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The nonstationary ideal on $P_\kappa(\lambda)$ for λ singular

Pierre Matet[1] · **Saharon Shelah**[2,3]

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Abstract We give a new characterization of the nonstationary ideal on $P_\kappa(\lambda)$ in the case when κ is a regular uncountable cardinal and λ a singular strong limit cardinal of cofinality at least κ.

Keywords $P_\kappa(\lambda)$ · Nonstationary ideal · Precipitous ideal

Mathematics Subject Classification 03E05 · 03E55

1 Introduction

Let κ be a regular uncountable cardinal and $\lambda \geq \kappa$ be a cardinal.
As [10] and [11] of which it is a continuation, this paper investigates ideals on $P_\kappa(\lambda)$ with some degree of normality. For $\delta \leq \lambda$, let $\mathrm{NS}^\delta_{\kappa,\lambda}$ denotes the least δ-normal ideal on $P_\kappa(\lambda)$. Thus $\mathrm{NS}^\delta_{\kappa,\lambda}$ = the noncofinal ideal $I_{\kappa,\lambda}$ for any $\delta < \kappa$, and $\mathrm{NS}^\lambda_{\kappa,\lambda}$ = the nonstationary ideal $\mathrm{NS}_{\kappa,\lambda}$. $\mathrm{NSS}_{\kappa,\lambda}$ denotes the least seminormal ideal on $P_\kappa(\lambda)$. It is

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✉ Pierre Matet
pierre.matet@unicaen.fr

Saharon Shelah
shelah@math.huji.ac.il

1 Laboratoire de Mathématiques, Université de Caen - CNRS, BP 5186, 14032 Caen Cedex, France

2 Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel

3 Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA

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simple to see that $\mathrm{NSS}_{\kappa,\lambda} = \mathrm{NS}_{\kappa,\lambda}$ in case $\mathrm{cf}(\lambda) < \kappa$. If λ is regular, then by a result of Abe [1], $\mathrm{NSS}_{\kappa,\lambda} = \bigcup_{\delta<\lambda} \mathrm{NS}^{\delta}_{\kappa,\lambda}$.

One problem we address in the paper is whether for $\lambda > \kappa$ $\mathrm{NS}_{\kappa,\lambda}$ is the restriction of a smaller ideal, i.e. whether $\mathrm{NS}_{\kappa,\lambda} = J|A$ for some ideal $J \subset \mathrm{NS}_{\kappa,\lambda}$ and some $A \in \mathrm{NS}^{*}_{\kappa,\lambda}$. The question as stated has a positive answer (see [2]) with $J = \nabla^{\lambda} I_{\kappa,\lambda}$. By a result of Abe [1] we can also take $J = \mathrm{NSS}_{\kappa,\lambda}$ in case $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. We investigate the possibility of taking $J = \bigcup_{\delta<\xi} \mathrm{NS}^{\delta}_{\kappa,\lambda}$ for some $\xi \leq \lambda$. If λ is regular, no such J will work since then, by an argument of [11], there is no A such that $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NSS}_{\kappa,\lambda} \mid A$.

Let $\mathcal{H}_{\kappa,\lambda}$ assert that $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}\right) \leq \lambda$ for every cardinal τ with $\kappa \leq \tau < \lambda$, where $\overline{\mathrm{cof}}\left(\mathrm{NS}^{\tau}_{\kappa,\lambda}\right)$ denotes the reduced cofinality of $\mathrm{NS}^{\tau}_{\kappa,\lambda}$. Clearly, $\mathcal{H}_{\kappa,\lambda}$ follows from $2^{<\lambda} = \lambda$. But there are other situations in which $\mathcal{H}_{\kappa,\lambda}$ holds. For instance, if in V, GCH holds, λ is a limit cardinal, χ is a regular uncountable cardinal less than κ, and $\mathbb{P}$ is the forcing notion to add λ^{+} Cohen subsets of χ, then in $V^{\mathbb{P}}$, $2^{\chi} > \lambda$ but, by results of [11], for every cardinal τ with $\kappa \leq \tau < \lambda$, $\mathrm{cof}\left(\mathrm{NS}_{\kappa,\tau}\right) = \tau^{+}$ and hence $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}\right) \leq \lambda$.

It is known [10,16] that if $\mathrm{cf}(\lambda) < \kappa$, then $\mathcal{H}_{\kappa,\lambda}$ holds just in case $\mathrm{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|A$ for some A. We will prove the following.

Theorem 1.1 *Suppose that* $\kappa \leq \mathrm{cf}(\lambda) < \lambda$ *and* $\mathcal{H}_{\kappa,\lambda}$ *holds. Then (a)* $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\mathrm{cf}(\lambda)}_{\kappa,\lambda}|A$ *for some* A*, but (b) there is no* B *such that* $\mathrm{NS}_{\kappa,\lambda} = \left(\bigcup_{\delta<\mathrm{cf}(\lambda)} \mathrm{NS}^{\delta}_{\kappa,\lambda}\right)|B$.

It is not known whether the converse holds:

Question Suppose that $\kappa \leq \mathrm{cf}(\lambda) < \lambda$ and $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\mathrm{cf}(\lambda)}_{\kappa,\lambda}|A$ for some A. Does it follow that $\mathcal{H}_{\kappa,\lambda}$ holds ?

If λ is singular and $\mathcal{H}_{\kappa,\lambda}$ holds, then by the results above $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\mathrm{cf}(\lambda)}_{\kappa,\lambda}|A$ for some A. The following problem is open.

Question Is it consistent that "λ is singular but $\mathrm{NS}_{\kappa,\lambda} \neq \mathrm{NS}^{\delta}_{\kappa,\lambda}|A$ for every $\delta < \lambda$ and every $A \in \mathrm{NS}^{*}_{\kappa,\lambda}$" ?

For any infinite cardinal $\tau < \lambda$, let $u(\tau,\lambda) =$ the least size of any cofinal subset of $(P_{\tau}(\lambda), \subset)$.

Now suppose $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. Then by results of [10], there is no A such that $\mathrm{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|A$. And it is shown in [11] that for any δ such that $\kappa \leq \delta < \mathrm{cf}(\lambda)$ and $u(|\delta|^{+},\lambda) = \lambda$, there is no A such that $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\delta}_{\kappa,\lambda}|A$. Thus assuming Shelah's Strong Hypothesis (SSH), $\mathrm{NS}_{\kappa,\lambda} \neq \mathrm{NS}^{\delta}_{\kappa,\lambda}|A$ for every $\delta < \mathrm{cf}(\lambda)$ and every $A \in \mathrm{NS}^{*}_{\kappa,\lambda}$.

Question Is it consistent relative to some large cardinal that "$\kappa < \mathrm{cf}(\lambda) < \lambda$ and $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^{\delta}_{\kappa,\lambda}|A$ for some $\delta < \mathrm{cf}(\lambda)$ and some $A \in \mathrm{NS}^{*}_{\kappa,\lambda}$"?

Another problem we consider is whether $\mathrm{NS}^{\delta}_{\kappa,\lambda}$ is nowhere precipitous, where $\delta \leq \lambda$. As shown by Matsubara and Shioya [14], $I_{\kappa,\lambda}$ is nowhere precipitous, and in fact so is any ideal J on $P_{\kappa}(\lambda)$ of cofinality $u(\kappa,\lambda)$. Thus for every ideal J on $P_{\kappa}(\lambda)$,

$$\overline{\mathrm{cof}}(J) \leq \lambda \Rightarrow \mathrm{cof}(J) = u(\kappa,\lambda) \Rightarrow J \text{ is nowhere precipitous.}$$

We establish the following.

Proposition 1.2 *Suppose that $\mathcal{H}_{\kappa,\lambda}$ holds, and let $\xi > \kappa$ be such that*

- *ξ is either a successor ordinal, or a limit ordinal of cofinality at least κ;*
- *$\xi \leq \eta$, where η equals $\lambda + 1$ if $\mathrm{cf}(\lambda) < \kappa$, and $\mathrm{cf}(\lambda)$ otherwise.*

Then $\overline{\mathrm{cof}}\left(\bigcup_{\delta<\xi} \mathrm{NS}^{\delta}_{\kappa,\lambda}\right) \leq \lambda$.

It follows from Theorem 1.1 and Proposition 1.2 that if $\mathcal{H}_{\kappa,\lambda}$ holds, then $\mathrm{NSS}_{\kappa,\lambda}|A = \mathrm{NS}^{\delta}_{\kappa,\lambda}|A$ for some $A \in \mathrm{NS}^{*}_{\kappa,\lambda}$, where δ equals $\mathrm{cf}(\lambda)$ if $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, and 0 otherwise.

Let us next consider cases when $\kappa \leq \delta \leq \lambda$ and $\mathrm{cof}\left(\mathrm{NS}^{\delta}_{\kappa,\lambda}\right) > u(\kappa, \lambda)$. Goldring [7] and the second author proved that if λ is regular and $\mu > \lambda$ is Woodin, then in $V^{\mathrm{Col}(\lambda,<\mu)}$ $\mathrm{NS}_{\kappa,\lambda}$ is precipitous. On the other hand Matsubara and the second author [13] showed[1] that if λ is a strong limit cardinal with $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, then $\mathrm{NS}_{\kappa,\lambda}$ is nowhere precipitous. We establish the following.

Theorem 1.3 *Let σ be a cardinal such that $\kappa \leq \mathrm{cf}(\lambda) \leq \sigma < \lambda$ Then the following hold:*

(i) *If $\sigma = \mathrm{cf}(\lambda)$ and $\tau^{\mathrm{cf}(\lambda)} < \lambda$ for every cardinal $\tau < \lambda$, then $\mathrm{NS}^{\sigma}_{\kappa,\lambda}$ is nowhere precipitous.*
(ii) *If $\mathrm{cf}(\lambda) < \sigma$ and $\tau^{c(\kappa,\sigma)} < \lambda$ for every cardinal $\tau < \lambda$, where $c(\kappa, \sigma)$ denotes the least size of any closed unbounded subset of $P_\kappa(\sigma)$, then $\mathrm{NS}^{\sigma}_{\kappa,\lambda}$ is nowhere precipitous.*

Note that if $\kappa \leq \mathrm{cf}(\lambda) \leq \sigma < \lambda$ and the hypothesis of (i) [respectively, (ii)] of Theorem 1.3. holds, then $\lambda^{<\mathrm{cf}(\lambda)} = \lambda$, so by results of [10],

$$\mathrm{cof}\left(\mathrm{NS}^{\sigma}_{\kappa,\lambda}\right) \geq \overline{\mathrm{cof}}\left(\mathrm{NS}^{\sigma}_{\kappa,\lambda}\right) > \lambda = u(\kappa, \lambda).$$

By combining Theorems 1.1 and 1.3, we obtain the following.

Theorem 1.4 *Suppose that $\mathcal{H}_{\kappa,\lambda}$ holds, $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, and $\tau^{\mathrm{cf}(\lambda)} < \lambda$ for every cardinal $\tau < \lambda$. Then $\mathrm{NS}_{\kappa,\lambda}$ is nowhere precipitous.*

It is not clear whether Theorem 1.4 constitutes a real improvement in comparison to the result of Matsubara and the second author quoted above.

Question Suppose that $\mathcal{H}_{\kappa,\lambda}$ holds, $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, and $\tau^{\mathrm{cf}(\lambda)} < \lambda$ for every cardinal $\tau < \lambda$. Does it then follow that λ is a strong limit cardinal ?

The paper is organized as follows. Section 2 collects basic definitions and facts concerning ideals on $P_\kappa(\lambda)$. It is shown in Sect. 3 that $\overline{\mathrm{cof}}\left(\mathrm{NS}^{\pi}_{\kappa,\lambda}\right)$ is a nondecreasing function of π. In Sect. 4 we establish that if λ is regular, then $\overline{\mathrm{cof}}\left(\mathrm{NSS}_{\kappa,\lambda}\right) = \lambda$ just

[1] At some point the first author claimed to have found an error in the proof but it turned out that the mistake was his.

in case $\mathcal{H}_{\kappa,\lambda}$ holds. In Sect. 5, Proposition 1.2 is proved. In Sect. 6 we show that it is consistent relative to a large cardinal that "λ is regular and $\overline{\text{cof}}\left(\text{NS}_{\kappa,\lambda}|A\right) < \lambda$ for some A". It is shown in Sect. 7 that if λ is singular and $\mathcal{H}_{\kappa,\lambda}$ holds, then $\text{NS}_{\kappa,\lambda} = \text{NS}_{\kappa,\lambda}^{\text{cf}(\lambda)}|A$ for some A. Finally in Sect. 8 we prove Theorem 1.3 and note that it is consistent relative to a large cardinal that "there is an ideal J on $P_\kappa(\lambda)$ such that $\overline{\text{cof}}(J) > \lambda$ but $\text{cof}(J) = u(\kappa, \lambda)$."

2 Ideals on $P_\kappa(\lambda)$

In this section we collect basic material concerning ideals on $P_\kappa(\lambda)$.

NS_κ denotes the nonstationary ideal on κ.

For a set A and a cardinal ρ, let $P_\rho(A) = \{a \subseteq A :| a |< \rho\}$.

Given four cardinals τ, ρ, χ and σ, we define $\text{cov}(\tau, \rho, \chi, \sigma)$ as follows. If there is $X \subseteq P_\rho(\tau)$ with the property that for any $a \in P_\chi(\tau)$, we may find $Q \in P_\sigma(X)$ with $a \subseteq \bigcup Q$, we let $\text{cov}(\tau, \rho, \chi, \sigma)$ = the least cardinality of any such X. Otherwise we let $\text{cov}(\tau, \rho, \chi, \sigma) = 0$.

We let $\text{cov}(\tau, \rho, \chi, \sigma) = u(\tau, \chi)$ in case $\rho = \chi$ and $\sigma = 2$.

