from ${}^{lg(\bar{y})}M$ onto (not just into) $\mathbb{B} = \mathbb{B}^{\mathrm{c}}_{I_1}$ such that clause (α), (β) are satisfied.

Proof (Claim 2.3). Clearly $\mathscr{P}(M)$ is a Boolean algebra and $\{\varphi(M, \bar{b}_t) : t \in M\}$ generates freely a subalgebra of $\mathscr{P}(M)$ which we call $\mathbb{B}'$. So there is a homomorphism h from $\mathbb{B}'$ into $\mathbb{B}$ mapping $\varphi(M, \bar{b}_t)$ to e_t (moreover h is unique and is an isomorphism from $\mathbb{B}'$ onto $\mathbb{B}_I \subseteq \mathbb{B}^{\mathrm{c}}_I$). So h is a homomorphism from $\mathbb{B}' \subseteq \mathscr{P}(M)$ into $\mathbb{B}^{\mathrm{c}}_I$, which is a complete Boolean algebra hence there is a homomorphism h^+ from the Boolean algebra $\mathscr{P}(M)$ into $\mathbb{B}^{\mathrm{c}}$ extending h.

Lastly, define $F\colon {}^{lg(\bar{y})}M \to \mathbb{B}^{\mathrm{c}}$ by $F(\bar{b}) = h^+(\varphi(M, \bar{b}))$. Now check. $\square$

Conclusion 2.5. *Assume* $\circledast$ *from Claim* 2.3 *and*

$\boxdot$ (a) $I = \lambda$ *is regular uncountable*
(b) $|M| \subseteq \mathscr{U} \subseteq \lambda$
(c) D_α *is an ultrafilter of* $\mathbb{B}^{\mathrm{c}}_I$ *for* $\alpha < \lambda$
(d) $\mathscr{U} \backslash |M|$ *is unbounded in* λ.

Then *we can find* $\langle a_\alpha : \alpha < \lambda \rangle$ *and* N *such that*

(α) $M \prec N$
(β) $|N| \subseteq \mathscr{U}$
(γ) $a_\alpha \in N$ *for* $\alpha < \lambda$
(δ) a_α *realizes* $p_{D_\alpha} \in \mathbf{S}_\varphi(M)$.

Remark 2.6. Conclusion 2.5 is easy but intended to clarify how we shall use the ultrafilters, so is quoted toward the end of the section.

Proof (Conclusion 2.5). Should be clear. $\square$

Discussion 2.7. Note that compared to Section 1 instead $\bar{x}$, $\bar{y}$, $\bar{a}_\alpha$, $\bar{b}_\beta$ we have x, $\bar{y}$, a_α, $\bar{b}_\beta$. Compared to Section 1, we have less control over $\{\mathrm{tp}(a, M, N) : a \in N\}$. There, for the sequences $\bar{b}$ of M which are not among $\{\bar{b}_\gamma : \gamma < \lambda\}$, we can demand $N \models \neg\varphi[\bar{a}_\gamma, \bar{b}]$ for $\gamma < \lambda$ so $\mathrm{tp}_\varphi(\bar{a}_\gamma, M, N)$ can be clearly read. Here the complete Boolean algebra $\mathbb{B}^{\mathrm{c}}_I$ is helping, a small price is that we need $\theta > \aleph_0$.

In order to try to keep track of what is going on we shall use only $\mathrm{tp}(a_\gamma, M, N)$ of the form p_D for ultrafilter D on $\mathbb{B}^{\mathrm{c}}_I$. Further, we better have, e.g., a nice function π from ${}^\lambda 2$ to $\mathrm{uf}(\mathbb{B}^{\mathrm{c}}_I)$ such that $(e_\alpha \in \pi(\eta)) \Leftrightarrow \eta(\alpha) = 1$.

A possible approach is: we define $\langle M_{\eta,u} : \eta \in \mathscr{T} \subseteq \mathrm{des}(\lambda), u \in \mathscr{P}(n_\eta) \rangle$ as in [5, Section 3] and we define $D_\eta \in \mathrm{uf}(\mathbb{B}^{\mathrm{c}} \cap M)$ such that $\alpha \in M_\eta \cap \lambda \Rightarrow [e^{\eta(\alpha)}_\eta \in D_\eta]$ and $\bigcup_\eta \mathbb{D}_\eta \in \mathrm{uf}(\mathbb{B}^{\mathrm{c}})$.

