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ON ALMOST CATEGORICAL THEORIES

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Our main result is:

THEOREM 1. *If T is a countable superstable unidimensional theory then one of the following occurs.*

1) $I(\lambda, T) = 1$ *for all* $\lambda > \aleph_0$,

2) $I(\lambda, T) = \min(2^\lambda, 2^{2^{\aleph_0}})$ *for all* $\lambda > \aleph_0$

3) $I(\lambda, T) = 2^\lambda$ *for all* $\lambda > \aleph_0$.

If T is w-stable we are in case 1 (Chapter IX of [Shelah 78]); if T has XIII we are in case 3) (Chapter XIII of [Shelah 86]). Thus we can assume that T does not have *otop*. Clearly a unidimensional theory does not have *dop*.

THEOREM 2. *If a countable superstable unidimensional theory T has a trivial regular type then T is $\aleph_1$-categorical.*

PROOF: It suffices to show that a finite inessential extension of T is $\aleph_1$-categorical. Given $M \models T$ with $\|M\| = \aleph_1$ we can choose $\overline{a} \in M$ and $\theta(x, \overline{y})$ such that $R(\theta(x, \overline{a}), L, \infty) = 1$ (see IX.1.11 of [Shelah 78]). Suppose p is a strong type over $\overline{a}$ with $\theta(x, \overline{a}) \in p$. Then p is regular and (since triviality is preserved by nonorthogonality [X.7.3 of [Shelah 86], XVI.2 of [Baldwin 1986]) p is trivial.

Claim 3. If $\overline{a} \subseteq N \subseteq M$ and $N \models T$ contains a maximal (in M) sequence of indiscernibles based on p then $N = M$.

PROOF: Let $I \subseteq M$ be a maximal sequence of indiscernibles based on p. Suppose for contradiction that $N \neq M$ and choose $b \in M - N$. Let $q = t(b, N)$. Then $p \not\perp q$ (by unidimensionality). Since p is trivial $P \not\perp^a q$ (e.g. X.7.3 of [Shelah 86]). Thus, there is a $c \in M$ realizing p^N and with $c \not\downarrow_M b$. Then since $R(\theta(x, \overline{a}), L, \infty) = 1$, $c \in acl(M\overline{b})$ so $c \in M$. But c realizes p^M implies $c \not\downarrow_{\overline{a}} M$ so c contradicts the maximality of I.

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We deduce immediately from Claim 3 and the Lowenheim Skolem theorem that if I is a maximal indiscernibles set of realizations of p, then $|I| = \aleph_1$. Fixing such an I, we now show

Claim 4. $M = acl(\overline{a} \cup I)$.

PROOF: By Claim 3 it suffices to show $A =_{\text{def}} acl(\overline{a}I)$ is the universe of a model of T. If not, choose a formula $\psi(x, \overline{c})$ to minimize $R(\psi(x,\overline{c}), L, \infty)$ subject to the requirements i) $\psi(x,\overline{c})$ is defined over A ii) $M \models (\exists x)\psi(x,\overline{c})$ iii) $\psi(x,\overline{c})$ has no solution in A. Choose $b \in M$ such that $\models \psi(b,\overline{c})$. Then there exists a finite $A' \subseteq A$ with $b \not\downarrow_{A'} A$ and $\overline{c} \in A'$. Further, there is a finite $I_o \subseteq I$ with $A' \subseteq acl(\overline{a}\ I_0)$. Now if $i \in I - I_0$, $t(b; A') \not\perp t(i, A')$ and so, by triviality again, there is a b' realizing $t(b; A')$ with $b' \not\downarrow_{A'} i$. But then $R(t(b'; A), L, \infty) < R(t(b, A'), L, \infty)$ contradicting the minimal rank of Ψ unless $b' \in A$. But if $b' \in A$, ii) in the definition of Ψ is contradicted. Thus, A is the universe of a model of T.

From Claim 4 we immediately deduce Theorem 2. Let us consider the proof of Theorem 1. If T is ω-stable or has a trivial type we are in case 1. If T has otop we are in case 3. For case 2, note first that Theorem IX.1.20 of [Shelah 78] implies (since T is not ω-stable) that $I(\lambda, T) \geq \min(2^\lambda, 2^{2^{\aleph}})$. The full result is now immediate from

Lemma 5. Let T be a countable stable unidimensional theory with notop and a nontrivial regular type. If $\lambda > \aleph_0$, and $I(\lambda, T) < 2^\lambda$ then $I(\lambda, T) \geq 2^{2^{\aleph_0}}$.

PROOF: The conclusion is obvious if $\lambda \leq 2^{\aleph_0}$. Suppose $\lambda > 2^{\aleph_0}$ and N is a model of T with power λ. Choose a model M which is relatively a-saturated in N and with $|M| \leq 2^{\aleph_0}$. We will show that the isomorphism type of M determines that of N. Since T is unidimensional and nonorthogonality preserves triviality all regular types in $S(M)$ are nontrivial. Let $\langle \overline{a}_i : i < |N| \rangle$ be a maximal independent set of realizations of regular types over M in N. Choose M_i with $M \cup a_i \leq M_i \leq N$ such that a_i dominates M_i over M. Since each $t(a_i, M)$ is nontrivial and T has notop,Theorem XIII.4.5 of [Shelah 86] implies $M_i \cong M_j$ over M. Since T is unidimensional it has no depth 2 types. Thus by the Decomposition Lemma (Chapters XII, XIII of [Shelah 86]), N is constructible over $\cup_{i<|N|} M_i$. Thus the isomorphism type of N is determined by that of M and we conclude the Lemma.

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Editors note: Several related results have come to my attention which clarify the role of the classification of nontrivial types in this context.

First, Steve Buechler has suggested a simpler proof of Theorem 2 (in the superstable but not ω-stable case). It relies on the following:

Theorem. If T is countable, superstable, and unidimensional but not ω-stable theory then T has a (stationary) minimal type which is not modular.

Proof. Choose a formula ϕ of infinity rank 1 with parameters from a finite set A. Since T cannot have a formula of Morley rank one it is easy to show $P = \{p \in S(acl(A)) : \phi \in p\}$ has power $2^{\aleph_0}$. Suppose for contradiction that all members of P are modular. They are all nonorthogonal (by unidimensionality) and the modularity implies they are not weakly orthogonal. Since all members of P have infinity rank one each member of P is realized in $acl(Aa)$, if a realizes some $p \in P$. This contradicts the countability of the language.

To review the situation, let T be superstable countable and unidimensional. This last result shows that if every regular type is modular then T is $\aleph_1$-categorical. By an appropriate choice of the type p before Claim 4 we could show T is almost strongly minimal. Vaughn [Vaughn 1985] proved that a locally modular $\aleph_1$-categorical theory which is almost of modular type is almost strongly minimal. Ostensibly, Vaughn's result is somewhat stronger. For, to be almost of modular type demands only that each element admit a filtration by modular types.

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