Fact 2.1 [15, pp. 85–86] *Let τ, ρ, χ and σ be four cardinals such that $\tau \geq \rho \geq \chi \geq \omega$ and $\chi \geq \sigma \geq 2$. Then the following hold:*

(i) *If $\tau > \rho$, then* $\text{cov}(\tau, \rho, \chi, \sigma) \geq \tau$.
(ii) $\text{cov}(\tau, \rho, \chi, \sigma) = \text{cov}(\tau, \rho, \chi, \max\{\omega, \sigma\})$.
(iii) $\text{cov}(\tau^+, \rho, \chi, \sigma) = \max\{\tau^+, \text{cov}(\tau, \rho, \chi, \sigma)\}$.
(iv) *If $\tau > \rho$ and $\text{cf}(\tau) < \sigma = \text{cf}(\sigma)$, then* $\text{cov}(\tau, \rho, \chi, \sigma) = \sup\{\text{cov}(\tau', \rho, \chi, \sigma) : \rho \leq \tau' < \tau\}$.
(v) *If τ is a limit cardinal such that $\tau > \rho$ and $\text{cf}(\tau) \geq \chi$, then* $\text{cov}(\tau, \rho, \chi, \sigma) = \sup\{\text{cov}(\tau', \rho, \chi, \sigma) : \rho \leq \tau' < \tau\}$.

$I_{\kappa,\lambda}$ denotes the set of all $A \subseteq P_\kappa(\lambda)$ such that $\{a \in A : b \subseteq a\} = \emptyset$ for some $a \in P_\kappa(\lambda)$.

By an *ideal* on $P_\kappa(\lambda)$, we mean a collection J of subsets of $P_\kappa(\lambda)$ that is closed under subsets (i.e. $P(A) \subseteq J$ for all $A \in J$), κ-complete (i.e. $\bigcup X \in J$ for every $X \in P_\kappa(J)$), fine (i.e. $I_{\kappa,\lambda} \subseteq J$) and proper (i.e. $P_\kappa(\lambda) \notin J$).

Given an ideal J on $P_\kappa(\lambda)$, let $J^+ = \{A \subseteq P_\kappa(\lambda) : A \notin J\}$ and $J^* = \{A \subseteq P_\kappa(\lambda) : P_\kappa(\lambda) \setminus A \in J\}$. For $A \in J^+$, let $J|A = \{B \subseteq P_\kappa(\lambda) : B \cap A \in J\}$. Given a cardinal $\chi > \lambda$ and $f : P_\kappa(\lambda) \to P_\kappa(\chi)$, we let

$$f(J) = \left\{X \subseteq P_\kappa(\chi) : f^{-1}(X) \in J\right\}.$$

$\mathcal{M}_J$ denotes the collection of all maximal antichains in the partially ordered set $(J^+, \subseteq)$, i.e. of all $Q \subseteq J^+$ such that

- $A \cap B \in J$ for any distinct $A, B \in Q$;
- for every $C \in J^+$, there is $A \in Q$ with $A \cap C \in J^+$.

For a cardinal ρ, J is ρ-*saturated* if $|Q| < \rho$ for every $Q \in \mathcal{M}_J$.

$\mathrm{cof}(J)$ denotes the least cardinality of any $X \subseteq J$ such that $J = \bigcup_{A \in X} P(A)$. $\overline{\mathrm{cof}}(J)$ denotes the least size of any $Y \subseteq J$ with the property that for every $A \in J$, there is $y \in P_\kappa(Y)$ with $A \subseteq \bigcup y$.

$\mathrm{non}(J)$ denotes the least cardinality of any $A \in J^+$.

Note that $\mathrm{cof}(J) \geq \mathrm{non}(J) \geq \mathrm{non}(I_{\kappa,\lambda}) = u(\kappa, \lambda)$.

The following is well-known (see e.g. [10] and [11]).

Fact 2.2 (i) $\lambda^{<\kappa} = \max\{2^{<\kappa}, u(\kappa, \lambda)\}$.
(ii) $\overline{cof}(I_{\kappa,\lambda}) = \lambda$.
(iii) *Let J be an ideal on $P_\kappa(\lambda)$ such that* $\overline{\mathrm{cof}}(J) \leq \lambda$. *Then* $\mathrm{cof}(J) = u(\kappa, \lambda)$.

Shelah's Strong Hypothesis (SSH) asserts that for any two uncountable cardinals χ and ρ with $\chi \geq \rho = \mathrm{cf}(\rho)$, $u(\rho, \chi)$ equals χ if $\mathrm{cf}(\chi) \geq \rho$, and χ^+ otherwise.

Fact 2.3 [8]

(i) *Suppose that there is a π-saturated ideal on $P_\nu(\lambda)$, where π and ν are two cardinals such that $\omega < \nu = \mathrm{cf}(\nu) \leq \lambda$ and $\pi < \nu \cap \kappa^+$. Then $u(\kappa, \lambda)$ equals λ if $\mathrm{cf}(\lambda) \geq \kappa$, and λ^+ otherwise.*
(ii) *Suppose that there is a regular uncountable cardinal $\nu < \lambda$ that is mildly π^+-ineffable for every cardinal π with $\nu \leq \pi < \lambda$. Then the following hold:*
  - *$u(\kappa, \lambda)$ equals λ if $\mathrm{cf}(\lambda) \geq \kappa$, and λ^+ if $\omega < \mathrm{cf}(\lambda) < \kappa$.*
  - $\mathrm{cov}(\lambda, \kappa, \kappa, \omega_1) = \lambda$ *if* $\mathrm{cf}(\lambda) = \omega$.

Numerous variations on the original notion of ideal normality have been considered over the years. One such variant is the concept of δ-normality which has been studied by Abe [1].

Let $\delta \leq \lambda$. An ideal J on $P_\kappa(\lambda)$ is δ-*normal* if given $A \in J^+$ and $f : A \to \delta$ with the property that $f(a) \in a$ for all $a \in A$, there exists $B \in J^+ \cap P(A)$ such that f is constant on B.

$\mathrm{NS}^\delta_{\kappa,\lambda}$ denotes the smallest δ-normal ideal on $P_\kappa(\lambda)$.

Note that λ-normality is the same as normality, so $\mathrm{NS}^\lambda_{\kappa,\lambda} = \mathrm{NS}_{\kappa,\lambda}$.

$c(\kappa, \lambda)$ denotes the least size of any closed unbounded subset of $P_\kappa(\lambda)$.

Fact 2.4 (i) [1] *Let δ be an ordinal such that $\delta + \kappa \leq \lambda$. Then $\mathrm{NS}^{\delta+\kappa}_{\kappa,\lambda} \setminus \mathrm{NS}^\delta_{\kappa,\lambda} \neq \emptyset$.*
(ii) [11] *Suppose $\kappa \leq \delta < \lambda$. Then $\mathrm{NS}^\delta_{\kappa,\lambda} = \mathrm{NS}^{|\delta|}_{\kappa,\lambda} | A$ for some A.*
(iii) [11] *Let δ and η be two ordinals such that $|\delta| < |\eta| < \lambda$ and $\kappa \leq \eta$. Then there is no A such that $NS^\eta_{\kappa,\lambda} = \mathrm{NS}^\delta_{\kappa,\lambda} | A$.*

Fact 2.5 (i) [10] $\overline{\mathrm{cof}}\left(\mathrm{NS}^\delta_{\kappa,\lambda}\right) \geq \lambda$ *for every* $\delta \leq \lambda$.
(ii) [8,10] *Let $\delta \leq \lambda$. Then* $\overline{\mathrm{cof}}\left(\mathrm{NS}^\delta_{\kappa,\lambda} | A\right) = \overline{\mathrm{cof}}\left(\mathrm{NS}^\delta_{\kappa,\lambda}\right)$ *for every* $A \in \mathrm{NS}^*_{\kappa,\lambda}$.
(iii) [10] $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}\right) \geq \overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\rho}\right)$ *for every cardinal ρ with $\kappa \leq \rho < \lambda$.*
(iv) [10] *Suppose* $\mathrm{cf}(\lambda) \geq \kappa$. *Then* $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}\right) > \lambda$.

The concept of $[\delta]^{<\theta}$-normality generalizes that of δ-normality.

Let $\delta \leq \lambda$, and let θ be a cardinal with $\theta \leq \kappa$. An ideal J on $P_\kappa(\lambda)$ is $[\delta]^{<\theta}$-*normal* if given $A \in J^+$ and $f : A \to P_\theta(\delta)$ with the property that $f(a) \in P_{|a \cap \theta|}(a \cap \delta)$ for all $a \in A$, there exists $B \in J^+ \cap P(A)$ such that f is constant on B.

Note that for $\theta = \kappa$, $[\lambda]^{<\theta}$-normality is the same as strong normality.

We set $\overline{\theta} = \theta$ if $\theta < \kappa$, or $\theta = \kappa$ and κ is a limit cardinal, and $\overline{\theta} = \nu$ if $\theta = \kappa = \nu^+$.

Fact 2.6 [11]

(i) *Suppose that $\delta < \kappa$, or $\theta < \kappa$, or κ is not a limit cardinal. Then there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ if and only if $|P_{\overline{\theta}}(\rho)| < \kappa$ for every cardinal $\rho < \kappa \cap (\delta + 1)$.*

(ii) *Suppose that $\delta \geq \kappa, \theta = \kappa$ and κ is a limit cardinal. Then there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$ if and only if κ is a Mahlo cardinal.*

(iii) *Suppose that there exists a $[\kappa]^{<\theta}$- normal ideal on $P_\kappa(\lambda)$. Then $\kappa^{<\overline{\theta}} = \kappa$, and $(\pi^{<\overline{\theta}})^{<\overline{\theta}} = \pi^{<\overline{\theta}}$ for every cardinal $\pi > \kappa$.*

Assuming that there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$, $\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}$ denotes the smallest such ideal.

Fact 2.7 [11]

(i) *Suppose that $\theta < 2$ or $\delta < \kappa$. Then $\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} = I_{\kappa,\lambda}$.*

(ii) *Suppose that $2 \leq \theta \leq \omega$. Then $\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} = \mathrm{NS}_{\kappa,\lambda}^{\delta}$.*

(iii) *Suppose that $|\delta|^{<\overline{\theta}} = |\eta|^{<\overline{\pi}}$, where $\kappa \leq \eta \leq \lambda$ and π is a cardinal with $2 \leq \pi \leq \kappa$. Then $\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} | A = \mathrm{NS}_{\kappa,\lambda}^{[\eta]^{<\pi}} | A$ for some $A \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\gamma]^{<\rho}}\right)^*$, where $\gamma = \max\{\delta, \eta\}$ and $\rho = \max\{\theta, \pi\}$.*

Given an ordinal η, a cardinal π and $f : P_\pi(\eta) \to P_\kappa(\lambda)$, let $C(f, \kappa, \lambda)$ be the set of all $a \in P_\kappa(\lambda)$ such that $a \cap \pi \neq \emptyset$ and $f(e) \subseteq a$ for every $e \in P_{|a\cap\pi|}(a \cap \eta)$.

Fact 2.8 [11] *Suppose that $A \subseteq P_\kappa(\lambda)$, $\kappa \leq \delta \leq \lambda$, and θ is a cardinal with $2 \leq \theta \leq \kappa$. Then the following are equivalent:*

(i) $A \in \mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}$.

(ii) $A \cap C(f, \kappa, \lambda) = \emptyset$ *for some* $f : P_{\max\{\overline{\theta},3\}}(\delta) \to P_\kappa(\lambda)$.

(iii) $A \cap \{a \in C(g, \kappa, \lambda) : a \cap \kappa \in \kappa\} = \emptyset$ *for some* $g : P_{\max\{\overline{\theta},3\}}(\delta) \to P_3(\lambda)$.

Fact 2.9 [10] *Let χ and θ be two cardinals such that $2 \leq \theta \leq \kappa \leq \chi \leq \lambda$. Then the following hold:*

(i) *Let J be a $[\chi]^{<\theta}$- normal ideal on $P_\kappa(\lambda)$. Then either $\mathrm{cf}\left(\overline{\mathrm{cof}}(J)\right) < \kappa$, or $\mathrm{cf}\left(\overline{\mathrm{cof}}(J)\right) > \chi^{<\overline{\theta}}$.*

(ii) *If $\chi^{<\overline{\theta}} < \lambda$, then $\overline{\mathrm{cof}}\left(NS_{\kappa,\lambda}^{[\chi]^{<\theta}}\right) \geq \lambda$.*

Fact 2.10 [10,11] *Suppose that $\kappa \leq \delta < \lambda$, and θ is a cardinal with $2 \leq \theta \leq \kappa$. Then the following hold:*

(i) $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}\right) = \max\{\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,|\delta|}^{|\delta|^{<\theta}}\right), \mathrm{cov}\left(\lambda, (|\delta|^{<\overline{\theta}})^+, (|\delta|^{<\overline{\theta}})^+, \kappa\right)\}$ *and* $\mathrm{cof}\left(\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}\right) = \max\{\mathrm{cof}\left(\mathrm{NS}_{\kappa,|\delta|}^{|\delta|^{<\theta}}\right), \mathrm{cov}\left(\lambda, (|\delta|^{<\overline{\theta}})^+, (|\delta|^{<\overline{\theta}})^+, 2\right)\}$.

(ii) *If λ is a limit cardinal and either* $\mathrm{cf}(\lambda) < \kappa$ *or* $\mathrm{cf}(\lambda) > |\delta|^{<\overline{\theta}}$, *then* $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}\right) = \sup\{\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}^{[\delta]^{<\theta}}\right) : \delta < \tau < \lambda\}$.

For a cardinal τ, $\mathfrak{d}_{\kappa,\lambda}^{\tau}$ denotes the smallest cardinality of any family F of functions from τ to $P_\kappa(\lambda)$ with the property that for any $g : \tau \to P_\kappa(\lambda)$, there is $f \in F$ such that $g(\alpha) \subseteq f(\alpha)$ for every $\alpha < \tau$.

Fact 2.11 [11]

(i) *For any cardinal $\tau > 0$,* $\mathrm{cf}(\mathfrak{d}_{\kappa,\lambda}^{\tau}) > \tau$.
(ii) *Suppose that $0 < \delta \le \lambda$, and θ is a cardinal with $0 < \theta \le \kappa$. Then* $\mathrm{cof}\left(\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}} \mid A\right) = \mathfrak{d}_{\kappa,\lambda}^{|P_{\overline{\theta}}(\delta)|}$ *for every* $A \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}\right)^+$.