We need some continuity so each "$e \in D_\eta$" ($e \in \mathbb{B}^{\mathrm{c}}$) depends on $\eta \restriction u_e$ for some "small" $u_e \subseteq \lambda$.

Theorem 2.8. *In Theorem* 1.9 *it suffices to assume* $\boxtimes'$ *which means clauses* (b), (c), (d) *of* $\boxtimes$ *and*

(a)$'$ T *has the independence property.*

Theorem 2.9. *In Theorem* 1.10 *it suffices to assume* $\boxtimes'$ *of Theorem* 2.8.

Proof (Theorem 2.8). Just combine the proofs of Theorem 1.9 from Section 1 and Theorem 2.9 below. $\square$

Proof (Theorem 2.9). As in the proof of Theorem 1.9 we can assume λ is strongly inaccessible though the proof is just easier otherwise. We let

$\circledast_1$ (a) $E_* = \{\delta < \lambda : \delta = \beth_\delta\}$, a club of λ

(b) $S_* = \{\beth_{\alpha+\omega} : \alpha < \lambda\}$

(c) choose a regular uncountable $\theta < \lambda$

and let

$\circledast_2$ (a) $S = \{\delta \in E_* : \mathrm{cf}(\delta) = \theta\} = S^\lambda_\theta \cap E_*$

(b) let $\overline{C}$ be as in $\circledast_1$ of the proof of Theorem 1.9, in particular $\overline{C} = \langle C_\alpha : \alpha < \lambda\rangle$, $C_\alpha \subseteq S_*$, $\mathrm{otp}(C_\alpha) \le \theta$, $\alpha \in C_\beta \Rightarrow C_\alpha = C_\beta \cap \alpha$ and $\alpha \in S \Leftrightarrow \alpha = \sup(C_\alpha) \Leftrightarrow \mathrm{otp}(C_\alpha) = \theta$ and $\alpha \in S \Rightarrow \sup(C_\alpha) = \alpha$

(c) for $\mu \in S$ let $A^*_\mu = \bigcup\{[\chi, 2^\chi] : \chi \in C_\mu\}$.

Let $\mathbb{D}_*$ be an ultrafilter of $\mathbb{B}^{\mathrm{c}}_\lambda$ such that $e_\alpha \notin \mathbb{D}_*$ for $\alpha < \lambda$.

Now for $\eta \in {}^\lambda 2$ we choose $\mathbb{D}_\eta$ such that

$\circledast_3$ (a) $\mathbb{D}_\eta$ is an ultrafilter of $\mathbb{B}^{\mathrm{c}}_\lambda$

(b) if $e \in \mathbb{D}_* \subseteq \mathbb{B}^{\mathrm{c}}_\lambda$ belongs to $\mathbb{B}^{\mathrm{c}}_{\lambda,\eta^{-1}\{0\}}$ (see Definition 2.2, the completion of the subalgebra of $\mathbb{B}^{\mathrm{c}}_\lambda$ generated by $\{e_\alpha : \eta(\alpha) = 0\}$) then $e \in \mathbb{D}_\eta$.

(c) if $\alpha < \lambda$ and $\eta(\alpha) = 1$ then $e_\alpha \in \mathbb{D}_\eta$.

So

$\circledast_4$ (a) if $\eta \in {}^\lambda 2$ is constantly zero then $\mathbb{D}_\eta = \mathbb{D}_*$

(b) $e_\alpha \in \mathbb{D}_\eta \Leftrightarrow \eta(\alpha) = 1$ for $\alpha < \lambda$, $\eta \in {}^\lambda 2$.

Now let $\bar\eta = \langle \eta_\varepsilon : \varepsilon < \lambda\rangle$ be a sequence of members of ${}^\lambda 2$ and below we shall be interested mainly in the case $\alpha = \mu \in S$.

Define

$\circledast_5$ for $e \in \mathbb{B}^{\mathrm{c}}_\lambda$ and $\alpha \le \lambda$ we let $Y^\alpha_{\bar\eta,e} := \{\varepsilon < \alpha : e \in \mathbb{D}_{\eta_\varepsilon}\}$

$\circledast_6$ $\mathscr{P}_{\bar\eta,\alpha} := \{Y^\alpha_{\bar\eta,e} : e \in \mathbb{B}^{\mathrm{c}}_\lambda\}$.

