ERRATA TO: FIRST ORDER THEORY OF PERMUTATION GROUPS†

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Lemmas 4.1, 4.4 (of [1]) are incorrect, hence 0.1 and 4.2, 4.4, 4.5 fall. We give here a correct version.

DEFINITION 1. (A) For any α there is a unique representation

$$\alpha = \Omega^{\omega}\alpha_{\omega} + \cdots + \Omega^{n}\alpha_{[n]} + \cdots + \Omega\alpha_{[1]} + \alpha_{[0]}; \alpha_{[n]} < \Omega.$$

Let

$$\alpha[n] = \Omega^{\omega}\alpha_{\omega} + \cdots + \Omega^{n+1}\alpha_{[n+1]}; \alpha^{[n]} = \begin{cases} 1 + \mathrm{cf}\,\alpha[n]; \mathrm{cf}\,\alpha[n] < \Omega \\ 0 \qquad \text{otherwise.} \end{cases}$$

(B) Define K^{15} by

$$M_{\alpha}^{15} = \langle U_{\alpha}, \alpha_{[0]}, \cdots, \alpha_{[n]}, \cdots, \alpha^{[0]}, \cdots, \alpha^{[n]}, \cdots, \cdots R_n(A_{\alpha}), \cdots; < \rangle$$

where $A_{\alpha} = U_{\alpha} \cup \bigcup_{n<\omega} \alpha_{[n]} \cup \bigcup_{n<\omega} \alpha^{[n]}$ in abuse of notation; this is a disjoint union.

Note that $|A_{\alpha}| \leqq 2^{\aleph_0}$.

(C) Define K^{16} by

$$M_{\alpha}^{16} = \langle \alpha + 1, U_{\alpha}, \cdots, R_n^{\Omega}((\alpha + 1) \cup U_{\alpha}), \cdots; < \rangle.$$

REMARK. In M_{α}^{15}, instead of one order sign $<$, we should have many: one for each $\alpha_{[n]}, \alpha^{[n]}$. Also we should have separate each $R_n(A_{\alpha})$ according to which place (in the relation) is designated for which domain.

Our main result is:

THEOREM 1. *K^1, K^{15}, K^{16} are bi-interpretable.*

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Hence instead of conclusion 0.1 we have:

COROLLARY 2. $\langle P_\alpha;\circ\rangle \equiv \langle P_\beta;\circ\rangle$ *iff*

$$\langle U_\alpha, \alpha_{[0]}, \alpha_{[1]}, \cdots, \alpha^{[0]}, \alpha^{[1]}, \cdots; <\rangle \equiv_{L_2} \langle U_\beta, \beta_{[0]}, \beta_{[1]}, \cdots, \beta^{[0]}, \beta^{[1]}, \cdots; <\rangle.$$

REMARK. A natural question is: in what is K^{15} better than K^7 or even K^1? A possible answer is

(1) Comparing the cardinals of the union of the domains, we get for K^1, $2^{\aleph_\alpha}$, for K^7 $\leq |\alpha| + 2^{\aleph_0}$, and for $K^{15} \leqq 2^{\aleph_0}$.

(2) In Corollary 2 the second part of the equivalence speaks on a well-known logic — L_2.

(3) In M_α^{15} much irrelevant information on α is thrown.

So for many α's we get isomorphic M_α^{15}'s hence it is clear that they have the same first-order theory of $\langle P_\alpha;\circ\rangle$.

LEMMA 3. K^1, K^{16} *are bi-interpretable.*

PROOF. Trivially K^{16}, K^7 are explicitly bi-interpretable. So by [1] 3.3 K^1, K^{16} are bi-interpretable.

LEMMA 4. K^{15} *is explicitly interpretable in* K^{16}.

PROOF. Clearly there are formulas in $L(K^{16})$ which define for $\beta \in \alpha + 1$; the following (in M_α^{16}):

$\beta = \alpha$, $|\beta| \geqq \Omega$, β is divisible by Ω, β is divisible by Ω^2 (i.e. $\{\gamma: \gamma < \beta, \gamma$ divisible by $\Omega\}$ has an order-type divisible by Ω), β is divisible by Ω^m;

$\gamma < \Omega$ is the cofinality of β; β is the maximal $\gamma \leqq \alpha$ which is divisible by Ω^n (for any fixed n).

From this the lemma is clear.

MAIN LEMMA 5. K^{16} *is interpretable in* K^{15}.

DEFINITION 2. For any ordinal i and set of ordinals I let $\gamma(i, I)$ = order type of $\{j \leqq i: (\forall \alpha \in I)(\alpha < i \rightarrow \alpha < j)\}$.