Next let us recall a few facts concerning the notion of precipitousness.

An ideal J on $P_\kappa(\lambda)$ is *precipitous* if whenever $A \in J^+$ and $\langle Q_n : n < \omega\rangle$ is a sequence of members of $\mathcal{M}_{J|A}$ such that $Q_{n+1} \subseteq \bigcup_{B \in Q_n} P(B)$ for all $n < \omega$, there exists $f \in \prod_{n\in\omega} Q_n$ such that $f(0) \supseteq f(1) \supseteq \ldots$ and $\bigcap_{n<\omega} f(n) \neq \emptyset$. J is *nowhere precipitous* if for each $A \in J^+$, $J|A$ is not precipitous.

Let $G(J)$ denote the following two-player game lasting ω moves, with player I making the first move: I and II alternately pick members of J^+, thus building a sequence $\langle X_n : n < \omega\rangle$, subject to the condition that $X_0 \supseteq X_1 \supseteq \ldots$. II wins $G(J)$ just in case $\bigcap_{n<\omega} X_n = \emptyset$.

Fact 2.12 [5] *An ideal J on $P_\kappa(\lambda)$ is nowhere precipitous if and only if* II *has a winning strategy in the game $G(J)$.*

The following is a straightforward generalization of a result of Foreman [4]:

Proposition 2.13 *Let χ and θ be two cardinals such that $\chi \le \lambda$ and $\theta \le \kappa$. Then every $[\chi]^{<\theta}$-normal, $(\chi^{<\overline{\theta}})^+$-saturated ideal on $P_\kappa(\lambda)$ is precipitous.*

Fact 2.14 [14] *Suppose that J is an ideal on $P_\kappa(\lambda)$ such that* $\mathrm{cof}(J) = non(J)$. *Then J is nowhere precipitous.*

Thus for an ideal J on $P_\kappa(\lambda)$,

$$\overline{\mathrm{cof}}(J) \le \lambda \Rightarrow \mathrm{cof}(J) = u(\kappa,\lambda) \Rightarrow J \quad \text{is nowhere precipitous.}$$

Let τ be a cardinal with $\kappa \le \tau \le \lambda$. It is simple to see that if GCH holds and either $\mathrm{cf}(\lambda) < \kappa$ or $\tau < \mathrm{cf}(\lambda)$, then $\mathrm{cof}\left(\mathrm{NS}_{\kappa,\lambda}^{\tau}\right) = u(\kappa,\lambda)$. Note that if SSH holds and $\kappa \le \mathrm{cf}(\lambda) \le \tau$, then by Facts 2.5(i) and 2.9, $\mathrm{cof}\left(\mathrm{NS}_{\kappa,\lambda}^{\tau}\right) > u(\kappa,\lambda)$.

Proposition 2.15 *Suppose that σ is a strong limit cardinal with* $\mathrm{cf}(\sigma) < \kappa < \sigma \le \lambda \le 2^\sigma$. *Then the following hold:*

(i) $\mathrm{cof}\left(\mathrm{NS}_{\kappa,\lambda}^{\tau}\right) = u(\kappa,\lambda)$ *for every cardinal τ with $\kappa \le \tau \le \sigma$.*

(ii) *Suppose* $2^\lambda = 2^\sigma$. *Then* $\operatorname{cof}\left(\mathrm{NS}^{\tau}_{\kappa,\lambda}\right) = u(\kappa, \lambda)$ *for every cardinal* τ *with* $\sigma < \tau \leq \lambda$.

Proof (i) Let τ be a cardinal with $\kappa \leq \tau \leq \sigma$. If $\tau = \lambda$, then

$$\operatorname{cof}\left(\mathrm{NS}^{\tau}_{\kappa,\lambda}\right) \leq 2^\lambda = \lambda^{<\kappa} = u(\kappa, \lambda).$$

Otherwise by Fact 2.10, $\operatorname{cof}\left(\mathrm{NS}^{\tau}_{\kappa,\lambda}\right) = \max\{\operatorname{cof}\left(\mathrm{NS}_{\kappa,\tau}\right), u(\tau^+, \lambda)\} \leq \lambda^\tau = \sigma^\tau = \sigma^{\mathrm{cf}(\sigma)} \leq \lambda^{<\kappa} = u(\kappa, \lambda)$.

(ii) Given a cardinal τ with $\sigma < \tau \leq \lambda$,

$$\operatorname{cof}\left(\mathrm{NS}^{\tau}_{\kappa,\lambda}\right) \leq 2^\lambda = 2^\sigma = \sigma^{\mathrm{cf}(\sigma)} = u(\kappa, \lambda). \qquad \square$$

3 $\overline{\operatorname{cof}}\left(\mathrm{NS}^{\chi}_{\kappa,\lambda}\right)$

By Fact 2.11(ii), $\operatorname{cof}\left(\mathrm{NS}^{\chi}_{\kappa,\lambda}\right) = \mathfrak{d}^{\chi}_{\kappa,\lambda}$ for any cardinal χ with $\kappa \leq \chi \leq \lambda$. We now derive a similar formula for $\overline{\operatorname{cof}}\left(\mathrm{NS}^{\chi}_{\kappa,\lambda}\right)$.

For a cardinal τ, $\overline{\mathfrak{d}}^{\tau}_{\kappa,\lambda}$ denotes the smallest cardinality of any family F of functions from τ to $P_\kappa(\lambda)$ with the property that for any $g : \tau \to P_\kappa(\lambda)$, there is $Z \in P_\kappa(F)$ such that $g(\alpha) \subseteq \bigcup_{f \in Z} f(\alpha)$ for every $\alpha < \tau$.

Lemma 3.1 *Let* θ *and* χ *be two cardinals such that* $2 \leq \theta \leq \kappa \leq \chi \leq \lambda$. *Then* $\overline{\operatorname{cof}}\left(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\lambda}\right) \leq \overline{\mathfrak{d}}^{\chi^{<\bar{\theta}}}_{\kappa,\lambda}$.

Proof Select a collection G of functions from $P_{\max\{\bar{\theta},3\}}(\chi)$ to $P_\kappa(\lambda)$ so that $|G| = \overline{\mathfrak{d}}^{\chi^{<\bar{\theta}}}_{\kappa,\lambda}$ and for any $k : P_{\max\{\bar{\theta},3\}}(\chi) \to P_\kappa(\lambda)$, there is $Z_k \in P_\kappa(G)$ such that $k(e) \subseteq \bigcup_{g \in Z_k} g(e)$ for all $e \in P_{\max\{\bar{\theta},3\}}(\chi)$. Then clearly for each $k : P_{\max\{\bar{\theta},3\}}(\chi) \to P_\kappa(\lambda)$, $\bigcap_{g \in Z_k} C(g, \kappa, \lambda) \subseteq C(k, \kappa, \lambda)$. Hence $\overline{\operatorname{cof}}\left(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\lambda}\right) \leq |G|$. $\square$

Lemma 3.2 *Let* θ *and* χ *be two cardinals such that* $\omega \leq \theta = \mathrm{cf}(\theta) < \kappa \leq \chi \leq \lambda$. *Then* $\overline{\mathfrak{d}}^{\chi^{<\theta}}_{\kappa,\lambda} \leq u\left(\theta, \overline{\operatorname{cof}}\left(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\lambda}\right)\right)$.

Proof Pick a collection H of functions from $P_\theta(\chi) \to P_3(\lambda)$ so that $|H| = \overline{\operatorname{cof}}\left(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\lambda}\right)$ and for any $A \in (NS^{[\chi]^{<\theta}}_{\kappa,\lambda})^*$, there is $Q \in P_\kappa(H) \setminus \{\emptyset\}$ with $\left\{b \in \bigcap_{h \in Q} C(h, \kappa, \lambda) : b \cap \kappa \in \kappa\right\} \subseteq A$. Select $\mathfrak{X} \subseteq P_\theta(H) \setminus \{\emptyset\}$ so that $|\mathfrak{X}| = u(\theta, |H|)$ and for any $Z \in P_\theta(H)$, there is $X \in \mathfrak{X}$ with $Z \subseteq X$. For $X \in \mathfrak{X}$, define $t_X : P_\theta(\chi) \to P_\kappa(\lambda)$ by $t_X(e) = \bigcap T_{X,e}$, where

$$T_{X,e} = \left\{b \in \bigcap\nolimits_{h \in X} C(h, \kappa, \lambda) : e \cup \theta \subseteq b \text{ and } b \cap \kappa \in \kappa\right\}.$$

Note that $t_X(e) \in T_{X,e}$, and $t_Y(e) \subseteq t_X(e)$ for all $Y \in \mathfrak{X} \cap P(X)$.

Now fix $f : P_\theta(\chi) \to P_\kappa(\lambda)$. We may find $W \in P_\kappa(\mathfrak{X})$ such that

$$\left\{b \in \bigcap\nolimits_{h\in\bigcup W} C(h,\kappa,\lambda) : b \cap \kappa \in \kappa\right\} \subseteq C(f,\kappa,\lambda),$$

$\theta \leq |W|$ and for any $K \in P_\theta(W)$, there is $Z \in W$ with $\bigcup K \subseteq Z$. For $e \in P_\theta(\chi)$, put $b_e = \bigcup_{X\in W} t_X(e)$. Note that $b_e \cap \kappa \in \kappa$.

Claim *Let $k \in \bigcup W$. Then $b_e \in C(k,\kappa,\lambda)$.*

Proof of Claim Fix $d \in P_\theta(b_e \cap \chi)$. Pick $\varphi : d \to W$ so that $\beta \in t_{\varphi(\beta)}(e)$ for every $\beta \in d$. Select $Y \in W$ with $k \in Y$. There must be $Z \in W$ such that $Y \cup \left(\bigcup_{\beta\in d} \varphi(\beta)\right) \subseteq Z$. Then $d \in P_\theta(t_Z(e))$ and $t_Z(e) \in C(k,\kappa,\lambda)$, so $k(d) \subseteq t_Z(e) \subseteq b_e$. This completes the proof of the claim. □

Thus $b_e \in \bigcap_{h\in\cup W} C(h,\kappa,\lambda)$. Hence $b_e \in C(f,\kappa,\lambda)$, and consequently $f(e) \subseteq b_e$. □

Proposition 3.3 *Let χ be a cardinal with $\kappa \leq \chi \leq \lambda$. Then* $\overline{\mathrm{cof}}\left(\mathrm{NS}^{\chi}_{\kappa,\lambda}\right) = \bar{\mathfrak{d}}^{\chi}_{\kappa,\lambda}$.

Proof By Lemmas 3.1 and 3.2. □

Corollary 3.4 *Let π and χ be two cardinals such that $\kappa \leq \pi < \chi \leq \lambda$. Then* $\overline{\mathrm{cof}}\left(\mathrm{NS}^{\pi}_{\kappa,\lambda}\right) \leq \overline{\mathrm{cof}}\left(\mathrm{NS}^{\chi}_{\kappa,\lambda}\right)$.

4 $\mathbf{NSS}_{\kappa,\lambda}$

An ideal J on $P_\kappa(\lambda)$ is *seminormal* if it is δ-normal for every $\delta < \lambda$.

$\mathrm{NSS}_{\kappa,\lambda}$ denotes the smallest seminormal ideal on $P_\kappa(\lambda)$.

Fact 4.1 (i) (Folklore) *Suppose* $\mathrm{cf}(\lambda) < \kappa$. *Then* $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NSS}_{\kappa,\lambda}$.
(ii) [1] *Suppose* $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. *Then* $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NSS}_{\kappa,\lambda}|A$ *for some* A.

Proposition 4.2 *Suppose* $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. *Then* $\overline{\mathrm{cof}}\left(\mathrm{NSS}_{\kappa,\lambda}\right) > \lambda$.

Proof By Facts 2.5(iv) and 4.1. □

We will see that "$\overline{\mathrm{cof}}\left(\mathrm{NSS}_{\kappa,\lambda}\right) > \lambda$" needs not hold in case λ is regular. Note that if λ is regular, then by Fact 2.5(iv), $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}\right) > \lambda$.

Fact 4.3 [1] *Suppose that λ is regular. Then* $\mathrm{NSS}_{\kappa,\lambda} = \bigcup_{\delta<\lambda} \mathrm{NS}^{\delta}_{\kappa,\lambda}$.

Proof It is immediate that $\bigcup_{\delta<\lambda} \mathrm{NS}^{\delta}_{\kappa,\lambda} \subseteq \mathrm{NSS}_{\kappa,\lambda}$. To show the reverse inclusion, fix $A \in \left(\bigcup_{\delta<\lambda} \mathrm{NS}^{\delta}_{\kappa,\lambda}\right)^+$, η with $\kappa \leq \eta < \lambda$, and $f : A \longrightarrow \eta$ with the property that $f(a) \in a$ for all $a \in A$. For ξ with $\eta \leq \xi < \lambda$, we may find $B_\xi \in \left(\mathrm{NS}^{\xi}_{\kappa,\lambda}\right)^+ \cap P(A)$ and $\gamma_\xi < \eta$ such that f takes the constant value γ_ξ on B_ξ. There must be $\beta < \eta$ and $Z \subseteq \{\xi : \eta \leq \xi < \lambda\}$ such that $|Z| = \lambda$ and $\gamma_\xi = \beta$ for all $\xi \in Z$. Now set $C = \bigcup_{\xi\in Z} B_\xi$. Then clearly $C \in \left(\bigcup_{\delta<\lambda} \mathrm{NS}^{\delta}_{\kappa,\lambda}\right)^+$. Moreover f is identically β on C. □

Fact 4.4 [10] *Suppose that θ is a cardinal with $2 \le \theta \le \kappa$, and J is an ideal on $P_\kappa(\lambda)$ such that $J \subseteq \mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}$ and $\overline{\mathrm{cof}}(J) \le \lambda^{<\overline{\theta}}$. Then $J|A = I_{\kappa,\lambda}|A$ for some $A \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$.*

In particular, if $J \subseteq \mathrm{NS}_{\kappa,\lambda}$ and $\overline{\mathrm{cof}}(J) \le \lambda$, then $J|D = I_{\kappa,\lambda}|D$ for some $D \in \mathrm{NS}^*_{\kappa,\lambda}$.

