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ITERATED FORCING AND NORMAL IDEALS ON ω_1

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ABSTRACT

We prove that suitable iteration does not collapse $\aleph_1$ [and does not add reals], i.e., that in such iteration, certain sealing of maximal antichains of stationary subsets of ω_1 is allowed. As an application, e.g., we prove from supercompact hypotheses, mainly, the consistency of: ZFC + GCH + "for some stationary set $S \subseteq \omega_1$, $\mathcal{P}(\omega_1)/(D_{\omega_1} + S)$ is the Levy algebra" (i.e., the complete Boolean Algebra corresponding to the Levy collapse Levy $(\aleph_0, < \aleph_2)$ (and we can add "a variant of PFA") and the consistency of the same, with "Ulam property" replacing "Levy algebra"). The paper assumes no specialized knowledge (if you agree to believe in the semi-properness iteration theorem and RCS iteration).

§0. Introduction

By Foreman, Magidor and Shelah [FMS 1], CON(ZFC + κ is supercompact) implies the consistency of ZFC + "D_{ω_1} is $\aleph_2$-saturated" [i.e., if $\mathfrak{B}$ is the Boolean algebra $P(\omega_1)/D_{\omega_1}$, "D_{ω_1} is $\aleph_2$-saturated" means "$\mathfrak{B}$ satisfies the $\aleph_2$-c.c."]; see there for previous history. This in fact was deduced from the Martin Maximum by [FMS 1] whose consistency was proved by RCS iteration of semi-proper forcings (see [Sh 1]). Note that [FMS] refutes the thesis: in order to get an elementary embedding j of V with small critical ordinal, into some transitive class M of some generic extension V^P of V, *you should start* with an elementary embedding of j of V' into some M' and then force over V'.

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This thesis was quite strongly rooted. Note that it is closely connected to the existence of normal filters D on λ which are λ^+-saturated or at least precipitous (use for P the set of non-zero members of $\mathscr{P}(\lambda)/D$ ordered by inverse inclusion, j the generic ultrapower). See [FMS 1] for older history.

In fact, it was proved directedly that $MM^+ \equiv SPFA^+$. Much later we prove that MM is equivalent to the Semi-Proper Forcing Axiom (in ZFC) [Sh 5].

Following [FMS 1, §1, §2] much activity follows. Woodin proves from CON(ZF + ADR + θ regular) the consistency of ZFC + "$\mathfrak{B} \restriction S$ is $\aleph_1$-dense", for some stationary $S \subseteq \omega_1$.

By Shelah and Woodin [ShW], if there is a supercompact cardinal, then every projective set of reals is Lebesgue measurable (etc.). This was obtained by combining (A) and (B) below which were proved simultaneously:

(A) The conclusion holds if there are weakly compact cardinal κ and a forcing notion P, $|P| = \kappa$, satisfying the κ-c.c., not adding reals and $\Vdash_P$ "there is a normal filter D on ω_1, $B = \mathscr{P}(\omega_1)/D$ satisfying the $\aleph_2$-c.c."

(B) There is a forcing as required in (A) (see [FMS 1, §3]).

This was improved to using cardinals κ satisfying: $\mathrm{Pr}_a(\kappa) \overset{\mathrm{def}}{=} \kappa$ is strongly inaccessible, and for every $f: \kappa \to \kappa$ there is an elementary embedding $j: V \to M$ (M a transitive class), κ the critical ordinal of j and $H(j(f)(\kappa)) \subseteq M$, or at least $\mathrm{Pr}_b(\kappa)^\dagger \overset{\mathrm{def}}{=} \kappa$ is strongly inaccessible, and for every $f: \kappa \to \kappa$ there is $\kappa_1 < \kappa$, $(\forall \alpha < \kappa_1) f(\alpha) < \kappa_1$ and for some elementary embedding $j: V \to M$ (M a transitive class), κ_1 is the critical ordinal of j and $H((j(f))(\kappa_1)) \subseteq M$ ($S \subseteq \omega_1$ stationary, co-stationary).

For the Lebesgue measurability of every projective set, we use approximately n such cardinals for sets Σ_n.

By [Sh 2] "$2^{\aleph_0} < 2^{\aleph_1} \Rightarrow D_{\omega_1}$ is not $\aleph_1$-dense", and if D is a layered filter on λ then $\mathscr{P}(\lambda)$ is the union of λ filters extending D.