Now what can we say on $\mathscr{P}_{\bar\eta,\mu}$ for $\mu \in S$? As we can consider $e \in \{e_\alpha : \alpha \in [\mu, 2^\mu)\}$, clearly

$\circledast_7$ $\{\{\varepsilon < \mu : \eta_\varepsilon(\alpha) = 1\} : \alpha \in [\mu, 2^\mu)\} \subseteq \mathscr{P}_{\bar\eta,\mu} \subseteq \mathscr{P}(\mu)$.

This may be looked at as a "lower bound" of $\mathscr{P}_{\bar\eta,\mu}$. Naturally we try to get also an "upper bound" to $\mathscr{P}_{\bar\eta,\mu}$; now note

$\circledast_8$ if $e \in \mathbb{B}^{\mathrm{c}}_\lambda$ then $Y^\mu_{\bar\eta,-e} = \mu \backslash Y^\mu_{\bar\eta,e}$.

Now define (recalling A^*_μ is from $\circledast_2$(c))

$\circledast_9$ Ξ is the set of $\bar\eta$ of the form $\langle \eta_\varepsilon : \varepsilon < \lambda\rangle$ such that:

(a) $\eta_\varepsilon \in {}^\lambda 2$ for every $\varepsilon < \lambda$,

(b) if $\eta_\varepsilon(\alpha) = 1$ *then* $(\exists \mu \in S)[\mu \le \alpha < 2^\mu \wedge \varepsilon \in A^*_\mu]$,

(c) if $\mu \in S$ and $u \subseteq [\mu, 2^\mu)$ is countable *then* $\{\varepsilon \in A^*_\mu$: if $\alpha \in u$ then $\eta_\varepsilon(\alpha) = 0\}$ is of cardinality μ.

Also (by our knowledge of the completion of a free Boolean algebra, $\mathbb{B}^{\mathrm{c}}_\lambda$ satisfies the c.c.c.) for every $e \in \mathbb{B}^{\mathrm{c}}_\lambda$ we can choose u_e such that:

$\boxplus_1$ (a) $u_e \subseteq \lambda$ is countable

(b) $e \in \mathbb{B}^{\mathrm{c}}_{\lambda,u_e}$.

So by clause (b) of $\circledast_3$ clearly

$\boxplus_2$ if $\bar\eta \in \Xi$, $e \in \mathbb{B}^{\mathrm{c}}_\lambda$, $\varepsilon < \mu \in S$ and $u_e \subseteq \eta_\varepsilon^{-1}\{0\}$ *then* $e \in \mathbb{D}_{\eta_\varepsilon} \Leftrightarrow e \in \mathbb{D}_*$

hence

$\boxplus_3$ if $\bar\eta \in \Xi$, $e \in \mathbb{B}^{\mathrm{c}}_\lambda \cap \mathbb{D}_*$ and $\mu \in S$ then $Y^\mu_{\bar\eta,e} \supseteq \{\varepsilon < \mu : u_e \subseteq \eta_\varepsilon^{-1}\{0\}\}$.

Next

$\boxplus_4$ for $\bar\eta \in \Xi$

(a) let $\mathscr{D}_{\bar\eta,\mu}$ be the filter on μ generated by $\{A^*_\mu\} \cup \{\{\varepsilon < \mu : u \subseteq \eta^{-1}_\varepsilon\{0\}$ and $\varepsilon > \zeta\} : \zeta < \mu$ and $u \subseteq \lambda$ is countable$\}$

(b) let $\mathscr{I}_{\bar\eta,\mu}$ be the dual ideal.

Clearly

$\boxplus_5$ (a) if $\bar\eta \in \Xi$, $\mu \in S$ and $\alpha \in \lambda \backslash [\mu, 2^\mu)$ then $\{\varepsilon < \mu : \eta_\varepsilon(\alpha) \neq 0\}$ is a bounded subset of μ,

(b) if $\bar\eta \in \Xi$ and $\mu \in S$ then $\mathscr{D}_{\bar\eta,\mu}$ is a uniform $\aleph_1$-complete filter on μ (recalling $\mathrm{cf}(\mu) = \theta > \aleph_0$ as $\mu \in S$) and $\varnothing \notin \mathscr{D}_{\bar\eta,\mu}$.

