On the Intersection of Closed Unbounded Sets

Author(s): U. Abraham and S. Shelah

Source: *The Journal of Symbolic Logic*, Vol. 51, No. 1 (Mar., 1986), pp. 180-189
Published by: Association for Symbolic Logic
Stable URL: https://www.jstor.org/stable/2273954

The Journal of Symbolic Logic
Volume 51, Number 1, March 1986

ON THE INTERSECTION OF CLOSED UNBOUNDED SETS

U. ABRAHAM AND S. SHELAH

Abstract. Forcing extensions yield models of ZFC in which a long sequence of club subsets of ω_1 has the following property: every subsequence of size $\aleph_1$ has a finite intersection.

Introduction. A theorem of F. Galvin says that if $\tau^{\underset{\smile}{\tau}} = \tau$, then Galvin's proposition holds: given any family of τ^+ many club (closed unbounded) subsets of τ, one can find a subfamily of cardinality τ the intersection of which contains a club. So, for example, if CH is assumed then for any family of $\aleph_2$ many club subsets of ω_1 there is a club which is contained in uncountably many members of that family. (See §3.2 in Baumgartner, Hajnal and Máté [3] for a proof.)

We shall show that the assumption $\tau \ = \tau$ in this theorem is necessary. Indeed we present two models of ZFC, in §1 and in §2, in which the negation of Galvin's proposition holds (so of course the continuum hypothesis does not).

The negation of Galvin's proposition is equivalent to the existence of a collection of size $\aleph_2$ of club subsets of $\aleph_1$ such that the intersection of any uncountable subcollection is countable. In our models we actually have a collection of size $\aleph_2$ of club subsets of $\aleph_1$ such that the intersection of any uncountable subcollection is even *finite*. In Remark 1.15 we show that those two forms are not equivalent. Galvin's proposition can fail in a model in which the following holds: There is family of size $\aleph_1$ of infinite subsets of $\aleph_1$ such that any uncountable subset of $\aleph_1$ contains a member of that family (and so any collection of $\aleph_2$ club subsets of $\aleph_1$ contains an uncountable subcollection with an infinite intersection).

In the second model in §2 Galvin's proposition fails in an absolute way, and so we could further extend the model to get Martin's axiom as well. Thus MA does not imply Galvin's proposition.

Theorem 1.1 is due to Abraham. Its proof uses a type of argument introduced by Abraham in his dissertation (see also [1]): to get a certain κ-closed forcing partial order to be κ^+-distributive, first prepare the ground by adding a Cohen generic subset of κ. This generic then guides the choice of conditions for a κ-sequence, the union of which is to be generic over an elementary substructure and is a genuine condition.

Theorem 2.1 and Remark 1.15 are due to Shelah and were proved several years later. An earlier result in this direction, due to Baumgartner [2, Theorem 3.4], gives a

Received February 28, 1985; revised April 23, 1985.

0022-4812/86/5101-0019/$02.00

model in which $2^{\aleph_0} = \aleph_2$ and, for every collection of size $\aleph_1$ of infinite subsets of ω_1, there is a club which does not contain any member of the collection. We do not know whether this entails the failure of Galvin's proposition.

§1.

1.1. THEOREM (G. C. H.). *Let κ be a regular cardinal, $\lambda > \kappa$, $\mathrm{cf}(\lambda) > \kappa^+$. Then there is a generic extension which does not add new κ sequences, does not collapse cardinals, and in which $2^{(\kappa^+)} = \lambda$ and the following holds*:

(*) *There is a family of λ many close unbounded subsets of κ^+ such that the intersection of each κ^+ of them is of cardinality less than κ.*

PROOF. Define first $Q = \{q \mid \text{for some } \alpha < \kappa, q: \alpha \to \kappa\}$. Then R is defined in V^Q as the poset of all closed subsets of κ^+ of cardinality $\leq \kappa$ which do not include any subset of cardinality κ which is an element of V. $Q * R = S$ is the building block. The final poset, defined below in 1.3, is a multiplication of λ copies of S with $< \kappa$ support on the left and κ support on the right. To be more precise, $(q, c) \in S$ means $q \in Q$ and $c \in V^Q$, $\varnothing \Vdash c$ is a closed subset, $\sup(c) \in c$, $\sup(c) < \kappa^+$ and c does not contain any subset of cardinality κ which is a member of V. S is partially ordered by $(q, c) \leq (q', c')$ iff $q \subseteq q'$, and $q' \Vdash c'$ is an end extension of c (i.e., $c' \cap \sup(c) + 1 = c$).

1.2. LEMMA. *Let $(q, c) \leq (q', c')$ be in S. Then there is $c^* \in V^Q$ such that*

a. $\varnothing \Vdash c^*$ *is an end extension of* c, *and*

b. $q' \Vdash c' = c^*$.

1.3. DEFINITIONS. a. For $(q, c) \in S$ define $\pi(q, c) = q$. π is a projection of S on Q.

b. The poset $\bar{S}$ is defined by letting $\sigma \in \bar{S}$ iff σ is a function, $\alpha \in \mathrm{Dom}(\sigma) \Rightarrow \alpha \in \lambda$ and $\sigma(\alpha) \in S$, $|\sigma| \leq \kappa$ and

$$|\{\alpha \in \mathrm{Dom}(\sigma) \mid \pi(\sigma(\alpha)) \neq \varnothing\}| < \kappa.$$

$\bar{S}$ is partially ordered naturally: $\sigma \leq \sigma'$ iff $\alpha \in \mathrm{Dom}(\sigma) \Rightarrow \alpha \leftarrow \mathrm{Dom}(\sigma')$ and $\sigma(\alpha) \leq \sigma'(\alpha)$ in S.

c. $\bar{Q}$ is defined by $\bar{q} \in \bar{Q}$ iff $\bar{q}$ is a function, $|\bar{q}| < \kappa$ and $\alpha \in \mathrm{Dom}(\bar{q}) \Rightarrow \alpha < \lambda$ and $\bar{q}(\alpha) \in Q$. The partial order is componentwise.

d. π is extended to a projection $\pi: \bar{S} \to \bar{Q}$ by $(\pi(\sigma))(\alpha) = \pi(\sigma(\alpha))$.

e. Define $\bar{R} \subseteq \bar{S}$ by $\sigma \in \bar{R}$ if $\sigma \in \bar{S}$ and $\pi(\sigma)(\alpha) = \varnothing$ for all α. $\bar{R}$ is a poset by the inherited partial order of $\bar{S}$; so for $\sigma, \sigma' \in \bar{R}$, $\sigma \leq \sigma'$ iff $\alpha \in \mathrm{Dom}(\sigma) \Rightarrow \alpha \in \mathrm{Dom}(\sigma')$ and $\varnothing \Vdash^R \sigma'(\alpha)$ is an end extension of $\sigma(\alpha)$.

f. In a natural way we can say $\bar{Q} \subseteq \bar{S}$ (by saying that $\sigma \in \bar{S}$ is in $\bar{Q}$ iff for all $\alpha \in \mathrm{Dom}(\sigma)$ we have $\sigma(\alpha) = (\bar{q}(\alpha), \varnothing)$, for some $\bar{q} \in \bar{Q}$). Now for $\bar{q} \in \bar{Q} \subset \bar{S}$ and $\bar{c} \in \bar{R}$ we write $(\bar{q}, \bar{c}) \in \bar{S}$ for the least upper bound of $\bar{q}$ and $\bar{c}$ in $\bar{S}$. Also, given any $\bar{\sigma} \in \bar{S}$ there are $\bar{q} \in \bar{Q}$ and $\bar{c} \in \bar{R}$ such that $\sigma = (\bar{q}, \bar{c})$.