The work presented here was done then, but was mistakenly held as incorrect for quite a time. Only we here replace part of the consistency proof of the Ulam assertion ($\mathscr{P}$, ω_1) is the union of $\aleph_1$ $\aleph_1$-complete non-trivial measures, by a deduction from a strong variant of layerness. We prove, from some

† Now usually called Woodin cardinals. After this work was written, the results of Martin Steel and Woodin clarify the connection between determinacy and large cardinals and Woodin has some consistency proofs for larger ideals on ω_1; very interesting among them is "$D_{\omega_1} + S$ is $\aleph_1$-dense" from a huge cardinal.

supercompacts, that we can force, for a stationary co-stationary $S \subseteq \omega_1$, that GCH + $\mathscr{P}(\omega_1)/(D_{\omega_1}+S)$ is the Levy algebra, i.e., the completion of Levy $(\aleph_0, <\aleph_2)$ (or $S=\omega_1$, but $2^{\aleph_0}=\aleph_2$) and related results. The method is (RCS) semi-proper iteration, hence it can be used, e.g., to prove consistency with additional statements. Relative to the progress of [ShW], we improved some results to the use of κ's satisfying Pr_b instead of supercompact; this is not written now as [ShW] hasn't materialized yet, but as a result we do not try to "save" in the use of large cardinals (one such λ suffices for D_{ω_1} $\aleph_1$-saturated).†

Note that the κ supercompact is seemingly necessary if we want a suitable variant of MA. Those points (i.e., using smaller large cardinals and getting also variants of MA (mainly without adding reals)) will be dealt with in a sequel paper. Around the time this was reasserted Woodin proved, from some κ's satisfying Pr_b, that CON(ZFC + $2^{\aleph_0}=\aleph_2$ + $\mathscr{P}(\omega_1)/D_{\omega_1}$) is the Levy algebra, by methods related to Steel forcing.

Also in Spring 1983, Foreman proved (in [FMS 2]), from the consistency of almost huge cardinals, CON(ZFC + GCH + on λ^+ + there is a layered ideal) for arbitrary regular λ. This was interesting as by [FMS 2], we can get a uniform ultrafilter D on λ which not only is not regular but even $\lambda^{\lambda^+}/D=\lambda^+$. Note that some years ago Magidor [Mg] proved the consistency of ZFC + GCH + "for some uniform ultrafilter on $\aleph_2$, $\aleph_0^{\aleph_2}/D=\aleph_2$". Another result of [FMS 1] is that we can get "for every λ, D_λ is precipitous" and we can get this even to Chang filters. Later Gitik, for λ a supercompact cardinal, gave a proof which does not use a supercompact above λ.

Much later the author proves, for D a layered filter on λ, that (if $2^\lambda=\lambda^+$, $\lambda=\lambda^{<\lambda}$) there is a homomorphism h from $\mathscr{P}(\lambda)/D$ into $\mathscr{P}(\lambda)$, $A/D=[h(A/D)]/D$; see [Sh 6].

Note that S-completeness is used for convenience; weaker notions can be used as well (see [Sh 1, VIII, §4], [Sh 4, §2]). We can also make that set of ordinals in which something nice occurs, a name.

Note that instead of one S, we can change it getting the result for a maximal antichain of S's.

Note that we can use 2.13 (3) much more extensively, e.g. in 2.19. Suppose κ is strongly inaccessible $\{\kappa_i, i<\kappa\}$ increasing continuous, κ_{i+1} is supercompact

† For "ZFC + D_{ω_1} is $\aleph_2$-saturated + MA(semi-proper)" one λ suffices.

For "ZFC + GCH + $(D_{\omega_1}+S_1)$ is layered + MA_{ω_1} (S_2-complete semi-proper forcing of power $\leqq \aleph_2$)" (S_1, S_2, S_3 a partition of ω_1 to a stationary set) it suffices to use $\{\lambda : \lambda<\kappa, \lambda$ satisfies $\mathrm{Pr}_b\}$ is stationary.

Similarly for the Levy algebra (with a suitably stronger assumption).

for i non-limit, $\kappa = \bigcup_{1<i} \kappa_i$ $\langle P_i, Q_i, t_i, B_i : i < \kappa \rangle$ an RCS iteration, $|P_i| < \kappa_{i+1}$, for i non-limit, $Q_i = S\,\mathrm{Seal}(\langle B^{P_j+1} : j+1 < i \rangle, S, \kappa_{i+1})$.

NOTATION AND BASIC FACTS.

(1) $\mathscr{P}(A)$ is the power set of A, $\mathrm{S}_{<\lambda}(A) = \{B : B \subseteq A, |B| < \lambda\}$, $<^*_\lambda$ is a well ordering of $H(\lambda)$, extending $<^*_\mu$ for $\mu < \lambda$.