[Why? See $\circledast_9$.]

Now by $\boxplus_3$ we have $\bar\eta \in \Xi \wedge e \in \mathbb{B}^{\mathrm{c}}_\lambda \cap \mathbb{D}_* \Rightarrow Y^\mu_{\bar\eta,e} \in \mathscr{D}_{\bar\eta,\mu}$ so recalling $\circledast_8$ we have $e \in \mathbb{B}^{\mathrm{c}}_\lambda \backslash \mathbb{D}_* \Rightarrow Y^\mu_{\bar\eta,e} = \varnothing \bmod \mathscr{D}_{\bar\eta,\mu}$ hence

$\boxplus_6$ $\mathscr{P}_{\bar\eta,\mu} \subseteq \{X \subseteq \mu : X \in \mathscr{D}_{\bar\eta,\mu}$ or $\mu \backslash X \in \mathscr{D}_{\bar\eta,\mu}\}$.

Now

$\odot_1$ if $\mu \in S$ then we can find $\bar A^\xi_\mu$ for $\xi < 2^{2^\mu}$ such that:

(a) $\bar A^\xi_\mu = \langle A^\xi_\gamma : \gamma \in [\mu, 2^\mu)\rangle$

(b) A^ξ_γ is an unbounded subset of A^*_μ

(c) $\mathscr{D}^\xi_\mu, \mathscr{I}^\xi_\mu$ are well defined, i.e. $\varnothing \in \mathscr{D}^\xi_\mu$ when we let

(α) $\mathscr{D}^\xi_\mu$ be the $\aleph_1$-complete filter of subsets of μ generated by $\{A^\xi_\gamma \backslash \beta : \gamma \in [\mu, 2^\mu)$ and $\beta < \mu\}$ so $A^*_\mu \backslash \beta \in \mathscr{D}^\xi_\mu$ for $\beta < \mu$

(β) $\mathscr{I}^\xi_\mu = \{\mu \backslash B : B \in \mathscr{D}^\xi_\mu\}$, i.e., the dual ideal

(d) moreover if $\xi^1 \neq \xi^2$ are $< 2^{2^\mu}$, then

$$\{A^*_\mu \backslash A^{\xi^1}_\gamma : \gamma \in [\mu, 2^\mu)\} \nsubseteq \mathscr{D}^{\xi^2}_\mu \cup \mathscr{I}^{\xi^2}_\mu.$$

[Why $\odot_1$ holds? As $|A^*_\mu| = |\mu|$ is a strong limit cardinal of cofinality $\theta > \aleph_0$ clearly $\mu = |A^*_\mu| = |A^*_\mu|^{\aleph_0}$ hence by [2] there is a sequence $\langle B_\gamma : \gamma \in [\mu, 2^\mu)\rangle$ of subsets of A^*_μ such that any nontrivial Boolean combination of countably many of them has cardinality μ. Let $\langle U_\xi : \xi < 2^{2^\mu}\rangle$ be a sequence of pairwise distinct subsets of $[\mu, 2^\mu)$ each of cardinality $2^{|\mu|}$ no one included in another and let $\langle A^\xi_\gamma : \gamma \in [\mu, 2^{|\mu|})\rangle$ list $\{B_\gamma : \gamma \in U_\xi\}$.

Now check.]

$\odot_2$ in $\odot_1$ it follows that

(e) for every $\mathscr{P} \subseteq \mathscr{P}(\mu)$ for at most one $\xi < 2^{2^\mu}$ we have

$$\{A^*_\mu \backslash A^\xi_\gamma : \gamma \in [\mu, 2^\mu)\} \subseteq \mathscr{P} \subseteq \mathscr{D}^\xi_\mu \cup \mathscr{I}^\xi_\mu.$$

$\odot_3$ for every $\bar\xi = \langle \xi(\mu) : \mu \in S\rangle \in \Pi\{2^{2^\mu} : \mu \in S\}$ there is $\bar\eta = \bar\eta_{\bar\xi}$ such that:

(a) $\bar\eta_{\bar\xi} \in \Xi$ so $\bar\eta_{\bar\xi} = \langle \eta_{\bar\xi,\varepsilon} : \varepsilon < \lambda\rangle$

(b) if $\mu \in S$, $\gamma \in [\mu, 2^\mu)$ then $\{\varepsilon \in \lambda : \eta_{\bar\xi,\varepsilon}(\gamma) = 1\} = A^*_\mu \backslash A^{\xi(\mu)}_\gamma$.