REMARK. The following set D is dense in S, and we assume for technical reasons that whenever we said $(q, c) \in S$ we mean $(q, c) \in D$. $(q, c) \in D$ iff for some ordinal $\gamma < \kappa^+$, $\varnothing \Vdash \gamma$ is $\sup(c)$. (Recall that we must have $\varnothing \Vdash \gamma \in c$, because c is a closed set.)

1.4. DEFINITIONS. For $A \subset \lambda$, $\bar{Q} \mid A = \{\bar{q} \in \bar{Q} \mid \mathrm{Dom}(\bar{q}) \subseteq A\}$; similarly we define $\bar{S} \mid A$. We have $\bar{S} \simeq \bar{S} \mid A \times \bar{S} \mid \lambda - A$. Also if $\dot{\bar{Q}}$ is V generic over $\bar{Q}$ then denote $\dot{\bar{Q}} \mid A = \dot{\bar{Q}} \cap (\bar{Q} \mid A)$; similarly we denote $\dot{\bar{S}} \mid A = \dot{\bar{S}} \cap (\bar{S} \mid A)$, for a V generic filter $\dot{\bar{S}}$ over $\bar{S}$. The well-known results for multiplication of two posets hold. If $\dot{S}$ is a V generic filter over S then $\pi''\dot{S} = \dot{Q}$ is V generic over Q and in $V[\dot{Q}]$, R is interpreted as a poset for adding a closed unbounded subset of κ^+; we call this generic closed unbounded subset *the closed unbounded subset of* κ^+ *given by* $\dot{S}$.

Now if $\dot{\bar{S}}$ is a V generic filter over $\bar{S}$, then for any $\xi \in \lambda$, $\dot{\bar{S}} | \{\xi\}$ is a V generic filter over (essentially) S, and we denote the closed unbounded subset of κ^+ given by that filter by C_ξ.

1.5. LEMMA. $\bar{S}$ *is* κ*-closed* (because κ is regular).

1.6. LEMMA. $\bar{S}$ *satisfies the* κ^{++}*-c.c.* (by a Δ-system argument).

From these lemmas it follows that κ and cardinals above κ^+ are not collapsed by $\bar{S}$. We still have to show κ^+ is not collapsed (this is the main difficulty), and that (*) holds. To show that κ^+ is not collapsed, we prove that any set of ordinals in $V^{\bar{S}}$ of cardinality κ is already in $V^{\bar{Q}}$ (this is meaningful because $V^{\bar{Q}} \subseteq V^{\bar{S}}$) and then use the fact that κ^+ is not collapsed in $V^{\bar{Q}}$ because $\bar{Q}$ satisfies the κ^+-c.c.

1.7. MAIN LEMMA. *Let* $h \in V^{\bar{S}}$ *be a name,* $\sigma_0 \in \bar{S}$ *and* $\sigma_0 \Vdash h\colon \kappa \to \mathrm{Ord}$. *Then there is* $\sigma_1 \in \bar{S}$, $\sigma_1 \geq \sigma_0$, *and* $h' \in V^{\bar{Q}}$ *such that* $\sigma_1 \Vdash h = h'$.

PROOF. Observe that $\bar{R}$ has a supremum for each increasing sequence of length $< \kappa$. But the union of an increasing κ sequence of closed sets might include a set of cardinality κ in V even if each member of the sequence does not. Set $\sigma_0 = (\bar{q}_0, \bar{c}_0)$.

Let $\tau > \lambda$ be a cardinal such that $h \in H(\tau)$ and let $M \prec H(\tau)$ be an elementary submodel of cardinality κ such that $h, \kappa, \lambda, \bar{R}, \bar{Q}, \sigma_0 \in M$, $M \cap \kappa^+ = \gamma$ is an ordinal $< \kappa^+$, and M is closed under union of subsets of cardinality less than κ. Enumerate now $M \cap \lambda = \{\xi_i | i < \kappa\}$. Let $\{\bar{q}_i | i < \kappa\}$ be an enumeration of $\bar{Q} \cap M$, and observe that

$$\bar{Q} \cap M = \bar{Q} | M \cap \lambda = \{\bar{q} \in \bar{Q} | \mathrm{Dom}(q) \subset M\}.$$

Let $\{D_i | i < \kappa\}$ be an enumeration of all dense sets of $\bar{S}$ in M. $\{r_i | i < \kappa\}$ is an enumeration of all $r \in V^Q \cap M$ with $\varnothing \Vdash r \in R$. Finally $\{f_i | i < \kappa\}$ is an enumeration of ${}^{\kappa}\kappa$ with the property that if $f_i \subset f_j$ then $i < j$.

Our aim is to define $T(f, \xi) \in V^Q \cap M$ for $f \in {}^{\kappa}\kappa$ and $\xi \in M \cap \lambda$ which has the following properties 1.7.1–1.7.3.

1.7.1. a. $\varnothing \Vdash T(f, \xi) \in R$.

b. For $\xi \in \mathrm{Dom}(\bar{c}_0)$, $\varnothing \Vdash T(\varnothing, \xi) \geq \bar{c}_0(\xi)$. (Recall that $(\bar{q}_0, \bar{c}_0) = \sigma_0$ is the condition we start from.)

c. For $f, f' \in {}^{\kappa}\kappa$, if $f \subset f'$ then, for each $\xi \in M \cap \lambda$, $\varnothing \Vdash T(f', \xi)$ is an end extension of $T(f, \xi)$.

1.7.2. For $\xi \in M \cap \lambda$, $f \in {}^{\alpha}\kappa$, $\alpha < \kappa$, $\nu < \kappa$, if $\varnothing \Vdash r_\nu$ is an end extension of $T(f, \xi)$, then $\varnothing \Vdash T(f^\frown(\alpha, \nu), \xi)$ is an end extension of r_ν.