Fact 4.5 [10] *Suppose that θ is a cardinal with $2 \le \theta \le \kappa$, and let σ be the least cardinal τ such that $\tau^{<\overline{\theta}} \ge \lambda$. Then $\overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) \ge \sigma$ for every $A \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$.*

Proposition 4.6 *Suppose that θ is a cardinal with $2 \le \theta \le \kappa$, and J is an ideal on $P_\kappa(\lambda)$ with $J \subseteq \mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}$. Let σ be the least cardinal τ such that $\tau^{<\overline{\theta}} \ge \lambda$. Then $\overline{\mathrm{cof}}(J) \ge \sigma$.*

Proof If $\overline{\mathrm{cof}}(J) > \lambda^{<\overline{\theta}}$, there is nothing to prove. Otherwise, there is by Fact 4.4 $A \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$ such that $J|A = I_{\kappa,\lambda}|A$. Then by Fact 4.5, $\sigma \le \overline{\mathrm{cof}}(I_{\kappa,\lambda}|A) \le \overline{\mathrm{cof}}(J)$. □

In particular, $\overline{\mathrm{cof}}(J) \ge \lambda$ for any ideal $J \subseteq \mathrm{NS}_{\kappa,\lambda}$.

Fact 4.7 [8]

(i) *Suppose that λ is a successor cardinal, say $\lambda = \nu^+$. Then $\mathrm{NSS}_{\kappa,\lambda}|C = I_{\kappa,\lambda}|C$ for some $C \in \mathrm{NS}^*_{\kappa,\lambda}$ if and only if $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\nu}\right) \le \lambda$.*
(ii) *Suppose that λ is a regular limit cardinal. Then $\mathrm{NSS}_{\kappa,\lambda}|C = I_{\kappa,\lambda}|C$ for some $C \in \mathrm{NS}^*_{\kappa,\lambda}$ if and only if $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}\right) \le \lambda = \mathrm{cov}(\lambda, \tau^+, \tau^+, \kappa)$ for every cardinal τ with $\kappa \le \tau < \lambda$.*

Recall from the introduction that $\mathcal{H}_{\kappa,\lambda}$ is said to hold if $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}\right) \le \lambda$ for every cardinal τ with $\kappa \le \tau < \lambda$.

Proposition 4.8 *Suppose that λ is a regular cardinal. Then the following are equivalent:*

(i) *$\mathcal{H}_{\kappa,\lambda}$ holds.*
(ii) *$\overline{\mathrm{cof}}\left(\mathrm{NSS}_{\kappa,\lambda}\right) = \lambda$.*
(iii) *$\mathrm{NSS}_{\kappa,\lambda}|C = I_{\kappa,\lambda} \mid C$ for some $C \in \mathrm{NS}^*_{\kappa,\lambda}$.*

Proof (i) $\longrightarrow$ (ii): By Proposition 4.6, $\overline{\mathrm{cof}}\left(\mathrm{NSS}_{\kappa,\lambda}\right) \ge \lambda$. For the reverse inequality, we consider two cases. First suppose that λ is a successor cardinal, say $\lambda = \nu^+$. Then by Fact 4.3 $\mathrm{NSS}_{\kappa,\lambda} = \bigcup_{\nu \le \delta < \lambda} \mathrm{NS}^\delta_{\kappa,\lambda}$. Now for $\nu \le \delta < \lambda$, $\overline{\mathrm{cof}}\left(\mathrm{NS}^\delta_{\kappa,\lambda}\right) \le \overline{\mathrm{cof}}\left(\mathrm{NS}^\nu_{\kappa,\lambda}\right) = \max\{\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\nu}\right), \mathrm{cov}(\lambda, \lambda, \lambda, \kappa)\} \le \max\{\lambda, \lambda\} = \lambda$ by Facts 2.4(ii) and 2.10. Hence $\overline{\mathrm{cof}}\left(\bigcup_{\nu \le \delta < \lambda} \mathrm{NS}^\delta_{\kappa,\lambda}\right) \le \lambda$.

Next suppose that λ is a limit cardinal. Given a cardinal χ with $\kappa \le \chi < \lambda$, by Corollary 3.4 $\overline{\mathrm{cof}}\left(\mathrm{NS}^\chi_{\kappa,\tau}\right) \le \lambda$ for every cardinal τ with $\chi \le \tau < \lambda$, so by Fact 2.10 $\overline{\mathrm{cof}}\left(\mathrm{NS}^\chi_{\kappa,\lambda}\right) \le \lambda$. It follows that $\overline{\mathrm{cof}}\left(\mathrm{NSS}_{\kappa,\lambda}\right) \le \lambda$ since by Fact 4.3 $\mathrm{NSS}_{\kappa,\lambda} = \bigcup_{\kappa \le \chi < \lambda} \mathrm{NS}^\chi_{\kappa,\lambda}$.

(ii) $\longrightarrow$ (iii): By Fact 4.4.

(iii) $\longrightarrow$ (i): By Facts 2.5(iii) and 4.7. □

5 Ideals J on $P_\kappa(\lambda)$ with $\overline{\mathrm{cof}}(J) = \lambda$

In this section we look for cases when $\overline{\mathrm{cof}}\big(\bigcup_{\delta<\xi} \mathrm{NS}^{\delta}_{\kappa,\lambda}\big) = \lambda$, where $\kappa < \xi \leq \lambda + 1$. We start with the following observation.

Lemma 5.1 *Suppose that $K \subseteq \mathrm{NS}_{\kappa,\lambda}$ is an ideal on $P_\kappa(\lambda)$ with $\overline{\mathrm{cof}}(K) \leq \lambda$, and ξ is an ordinal such that*

- $\kappa < \xi \leq \lambda + 1$*;*
- *ξ is either a successor ordinal, or a limit ordinal of cofinality at least κ;*
- $\bigcup_{\delta<\xi} \mathrm{NS}^{\delta}_{\kappa,\lambda} \subseteq K$.

Then $\overline{\mathrm{cof}}\big(\bigcup_{\delta<\xi} \mathrm{NS}^{\delta}_{\kappa,\lambda}\big) = \lambda$.

Proof By Fact 4.5 we may find $A \in \mathrm{NS}^{*}_{\kappa,\lambda}$ such that $K|A = I_{\kappa,\lambda}|A$. For any cardinal χ with $\kappa \leq \chi < \xi$, $\mathrm{NS}^{\chi}_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$, so by Lemma 2.5(ii) $\overline{\mathrm{cof}}\left(\mathrm{NS}^{\chi}_{\kappa,\lambda}\right) \leq \lambda$. Hence by Fact 2.4(ii) $\overline{\mathrm{cof}}\left(\mathrm{NS}^{\delta}_{\kappa,\lambda}\right) \leq \lambda$ for every δ with $\kappa \leq \delta < \xi$. It easily follows that $\overline{\mathrm{cof}}\big(\bigcup_{\delta<\xi} \mathrm{NS}^{\delta}_{\kappa,\lambda}\big) \leq \lambda$. The reverse inequality holds by Proposition 4.6. □

So we are looking for a large $K \subseteq \mathrm{NS}_{\kappa,\lambda}$ with $\overline{\mathrm{cof}}(K) \leq \lambda$. Assuming that $\mathcal{H}_{\kappa,\lambda}$ holds, we can take $K = \bigcup_{\delta<\mathrm{cf}(\lambda)} \mathrm{NS}^{\delta}_{\kappa,\lambda}$ if λ is a singular cardinal of cofinality at least κ, and $K = \mathrm{NSS}_{\kappa,\lambda}$ otherwise.

Fact 5.2 [10] *Let θ be a cardinal with $2 \leq \theta \leq \kappa$. Suppose $\overline{\theta} \leq \mathrm{cf}(\lambda) < \kappa$. Then for any cardinal ν with $\kappa \leq \nu < \lambda$, $\overline{\mathrm{cof}}\left(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\right) \leq \bigcup_{\nu\leq\tau<\lambda} \overline{\mathrm{cof}}\left(\mathrm{NS}^{[\tau]^{<\theta}}_{\kappa,\tau}\right)$.*

Proposition 5.3 *Let θ be a cardinal with $2 \leq \theta \leq \kappa$. Suppose that $\overline{\theta} \leq \mathrm{cf}(\lambda) < \kappa$ and there is a cardinal ν with $\kappa \leq \nu < \lambda$ such that for any cardinal τ with $\nu \leq \tau < \lambda$, $\overline{\mathrm{cof}}\left(\mathrm{NS}^{[\tau]^{<\theta}}_{\kappa,\tau}\right) \leq \lambda$ and $\tau^{<\overline{\theta}} < \lambda$. Then $\overline{\mathrm{cof}}\left(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\right) = \lambda$.*

Proof By Proposition 4.6 and Fact 5.2. □

In particular, if $\mathrm{cf}(\lambda) < \kappa$ and $\mathcal{H}_{\kappa,\lambda}$ holds, then $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}\right) = \lambda$.

Note that if $\overline{\mathrm{cof}}\left(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\right) = \lambda$, then by Fact 4.4 $\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda} = I_{\kappa,\lambda}|C$ for some C.

Fact 5.4 [11] *Let $A \in I^{+}_{\kappa,\lambda}$ be such that $|\{a \in A : b \subseteq a\}| = |A|$ for every $b \in P_\kappa(\lambda)$. Then A can be decomposed into $|A|$ pairwise disjoint members of $I^{+}_{\kappa,\lambda}$.*

Proposition 5.5 *Let θ be a cardinal with $2 \leq \theta \leq \kappa$. Suppose that there is C such that $\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda} = I_{\kappa,\lambda}|C$. Then $P_\kappa(\lambda)$ can be split into π members of $\left(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\right)^{+}$, where π is the least size of any member of $\left(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\right)^{*}$.*

Proof Pick $D \in \left(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\right)^{*}$. Then by Fact 5.4, $C \cap D$ can be decomposed into π pairwise disjoints members of $\left(\mathrm{NS}^{[\lambda]^{<\theta}}_{\kappa,\lambda}\right)^{+}$. □

In particular, if $\mathrm{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|C$ for some C, then $P_\kappa(\lambda)$ can be split into $c(\kappa,\lambda)$ disjoint stationary sets.

Proposition 5.6 *Suppose that θ and ρ are two cardinals such that $\omega \leq \theta = \mathrm{cf}(\theta) < \kappa \leq \rho \leq \lambda$, $u(\theta,\lambda) = \lambda$, and either $\mathrm{cf}(\lambda) < \kappa$ or $\mathrm{cf}(\lambda) > \rho^{<\theta}$. Suppose further that for every cardinal τ with $\rho \leq \tau < \lambda$, $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}^{[\tau]^{<\theta}}\right) \leq \lambda$. Then $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}^{[\rho]^{<\theta}}\right) \leq \lambda$.*

Proof It suffices to show that $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}^{[\rho]^{<\theta}}\right) \leq \lambda$ for any cardinal τ with $\rho \leq \tau < \lambda$ since by Facts 2.1 and 2.10 $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}^{[\rho]^{<\theta}}\right) = \bigcup_{\rho<\tau<\lambda} \overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}^{[\rho]^{<\theta}}\right)$ if λ is a limit cardinal, and $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}^{[\rho]^{<\theta}}\right) = \max\{\lambda, \overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\nu}^{[\rho]^{<\theta}}\right)\}$ if $\lambda = \nu^+$. Now for any cardinal τ with $\rho \leq \tau < \lambda$,

$$\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}^{[\rho]^{<\theta}}\right) \leq \overline{\mathfrak{d}_{\kappa,\tau}^{\rho^{<\theta}}} \leq \overline{\mathfrak{d}_{\kappa,\tau}^{\tau^{<\theta}}} \leq u\left(\theta, \overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}^{[\tau]^{<\theta}}\right)\right) \leq u(\theta,\lambda) = \lambda$$

by Lemmas 3.1 and 3.2. □

Proposition 5.7 *Suppose that $\mathcal{H}_{\kappa,\lambda}$ holds, and ξ is an ordinal such that*

- *$\kappa < \xi \leq \eta$, where η equals $\lambda + 1$ if $\mathrm{cf}(\lambda) < \kappa$, and $\mathrm{cf}(\lambda)$ otherwise;*
- *ξ is either a successor ordinal, or a limit ordinal of cofinality at least κ.*

Then $\overline{\mathrm{cof}}\left(\bigcup_{\delta<\xi} \mathrm{NS}_{\kappa,\lambda}^{\delta}\right) = \lambda$.

Proof By Facts 2.4(ii) and 5.1 and Propositions 4.8, 5.3 and 5.6. □

In particular if $\mathcal{H}_{\kappa,\lambda}$ holds and $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, then $\overline{\mathrm{cof}}\left(\bigcup_{\delta<\mathrm{cf}(\lambda)} \mathrm{NS}_{\kappa,\lambda}^{\delta}\right) = \lambda$ (and hence by Fact 1.5(iv) there is no A such that $\mathrm{NS}_{\kappa,\lambda} = \left(\bigcup_{\delta<\mathrm{cf}(\lambda)} \mathrm{NS}_{\kappa,\lambda}^{\delta}\right)|A$).

6 Ideals J on $P_\kappa(\lambda)$ with $\overline{\mathrm{cof}}(J) < \lambda$

There may exist ideals J on $P_\kappa(\lambda)$ such that $\overline{\mathrm{cof}}(J) < \lambda$. Some examples were presented in [10]. We now give some more.

Given two cardinals $\pi \leq \kappa$ and $\chi \geq \lambda$, $\mathcal{A}_{\kappa,\lambda}(\pi,\chi)$ asserts the existence of $Z \subseteq P_\pi(\lambda)$ with $|Z| = \chi$ such that $|Z \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$.