[Why? Just read the definition of Ξ in $\circledast_9$ and $\bar A^\xi_\mu$ in $\odot_1$.]

$\odot_4$ if $\mu \in S$ then $\mathscr{D}_{\bar\eta_{\bar\xi},\mu} \cup \mathscr{I}_{\bar\eta_{\bar\xi},\mu} = \mathscr{D}^{\xi(\mu)}_\mu \cup \mathscr{I}^{\xi(\mu)}_\mu$.

[Why? Easy, recalling $\boxplus_5$(a).]

$\odot_5$ if $\gamma(*) < \lambda^+$ and $\bar{\mathbf{P}}^\gamma = \langle \mathbf{P}^\gamma_\mu : \mu \in S\rangle$, for $\gamma < \gamma(*)$ where $\mathbf{P}^\gamma_\mu \subseteq \mathscr{P}(\mathscr{P}(\mu))$ has cardinality $\leq 2^\mu$ for $\mu \in S$, $\gamma < \gamma(*)$ *then* we can find $\bar\xi = \langle \xi(\mu) :$

$\mu \in S\rangle \in \Pi\{2^{2^\mu} : \mu \in S\}$ such that for every $\gamma < \gamma(*)$ the following set is not stationary: $S_{\bar\eta,\gamma} = \{\mu \in S$: for some $\mathscr{P} \in \mathbf{P}^\gamma_\mu$ we have $\{A^{\xi(\mu)}_\gamma : \gamma \in [\mu, 2^\mu)\} \subseteq \mathscr{P} \subseteq \mathscr{D}_{\bar\eta,\mu} \cup \mathscr{I}_{\bar\eta,\mu}\}$.

[Why? Let $\langle u_\alpha : \alpha < \lambda\rangle$ be an increasing continuous sequence of subsets of $\gamma(*)$ with union $\gamma(*)$ such that $|u_\alpha| \leq |\alpha|$ for $\alpha < \lambda$. Now for each $\mu \in S$, the family $\bigcup\{\mathbf{P}^\gamma_\mu : \gamma \in u_\mu\}$ is a family of $\leq |u_\mu| \times 2^\mu$ subsets of $\mathscr{P}(\mu)$.

Now by clause (e) of $\odot_1$ for each $\mu \in S$, $\gamma \in u_\mu$, $\mathscr{P} \in \mathbf{P}^\gamma_\mu$ let $\xi_{\mu,\gamma,\mathscr{P}} < 2^{2^\mu}$ be such that: if for some $\xi < 2^{2^\mu}$ we have $\{A^\xi_\gamma : \gamma \in [\mu, 2^\mu)\} \subseteq \mathscr{P} \subseteq \mathscr{D}^\xi_\mu \cup \mathscr{I}^\xi_\mu$ then $\xi_{\mu,\gamma,\mathscr{P}}$ is the first such ξ. Choose $\xi(\mu) < 2^{2^\mu}$ which does not belong to $\{\xi_{\mu,\gamma,\mathscr{P}} : \gamma \in u_\mu$ and $\mathscr{P} \in \mathbf{P}^\gamma_\mu\}$.

So let $\bar\eta = \bar\eta_{\langle\xi(\mu):\mu\in S\rangle} \in \Xi$ be as in $\odot_3$ now $\bar\eta$ is as required by $\odot_2, \odot_3, \odot_4$.

Let us elaborate, why is it as required in $\odot_5$?

First, clearly $\eta_\varepsilon \in {}^\lambda 2$ for $\varepsilon < \lambda$. Second, fix $\gamma < \gamma(*)$, then there is $\alpha < \lambda$ such that $\gamma \in u_\alpha$, so it suffices to show that, for any $\mu \in S\backslash\alpha$, we have $\mu \notin S_{\bar\eta,\gamma}$. So assume $\mathscr{P} \in \mathbf{P}^\gamma_\mu$ satisfies clause (e) of $\odot_1$, and we should prove that $\neg[\{A^{\xi(\mu)}_\gamma : \gamma \in [\mu, 2^\mu)\} \subseteq \mathscr{P} \subseteq \mathscr{D}_{\bar\eta,\mu} \cup \mathscr{I}_{\bar\eta,\mu}]$; but if for some $\xi < 2^{2^\mu}$ we have $\{A^\xi_\gamma : \gamma \in [\mu, 2^\mu)\} \subseteq \mathscr{P} \subseteq \mathscr{D}_{\bar\eta,\mu} \cup \mathscr{I}_{\bar\eta,\mu}$ then necessarily $\xi = \xi_{\mu,\gamma,\mathscr{P}} \neq \xi(\mu)$, contradiction to $\odot_1$.]