1.7.3. Before formulating the third condition, let us define $T(\bar{q}) \in \bar{R}$ for $\bar{q} \in \bar{Q} \cap M$ in the following manner. $\mathrm{Dom}(T(\bar{q})) = M \cap \lambda$, and, for $\xi \in M \cap \lambda$,

$$T(\bar{q})(\xi) = \begin{cases} T(\bar{q}(\xi), \xi), & \text{if } \xi \in \mathrm{Dom}(\bar{q}), \\ T(\varnothing, \xi), & \text{otherwise.} \end{cases}$$

We demand that for each $\bar{q} \in \bar{Q} \cap M$ and $D \in M$ a dense open subset of $\bar{S}$ there exists $\bar{q}' \geq \bar{q}$, $\bar{q}' \in \bar{Q} \cap M$, such that $(\bar{q}', T(\bar{q}')) \in D$.

The construction of the two-variable function T will be done later on; for the moment assume we have such a function and see how to end the proof.

With the help of T define a function $\bar{c}$ such that $\mathrm{Dom}(\bar{c}) = M \cap \lambda$ (and $\bar{c} \in \bar{R}$, as we will soon show), and such that for any $\xi \in M \cap \lambda$ and $f \in Q$, $f \Vdash^{Q} \bar{c}(\xi)$ is a closed subset of $\gamma + 1$ and is an end extension of $T(f, \xi)$.

In defining $\bar{c}$ we are making use of 1.7.1.c (recall that $\gamma = M \cap \kappa^+$).

1.8. LEMMA. $\bar{c} \in \bar{R}$.

PROOF. Let $f \in Q$ and $A \in V$, $A \subseteq \kappa^+$, $|A| = \kappa$, be given; we will find $f' \geq f$ such that $f' \Vdash A \not\subseteq \bar{c}(\xi)$. Say $\beta < \kappa^+$ is the ordinal such that $\varnothing \Vdash \beta = \sup T(f, \xi)$. (See the remark following 1.3.) Then if $|A \cap \beta| = \kappa$, as $f \Vdash \bar{c}(\xi)$ is an end extension of $T(f, \xi)$, we get by $\varnothing \Vdash T(f, \xi) \in R$ that $f \Vdash A \cap \beta \not\subset c(\xi)$. So suppose that $|A - \beta| = \kappa$. If $(A - \beta + 1) \cap \gamma = \varnothing$ then $A \not\subseteq \gamma + 1$; but $\bar{c}(\xi) \subseteq \gamma + 1$, hence $A \not\subseteq \bar{c}(\xi)$. We can assume that there exists $a \in (A - \beta + 1) \cap \gamma$. Obviously $\varnothing \Vdash T(f, \xi) \cup \{a + 1\} \in R$, so for some $\nu < \kappa$, $T(f, \xi) \cup \{a + 1\} = r_\nu$, and of course $\varnothing \Vdash a \notin r_\nu$. Say $f \in {}^{\alpha}\kappa$; then, by property 1.7.2, denoting $f' = f^\frown(\alpha, \nu)$, $\varnothing \Vdash T(f', \xi)$ is an end extension of r_ν. Hence $f' \vdash a \in A$ but $a \notin \bar{c}(\xi)$. □

The "genericity" of $\bar{c}$ is expressed in the following.

1.9. LEMMA. *For each $\bar{q} \in \bar{Q}$ and $D \in M$ a dense open subset of $\bar{S}$, there is $\bar{q}' > \bar{q}$ such that $(\bar{q}', \bar{c})$ is in D.*

PROOF. Let $\bar{q}^* = \bar{q} \restriction (M \cap \lambda)$; then $\bar{q}^* \in M$. By property 1.7.3, there exists $\bar{q}' > \bar{q}^*$, $\bar{q}' \in \bar{Q} \cap M$, such that $(\bar{q}', T(\bar{q}')) \in D$. One can easily check that $(\bar{q}', \bar{c}) \geq (\bar{q}', T(\bar{q}'))$. $\bar{q}'$ and $\bar{q}$ are compatible in $\bar{Q}$, and $\bar{q}' \cup \bar{q} \in \bar{Q}$ is the desired condition. □

1.10. CONCLUSION. Set $\sigma_1 = (\bar{q}_0, \bar{c})$; then $\sigma_1 \Vdash h \in V^{\bar{Q}}$ (and this proves the Main Lemma assuming T is constructed).

PROOF. For any $\alpha < \kappa$, $D_\alpha = \{\sigma \in \bar{S} \mid \exists \zeta(\sigma \Vdash h(\alpha) = \zeta)\}$ is a dense open set above σ_0 in $\bar{S}$. Hence by Lemma 1.9 the set $E_\alpha = \{\bar{q} \in \bar{Q} \mid \exists \zeta((\bar{q}, \bar{c}) \Vdash h(\alpha) = \zeta)\}$ is dense open above $\bar{q}_0$ in $\bar{Q}$. From this the conclusion follows. □

Now we construct T. First we construct by induction on $\alpha < \kappa$ the functions $T_\alpha \in M$ and $\bar{p}_\alpha \in \bar{R} \cap M$. The T_α are thought of as approximations of size $< \kappa$ to T, and finally we will set $T = \bigcup_{\alpha < \kappa} T_\alpha$. The p_α's are commitments of size κ. We first list some properties that the T_α should satisfy. Then we prove the consequence that $T = \bigcup_{\alpha<\kappa} T_\alpha$ satisfies 1.7.1–1.7.3, and finally we construct the T_α's and p_α's. The reader is advised to look at the enumerations of some subsets of M given after the definition of M so that the meanings of ξ_i, $\bar{q}_i$ and D_i are clear.

Properties of T_α and p_α.

1′. a. $\bar{p}_0 = \bar{c}_0$; $\{\xi_i \mid i < \alpha\} \subseteq \mathrm{Dom}(\bar{p}_\alpha)$. If $\beta < \alpha < \kappa$ then $\mathrm{Dom}(\bar{p}_\beta) \subseteq \mathrm{Dom}(\bar{p}_\alpha)$ and, for all $\xi \in \mathrm{Dom}(p_\beta)$, $\varnothing \Vdash \bar{p}_\alpha(\xi)$ is an end extension of $\bar{p}_\beta(\xi)$.

If $i < \beta < \alpha$ then $\bar{p}_\beta(\xi_i) = \bar{p}_\alpha(\xi_i)$.

b. $\mathrm{Dom}(T_\alpha) = E_\alpha \times \{\xi_i \mid i < \alpha\}$, where $E_\alpha \subseteq {}^{\kappa}\!\kappa$ is closed under restriction (i.e., $f \in E_\alpha$ and $f' = f \restriction \beta \Rightarrow f' \in E_\alpha$) and $|E_\alpha| < \kappa$, and $\{f_i \mid i < \alpha\} \subset E_\alpha$. $\varnothing \Vdash T_\alpha(f, \xi) \in R$, for f, ξ in the domain of T_α. If $f, f' \in E_\alpha$ and $f \subseteq f'$, then, for $i < \alpha$, $\varnothing \Vdash T_\alpha(f', \xi_i)$ is an end extension of $T_\alpha(f, \xi_i)$.

c. If $\beta < \alpha$ then $E_\beta \subset E_\alpha$. For any $f \in E_\beta$ and $i < \beta$,

$$T_\beta(f, \xi_i) = T_\alpha(f, \xi_i).$$

d. For any $i < \alpha$, $\varnothing \Vdash \bar{p}_\alpha(\xi_i) \leq T_\alpha(\varnothing, \xi_i)$.