Fact 6.1 [10] *Let θ and χ be two cardinals such that*

- *$2 \leq \theta \leq \kappa$, $\lambda < \chi$ and there is a $[\chi]^{<\theta}$-normal ideal on $P_\kappa(\chi)$;*
- *$\mathcal{A}_{\kappa,\lambda}(\pi,\chi)$ holds for some regular uncountable cardinal $\pi < \kappa$.*

Then $\overline{\mathrm{cof}}(I_{\kappa,\chi}|A) \leq \lambda$ for some $A \in \left(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}\right)^+$.

Fact 6.2 [9] *Let τ be the largest limit cardinal less than or equal to κ. Assume $\mathrm{cf}(\lambda) < \kappa$ and one of the following conditions is satisfied:*

(a) $\tau = \kappa$.
(b) $\tau > \mathrm{cf}(\lambda)$ *and* $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\tau)$.
(c) $\tau > \mathrm{cf}(\lambda) = \mathrm{cf}(\tau)$ *and* $\min\{\mathrm{pp}(\tau), \tau^{+3}\} < \kappa$.
(d) $\tau \leq \mathrm{cf}(\lambda)$ *and* $\min\{2^{\mathrm{cf}(\lambda)}, (\mathrm{cf}(\lambda))^{+3}\} < \kappa$.

Then $\mathcal{A}_{\kappa,\lambda}((\mathrm{cf}(\lambda))^+, \lambda^+)$ *holds.*

Suppose for instance that κ is a limit cardinal and $\mathrm{cf}(\lambda) < \kappa$. Then by Facts 6.1 and 6.2, $\overline{\mathrm{cof}}(I_{\kappa,\lambda^+}|B) \leq \lambda$ for some $B \in \mathrm{NS}^+_{\kappa,\lambda^+}$.

Note that in case κ is the successor of a cardinal of cofinality $\mathrm{cf}(\lambda)$, Fact 6.2 does not apply, as none of the conditions (a)–(d) is satisfied. To handle this case, we introduce the following principle.

Given a cardinal $\chi \geq \lambda$, $\mathcal{B}_{\kappa,\lambda}(\chi)$ asserts the existence of $Z \subseteq P_\kappa(\lambda)$ with $|Z| = \chi$ such that for every $e \subseteq Z$ with $|e| = \kappa$, there is a $< \kappa$-to-one function in $\prod_{z\in e} z$.

Fact 6.3 [10] *Let* θ *and* χ *be two cardinals such that*

- $2 \leq \theta \leq \kappa$, $\lambda < \chi$ *and there is a* $[\chi]^{<\theta}$*-normal ideal on* $P_\kappa(\chi)$*;*
- $\mathcal{B}_{\kappa,\lambda}(\chi)$ *holds.*

Then $\overline{\mathrm{cof}}(I_{\kappa,\chi}|A) \leq \lambda$ *for some* $A \in \left(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\right)^+$.

Note that in case $\mathrm{cf}(\lambda < \kappa$, $\mathcal{B}_{\kappa,\lambda}(\lambda^+)$ follows from ADS_λ, where ADS_λ asserts the existence of $y_\alpha \subseteq \lambda$ for $\alpha < \lambda^+$ such that

- for any $\alpha < \lambda^+$, $\sup y_\alpha = \lambda$ and $\mathrm{o.t.}(y_\alpha) = \mathrm{cf}(\lambda)$;
- given $\beta < \lambda^+$, there is $g : \beta \longrightarrow \lambda$ such that

$$(y_\alpha \backslash g(\alpha)) \cap (y_{\alpha'} \backslash g(\alpha')) = \emptyset$$

for any $\alpha, \alpha' \in \beta$ with $\alpha \neq \alpha'$.

For more on the existence of $A \in \left(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\right)^+$ such that $\overline{\mathrm{cof}}(I_{\kappa,\chi}|A) < \chi$, see [9] and [10].

Proposition 6.4 *Suppose that* θ *and* χ *are two cardinals such that* $2 \leq \theta \leq \kappa$ *and* $\lambda < \chi$, *and* $A \in \left(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\right)^+$ *is such that* $\overline{\mathrm{cof}}(I_{\kappa,\chi}|A) \leq \lambda$. *Then there is* $B \in \left(\mathrm{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\right)^+$ *and a function* f *such that*

- f *is an isomorphism from* $(P_\kappa(\lambda), \subset)$ *onto* $(B, \subset)$*;*
- *for any* $\delta \leq \lambda$, $f(\mathrm{NS}^{[\delta]^{<\theta}}_{\kappa,\lambda}) = \mathrm{NS}^{[\delta]^{<\theta}}_{\kappa,\chi}|B$ *[and hence* $\overline{\mathrm{cof}}(\mathrm{NS}^{[\delta]^{<\theta}}_{\kappa,\chi}|B) \leq \overline{\mathrm{cof}}(\mathrm{NS}^{[\delta]^{<\theta}}_{\kappa,\lambda})$ *and* $\mathrm{cof}(\mathrm{NS}^{[\delta]^{<\theta}}_{\kappa,\chi}) \leq \mathrm{cof}(\mathrm{NS}^{[\delta]^{<\theta}}_{\kappa,\lambda})$*].*

Proof Select $x_\beta \in P_\kappa(\chi)$ for $\beta < \lambda$ so that for each $X \in I_{\kappa,\chi}$, there is $z \in P_\kappa(\lambda)$ with $X \cap \{y \in A : \bigcup_{\beta\in z} x_\beta \subseteq y\} = \emptyset$. For $\lambda \leq \alpha < \chi$, pick $z_\alpha \in P_\kappa(\lambda)$ with

$$\left\{y \in A : \bigcup\nolimits_{\beta\in z_\alpha} x_\beta \subseteq y\right\} \subseteq \{t \in P_\kappa(\chi) : \alpha \in t\}.$$

Let C be the set of all $x \in P_\kappa(\chi)$ such that $\left(\bigcup_{\beta\in x\cap\lambda} x_\beta\right) \cup \left(\bigcup_{\alpha\in x\backslash\lambda} z_\alpha\right) \subseteq x$. Note that $C \in \mathrm{NS}^*_{\kappa,\chi}$.

Claim 1 *Let $x \in A \cap C$. Then $x \backslash \lambda = \{\alpha \in \chi \backslash \lambda : z_\alpha \subseteq x \cap \lambda\}$.*

Proof of Claim 1 Since $x \in C$, $x \backslash \lambda \subseteq \{\alpha \in \chi \backslash \lambda : z_\alpha \subseteq x \cap \lambda\}$. To show the reverse inclusion, fix $\alpha \in \chi \backslash \lambda$ with $z_\alpha \subseteq x \cap \lambda$. Then $\bigcup_{\beta \in z_\alpha} x_\beta \subseteq x$, and hence $\alpha \in x$, which completes the proof of Claim 1. □

Claim 2 *Let $a \in P_\kappa(\lambda)$. Then $|\{\alpha \in \chi \backslash \lambda : z_\alpha \subseteq a\}| < \kappa$.*

Proof of Claim 2 Pick $x \in A \cap C$ with $a \subseteq x$. Then by Claim 1,

$$\{\alpha \in \chi \backslash \lambda : z_\alpha \subseteq a\} \subseteq \{\alpha \in \chi \backslash \lambda : z_\alpha \subseteq x \cap \lambda\} \subseteq x,$$

which completes the proof of Claim 2. □

Using Claim 2, define $f : P_\kappa(\lambda) \longrightarrow P_\kappa(\chi)$ by $f(a) = a \cup \{\alpha \in \chi \backslash \lambda : z_\alpha \subseteq a\}$. Put $B = \mathrm{ran}(f)$. By Claim 1, $x = f(x \cap \lambda)$ for any $x \in A \cap C$, so $A \cap C \subseteq B$. It follows that $B \in \left(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}\right)^+$.

As is easily seen, f is an isomorphism from $\big(P_\kappa(\lambda), \subset\big)$ onto $(B, \subset)$, and moreover $f^{-1}(X) \in I_{\kappa,\lambda}$ for any $X \in I_{\kappa,\chi}$. Now fix $\delta \leq \lambda$. Set $J = \mathrm{NS}_{\kappa,\lambda}^{[\delta]^{<\theta}}$. It is simple to see that $f(J)$ is an ideal on $P_\kappa(\chi)$ with the property that $B \in \big(f(J)\big)^*$.

Claim 3 *$f(J)$ is $[\delta]^{<\theta}$-normal.*

Proof of Claim 3 Fix $X \in \big(f(J)\big)^+ \cap P(B)$ and $h : X \longrightarrow P_\theta(\delta)$ such that $h(x) \in P_{|x \cap \theta|}(x)$ for every $x \in X$. Define $k : f^{-1}(X) \longrightarrow P_\theta(\delta)$ by $k(a) = h\big(f(a)\big)$. There must be $A \in J^+ \cap P\big(f^{-1}(X)\big)$ such that k is constant on A. Then clearly $f``A \in \big(f(J)\big)^+ \cap P(X)$, and moreover $h$ is constant on $f``A$, which completes the proof of the claim. □

It immediately follows from Claim 3 that $\mathrm{NS}_{\kappa,\chi}^{[\delta]^{<\theta}} | B \subseteq f(J)$.

To establish the reverse inclusion fix $Y \in f(J)$. Since $f^{-1}(Y \cap B) \in J$, we may find $g : P_\theta(\delta) \longrightarrow P_\kappa(\lambda)$ such that $f^{-1}(Y \cap B) \cap C(g, \kappa, \lambda) = \emptyset$. Then clearly $(Y \cap B) \cap C(g, \kappa, \chi) = \emptyset$ and hence $Y \cap B \in \mathrm{NS}_{\kappa,\chi}^{[\delta]^{<\theta}}$. □

Let $\kappa = (2^\rho)^+$, where ρ is an infinite cardinal, and suppose that λ is a strong limit cardinal with $\mathrm{cf}(\lambda) \leq \rho$. Then $\mathcal{A}_{\kappa,\lambda}(\rho^+, 2^\lambda)$ holds, since $|P_{\rho^+}(\lambda) \cap P(a)| \leq 2^\rho$ for any $a \in P_\kappa(\lambda)$. Hence by Facts 5.2 and 6.1 and Proposition 6.4, $\overline{\mathrm{cof}}\big(\mathrm{NS}_{\kappa,2^\lambda}^\lambda | B\big) \leq \lambda$ for some $B \in \mathrm{NS}_{\kappa,2^\lambda}^+$.

Proposition 6.5 *Suppose that $\overline{\mathrm{cof}}\,\big(\mathrm{NS}_{\kappa,\lambda}\big) \leq \lambda^+$, and there is $A \in \mathrm{NS}_{\kappa,\lambda^+}^+$ such that $\overline{\mathrm{cof}}(I_{\kappa,\lambda^+} | A) \leq \lambda$. Then $\overline{\mathrm{cof}}\,\big(\mathrm{NSS}_{\kappa,\lambda^+} | B\big) < \lambda^+$ for some $B \in \mathrm{NS}_{\kappa,\lambda^+}^+$.*

Proof By Fact 4.7(i), there is $C \in \mathrm{NS}_{\kappa,\lambda^+}^*$ such that $\mathrm{NSS}_{\kappa,\lambda^+} | C = I_{\kappa,\lambda^+} | C$. Then $B = A \cap C$ is as desired. □

For example, suppose that $\kappa = \omega_4$ and $\lambda = \beth_\alpha$ for some infinite limit ordinal α of cofinality ω. Then by Facts 5.2, 6.1 and 6.2 and Proposition 6.5, $\overline{\text{cof}}(\text{NSS}_{\kappa,\lambda^+}|B) \le \lambda$ for some $B \in \text{NS}^+_{\kappa,\lambda^+}$.

If λ is singular, then by Fact 4.1 $\text{NS}_{\kappa,\lambda} = \text{NSS}_{\kappa,\lambda}|B$ for some B, so $\overline{\text{cof}}\left(\text{NSS}_{\kappa,\lambda}|A\right) < \lambda$ for some $A \in \text{NS}^+_{\kappa,\lambda}$ just in case $\overline{\text{cof}}\left(\text{NS}_{\kappa,\lambda}|D\right) < \lambda$ for some $D \in \text{NS}^+_{\kappa,\lambda}$.

Suppose that $\overline{\text{cof}}\left(\text{NS}_{\kappa,\lambda}|D\right) < \lambda$ for some $D \in \text{NS}^+_{\kappa,\lambda}$. Then setting $\sigma = \overline{\text{cof}}\left(\text{NS}_{\kappa,\lambda}|D\right)$,

$$\text{cof}\left(\text{NS}_{\kappa,\lambda}\right) \le u(\kappa,\sigma) \le u(\kappa,\lambda) \le \text{cof}\left(\text{NS}_{\kappa,\lambda}\right)$$

by Fact 2.11(ii), so $\text{cof}\left(\text{NS}_{\kappa,\lambda}\right) = u(\kappa,\sigma) = u(\kappa,\lambda)$. Hence by Fact 2.5(iv), SSH does not hold.

Proposition 6.6 *Let θ and χ be two cardinals such that $2 \le \theta \le \kappa$ and $\lambda < \chi$. Suppose that $\overline{\text{cof}}\left(\text{NS}_{\kappa,\chi}\right) \le \chi^{<\overline{\theta}}$, and there is $A \in \left(\text{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\right)^+$ such that $\overline{\text{cof}}(I_{\kappa,\chi}|A| \le \lambda$. Then $\overline{\text{cof}}\left(\text{NS}_{\kappa,\chi}|B\right) \le \lambda$ for some $B \in \left(\text{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\right)^+$.*

Proof By Fact 4.4 there is $C \in \left(\text{NS}^{[\chi]^{<\theta}}_{\kappa,\chi}\right)^*$ such that $\text{NS}_{\kappa,\chi}|C = I_{\kappa,\chi}|C$. Then $B = A \cap C$ is as desired. □

Here is an example of a situation where Proposition 6.6 applies. Starting from a $\mathcal{P}^3(\nu)$-hypermeasurable, Cummings [3] constructs a generic extension W of V in which for any infinite cardinal ρ, 2^ρ equals ρ^+ if ρ is a successor cardinal, and ρ^{++} otherwise. In W, let σ be a regular uncountable cardinal, and $\mu > \sigma$ be a cardinal of cofinality less than σ. Suppose that

- σ is not the successor of a cardinal τ with $\text{cf}(\tau) \le \text{cf}(\mu)$;
- σ is not the successor of the successor of a limit cardinal π with $\text{cf}(\pi) \le \text{cf}(\mu)$.