$\odot_6$ if $\langle M_\gamma : \gamma \leq \gamma(*)\rangle$ is a $\prec$-increasing continuous and $M_\gamma \in \mathrm{EC}_\lambda(T)$ and $\bar b_\alpha \in {}^{\ell g(\bar y)}(M_0)$ for $\alpha < \lambda$ are such that $\langle\varphi(x, \bar b_\alpha) : \alpha < \lambda\rangle$ is independent, *then* we can find N such that
 (α) $M_{\gamma(*)} \prec N \in \mathrm{EC}_\lambda(T)$
 (β) if $N \prec N' \in \mathrm{EC}_\lambda(T)$ and $\gamma < \gamma(*)$ *then*[3] we have
$$\mathrm{inv}^\varphi_6(M_\gamma, N') \notin \{\mathrm{inv}^\varphi_6(M_{\gamma_1}, M_{\gamma_2}) : \gamma \leq \gamma(*) \text{ and } \gamma_1 < \gamma_2 \leq \gamma(*)\}.$$

[Why? Without loss of generality the universe of $M_{\gamma(*)}$ is $\mathscr{U}_1 \in [\lambda]^\lambda$ such that $\lambda\backslash\mathscr{U}_1$ has cardinality λ. Let $\langle u_\alpha : \alpha < \lambda\rangle$ be as in the proof of $\odot_5$.

For $\delta \leq \lambda$ and $\gamma(1) < \gamma(2) \leq \gamma(*)$ let $\mathbf{P}^{\gamma(1),\gamma(2)}_\delta = \mathrm{inv}^5_\varphi(\delta, \mathrm{id}_{N_{\gamma(2)}}, M_{\gamma(1)}, M_{\gamma(2)})$, see Definition 1.4, clearly $\mathrm{inv}^\varphi_6(M_{\gamma(1)}, M_{\gamma(2)}) = \langle\mathbf{P}^{\gamma(1),\gamma(2)}_\delta : \delta < \lambda\rangle/\mathscr{D}_\lambda$. So it is enough[4] to find N and sequence $\langle a_\alpha : \alpha < \lambda\rangle$ of elements of N such that $M_{\gamma(*)} \prec N$, $|N| = \lambda$ and for each $\gamma(0) \leq \gamma(*)$, for every $\mu \in S$ except nonstationarily many, the family
$$\{\{\gamma < \mu : N \models \varphi[a_\gamma, \bar b]\} : \bar b \in {}^{\ell g(\bar y)}(M_{\gamma(0)})\}$$
is not in $\mathbf{P}_\mu := \bigcup\{\mathbf{P}^{\gamma(1),\gamma(2)}_\mu : \gamma(1) < \gamma(2) \leq \gamma(*)$ are from $u_\mu\}$.

We choose $\bar\xi = \langle\xi(\mu) : \mu \in S\rangle$ as in $\odot_5$; let $\bar\eta = \bar\eta_{\bar\xi}$, see $\odot_3$, so recalling $\circledast_3$ clearly $\langle\mathbb{D}_{\eta_\varepsilon} : \varepsilon < \lambda\rangle$ is well defined. Now for each $\varepsilon < \alpha$ letting F be from Claim 2.3 for the model $M_{\gamma(*)}$ and the sequence $\langle\varphi(x, \bar b_\alpha) : \alpha < \lambda\rangle$, let $p_\varepsilon \in \mathbf{S}_\varphi(M_{\gamma(*)})$ be such that for every $\bar b \in {}^{\ell g(\bar y)}(M_{\gamma(*)})$ we have $\varphi(x, \bar b) \in p_\varepsilon \Leftrightarrow F(\bar b) \in \mathbb{D}_{\eta_\varepsilon}$ so $\neg\varphi(x, \bar b) \in p_\varepsilon \Leftrightarrow F(\bar b) \notin \mathbb{D}_{\eta_\varepsilon}$.