2′. For each $i < \alpha$ and $f \in E_\alpha$ (say $\beta = \mathrm{Dom}(f)$), if $\varnothing \Vdash r_\nu$ is an end extension of $T_\alpha(f, \xi_i)$ and if $f' = f^\frown(\beta, \nu) \in E_\alpha$ then

$$\varnothing \Vdash T_\alpha(f', \xi_i) \text{ is an end extension of } r_\nu.$$

3′. For each $\bar{q}_i$ and D_j, $i, j < \kappa$, there exist $\bar{q}' \geq \bar{q}_i$, $\bar{q}' \in \bar{Q} \cap M$, and $\alpha < \kappa$ such that $(\bar{q}'(\xi), \xi) \in \mathrm{Dom}(T_\alpha)$ for all $\xi \in \mathrm{Dom}(\bar{q}')$ and such that $(\bar{q}', \bar{p}) \in D_j$ when $\bar{p}$ is defined as follows. $\mathrm{Dom}(\bar{p}) = \mathrm{Dom}(\bar{q}') \cup \mathrm{Dom}(\bar{p}_\alpha)$. For $\xi \in \mathrm{Dom}(\bar{q}')$, $\bar{p}(\xi) = T_\alpha(\bar{q}'(\xi), \xi)$; and for $\xi \in \mathrm{Dom}(\bar{p}_\alpha) - \mathrm{Dom}(\bar{q}')$, $\bar{p}(\xi) = \bar{p}_\alpha(\xi)$.

1.11. LEMMA. *If the* T_α, $\alpha < \kappa$, *satisfy* 1′, 2′ *and* 3′ *then* $T = \bigcup_{\alpha<\kappa} T_\alpha$ *satisfies* 1, 2 *and* 3 *of* 1.7.

PROOF. From 1′.b we get that $\mathrm{Dom}(t) = {}^{\underline{\kappa}}\kappa \times M \cap \lambda$. That T is a function follows from 1′.c.

1.a is due to 1′.b.

1.b is due to 1′.d and 1′.a.

1.c is due to 1′.b.

2 follows from 2′.

3 follows thus: Let $\bar{q} \in \bar{Q} \cap M$, and let $D \in M$ be a dense open subset of $\bar{S}$. Then $\bar{q} = \bar{q}_i$ and $D = D_j$ for some $i, j < \kappa$. By 3′, there exist $\bar{q}' \geq \bar{q}$, $\bar{q}' \in \bar{Q} \cap M$, and $\alpha < \kappa$ such that $(\bar{q}'(\xi), \xi) \in \mathrm{Dom}(T_\alpha)$ for all $\xi \in \mathrm{Dom}(\bar{q}')$ and such that $(\bar{q}', \bar{p}) \in D$, where $\bar{p}$ was defined in 3′. To prove 3 we need to show that $(\bar{q}', T(\bar{q}')) \geq (\bar{q}', \bar{p})$. Well, if $\xi \in \mathrm{Dom}(\bar{p})$ there are two cases: $\xi \in \mathrm{Dom}(\bar{q}')$ and $\xi \in \mathrm{Dom}(\bar{p}_\alpha) - \mathrm{Dom}(\bar{q}')$. In the first case $\bar{p}(\xi) = T_\alpha(\bar{q}'(\xi), \xi)$ and $T(\bar{q}')(\xi) = T(\bar{q}'(\xi), \xi)$, so that $\varnothing \Vdash T(\bar{q}')(\xi) \geq \bar{p}(\xi)$. In the second case, $\bar{p}(\xi) = \bar{p}_\alpha(\xi)$ and $T(\bar{q}')(\xi) = T(\varnothing, \xi)$. Now pick any $\beta < \kappa$ such that $\alpha \leq \beta$ and $T(\varnothing, \xi) = T_\beta(\varnothing, \xi)$. Then $\varnothing \Vdash T(\bar{q}')(\xi) \geq \bar{p}(\xi)$ follows from $\varnothing \Vdash T_\beta(\varnothing, \xi) \geq \bar{p}_\beta(\xi)$ and $\varnothing \Vdash \bar{p}_\beta(\xi) \geq \bar{p}_\alpha(\xi)$, which are due to 1′.d and 1′.a.

1.12. *Construction of* T_α *and* $\bar{p}_\alpha$. We construct T_α and $\bar{p}_\alpha$ by induction on $\alpha < \kappa$ such that 1′ and 2′ hold for T_α and $\bar{p}_\alpha$. As for 3′, it asks that for each $\bar{q}_i$ and D_j, $i, j < \kappa$, there exist suitable $\bar{q}'$ and α; we will take care of that by performing κ many tasks which are presented to us at successor stages by some enumeration. We put $\bar{p}_0 = \bar{c}_0$ and $T_0 = \varnothing$. If $\delta < \kappa$ is limit, then $E_\delta = \bigcup_{\alpha<\delta} E_\alpha$, $T_\delta = \bigcup_{\alpha<\delta} T_\alpha$ and $\bar{p}_\delta$ is defined as the supremum of $\langle \bar{p}_\alpha \,|\, \alpha < \delta \rangle$, which exists by Lemma 3.4. It is easy to check that 1′ and 2′ hold for T_δ and $\bar{p}_\delta$, and that T_δ, $\bar{p}_\delta \in M$. (M is closed under subsets of cardinality less than κ.)