Then by Facts 6.1 and 6.2 and Proposition 6.6, $\overline{\text{cof}}\left(\text{NS}_{\sigma,\mu^+}|B\right) \le \mu$ for some $B \in \left(\text{NS}^{[\mu^+]^{<(\text{cf}(\mu))^+}}_{\sigma,\mu^+}\right)^+$.

7 Cases when $\text{NS}_{\kappa,\lambda} = \text{NS}^{\text{cf}(\lambda)}_{\kappa,\lambda}|A$ for some A

In this section we establish that if $\kappa \le \text{cf}(\lambda) < \lambda$ and $\mathcal{H}_{\kappa,\lambda}$ holds, then $\text{NS}_{\kappa,\lambda} = \text{NS}^{\text{cf}(\lambda)}_{\kappa,\lambda}|A$ for some A. Note that if $\text{cf}(\lambda) < \kappa$ and $\mathcal{H}_{\kappa,\lambda}$ holds, then by Facts 4.5 and 5.1, $\text{NS}_{\kappa,\lambda} = I_{\kappa,\lambda}|A$ for some A. Note further that if λ is regular, then trivially $\text{NS}_{\kappa,\lambda} = \text{NS}^\lambda_{\kappa,\lambda}|P_\kappa(\lambda)$. By combining the three cases, we obtain that if $\overline{\text{cof}}\left(\text{NS}_{\kappa,\tau}\right) \le \lambda$ for every cardinal τ with $\max\{\kappa, \text{cf}(\lambda)\} \le \tau < \lambda$, then $\text{NS}_{\kappa,\lambda} = \text{NS}^{\text{cf}(\lambda)}_{\kappa,\lambda}|A$ for some A.

Theorem 7.1 *Let π, θ and χ be three cardinals with $\kappa \le \pi < \lambda$ and $2 \le \theta \le \kappa \le \chi \le \lambda$. Suppose that*

- *λ is singular;*
- $\overline{\theta} \leq \mathrm{cf}(\lambda)$ *in case* $\chi = \lambda$*;*
- $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{[\min\{\chi,\tau\}]^{<\theta}}) \leq \lambda^{<\overline{\theta}}$ *for every cardinal* τ *with* $\pi \leq \tau < \lambda$.

Then there is $A \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$ *such that* $\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}} \subseteq \mathrm{NS}_{\kappa,\lambda}^{\mathrm{cf}(\lambda)}|A$.

Proof Set $\mu = \mathrm{cf}(\lambda)$ and select an increasing sequence of cardinals $\langle \lambda_\eta : \eta < \mu \rangle$ so that

- $\sup\{\lambda_\eta : \eta < \mu\} = \lambda$;
- $\lambda_0 > \max\{\pi, \mu\}$;
- $\lambda_0 \geq \chi$ in case $\chi < \lambda$.

For $\eta < \mu$, pick a family G_η of functions from $P_{\max\{\overline{\theta},3\}}(\min\{\chi, \lambda_\eta\})$ to $P_3(\lambda_\eta)$ so that $|G_\eta| \leq \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda_\eta}^{[\min\{\chi,\lambda_\eta\}]^{<\theta}})$ and for every $H \in \left(\mathrm{NS}_{\kappa,\lambda_\eta}^{[\min\{\chi,\lambda_\eta\}]^{<\theta}}\right)^*$, there is $y \in P_\kappa(G_\eta)\backslash\{\emptyset\}$ such that $\{b \in \bigcap_{g\in y} C(g, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa\} \subseteq H$. Let $\bigcup_{\eta<\mu} G_\eta = \{g_e : e \in P_{\overline{\theta}}(\lambda)\}$.

Let A be the set of all $a \in P_\kappa(\lambda)$ such that

- $\overline{\theta} \subseteq a$ in case $\overline{\theta} < \kappa$;
- $\omega \subseteq a$;
- $a \cap \kappa \in \kappa$;
- $k(\alpha) \in a$ for every $\alpha \in a$, where $k : \lambda \to \mu$ is defined by $k(\alpha) =$ the least $\eta < \mu$ such that $\alpha \in \lambda_\eta$;
- if $\chi = \lambda$, then $i(v) \in a$ for every $v \in P_{|a\cap\max\{\overline{\theta},3\}|}(a)$, where $i : P_{\max\{\overline{\theta},3\}}(\lambda) \to \mu$ is defined by $i(v) =$ the least $\eta < \mu$ such that $v \subseteq \lambda_\eta$;
- $g_e(u) \subseteq a$ whenever $e \in P_{|a\cap\overline{\theta}|}(a)$ and $u \in P_{|a\cap\max\{\overline{\theta},3\}|}(a) \cap \mathrm{dom}(g_e)$.

It is immediate that $A \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$. Let us check that A is as desired. Thus fix $f : P_{\max\{\overline{\theta},3\}}(\chi) \to P_3(\lambda)$. Given $\eta < \mu$, define $p_\eta : P_{\max\{\overline{\theta},3\}}(\min\{\chi, \lambda_\eta\}) \to P_2(\lambda_\eta)$ by $p_\eta(v) = \{\zeta\}$, where $\zeta =$ the least σ such that $\eta \leq \sigma < \mu$ and $f(v) \subseteq \lambda_\sigma$. Also define $q_\eta : P_{\max\{\overline{\theta},3\}}(\min\{\chi, \lambda_\eta\}) \to P_3(\lambda_\eta)$ by $q_\eta(v) = \lambda_\eta \cap f(v)$. Select $x_\eta, y_\eta \in P_\kappa(P_{\overline{\theta}}(\lambda))\backslash\{\emptyset\}$ so that

- $\{g_e : e \in x_\eta \cup y_\eta\} \subseteq G_\eta$;
- $\{b \in \bigcap_{e\in x_\eta} C(g_e, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa\} \subseteq C(p_\eta, \kappa, \lambda_\eta)$;
- $\{b \in \bigcap_{e\in y_\eta} C(g_e, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa\} \subseteq C(q_\eta, \kappa, \lambda_n)$.

Finally define $u : \mu \to P_\kappa(\lambda)$ by $u(\eta) = \bigcup(x_\eta \cup y_\eta)$, and $t : P_2(\mu) \to P_\kappa(\lambda)$ so that for any $\eta \in \mu$, $t(\{\eta\})$ equals $u(\eta)$ if $\overline{\theta} < \kappa$, and $u(\eta) \cup |u(\eta)|^+$ otherwise.

We claim that $A \cap C_t^{\kappa,\lambda} \subseteq C_f^{\kappa,\lambda}$. Thus let $a \in A \cap C_t^{\kappa,\lambda}$ and $v \in P_{|a\cap\max\{\overline{\theta},3\}|}(a \cap \chi)$. There must be $\eta \in a \cap \mu$ such that $v \subseteq \lambda_\eta$. Then $a \cap \lambda_\eta \in C(p_\eta, \kappa, \lambda_\eta)$ since $x_\eta \subseteq P_{|a\cap\overline{\theta}|}(a)$. It follows that $v \cup f(v) \subseteq \lambda_\sigma$ for some $\sigma \in a \cap \mu$. Now $a \cap \lambda_\sigma \in C(q_\sigma, \kappa, \lambda_\sigma)$, since $y_\sigma \subseteq P_{|a\cap\overline{\theta}|}(a)$, so $f(v) \subseteq a$. □

In Theorem 7.1 we assumed that $\overline{\theta} \leq \mathrm{cf}(\lambda)$ in case $\chi = \lambda$. Some condition of this kind is necessary. In fact if $\mathrm{cf}(\lambda) < \kappa$ and $u(\kappa, \lambda^{<\overline{\theta}}) = \lambda^{<\overline{\theta}}$, then for each $A \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$, $\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}} \neq \mathrm{NS}_{\kappa,\lambda}^{\mathrm{cf}(\lambda)}|A$ since by Fact 2.11,

$$\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right) > \lambda^{<\overline{\theta}} \geq \lambda \geq \overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}^{\mathrm{cf}(\lambda)} \mid A\right).$$

Corollary 7.2 *Suppose that one of the following holds:*

(i) SSH *holds.*
(ii) *There exists a σ-saturated ideal on $P_\nu(\lambda)$, where σ and ν are two cardinals such that $\omega < \nu = \mathrm{cf}(\nu) < \lambda$ and $\sigma < \nu$.*
(iii) *There is a regular uncountable cardinal $\tau < \lambda$ that is mildly π-ineffable for every cardinal π with $\tau \leq \pi < \lambda$.*

Let θ and χ be two cardinals such that $2 \leq \theta \leq \kappa$, $\max\{\kappa, \mathrm{cf}(\lambda)\} \leq \chi < \lambda$ and $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \lambda^{<\overline{\theta}}$. Then $\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}|A = \mathrm{NS}_{\kappa,\lambda}^{\mathrm{cf}(\lambda)}|A$ for some $A \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$.

Proof Use Facts 2.3 and 2.10. □

Corollary 7.3 *Suppose that $\mathrm{cf}(\lambda) < \kappa$, and θ and χ are two cardinals such that $2 \leq \theta \leq \kappa \leq \chi < \lambda$ and $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{[\chi]^{<\theta}}) \leq \lambda^{<\overline{\theta}}$ for every cardinal τ with $\chi \leq \tau < \lambda$. Then $\mathrm{cof}(\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}) = u(\kappa, \lambda)$.*

Proof By Theorem 7.1 there is $A \in \left(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}}\right)^*$ such that $\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}|A = I_{\kappa,\lambda}|A$. Then by Fact 2.11(ii), $\mathrm{cof}(\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}) = \mathrm{cof}(\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}|A) = \mathrm{cof}(I_{\kappa,\lambda}|A) = \mathrm{cof}(I_{\kappa,\lambda}) = u(\kappa, \lambda)$. □

Corollary 7.4 (i) *Suppose that λ is singular and $\mathcal{H}_{\kappa,\lambda}$ holds. Then $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}_{\kappa,\lambda}^{\mathrm{cf}(\lambda)}|A$ for some A.*
(ii) *Let χ be a cardinal such that $\max\{\kappa, \mathrm{cf}(\lambda)\} \leq \chi < \lambda$ and $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}^{\chi}\right) \leq \lambda$ for every cardinal τ with $\chi \leq \tau < \lambda$. Then $\mathrm{NS}_{\kappa,\lambda}^{\chi}|A = \mathrm{NS}_{\kappa,\lambda}^{\mathrm{cf}(\lambda)} \mid A$ for some $A \in \mathrm{NS}_{\kappa,\lambda}^*$.*

Corollary 7.5 (i) *Let $\chi \geq \kappa$ be a cardinal, and $\alpha < \kappa$ be a limit ordinal such that $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\chi}\right) \leq \chi^{+\alpha}$. Then $\mathrm{NS}_{\kappa,\chi^{+\alpha}}^{\chi}|A = I_{\kappa,\chi^{+\alpha}}|A$ for some $A \in \mathrm{NS}_{\kappa,\chi^{+\alpha}}^*$.*
(ii) *Let $\chi > \kappa$ be a cardinal such that $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\chi}\right) < \chi^{+\kappa}$. Then $\mathrm{NS}_{\kappa,\chi^{+\kappa}}^{\chi}|A = \mathrm{NS}_{\kappa,\chi^{+\kappa}}^{\kappa}|A$ for some $A \in \mathrm{NS}_{\kappa,\chi^{+\kappa}}^*$.*

Proof Use Facts 2.1 and 2.10. □

Note that we do get a better result by considering the reduced cofinality ($\overline{\mathrm{cof}}$) instead of the usual one (cof). For example, suppose that GCH holds in V. By a result of [12], there is a $< \kappa$-closed, κ^+-cc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$, $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\kappa}\right) = \kappa^{+\omega}$ and $\mathrm{cof}(\mathrm{NS}_{\kappa,\kappa}) = \kappa^{+(\omega+1)}$. Then in $V^{\mathbb{P}}$, there is by Corollary 7.5(i) $A \in \mathrm{NS}_{\kappa,\kappa^{+\omega}}^*$ such that $\mathrm{NS}_{\kappa,\kappa^{+\omega}}^{\kappa}|A = I_{\kappa,\kappa^{+\omega}}|A$.

Let us next discuss the condition in Theorem 7.1 that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{[\tau]^{<\theta}}) \leq \lambda^{<\overline{\theta}}$ for almost all cardinals $\tau < \lambda$.

Proposition 7.6 *Let θ and χ be two cardinals such that $2 \leq \theta \leq \kappa \leq \chi < \lambda$. Suppose that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \lambda^{<\overline{\theta}}$ and $\chi^{<\overline{\theta}} \geq \lambda$. Then $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}) = \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \chi$.*

Proof By Fact 2.6(iii) $\chi^{<\overline{\theta}} = \lambda^{<\overline{\theta}}$, so by Fact 4.5 $\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}} = I_{\kappa,\chi}|A$ for some A. It follows that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \chi$. Moreover by Fact 2.10

$$\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}) = \max\left\{\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}), \mathrm{cov}(\lambda, (\lambda^{<\overline{\theta}})^+, (\lambda^{<\overline{\theta}})^+, \kappa)\right\} = \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}).$$

□

Corollary 7.7 *Let θ and χ be two cardinals such that $2 \leq \theta \leq \kappa \leq \chi < \chi^{<\overline{\theta}} = \lambda$. Suppose $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \chi^{<\overline{\theta}}$. Then there is $A \in (\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}})^*$ such that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda} \mid A) \leq \chi$.*

Proof By Fact 2.7 we may find $A \in (\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\theta}})^*$ such that $\mathrm{NS}_{\kappa,\lambda}|A = \mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}|A$. Then by Proposition 7.6, $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}|A) \leq \overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}^{[\chi]^{<\theta}}) \leq \chi$. □

Question Suppose that θ and χ are two cardinals such that $2 \leq \theta \leq \kappa \leq \chi$ and $\overline{\mathrm{cof}}(NS_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \chi^{<\overline{\theta}}$. Does then $\chi^{<\overline{\theta}} = \chi$ hold ?