(2) D_λ is the club filter on a regular $\lambda < \aleph_0$ and $D_{<\lambda}(A)$ is the club filter on $\mathrm{S}_{<\lambda}(A)$.

(3) (a) $\mathfrak{B}$ is the Boolean Algebra $\mathscr{P}(\omega_1)/D_\lambda$; we do not distinguish strictly between $A \in \mathscr{P}(A)$ and A/D_{ω_1}.

(b) $\mathfrak{B}$ of course depends on the universe, so we may write $\mathfrak{B}^{V^1}$ or $\mathfrak{B}[V^1]$; instead of $\mathfrak{B}[V^P]$ we may write $\mathfrak{B}^P$ or $\mathfrak{B}[P]$.

(c) If $V^1 \subseteq V^2$, $\omega_1^{V^1} = \omega_1^{V^2}$, then $\mathfrak{B}[V^1]$ is a weak subalgebra of $\mathfrak{B}[V^2]$ (i.e., maybe distinct elements in $\mathfrak{B}[V^1]$ are identified in $\mathfrak{B}[V^2]$).

(d) If $P \in V$ is a forcing notion preserving stationary subsets of ω_1 then $\mathfrak{B} = \mathfrak{B}[V]$ is a subalgebra of $\mathfrak{B}^P$ (identifying $(A/D_{\omega_1})^V$ and $(A/D_{\omega_1})^{V^P}$ for $A \in \mathscr{P}(\omega_1)^V$). If $\bar{Q} = \langle P_i, Q_j : i < \alpha \rangle$ is an iteration (with limit P_α, so $i < j \leqq \alpha \Rightarrow P_i \lessdot P_j$), we let $\underset{\sim}{B}^{\bar{Q}} = \bigcup_{i<\alpha} B^{P_{i+1}}$.

(4) (a) Let us, for Boolean algebras B_1, B_2, say $B_1 \lessdot B_2$ if $B_1 \subseteq B_2$ (i.e., B_1 is a subalgebra of B_2) and every maximal antichain of B_1 is a maximal antichain of B_2.

(b) Note that $B_1 \lessdot B_1$, *iff* B_1, B_2 are Boolean algebras, $B_1 \subseteq B_2$ and $(\forall x \in B_2 - \{0\})\,(\exists y \in B_1 - \{0\})\,(\forall z \in B_1)[z \cap y \neq 0 \rightarrow z \cap x \neq 0]$. However $B_1 \lessdot B_3$, $B_1 \subseteq B_2 \subseteq B_3$ implies $B_1 \lessdot B_2$.

(c) Hence, the satisfaction of "$B_2 \lessdot B_2$" does not depend on the universe of set theory, i.e., if $V \vDash B_1 \lessdot B_2$, $V \subseteq V^1$ then $V^1 \vDash B_1 \lessdot B_2$.

(d) By Solovay–Tenenbaum [ST] $\lessdot$ is transitive, and if $\langle B_i : i < \alpha \rangle$ is $\lessdot$-increasing and continuous then $B_i \lessdot \bigcup_{j<\alpha} B_j$.

Also, if $\langle B_\zeta : \zeta < \xi \rangle$ is a $\subseteq$-increasing sequence of Boolean algebras and $B_0 \lessdot B_\zeta$ for $\zeta < \xi$, then $B_0 \lessdot \bigcup_{\zeta<\xi} B_\zeta$.

(5) Also, if in V $P_1 \lessdot P_2 \lessdot P_3$, in V^{P_2} $\mathfrak{B}^{P_1} \lessdot \mathfrak{B}^{P_2}$, and in V^{P_3} $\mathfrak{B}^{P_2} \lessdot \mathfrak{B}^{P_3}$ *then* in V^{P_3} $\mathfrak{B}^{P_1} \lessdot \mathfrak{B}^{P_3}$;

(6) For a set a and forcing notion P, $\underset{\sim}{G}_P$ is the P-name of the generic set and $a[\underset{\sim}{G}_P] = a \cup \{\underset{\sim}{x}[\underset{\sim}{G}_P] : \underset{\sim}{x} \in a$ a P-name$\}$. So $a[\underset{\sim}{G}_P]$ is a P-name of a set, and for $G \subseteq P$ generic over V its interpretation is $a[G] = a \cup \{\underset{\sim}{x}[G] : \underset{\sim}{x} \in a$ a P-name$\}$ ($\underset{\sim}{x}[G]$ is the interpretation of the P-name V).

(7) We sometimes do not strictly distinguish between a model and its universe.