Successor stage: $\alpha = \gamma + 1$. We might be presented with a task: $\bar{q}_i$ and D_j are given, with $i, j < \gamma$ and $(\bar{q}_i(\xi), \xi) \in \mathrm{Dom}(T_\gamma)$ for $\xi \in \mathrm{Dom}(\bar{q}_i)$; we have to find $\bar{q}' \geq \bar{q}_i$ such that $(\bar{q}', \bar{p}) \in D_j$ for $\bar{p}$ as described in 3′. We do this in two steps. *In the first step* we define T'_γ to make sure 1′ and 2′ will hold for $T_{\gamma+1}$. T'_γ is defined by

$$\mathrm{Dom}(T'_\gamma) = (E_\gamma \cup \{f_\gamma\}) \times \{\xi_j \,|\, j \leq \gamma\} \quad \text{and} \quad T'_\gamma \supset T_\gamma.$$

So we have to define $T'_\gamma(f, \xi_\gamma)$ for $f \in E_\gamma \cup \{f_\gamma\}$ and $T'_\gamma(f_\gamma, \xi_\eta)$ for $\eta < \gamma$. $E_\gamma \cup \{f_\gamma\}$ is a subtree of ${}^{\underline{\kappa}}\kappa$ closed under restrictions, of cardinality $< \kappa$. $T'_\gamma(f, \xi_\gamma)$ is defined for $f \in E_\gamma \cup \{f_\gamma\}$ by induction on the rank of f (where the rank of f is simply $\mathrm{Dom}(f)$) such that $\varnothing \Vdash T'_\gamma(\varnothing, \xi_\gamma) \geq \bar{p}_\gamma(\xi_\gamma)$ (if $\xi_\gamma \in \mathrm{Dom}\ \bar{p}_\gamma$) and $f \subseteq f'$ in $E_\gamma \cup \{f_\gamma\}$ implies $\varnothing \Vdash T'_\gamma(f, \xi_\gamma) \leq T'_\gamma(f', \xi_\gamma)$. And if $\varnothing \Vdash r_\nu \geq T'_\gamma(f, \xi_\gamma)$, and $f^\frown(\beta, \nu) \in E_\gamma \cup \{f_\gamma\}$ where $\beta = \mathrm{Dom}(f)$, then $\varnothing \Vdash T'_\gamma(f^\frown(\beta, \nu), \xi_\gamma) \geq r_\nu$.

Now for $T'_\gamma(f_\gamma, \xi_\eta)$, $\eta < \gamma$, all the restrictions $f_\gamma \restriction \zeta$, $\zeta < \beta = \mathrm{Dom}(f_\gamma)$, are in E_γ. In case β is limit we use Lemma 1. 5; in case β is a successor we have to take care of 2′.

Second step. For each $\zeta \leq \gamma$ find a function $g_\zeta \in {}^{\underline{\kappa}}\kappa$ such that the following conditions are met:

a) $g_\zeta \notin E_\gamma \cup \{f_\gamma\}$ but for any $\nu < \mathrm{Dom}(g_\zeta)$, $g_\zeta \restriction \nu \in E_\gamma \cup \{f_\gamma\}$.

b) If $\xi_\zeta \in \mathrm{Dom}(\bar{q}_i)$, then $\bar{q}_i(\xi_\zeta) \subset g_\zeta$.

Now define $T''_\gamma \supset T'_\gamma$ by giving values to $T''_\gamma(g_\zeta, \xi_\zeta), \zeta \leq \gamma$, such that 2′ continues to hold. Then define $\bar{g} \in \bar{Q} \cap M$ by $\mathrm{Dom}(\bar{g}) = \{\xi_\zeta | \zeta \leq \gamma\}$ and $\bar{g}(\xi_\zeta) = g_\zeta$, so $\bar{g} \geq \bar{q}_i$. Define $\bar{p}' \in \bar{R} \cap M$ by $\mathrm{Dom}(\bar{p}') = \{\xi_\zeta | \zeta \leq \gamma\} \cup \mathrm{Dom}(\bar{p}_\gamma)$. Set $\bar{p}'(\xi_\zeta) = T''_\gamma(g_\zeta, \xi_\zeta)$ for $\zeta \leq \gamma$; and for $\xi_\zeta \in \mathrm{Dom}(\bar{p}_\gamma)$, $\zeta > \gamma$, set $\bar{p}'(\xi_\zeta) = p_\gamma(\xi_\zeta)$. Then $(\bar{g}, \bar{p}') \in \bar{S} \cap M$; hence there exists $(\bar{q}', \bar{p}'') \geq (\bar{g}, \bar{p}')$, $(\bar{q}', \bar{p}'') \in D_j \cap M$. By 1.2 we can get that, for any $\xi \in \mathrm{Dom}(\bar{p}')$, $\varnothing \Vdash \bar{p}'(\xi) \leq \bar{p}''(\xi)$.

Finally define $E_{\gamma+1}$ closed under restrictions such that

$$E_\gamma \cup \{f_\gamma\} \cup \{g_\zeta | \zeta \leq \gamma\} \cup \{\bar{q}'(\xi_\zeta) | \zeta \leq \gamma\} \subseteq E_{\gamma+1}$$

and define $T_{\gamma+1} \supset T''_\gamma$ by first setting $T_{\gamma+1}(g_\xi, \xi_\zeta) = \bar{p}''(\xi_\zeta)$ for $\zeta \leq \gamma$ and then defining the other values such that 2′ holds. Next define $\bar{p}_{\gamma+1}$ by

$$\mathrm{Dom}(\bar{p}_{\gamma+1}) = \mathrm{Dom}(\bar{p}''), \qquad \bar{p}_{\gamma+1} \restriction \{\xi_\zeta | \zeta \leq \gamma\} = \bar{p}_\gamma \restriction \{\xi_\zeta | \zeta \geq \gamma\},$$

and for $\xi \in \mathrm{Dom}(\bar{p}'')$ but $\xi \notin \{\xi_\zeta | \zeta \leq \gamma\}$,

$$\bar{p}_{\gamma+1}(\xi) = \bar{p}''(\xi).$$

We want to check 3′. Pick $\alpha > \gamma$ such that $(\bar{q}'(\xi), \xi) \in \mathrm{Dom}(T_\alpha)$ for $\xi \in \mathrm{Dom}\,\bar{q}'(\xi)$, and show that $(\bar{q}', \bar{p}) \in D_j$ for $\bar{p}$ as defined in 3′. This follows from $(\bar{q}', \bar{p}) \geq (\bar{q}', \bar{p}'') \in D_j$, which we prove now. If $\xi_\zeta \in \mathrm{Dom}(\bar{p}'')$ then

$$\xi_\zeta \in \mathrm{Dom}(\bar{p}_{\gamma+1}) \subseteq \mathrm{Dom}(\bar{p}_\alpha) \subseteq \mathrm{Dom}(\bar{p}).$$

In case $\zeta \leq \gamma$,

$$\bar{p}''(\xi_\zeta) = T_{\gamma+1}(g_\zeta, \xi_\zeta) \quad \text{and} \quad \xi_\zeta \in \mathrm{Dom}(\bar{g}) \subseteq \mathrm{Dom}(\bar{q}').$$

So $\bar{p}(\xi_\zeta) = T_\alpha(\bar{q}'(\xi_\zeta), \xi_\zeta)$, and as $\bar{q}'(\xi_\zeta) \supseteq g_\zeta$ we get $\varnothing \Vdash \bar{p}(\xi_\zeta) \geq \bar{p}''(\xi_\zeta)$. In case $\zeta > \gamma$, $\bar{p}''(\xi_\zeta) = \bar{p}_{\gamma+1}(\xi_\zeta)$; and then, if $\xi_\zeta \in \mathrm{Dom}(\bar{q}')$, $\bar{p}(\xi_\zeta) = T_\alpha(\bar{q}'(\xi_\zeta), \xi_\zeta)$ and, by 1′,

$$\varnothing \Vdash T_\alpha(\bar{q}'(\xi_\zeta), \xi_\zeta) \geq T_\alpha(\varnothing, \xi_\zeta) \geq p_\alpha(\xi_\zeta) \geq \bar{p}_{\gamma+1}(\xi_\zeta).$$

Hence $\varnothing \Vdash \bar{p}(\xi_\zeta) \geq \bar{p}''(\xi_\zeta)$. But if $\xi_\zeta \notin \mathrm{Dom}(\bar{q}')$, then $\bar{p}(\xi_\zeta) = \bar{p}_\alpha(\xi_\zeta)$, so that again $\varnothing \Vdash \bar{p}(\xi_\zeta) \geq \bar{p}''(\xi_\zeta)$. □

This finishes the proof of the Main Lemma.