Proposition 7.8 (i) *Suppose that θ and σ are two cardinals such that $2 \leq \theta \leq \kappa \leq \sigma < \lambda$, $\overline{\theta} \leq \mathrm{cf}(\lambda)$ and $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}^{[\tau]^{<\theta}}) \leq \lambda^{<\overline{\theta}}$ for every cardinal τ with $\sigma \leq \tau < \lambda$. Then there is a cardinal π with $\sigma \leq \pi < \lambda$ such that $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \lambda$ for every cardinal χ with $\pi \leq \chi < \lambda$.*

(ii) *Let θ and π be two cardinals with $2 \leq \theta \leq \kappa \leq \pi < \lambda$. Suppose that $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, and $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \lambda$ for every cardinal χ with $\pi \leq \chi < \lambda$. Then $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\rho}^{[\rho]^{<\theta}}) < \lambda$ for every cardinal ρ with $\max\{\pi, \mathrm{cf}(\lambda)\} \leq \rho < \lambda$.*

Proof (i) If $\nu^\rho < \lambda$ for every cardinal $\nu < \lambda$ and every cardinal $\rho < \overline{\theta}$, then $\lambda^{<\overline{\theta}} = \lambda$, and $\pi = \sigma$ is as desired. Now suppose there are two cardinals $\nu < \lambda$ and $\rho < \overline{\theta}$ such that $\nu^\rho \geq \lambda$. Set $\pi = \max\{\nu, \sigma\}$. Let χ be a cardinal with $\pi \leq \chi < \lambda$. Then $\chi^{<\overline{\theta}} = \lambda^{<\overline{\theta}}$, so by Proposition 7.6 $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\chi}^{[\chi]^{<\theta}}) \leq \chi$.

(ii) By Fact 2.9.

□

In particular, if $\kappa \leq \mathrm{cf}(\lambda) < \lambda$, then $\mathcal{H}_{\kappa,\lambda}$ holds just in case $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}) < \lambda$ for every cardinal τ with $\kappa \leq \tau < \lambda$.

Suppose that λ is a limit cardinal and χ is a cardinal with $\kappa \leq \chi \leq \lambda$. If either $\mathrm{cf}(\lambda) < \kappa$ or $\mathrm{cf}(\lambda) > \chi$, then by Fact 2.10 and Lemma 5.1,

$$\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}^{\chi}\right) \leq \sup\left\{\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}^{\min\{\chi,\tau\}}\right) : \pi \leq \tau < \lambda\right\},$$

where π equals κ if $\chi = \lambda$, and χ otherwise. We will now deal with the case when $\kappa \leq \mathrm{cf}(\lambda) \leq \chi$. The proof of the following is a modification of that of Theorem 7.1.

Proposition 7.9 *Let χ be a cardinal such that $\max\{\kappa, \mathrm{cf}(\lambda)\} \leq \chi \leq \lambda$. Set $\pi = \kappa$ if $\chi = \lambda$, and $\pi = \chi$ otherwise. Then $\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\lambda}^{\chi}\right) \leq \overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\rho}^{\mathrm{cf}(\lambda)}\right)$ and $\mathrm{cof}\left(\mathrm{NS}_{\kappa,\lambda}^{\chi}\right) \leq \mathrm{cof}\left(\mathrm{NS}_{\kappa,\rho}^{\mathrm{cf}(\lambda)}\right)$ where $\rho = \sup\{\overline{\mathrm{cof}}\left(\mathrm{NS}_{\kappa,\tau}^{\min\{\chi,\tau\}}\right) : \pi \leq \tau < \lambda\}$.*

Proof We can assume that $\mathrm{cf}(\lambda) < \chi$ since otherwise the result is trivial. We show that $\overline{\mathrm{cof}}\left(\mathrm{NS}^{\chi}_{\kappa,\lambda}\right) \leq \overline{\mathrm{cof}}\left(\mathrm{NS}^{\mathrm{cf}(\lambda)}_{\kappa,\rho}\right)$ and leave the proof of the other assertion to the reader. Put $\mu = \mathrm{cf}(\lambda)$ and pick an increasing sequence $\langle \lambda_\eta : \eta < \mu \rangle$ of cardinals cofinal in λ so that $\lambda_0 > \max\{\kappa, \mu\}$, and $\lambda_0 \geq \chi$ in case $\chi < \lambda$. For $\eta < \mu$, select a family G_η of functions from $P_3(\min\{\chi, \lambda_\eta\})$ to $P_3(\lambda_\eta)$ so that $|G_\eta| \leq \overline{\mathrm{cof}}\left(\mathrm{NS}^{\min\{\chi,\lambda_\eta\}}_{\kappa,\lambda_\eta}\right)$ and for any $H \in \left(\mathrm{NS}^{\min\{\chi,\lambda_\eta\}}_{\kappa,\lambda_\eta}\right)^*$, there is $y \in P_\kappa(G_\eta)\setminus\{\emptyset\}$ with $\left\{b \in \bigcap_{g\in y} C(g, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa\right\} \subseteq H$. Let $\bigcup_{\eta<\mu} G_\eta = \{g_\xi : \xi < \rho\}$. For $\xi < \rho$, let $g_\xi \in G_{\eta_\xi}$. Let A be the set of all $a \in P_\kappa(\lambda)$ such that $\omega \subseteq a$, $a \cap \kappa \in \kappa$ and $k(\alpha) \in a$ for all $\alpha \in a$, where $k : \lambda \to \mu$ is defined by $k(\alpha) =$ the least $\eta < \mu$ such that $\alpha \in \lambda_\eta$. Clearly $A \in \mathrm{NS}^*_{\kappa,\lambda}$, so by Fact 2.5(ii) $\overline{\mathrm{cof}}\left(\mathrm{NS}^{\chi}_{\kappa,\lambda}|A\right) = \overline{\mathrm{cof}}\left(\mathrm{NS}^{\chi}_{\kappa,\lambda}\right)$.

By Proposition 2.3 we may find a collection T of functions from μ to $P_\kappa(\rho)$ such that $|T| = \overline{\mathrm{cof}}\left(\mathrm{NS}^{\mu}_{\kappa,\rho}\right)$ and for any $u : \mu \to P_\kappa(\rho)$, there is $z \in P_\kappa(T)$ with the property that $u(\eta) \subseteq \bigcup_{t\in z} t(\eta)$ for every $\eta \in \mu$. For $t \in T$, let D_t be the set of all $a \in P_\kappa(\lambda)$ such that for any $\eta \in a \cap \mu$ and any $\xi \in t(\eta)$, $a \cap \lambda_{n_\xi} \in C(g_\xi, \kappa, \lambda_{\eta_\xi})$. Note that $D_t \in \left(\mathrm{NS}^{\chi}_{\kappa,\lambda}\right)^*$.

Now fix $f : P_3(\chi) \to P_3(\lambda)$. Given $\eta < \mu$, define $p_\eta : P_3(\min\{\chi, \lambda_\eta\}) \to P_2(\lambda_\eta)$ by $p_n(v) = \{\zeta\}$, where $\zeta =$ the least σ such that $\eta \leq \sigma < \mu$ and $f(v) \leq \lambda_\sigma$, and $q_n : P_3(\min\{\chi, \lambda_\eta\}) \to P_3(\lambda_\eta)$ by $q_\eta(v) = \lambda_\eta \cap f(v)$. Select $x_\eta, y_\eta \in P_\kappa(\rho)\setminus\{\emptyset\}$ so that

- $\{g_\xi : \xi \in x_\eta \cup y_\eta\} \subseteq G_\eta$;
- $\{b \in \bigcap_{\xi\in x_\eta} C(g_\xi, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa\} \subseteq C(p_\eta, \kappa, \lambda_\eta)$;
- $\{b \in \bigcap_{\xi\in y_\eta} C(g_\xi, \kappa, \lambda_\eta) : b \cap \kappa \in \kappa\} \subseteq C(q_\eta, \kappa, \lambda_\eta)$.

We may find $z \in P_\kappa(T)$ such that $x_\eta \cup y_\eta \subseteq \bigcup_{t\in z} t(\eta)$ for every $\eta \in \mu$.

Let us show that $A \cap \left(\bigcap_{t\in z} D_t\right) \subseteq C(f, \kappa, \lambda)$. Thus let $a \in A \cap \left(\bigcap_{t\in z} D_t\right)$ and $v \in P_3(a \cap \chi)$. There must be $\eta \in a \cap \mu$ such that $v \subseteq \lambda_\eta$. Then $a \cap \lambda_\eta \in \bigcap_{\xi\in x_\eta} C(g_\xi, \kappa, \lambda_\eta)$, so $v \cup f(v) \subseteq \lambda_\sigma$ for some $\sigma \in a \cap \mu$. Now $a \cap \lambda_\sigma \in \bigcap_{\xi\in y_\sigma} C(g_\xi, \kappa, \lambda_\sigma)$, and therefore $f(v) \subseteq a$. □

8 Nowhere precipitousness of $\mathrm{NS}^{\nu}_{\kappa,\lambda}$

Throughout this section it is assumed that $\kappa \leq \mathrm{cf}(\lambda) < \lambda$. Let ν be a cardinal with $\mathrm{cf}(\lambda) \leq \nu < \lambda$. We will show that under certain conditions, $\mathrm{NS}^{\nu}_{\kappa,\lambda}$ is nowhere precipitous. Our proof will follow that of Theorem 2.1 in [13], except that we do not appeal to pcf theory.

Set $\mu = \mathrm{cf}(\lambda)$. We assume that $c(\kappa, \nu) < \lambda$ in case $\nu > \mu$. Let $\rho < \lambda$ be a regular cardinal such that $\rho > \mu$ if $\nu = \mu$, and $\rho > c(\kappa, \nu)$ otherwise. Select a continuous, increasing sequence $\langle \lambda_\beta : \beta < \mu \rangle$ of cardinals so that $\sup\{\lambda_\beta : \beta < \mu\} = \lambda$ and $\lambda_0 > \rho$. Let E be the set of all infinite limit ordinals $\alpha < \mu$ with $\mathrm{cf}(\alpha) < \kappa$. We define D as follows. If $\nu = \mu$, we set $D = E$. Otherwise we pick D in $\mathrm{NS}^*_{\kappa,\nu}$ so that

- for any $d \in D$, $\sup(d \cap \mu)$ is an infinite limit ordinal;
- $|D| = c(\kappa, \nu)$.

We will show that if $\tau^{|D|} < \lambda$ for every cardinal $\tau < \lambda$, then $\mathrm{NS}^{\nu}_{\kappa,\lambda}$ is nowhere precipitous.

For $d \in D$, put $\alpha(d) = \sup(d \cap \mu)$. Note that $\alpha(d) \in E$. Moreover $\alpha(d) = d$ in case $\nu = \mu$.

Let W be the set of all $a \in P_\kappa(\lambda)$ such that

- $0 \in a$;
- $\gamma + 1 \in a$ for every $\gamma \in a \cap \nu$;
- $a \cap \kappa \in \kappa$;
- $\lambda_\beta \in a$ for every $\beta \in a \cap \mu$;
- $a \cap \nu \in D$ in case $\nu > \mu$.

Then clearly, $W \in \left(\mathrm{NS}^{\nu}_{\kappa,\lambda}\right)^*$. For $d \in D$, define W_d by letting $W_d = \{a \in W : \sup(a \cap \mu) = d\}$ if $\nu = \mu$, and $W_d = \{a \in W : a \cap \nu = d\}$ otherwise. Note that W is the disjoint union of the W_d's. Moreover, $\sup(a \cap \lambda_{\alpha(d)}) = \lambda_{\alpha(d)}$ for every $a \in W_d$.

Lemma 8.1 *Suppose that there is $T \subseteq P_\kappa(\lambda)$ such that*

(a) $|T \cap P(a)| < \rho$ *for any* $a \in P_\kappa(\lambda)$;
(b) $u(\rho, \tau) \leq |T|$ *for every cardinal* τ *with* $\rho \leq \tau < \lambda$.

Then for every $R \in \left(\mathrm{NS}^{\nu}_{\kappa,\lambda}\right)^+$,

$$\{d \in D : |\{a \cap \lambda_{\alpha(d)} : a \in R \cap W_d\}| \geq u(\rho, \lambda_{\alpha(d)})\}$$

lies in NS^+_μ *if* $\nu = \mu$, *and in* $\mathrm{NS}^+_{\kappa,\nu}$ *otherwise.*

Proof For $\beta \in \mu$, select $Z_\beta \in I^+_{\rho,\lambda_\beta}$ with $|Z_\beta| \leq |T|$. Then clearly there is $Q \subseteq T$ with $|\bigcup_{\beta<\mu} Z_\beta| = |Q|$. Pick a bijection $i : \bigcup_{\beta<\mu} Z_\beta \to Q$ and let j denote the inverse of i. For $\alpha \in E$, define $k_\alpha : P_\kappa(\lambda_\alpha) \to P_\rho(\lambda_\alpha)$ by $k_\alpha(b) = \bigcup_{e \in Q \cap P(b)} (j(e) \cap \lambda_\alpha)$.