DEFINITION 3. A k-representation of $\langle \bar{a}, \bar{b}, \bar{r}\rangle = \langle a_1, \cdots, b_1, \cdots, r_1, \cdots\rangle$, where $a_i \in (\alpha + 1)$, $b_i \in U_\alpha$, $r_i \in R^\Omega_{n_i}((\alpha + 1) \cup U_\alpha)$ is a sequence,

$$\langle A, B, \bar{g}, <^*, \bar{a}', \bar{b}', \bar{r}'\rangle = \langle A, B, g^0, \cdots, g^k, g_0, \cdots, g_k, <^*, a'_1, \cdots, b'_1, \cdots, r'_1, \cdots\rangle$$

such that, for some function F:

(1) A, B are disjoint subsets of U_α [more exactly $A, B \in R_1(A_\alpha)$].

(2) $a'_i \in A$, $b'_i \in B$, $r'_i \in R_{n_i}$ $(A \cup B) \subseteq R_{n_i}((\alpha+1) \cup U_\alpha)$, $<^*$ a well ordering of U_α, the g_n, g^n's are one-place functions from A into U_α.

(3) F is one-to-one, with domain $A \cup B$ and range $\subseteq (\alpha+1) \cup U_\alpha$,

(4) the a_i, b_i's and α belong to the range of F, and U_α and the domains of the r_i's are included in it,

(5) $F(a'_i) = a_i$, $F(b'_i) = b_i$, F maps r'_i onto r_i, F maps A into $(\alpha+1)$, B onto U_α; and for $a, b \in A$, $a <^* b$ if $F(a) < F(b)$,

(6) for any $a \in A$,

order type of $\{c \in U_\alpha : c <^* g_l(a)\} = \gamma(F(a), F(A))_{[l]}$
order type of $\{c \in U_\alpha : c <^* g^l(a)\} = \gamma(F(a), F(a))^{[l]}$.

REMARK. The definition depends on α.

DEFINITION 4. $\alpha \sim_k \beta$ if $\alpha_{[l]} = \beta_{[l]}$, $\alpha^{[l]} = \beta^{[l]}$ for $l \leqq k$.

LEMMA 6. A) *$\sim_k$ is an equivalence relation between ordinals; for each α there is $\beta < \Omega^{k+2}$ such that $\alpha \sim_k \beta$; and if $\alpha \sim_k \beta$, then $\alpha < \Omega^{k+1} \Leftrightarrow \beta < \Omega^{k+1} \Rightarrow \alpha = \beta$.*

B) *If $\alpha_i \sim_k \beta_i$ for $i < \gamma$, then $\alpha =^{df} \sum_{i<\gamma} \alpha_i \sim_k \sum_{i<\gamma} \beta_i =^{df} \beta$.*

C) *If $\alpha \sim_{k+1} \beta$, $A \subset \alpha$, $|A| < \Omega$ then there is an order-preserving $F: A \to \beta$ such that for every $a \in A \cup \{\alpha\}$ $\gamma(a, A) \sim_k \gamma(F(a), F(A))$ (where we stipulate $F(\alpha) = \beta$).*

REMARK. This lemma is not new, in fact, see e.g. Kino [2].

PROOF. A) Trivial.

B) We prove by induction on γ.

(I) For $\gamma = 0, 1$ there is nothing to prove.

(II) For $\gamma + 1$, if $\alpha_\gamma \geqq \Omega^{k+1}$ then $\beta_\gamma \geqq \Omega^{k+1}$ (and vice versa) and then $\alpha \sim_k \alpha_\gamma$, $\beta \sim_k \beta_\gamma$ hence $\alpha \sim_k \beta$. So assume $\alpha_\gamma < \Omega^{k+1}$ so $\alpha_\gamma = \beta_\gamma$ and it is easy to check that $\alpha \sim_k \beta$.

(III) γ a limit ordinal.

We can assume each α_i, β_i is $\neq 0$, hence $\mathrm{cf}\,\alpha = \mathrm{cf}\,\beta$. If $\{i < \gamma : \alpha_i \geqq \Omega^{k+1}\}$ is unbounded, so is $\{i < \gamma : \beta_i \geqq \Omega^{k+1}\}$ hence α and β are divisible by Ω^{k+1}, together this implies $\alpha \sim_k \beta$. So we can assume that there is $i_0 < \gamma$ such that $i_0 \leqq i < \gamma \Rightarrow \alpha_i < \Omega^{k+1}$ hence $\alpha_i = \beta_i$. So $\sum_{\gamma > i \geqq i_0} \beta_i = \sum_{\gamma > i \geqq i_0} \alpha_i$ hence $\alpha = \sum_{i \leqq i_0} \alpha_i + (\sum_{\gamma > i > i_0} \alpha_i) \sim_k \sum_{i \leqq i_0} \beta_i + (\sum_{\gamma > i > i_0} \beta_i) = \beta$.