1.13. Lemma. *Let U, $T \in V$ be two posets. $U \times T$ is the Cartesian product and $\dot{U} \times \dot{T}$ is a V generic filter over $U \times T$. (As is known, $\dot{U}$ is then V generic over U and $\dot{T}$ is $V[\dot{U}]$ generic over T.) κ is a cardinal in $V[\dot{U} \times \dot{T}]$.*

Let $E \in V[\dot{T}]$ be a set of ordinals which does not include any subset of cardinality κ which is a member of V. Then E does not have any subset of cardinality κ which is a member of $V[\dot{U}]$.

Proof. Let $t \in \dot{T}$ be such that $t \Vdash \boldsymbol{E}$ does not have any subset of cardinality κ which is in V. Assume for a contradiction that $A \in V[\dot{U}]$, $A \subseteq E$ and $|A| = \kappa$. Pick $(s', t') \in \dot{S} \times \dot{T}$, $t' \geq t$, which forces this information, and $s' \Vdash^s |\boldsymbol{A}| \geq \kappa$. The set $B = \{a | \text{for some } s^* \geq s', s^* \Vdash a \in \boldsymbol{A}\}$ is in V and of cardinality $\geq \kappa$, so $t' \Vdash B \nsubseteq \boldsymbol{E}$; hence for some $t'' \geq t'$ and $a \in B$, $t'' \Vdash a \notin \boldsymbol{E}$. But then there is $s^* \geq s'$, $s^* \Vdash a \in \boldsymbol{A}$, so that $(s^*, t'') \Vdash \boldsymbol{A} \nsubseteq \boldsymbol{E}$. Contradiction.

1.14. Conclusion of the Proof of Theorem 1.1. Let $\dot{\bar{S}}$ be a V generic filter over $\bar{S}$, and let $\dot{\bar{Q}}$ be the V generic filter over $\bar{Q}$ induced by $\dot{\bar{S}}$. Let C_ξ be the closed unbounded subsets of κ^+ described in 1.4. To show that 1.1 holds, let $H \in V[\dot{\bar{S}}]$ be any set of ordinals of cardinality κ; we will show that $H \subseteq C_\xi$ only for $\leq \kappa$ many

$\xi < \lambda$. By the Main Lemma, $H \in V[\dot{\bar{Q}}]$. So $H \in V[\dot{\bar{Q}} \mid A]$ for some $A \subseteq \lambda$, $|A| = \kappa$ (because $\bar{Q}$ satisfies the κ^+-c.c).

Claim. If $\xi \notin A$, then $H \nsubseteq C_\xi$.

PROOF. This follows from Lemma 3.3 for $U = \bar{S} \mid \lambda - \{\xi\}$, $V = \bar{S} \mid \{\xi\}$ and $E = C_\xi$.

1.15. REMARK. It is possible that there are $\aleph_2$ many club subsets of ω_1 such that the intersection of any uncountable subcollection is countable, and yet there is no family of size $\aleph_2$ of clubs of ω_1 such that the intersection of any uncountable subcollection is finite. To get this use Theorem 1.1 for $\kappa = \aleph_1$ and $\lambda = \aleph_3$, and then collapse $\aleph_1$ to be countable.

§2. In this section we get a universe of ZFC in which there are $\lambda \geq \aleph_2$ many club subsets of ω_1 such that the intersection of any uncountable subfamily is finite and, moreover, this property of the clubs remains in any extension of the universe which does not collapse $\aleph_1$. The clubs satisfy a certain combinatorial property which assures in an absolute way the finite intersection property.

Let us recall that a *graph* on λ is a set of unordered pairs from λ, and a complete subgraph (a clique) is a set all of whose pairs are in the graph.

First we need the following, which will be obtained by forcing.

2.1. Let E be a set of indices of size $\aleph_1$. For every $a \in E$ there is a graph R_e on λ such that:

(a) For every countable $X \subseteq \lambda$, there is $e \in E$ such that X is a complete subgraph (clique) in R_e.

(b) Any complete subgraph of R_e is countable; and this remains true in any extension of V.

Our aim is the following.

2.2. THEOREM. *The following is consistent with ZFC (via forcing): There is a collection $\langle R_e \mid e \in E \rangle$ satisfying* 2.1, *and there are clubs $C_\xi \subseteq \omega_1$, and $f_\xi: C_\xi \to E$, for $\xi \in \lambda$, such that*:

If $\xi \neq \zeta$ and $C_\xi \cap C_\zeta$ is unbounded below $\delta < \omega_1$,
then $f_\xi(\delta) = f_\zeta(\delta) = e$ and $R_e(\xi, \zeta)$.

A moment of thought reveals not only that the intersection of uncountably many C_ξ's must be finite, but in fact for any $\delta < \omega_1$ and subfamily $\langle C_\xi \mid \xi \in X \rangle$ with pairwise unbounded intersection below δ, $\|X\| < \aleph_1$.

We turn to the proof of the theorem. Start for example in L, and fix a ♣ sequence $\langle \varepsilon(\delta, n) \mid \delta \in \omega_1' \text{ and } n \in \omega \rangle$. That is, $\varepsilon(\delta, n)$, $n \in \omega$, is an increasing sequence unbounded in δ, and for any club $C \subseteq \omega_1$ there is δ such that $(\forall n)\ \varepsilon(\delta, n) \in C$ (see [4]). It is easy to see that the ♣ property remains in any countably closed forcing: Given any name τ of a club, construct an increasing sequence of conditions, p_δ, for $\delta \in \omega_1'$, such that for some countable closed set $C_\delta \subseteq \omega_1$, p_δ forces "C_δ is the set of the first δ members of τ". Now $C = \bigcup_{\delta < \omega_1} C_\delta$ is a club subset of ω_1 in the ground model, and so there is δ such that $\varepsilon(\delta, n) \in C$ for all n. So p_δ forces $\varepsilon(\delta, n) \in \tau$ for all n.