Claim *Let $S \in \left(\mathrm{NS}^{\nu}_{\kappa,\lambda}\right)^+$. Then there is $d \in D$ such that*

$$\{k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}) : a \in S \cap W_d\} \in I^+_{\rho,\lambda_{\alpha(d)}}.$$

Proof Assume otherwise. For $d \in D$, select $y_d \in P_\rho(\lambda_{\alpha(d)})$ so that $y_d \setminus k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}) \neq \emptyset$ for every $a \in S \cap W_d$. Set $y = \bigcup_{d \in D} y_d$. Note that $y \in P_\rho(\lambda)$. For $\beta \in \mu$, pick $z_\beta \in Z_\beta$ so that $y \cap \lambda_\beta \subseteq z_\beta$. Now let H be the set of all $a \in P_\kappa(\lambda)$ such that $i(z_\beta) \in \bigcup_{\zeta \in a \cap \mu} P(a \cap \lambda_\zeta)$ for every $\beta \in a \cap \mu$. Since $H \in \left(\mathrm{NS}^{\mu}_{\kappa,\lambda}\right)^*$, we can find a in $S \cap B \cap H$. Set $d = \sup(a \cap \mu)$ if $\nu = \mu$, and $d = a \cap \nu$ otherwise. Then $a \in W_d$ and $y_d \subseteq y \cap \lambda_{\alpha(d)} = \bigcup_{\beta \in a \cap \mu} (y \cap \lambda_\beta) \subseteq \bigcup_{\beta \in a \cap \mu} z_\beta = \bigcup_{\beta \in a \cap \mu} j(i(z_\beta)) \subseteq k_{\alpha(d)}(a \cap \lambda_{\alpha(d)})$. This contradiction completes the proof of the claim. □

It is now easy to show that the conclusion of the lemma holds: Fix $R \in \left(\mathrm{NS}^{\nu}_{\kappa,\lambda}\right)^+$, and A such that $A \in \mathrm{NS}^*_\mu$ if $\nu = \mu$, and $A \in \mathrm{NS}^*_{\kappa,\nu}$ otherwise. Set $Y = \bigcup_{d \in D \cap A} W_d$. Since $Y \in \left(\mathrm{NS}^{\nu}_{\kappa,\lambda}\right)^*$, there must be some $d \in D$ such that

$$\{k_{\alpha(d)}(a \cap \lambda_{\alpha(d)}) : a \in (R \cap Y) \cap W_d\} \in I^+_{\rho,\lambda_{\alpha(d)}}.$$

Then clearly, $d \in A$ and $|\{a \cap \lambda_{\alpha(d)} : a \in R \cap W_d\}| \geq u(\rho, \lambda_{\alpha(d)})$. □

Consider for instance the following situation: In V, GCH holds, σ is a strong cardinal with $\rho < \sigma < \lambda$, and η a cardinal greater than λ. Then by a result of Gitik and Magidor [6], there is a cardinal preserving, σ^+-cc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$,

- no new bounded subsets of σ are added;
- σ changes its cofinality to ω;
- $2^\sigma \geq \eta$.

Now working in $V^{\mathbb{P}}$, let $T = P_{\omega_1}(\sigma)$. Then clearly $|T \cap P(a)| \leq 2^{|a|} \leq \kappa < \rho$ for any $a \in P_\kappa(\lambda)$. Moreover for any two uncountable cardinals χ and θ with $\text{cf}(\chi) = \chi < \sigma \leq \theta \leq \eta$,

$$u(\chi, \theta) = \max\{2^{<\chi}, u(\chi, \theta)\} = \theta^{<\chi} = \sigma^{<\chi} = \sigma^{\aleph_0} = |T|.$$

Hence $u(\rho, \tau) \leq |T|$ for every cardinal τ with $\rho \leq \tau < \lambda$, so by Lemma 8.1 for any $R \in \left(\text{NS}^\nu_{\kappa,\lambda}\right)^+$,

$$\left\{d \in D : |\{a \cap \lambda_{\alpha(d)} : a \in R \cap W_d\}| \geq u(\rho, \lambda_{\alpha(d)})\right\}$$

lies in NS^+_μ if $\nu = \mu$, and in $\text{NS}^+_{\kappa,\nu}$ otherwise.

Note that for any cardinal χ with $\kappa \leq \chi \leq \sigma$, $\text{cof}(\text{NS}^\chi_{\kappa,\lambda}) = u(\kappa, \lambda)$ since $\text{cof}\left(\text{NS}^\chi_{\kappa,\lambda}\right) \leq (\lambda^{<\kappa})^\chi = (2^\sigma)^\chi = 2^\sigma$, and moreover, by Fact 2.9 and Proposition 4.6, $\overline{\text{cof}}\left(\text{NS}^\chi_{\kappa,\lambda}\right) > \lambda$ in case $\mu \leq \chi$.

Let us observe that if $T \subseteq P_\kappa(\lambda)$ is, as in condition (a) of Lemma 8.1, such that $|T \cap P(a)| \leq u(\kappa, \lambda)$ for any $a \in P_\kappa(\lambda)$, then it is easy to see that $|T| \leq u(\kappa, \lambda)$.

Proposition 8.2 *Suppose that there is $T \subseteq P_\kappa(\lambda)$ and a cardinal π with $\rho \leq \pi < \lambda$ such that*

- *$|T \cap P(a)| < \rho$ for any $a \in P_\kappa(\lambda)$;*
- *$\tau^\nu \leq u(\rho, \tau) \leq |T|$ for every cardinal τ with $\pi < \tau < \lambda$.*

Then $\text{NS}^\nu_{\kappa,\lambda}$ is nowhere precipitous.

Proof By Fact 2.12 it suffices to show that II has a winning strategy in the game $G\left(\text{NS}^\nu_{\kappa,\lambda}\right)$. We can assume without loss of generality that $\lambda_0 > \pi$. For $g : P_3(\nu) \to P_3(\lambda)$ and $\alpha < \mu$, define $g_\alpha : P_3(\nu) \to P_3(\lambda_\alpha)$ by $g_\alpha(e) = g(e) \cap \lambda_\alpha$.

Claim 1 *Let $g : P_3(\nu) \to P_3(\lambda)$. Then*

$$\{d \in D : \left\{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(g_{\alpha(d)}, \kappa, \lambda_{\alpha(d)})\right\} \subseteq C(g, \kappa, \lambda)\}$$

lies in $\left(\text{NS}_\mu | E\right)^*$ if $\nu = \mu$, and in $\text{NS}^*_{\kappa,\nu}$ otherwise.

Proof of Claim 1 We prove the claim in the case when $\nu > \mu$, and leave the proof in the case when $\nu = \mu$ to the reader. Define $h : P_3(\nu) \to \mu$ by $h(e) =$ the least $\beta < \mu$ such that $g(e) \subseteq \lambda_\beta$. Let Q be the set of all $d \in D$ such that $h(e) \in d$ for every $e \in P_3(d)$. Then clearly $Q \in \mathrm{NS}^*_{\kappa,\nu}$. Now fix $d \in Q$ and $a \in W_d$ such that $a \cap \lambda_{\alpha(d)} \in C(g_{\alpha(d)}, \kappa, \lambda_{\alpha(d)})$. Let $e \in P_3(a \cap \nu)$. Then $h(e) \in d$, so $g(e) \subseteq \lambda_{\alpha(d)}$. It follows that $g(e) \subseteq a$, since $g(e) \cap \lambda_{\alpha(d)} \subseteq a$. Thus $a \in C(g, \kappa, \lambda)$. This completes the proof of Claim 1. □

Claim 2 *Let* $X \in \left(\mathrm{NS}^\nu_{\kappa,\lambda}\right)^+$ *and* $Y \subseteq W$. *Suppose that*

$$Y \cap \{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(k, \kappa, \lambda_{\alpha(d)})\} \neq \emptyset$$

whenever $d \in D$ *and* $k : P_3(\nu) \to P_3(\lambda_{\alpha(d)})$ *are such that*

$$|\{a \cap \lambda_{\alpha(d)} : a \in X \cap W_d \text{ and } a \cap \lambda_{\alpha(d)} \in C(k, \kappa, \lambda_{\alpha(d)})\}| \geq u(\rho, \lambda_{\alpha(d)}).$$

Then $Y \in \left(\mathrm{NS}^\nu_{\kappa,\lambda}\right)^+$.

Proof of Claim 2 Fix $g : P_3(\nu) \to P_3(\lambda)$. By Lemma 8.1 and Claim 1, there must be $d \in D$ such that

$$|\{a \cap \lambda_{\alpha(d)} : a \in \left(X \cap C(g, \kappa, \lambda)\right) \cap W_d\}| \geq u(\rho, \lambda_{\alpha(d)})$$

and

$$\left\{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(g_{\alpha(d)}, \kappa, \lambda_{\alpha(d)})\right\} \subseteq C(g, \kappa, \lambda).$$

Then

$$Y \cap \left\{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(g_{\alpha(d)}, \kappa, \lambda_{\alpha(d)})\right\} \neq \emptyset$$

since $a \cap \lambda_{\alpha(d)} \in C(g_{\alpha(d)}, \kappa, \lambda_{\alpha(d)})$ for every $a \in X \cap C(g, \kappa, \lambda) \cap W_d$. Hence $Y \cap C(g, \kappa, \lambda) \neq \emptyset$. This completes the proof of the claim. □

Now to describe a strategy τ for player II in the game $G\left(\mathrm{NS}^\nu_{\kappa,\lambda}\right)$, let

$$X_0, Y_0, X_1, \ldots, Y_{n-1}, X_n$$

be a partial play of the game. We may assume $X_0 \subseteq W$. We define a subset of X_n, $Y_n \in \left(\mathrm{NS}^\nu_{\kappa,\lambda}\right)^+$ and its 1–1 enumeration $\langle y^n_{d,\xi} : d \in D \text{ and } \xi < |K^n_d|\rangle$. Here K^n_d is the set of all $k : P_3(\nu) \to P_3(\lambda_{\alpha(d)})$ such that

$$|X_n \cap \{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(k, \kappa, \lambda_{\alpha(d)})\}| \geq u(\rho, \lambda_{\alpha(d)}).$$

Fix $d \in D$ with K_d^n nonempty. Enumerate K_d^n as $\langle k_{d,\xi}^n : \xi < |K_d^n|\rangle$. Note that $|K_d^n| \leq \lambda_{\alpha(d)}^\nu \leq u(\rho, \lambda_{\alpha(d)})$ (and $K_d^n \subseteq K_d^{n-1}$ by $X_n \subseteq X_{n-1}$). So by induction on $\xi < |K_d^n|$ we can choose $y_{d,\xi}^n$ from

$$X_n \cap \{a \in W_d : a \cap \lambda_{\alpha(d)} \in C(k_{d,\xi}^n, \kappa, \lambda_{\alpha(d)})\} \backslash \big(\{y_{d,\zeta}^n : \zeta < \xi\} \cup \{y_d^{n-1} : \zeta \leq \xi\}\big).$$

Define $Y_n = \{y_{d,\xi}^n : d \in D \text{ and } \xi < |K_d^n|\}$. Then Y_n is a subset of X_n by construction, and is an element of $\left(\mathrm{NS}_{\kappa,\lambda}^\nu\right)^+$ by Claim 2. Moreover the enumeration is 1–1 by construction and the definition of W_d.

To see that τ is a winning strategy, suppose that $X_0, Y_0, X_1, \ldots$ is a play during which player II obeyed the strategy τ. We claim that $\bigcap_{n<\omega} Y_n = \emptyset$. Suppose to the contrary that $x \in \bigcap_{n<\omega} Y_n$. Let d be $\sup(x \cap \mu)$ if $\nu = \mu$, and $x \cap \nu$ otherwise. Then $d \in D$ and for each $n < \omega$, there is $\xi(n)$ such that $x = y_{d,\xi(n)}^n$. By the choice of $y_{d,\xi}^n$ we have $\xi(n) < \xi(n-1)$ for each $0 < n < \omega$, a contradiction. □

Let us observe the following. Suppose that there exist T and π as in the statement of Proposition 8.2. Then either $\mathrm{cof}\left(\mathrm{NS}_{\kappa,\lambda}^\nu\right) = u(\kappa, \lambda)$, or $\lambda^{<\mu} = \lambda$. To establish this, note that $u(\kappa, \lambda) \leq \lambda^{<\kappa} \leq \lambda^{<\mu} \leq |T| \leq u(\kappa, \lambda)$, so $|T| = u(\kappa, \lambda) = \lambda^{<\mu}$. It is now simple to see that $|T| = \lambda$ if $\tau^\nu < \lambda$ for every cardinal $\tau < \lambda$, and $|T| = \lambda^\nu$ otherwise.

Theorem 8.3 (i) *Suppose that $\tau^\mu < \lambda$ for every cardinal $\tau < \lambda$. Then $\mathrm{NS}_{\kappa,\lambda}^\mu$ is nowhere precipitous.*

(ii) *Suppose that $\nu > \mu$, and $\tau^{c(\kappa,\nu)} < \lambda$ for every cardinal $\tau < \lambda$. Then $\mathrm{NS}_{\kappa,\lambda}^\nu$ is nowhere precipitous.*

(iii) *Suppose that $\mathcal{H}_{\kappa,\lambda}$ holds, and $\tau^\mu < \lambda$ for every cardinal $\tau < \lambda$. Then $\mathrm{NS}_{\kappa,\lambda}$ is nowhere precipitous.*

Proof (i) Put $\nu = \mu$, $\rho = \mu^+$, $\pi = 2^\mu = 2^{<\rho}$ and $T = P_2(\lambda)$. Then clearly, $|T \cap P(a)| \leq |a| < \kappa < \rho$ for any $a \in P_\kappa(\lambda)$. Moreover, $\tau^\nu = \tau^{<\rho} = u(\rho, \tau) < \lambda = |T|$ for any cardinal τ with $\pi < \tau < \lambda$. Hence by Proposition 8.2, $\mathrm{NS}_{\kappa,\lambda}^\nu$ is nowhere precipitous.

(ii) Put $\rho = c(\kappa, \nu)^+$, $\pi = 2^{<\rho}$ and $T = P_2(\lambda)$. Then clearly, $|T \cap P(a)| \leq |a| < \kappa < \rho$ for all $a \in P_\kappa(\lambda)$. Moreover, $\tau^\nu \leq \tau^{<\rho} = u(\rho, \tau) < \lambda = |T|$ for every cardinal τ with $\pi < \tau < \lambda$. Now apply Proposition 8.2.

(iii) Use (i) and Corollary 7.4(i).

□

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