C) As $\alpha \sim_{k+1} \beta$, $\alpha = \alpha^1 + \xi, \beta = \beta^1 + \xi$, $\alpha_1 \beta_1$ are divisible by Ω^{k+2} and have equal cofinalities or cofinalities $\geqq \Omega$. It suffices to prove for the case $\xi = 0$, because we can define for $\alpha^1 \leqq i < \alpha$, $F(i) = \beta^1 + (i - \alpha^1)$. If $\lambda = \mathrm{cf}\,\alpha = \mathrm{cf}\,\beta < \Omega$, then $\alpha = \Sigma_{i<\lambda}\alpha_i$, $\beta = \Sigma_{i<\lambda}\beta_i$, each $\alpha_i\beta_i$ is divisible by Ω^{k+2} and has cofinality $\geqq \Omega$. We can now, for each $i < \lambda$, define F on

$$A \cap \{\xi: \sum_{j<i} \alpha_j \leqq \xi < \sum_{j\leqq i} \alpha_j\} \text{ into } \{\xi: \sum_{j<i} \beta_j \leqq \xi < \sum_{j\leqq i} \beta_j\}.$$

So we reduce the problem to the case $\mathrm{cf}\,\alpha$, $\mathrm{cf}\,\beta \geqq \Omega$. In this case define F inductively, so that for each $a \in A$, $\gamma(a, A) \sim_k \gamma(F(a), F(A))$ and $\gamma(F(a), F(A)) < \Omega^{k+2}$. As Ω is regular $|A| < \Omega$, this implies, by induction, that $F(a) < \Omega^{k+2} \leqq \beta$.

Claim 7. *In $L(K^{15})$ there are formulas ϕ_k such that $M_\alpha^{15} \models \phi_k[A, <^*, \bar{g}, b]$ iff $<^*$ is a well-ordering of U_α, $A \subseteq U_\alpha$,*

$\bar{g} = \langle g_0, \cdots, g_k, g^0, \cdots, g^k \rangle$,

g_l, g^l are one-place functions from A into U_α

and if $A = \{a_i: i < i_0\}$, $U_\alpha = \{b(j): j < j_0\}$, $b = b(j_1)$, $i<j \Rightarrow a_i <^* a_j$, $b_i <^* b_j$

and α_i is an ordinal and $b((\alpha_i)_{[l]}) = g_l(a_i)$, $b((\alpha_i)^{[l]}) = g^l(a_i)$ for each $i < i_0$, $l \leqq k$ then

$$\Sigma_{i<i_0}\alpha_i = j_1.$$

Remark. Remember 6B.

Proof. Just formalize what was said in the proof of Lemma 6B. As we have second order quantifiers, in fact, on U_α (in M_α^{15}) this is easy.

Claim 8. *For every kind of sequence $\langle a_1, \cdots, b_1, \cdots, r_1, \cdots \rangle$ (i.e. the number of a_i's, b_i's and r_i's; and the number of places of each r_l) for every k;*

(A) *there is a formula $\psi_k \in L(K^{15})$ such that for any sequence $\langle A, B, \bar{g}, <^*, \bar{a}', \bar{b}', \bar{r}' \rangle$ of the right kind (for being a suitable presentation)*

$M_\alpha^{15} \models \psi_k[A, B, \bar{g}, <^*, \bar{a}', \bar{b}', \bar{r}']$ iff $\langle A, B, \bar{g}, <^*, \bar{a}', \bar{b}', \bar{r} \rangle$ is a k-representation (in M_α^{15}).

(B) *Similarly there is a formula $\theta_k \in L(K^5)$ saying that two k-representations have a common source.*

Proof. (A) Just go through Definition 3 and see that it can be done with the help of Claim 7.

(B) Go through the representation with minimal A: that is let $A' = \{a'_1, \cdots,\}$ $\cup$ {last element of A} $\cup$ {the domain of the r_i's intersection with A}.

Lemma 9. *For each formula $\phi(\bar{x}, \bar{y}, \bar{z})$ in $L(K^{16})$ we can effectively find*

$k(\psi)$ and $\phi^* \in L(K^{15})$ such that for any α, and any $a_1, \cdots, \in \alpha + 1$, $b_1, \cdots, \in U_\alpha$, $r_i \in R^{\Omega}_{n_i}((\alpha + 1) \cup U_\alpha)$ and any $k(\psi)$-representation $\langle A, B, \bar{g}, <, \bar{a}', \bar{b}', \bar{r} \rangle$, $M^{15}_\alpha \models \phi^*[A, B, \bar{g}, <, \bar{a}', b', \bar{r}']$ iff $M^{16}_\alpha \models \phi[a_1, \cdots, b_1, \cdots, r_1, \cdots\}$.

REMARK. The proof is similar to 3.2, except the added parameter k.

PROOF. We prove it by induction of ϕ. For atomic formulas conjunction and negation there is no problem (remember that from k_1-representation we can get a k_2-representation, if $k_1 > k_2$, by omitting some g's). So we are left with the case of existential quantifiers. Now note that two k-representations may have a common source, but nevertheless not all their sources are common. But by Lemma 6(C) if two $(k + 1)$-representations have a common source, then any source of one is k-represented by the other after omitting the suitable g's. Now the proof should be clear.

PROOF OF LEMMA 5. By Lemma 9 it is immediate.

REFERENCES

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