Now we get a family $\langle R_e \mid e \in E \rangle$ satisfying (2.1). We can get this by forcing with the poset that adds for each $\xi \in \lambda$ a generic function $g_\xi: \omega_1 \to \omega$, with countable conditions. (This is the same as adding λ many Cohen subsets of ω_1.) If V is the universe obtained, and we set for $e \in \omega_1 = E$

$$R_e(\xi, \zeta) \quad \text{iff} \quad g_\xi(e) \neq g_\zeta(e),$$

then (a) and (b) of 2.1 hold.

Remark that, as we got V by a σ-closed forcing, the ♣ sequence retains its property in V.

Then we add to V $\aleph_1$ many Cohen generic reals: s_i, $i \in \omega_1$. Let V' be the universe thus obtained, $V' = V[\langle s_i \mid i \in \omega_1 \rangle]$. The CH holds in V', and $2^{\aleph_1} \geq \lambda$.

Next we shall get by forcing a collection $\langle r_{\xi,e} \mid \xi \in \lambda \text{ and } e \in E \rangle$ of subsets of ω such that:

2.3. (A) For every $e \in E$ and $\xi_0, \ldots, \xi_{n-1} \in \lambda$

$$[(\forall i \neq j < n) R_e(\xi_i, \xi_j)] \leftrightarrow \bigcap_{i<n} r_{\xi_i,e} \text{ is infinite.}$$

(B) If $\xi_1 \neq \xi_2$ and $e_1 \neq e_2$, then $r_{\xi_1,e_1} \cap r_{\xi_2,e_2}$ is finite.

To get this we define a forcing notion $\mathscr{P}$. The conditions in $\mathscr{P}$ are finite approximations. That is, $p \in \mathscr{P}$ iff p consists of a finite set of indexes and, for ξ and e in this set, p describes $r_{\xi,e} \restriction n$ for some n. The condition p' extends p iff p' gives more information and the following two conditions are met.

(A) If $r_{\xi,e}$ and $r_{\xi',e}$ appear in p and $\neg R_e(\xi, \xi')$, then p' does not add more points to $r_{\xi,e} \cap r'_{\xi,e}$ (other than those already existing by p).

(B) If $\xi_1 \neq \xi_2$ and $e_1 \neq e_2$ and $r_{\xi_1,e_1}, r_{\xi_2,e_2}$ appear in p, then the numbers which p' puts in $r_{\xi_1,e_1} \cap r_{\xi_2,e_2}$ are the same which p puts there.

$\mathscr{P} \in V$ readily satisfies the c.c.c. in V' (by the Δ-lemma). Let V'' be the generic extension of V' obtained via $\mathscr{P}$. In $V'' = V'[\langle r_{\xi,e} \mid \xi \in \lambda, e \in E \rangle]$, $2^{\aleph_0} \geq \lambda$.

It can be seen that, for a fixed $\xi \in \lambda$, the sequence $\langle r_{\xi,e} \mid e \in E \rangle$ is V' generic over the Cohen poset for adding $\aleph_1$ many reals.

If $\bar{r} = \langle r_e \mid e \in E \rangle$ is any sequence of Cohen generic reals over V', we define in $V'[\bar{r}]$ the poset $\mathscr{Q}(\bar{r})$ which will introduce a club set $C \subseteq \omega_1$ such that, for every limit $\delta \in \omega_1$,

2.4. There is $e \in E$, and $k \in \omega$, so that $n \geq k$

$$n \notin r_e \Rightarrow [\varepsilon(\delta, n), \varepsilon(\delta, n+1)) \cap C = \varnothing.$$

A condition is therefore a countable characteristic function of a closed subset of some successor of limit ordinal (called the height of the condition) such that for every limit δ in that successor ordinal the above requirement holds.

Now we are ready to define in V'' the poset $\mathscr{R}$ which gives the desired model.

2.5. Definition. $q \in \mathscr{R}$ iff q satisfies the following conditions: a) q is a function, and $\mathrm{Dom}(q) \subseteq \lambda$ is countable and is in V. b) For $\xi \in \mathrm{Dom}(q)$, $q(\xi) \in \mathscr{Q}(\langle r_{\xi,e} \mid e \in E \rangle)$. c) For some $\alpha = \alpha(q)$ and for all ξ, $q(\xi)$ is defined on $\alpha + 1$. d) There is a function p in V such that $\mathrm{Dom}(p) = \mathrm{Dom}(q)$ and, for every $\xi \in \mathrm{Dom}(p)$, $p(\xi)$ is a name of $q(\xi)$ in the Cohen extension

$$V[\langle s_i \mid i \in \omega_1 \rangle][\langle r_{\xi,e} \mid e \in E \rangle]$$

(over the poset for adding $\omega_1 + \omega_1$ many Cohen reals). We call p a "function name of q".

The partial order of $\mathscr{R}$ is defined naturally: q' extends q iff $\mathrm{Dom}(q') \supseteq \mathrm{Dom}(q)$, $\alpha(q') \geq \alpha(q)$ and for $\xi \in \mathrm{Dom}(q)$

$$q'(\xi) \restriction \alpha(q) = q(\xi).$$

Some properties of $\mathscr{R}$ are quite obvious, and we shall just list them; then we shall prove that $\mathscr{R}$ does not collapse $\aleph_1$.

2.6. *Properties of* $\mathscr{R}$. (1) For any $q \in R$ and $\beta > \alpha(q)$ there is an extension p of q with $\beta = \alpha(p)$. (A trivial way is to let $p(\xi)$ be constant zero on $(\alpha(q), \beta]$; that is, the intersection of the closed set with that interval is emtpy—we say p "is obtained by adding zeros to q".)

(2) $\mathscr{R}$ satisfies the $\aleph_2$-c.c. in V'' (by the standard Δ-argument).

(3) Let G be a V'' generic filter over $\mathscr{R}$, and for $\xi \in \lambda$ define $C_\xi = \bigcup\{q(\xi) \mid q \in G\}$. Let $f_\xi(\delta) = e$ be given by (2.4) when $\delta \in \mathrm{Dom}(q(\xi))$. Then the property required by Theorem 2.2 holds: If $C_\xi \cap C_\zeta$ is unbounded in $\delta \in \lim(\omega_1)$, then $e = f_\xi(\delta) = f_\zeta(\delta)$ and $R_e(\xi, \zeta)$.

2.7. MAIN LEMMA. *$\mathscr{R}$ does not collapse $\aleph_1$; in fact, $\mathscr{R}$ does not add new countable subsets to V''.*

PROOF. Given a name τ in $\mathscr{R}$-forcing of a possible collapsing function, construct in V an increasing and continuous sequence of length ω_1 of countable elementary submodels of $(H(\lambda^+), \in)$ which contain the name of τ. The intersection of these submodels with ω_1 form a club set, and so we get a limit $\delta < \omega_1$ such that for every n, $\varepsilon(\delta, n) = M_n \cap \omega_1$, where $\langle M_n \mid n \in \omega\rangle$ is an increasing subsequence of the above sequence. Let $M = \bigcup_{n\in\omega} M_n$. Let $h\colon \omega \to M$ be an enumeration of M in V. Fix $e_0 \in E$ such that $R_{e_0}(\xi, \zeta)$ holds for $\xi, \zeta \in M \cap \lambda$.