So by Conclusion 2.5 we can find an elementary extension N of $M_{\gamma(*)}$ and $a_\alpha \in N$ for $\alpha < \lambda$ such that a_α realizes p_α, and without loss of generality N has universe $\subseteq \lambda$ such that $\lambda\backslash|N|$ has cardinality λ. Concerning inv^6_φ our demand concerns what occurs for a club of $\delta < \lambda$ for this. Let $E \subseteq E_*$ be a club of λ such

[3]Really any pregiven set of $\leq \lambda$ "forbidden" inv^φ_6 is O.K. and can make it work for $\mathrm{inv}^\varphi_6(N_\gamma, N')$ for every $\gamma < \gamma(*)$.

[4]Can demand $\alpha < \lambda \Rightarrow {}^{\omega>}(\omega(\alpha+1))$ if $(N\backslash M_{\gamma(*)}) \cap [\omega\alpha, \omega\alpha+\omega)$ is infinite for every $\alpha < \lambda$.

that $\gamma < \delta \in E \Rightarrow a_\gamma \in N \cap \delta$. Now in $\odot_6(\beta)$ we promise something (given N) on "every N' such that...," so let $N \prec N' \in \mathrm{EC}_\lambda(T)$, and without loss of generality the universe of N' is $\subseteq \lambda$ and let $\delta \in S \cap E$. For any $\gamma \le \gamma(*)$ by $\odot_5$, i.e., by the choice of $\bar{\xi}, \bar{\eta}_\xi$ above there is a club $E_\gamma \subseteq E$ of λ such that for any $\mu \in S \cap E_\gamma$, the set $S_{\bar{\eta},\mu}$ from $\odot_5$ is disjoint to E_γ, hence the set $\mathscr{P}_{\mu,\gamma} := \{\{\gamma < \mu : N' \models \varphi[a_\gamma, \bar{b}]\} : \bar{b} \in {}^{lg(\bar{y})}(M_\gamma)\}$ does not belong to $\bigcup\{\mathbf{P}_\mu^{\gamma(1),\gamma(n)} : \gamma(1) < \gamma(2) \le \gamma(*)$ are from $u_\mu\}$ so we are done.]

$\odot_7$ if $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is as in $\boxtimes'$ from Theorem 2.8 then for some club E of λ^+, we have: if $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$ are from E and $\alpha_2 \ne \beta_2$ then

$$(M_{\alpha_2}, M_{\alpha_1}) \ncong (M_{\beta_2}, M_{\beta_1}).$$

[Why? For every $\beta < \lambda^+$ we apply $\odot_6$ to $\langle M_\alpha : \alpha \le \beta \rangle$ and get N_β as there so $M_\beta \prec N_\beta \in \mathrm{EC}_\lambda(T)$. As $M = \bigcup\{M_\gamma : \gamma < \lambda^+\}$ is saturated, without loss of generality $N_\beta \prec M$ hence for some $\xi_\beta < \lambda^+$ we have $N_\beta \prec M_{\xi_\beta}$.

Let $E = \{\delta < \lambda^+ : \delta$ a limit ordinal such that $\beta < \delta \Rightarrow \xi_\beta < \delta\}$.

Let $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$ be from E such that $\alpha_2 \ne \beta_2$ and we shall prove that $(M_{\alpha_2}, M_{\alpha_1})$ is not isomorphic to $(M_{\beta_2}, M_{\beta_1})$. By symmetry without loss of generality $\alpha_2 < \beta_2$ and let $\gamma(*) = \max\{\alpha_2, \beta_1\}$ so $\gamma(*) < \beta_2$. Now we apply $\odot_6$ with $\langle M_\gamma : \gamma \le \gamma(*) \rangle$, N, N', γ, β_1, α_2, α_2 here standing for $\langle M_\gamma : \gamma \le \gamma(*) \rangle$, N, N', $\gamma(0)$, $\gamma(1)$, $\gamma(2)$ there so we are clearly done. $\square$

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Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel and Department of Mathematics, Rutgers University, Hill Center — Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

E-mail address: `shelah@math.huji.ac.il`

URL: `http://shelah.logic.at`