V'' was obtained from V by a c.c.c. forcing, so it is possible to extend M_n to an elementary submodel N_n of $H(\lambda^+)^{V''}$ without adding ordinals (using the restriction of the generic filter to M_n). Let $N = \bigcup_{n\in\omega} N_n$. We shall find a condition in $\mathscr{R}$ which is N generic, and this suffices.

Pick any $\delta \leq i < \omega_1$; so $s = s_i$ is a $V[N]$-generic Cohen real. Also $s\colon \omega \to \omega$. (Actually we should first collapse N to a countable transitive structure, but this complicates the notation—see[1] for a detailed treatment of this point.) We shall define the required $q \in \mathscr{R}$ by setting $\mathrm{Dom}(q) = M \cap \lambda$ and $\alpha(q) = \delta = M \cap \omega_1$. For $\xi \in M \cap \lambda$ we shall define $q(\xi)$ as the union of an increasing sequence $q_n(\xi)$, $n \in \omega$, in $N \cap \mathscr{Q}(\langle r_{\xi,e} \mid e \in E\rangle)$, and then add δ to the club set.

The definition of the sequence of $q_n(\xi)$'s is done in $V'[r_{\xi,e} \mid e \in E]$, and is done in such a uniform way that the function q thus obtained is indeed in $\mathscr{R}$.

Let $n(\xi)$ be the first n such that $h(s(n)) = p$ is a countable function and $\xi \in \mathrm{Dom}(p)$. Let $k(\xi)$ be the first $k \geq n(\xi)$ such that $\xi = h(s(m))$ for some $m \leq k$.

Now $q_n(\xi)$ is defined inductively. For $n = 0$, $q_0(\xi) = \varnothing$ (in fact we should start with some given condition). Suppose $q_n(\xi)$ is defined.

If $n < n(\xi)$ we set $q_{n+1}(\xi) = \varnothing$.

Suppose $n \geq n(\xi)$. If $n \leq k(\xi)$, or $n > k(\xi)$ but $n \in r_{\xi,e_0}$, we look at $h(s(n)) = p \in M$. If p is a countable function and $p(\xi)$ is a name in the Cohen forcing for adding $\omega_1 + \omega_1$ many reals, and if the interpretation X of $p(\xi)$ in $V[s_i \mid i \in \omega_1][V_{\xi,e} \mid e \in E]$ is in $\mathscr{Q}(\langle r_{\xi,e} \mid e \in E\rangle)$ and extends $q_n(\xi)$ and is of height $< \varepsilon(\delta, n+1)$, then we let $q_{n+1}(\xi)$ be the extension of X of height $\varepsilon(\delta, n+1)$

obtained by adding zeros. Otherwise $q_{n+1}(\xi)$ is the extension of $q_n(\xi)$ of height $\varepsilon(\delta, n+1)$ obtained by adding zeros.

If $n > k(\xi)$ and $n \notin r_{\xi,e_0}$, we let $q_{n+1}(\xi)$ be the extension of $q_n(\xi)$ of height $\varepsilon(\delta, n+1)$ obtained by adding zeros. Set

$$q(\xi) = \{\delta\} \cup \bigcup_{n \in \omega} q_n(\xi).$$

(2.4) holds naturally for ordinals $< \delta$, and explicitly for δ.

It is not hard to see that $q \in \mathscr{R}$.

Now for $n \in \omega$ define the function $\bar{q}_n$ by setting

$$\mathrm{Dom}(\bar{q}_n) = \{\xi \mid n(\xi) < n\} \quad \text{and} \quad \bar{q}_n(\xi) = q_n(\xi).$$

2.8. *Claim.* $\bar{q}_n \in \mathscr{R} \cap N$, $\bar{q}_n \leq q$, *is a generic sequence for* N; *that is, if* $D \in N$ *is dense open in* $\mathscr{R}$ *then* $\bar{q}_n \in D$ *for some* n.

PROOF. It can be checked that $\bar{q}_n \in \mathscr{R} \cap N$, and the sequence itself is defined in $V[N][s]$. So the genericity of the sequence is shown by a density argument for the Cohen forcing which gave s. Suppose $s \restriction n$ is a condition, and $D \in N$ is a dense open subset of $\mathscr{R}$. Then $s \restriction n$ determines $\bar{q}_n$ as a condition in $\mathscr{R} \cap N$ of height $\varepsilon(\delta, n)$. Let $\xi_0, \ldots, \xi_{m-1}$ be the ordinals ξ with $k(\xi) < n$. Find $k \geq n$ such that $k \in \bigcap_{i=0}^{m-1} r_{\xi_i, e_0}$ and $\bar{q}_n$, $D \in N_k$. Extend $s \restriction n$ to $s \restriction k$ so that $h(s(i)) = \varnothing$ for $n \leq i < k$. Then $s \restriction k$ determines that $\bar{q}_k$ is the extension of $\bar{q}_n$ of height $\varepsilon(\delta, k)$ obtained by adding zeros. Hence $\bar{q}_k \in N_{k+1}$. Pick $q \in D \cap N_{k+1}$ extending $\bar{q}_k$, and let $p \in M_{k+1}$ be a function name for q. There is l with $h(l) = p$. Put $s(k) = l$. Now $s \restriction k+1$ forces that $\bar{q}_{k+1}$ extends q.

REFERENCES

[1] U. ABRAHAM, *On forcing without the continuum hypothesis*, this JOURNAL, vol. 48 (1983), pp. 658–661.

[2] J. E. BAUMGARTNER, *Applications of the proper forcing axiom*, ***Handbook of set-theoretic topology*** (K. Kunen and J. E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 913–959.

[3] J. E. BAUMGARTNER. A. HAJNAL and A. MÁTÉ, *Weak saturation properties of ideals*, ***Infinite and finite sets (Colloquium, Keszthely, 1973, dedicated to Paul Erdös on his sixtieth birthday), Vol. I***, Colloquia Mathematica Societatis János Bolyai, no. 10, North-Holland, Amsterdam, 1975, pp. 137–158.

[4] A. J. OSTASZEWSKI, *On countably compact, perfectly normal spaces*, ***Journal of the London Mathematical Society***, ser. 2, vol. 14 (1976), pp. 505–516.

DEPARTMENT OF MATHEMATICS
BEN GURION UNIVERSITY OF THE NEGEV
BEER SHEVA, ISRAEL

INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY
JERUSALEM, ISRAEL