(8) P, Q, R denote forcing notions, $\emptyset_P$ denotes the minimal element of P (i.e., $\Vdash_P$ "$\cdots$" iff $\emptyset \Vdash_P$ "$\cdots$"); without loss of generality it exists.

(9) If $\lambda > \aleph_0$ is a cardinal, N a countable elementary submodel of $(H(\lambda), \in)$, $P \in N$, $G \subseteq P$ generic over V, then $N[G] \prec (H(\lambda)^{V^P}, \in)$ (as $H(\lambda)^{V^P} = \{\underset{\sim}{\tau}[G]: \underset{\sim}{\tau} \in H(\lambda)$ a P-name$\}$ and if $\Vdash_P$ "$(H(\lambda)^{V^P}, \in) \models x\phi(x\underset{\sim}{a})$" then for some P-name $\underset{\sim}{\tau} \in H(\lambda) \Vdash_P$ "$(H(\lambda)^{V^P}, \in) \models \phi(\underset{\sim}{\tau}, \underset{\sim}{a})$"). See [Sh 1].
Also $(N, G) \prec (H(\lambda)^V, \in, G)$ (i.e., G is an extra predicate, so you may write $(N, G \cap |N|)$). Also, if R is any relation (or sequence of relations) on $H(\lambda)^V$, $N \prec (H(\lambda)^V, \in, R)$ (and $P \in N$, $G \subseteq N$ generic over V) *then* $(N, G) \prec (H(\lambda)^V, \in, R, G)$. Usually we use a well ordering $<^*$ of $H(\lambda)$.

(10) Let $N <_\kappa M$ mean $M \subseteq M$ and $N \cap \kappa$ is an initial segment of $M \cap \kappa$ and $M \prec N$; if we use it for sets (rather than models), the last demand is omitted. Note that if $N \prec M \prec (H(\mu), \in)$, $\kappa < \mu$, $N \cap \kappa = M \cap \kappa$ then $N <_{\kappa^+} M$.

§1. Preliminaries

1.1. DEFINITION.

(1) A forcing notion P is semi-proper if: for every λ regular $> 2^{|P|}$, any countable $N \prec (H(\lambda), \in)$ to which P belongs, and $p \in P \cap N$ *there is* q, $p \leqq q \in P$, q (N, P)-semi-generic (see below).

(2) For a set a, a forcing notion P and $q \in P$, we say q is (a, P)-semi-generic *if*: for every P-name $\underset{\sim}{\alpha} \in a$ of a countable ordinal, $q \Vdash_P$ "$\underset{\sim}{\alpha} \in a$"; i.e., if $q \Vdash$ "$a[\underset{\sim}{G}_P] \cap \omega_1 = a \cap \omega_1$".

(3) We call $W \subseteq \mathbf{S}_{<\aleph_1}(A)$ (where $\omega_1 \subseteq A$) *semi-stationary* in A *if* for every model M with universe A and countably many relations and functions, there is a countable $N \prec M$, such that $(\exists a \in W)[N \cap \omega_1 \subseteq a \subseteq N]$ [equivalently, $\{a \in \mathbf{S}_{<\aleph_1}(A): (\exists b \in W)[a \cap \omega_1 \subseteq b \subseteq a]\}$ is a stationary subset of $\mathbf{S}_{<\aleph_1}(A)$ (i.e., $\neq \emptyset \bmod D_{<\aleph_1}(A)$)].

1.2. CLAIM.

(1) If $W \subseteq \mathbf{S}_{<\aleph_1}(A)$ is stationary then it is semi-stationary.

(2) If $\omega_1 \subseteq A \subseteq B$, and $W \subseteq \mathbf{S}_{<\aleph_1}(A)$ *then*: W is semi-stationary in A iff W is semi-stationary in B (so we can omit "in A").

(3) If $W_1 \subseteq W_2 \subseteq \mathbf{S}_{<\aleph_1}(A)$, W_1 semi-stationary, then W_2 is semi-stationary.

(4) If $|A| = \aleph_1$, $A = \bigcup_{i<\omega_1} a_i$, a_i increasingly continuous in i, with a_i countable, *then* $W \subseteq \mathbf{S}_{<\aleph_1}(A)$ is semi-stationary iff $S_W = \{i : (\exists b \in W) i \subseteq b \subseteq a_i\}$ is stationary (as a subset of ω_1).

(5) If $p \in P$ is (b, P)-semi-generic, $b \cap \omega_1 \subseteq a \subseteq b$ *then* p is (a, P)-semi-generic.

(6) If $W \subseteq \mathbf{S}_{<\aleph_1}(\lambda)$, $\mu > \lambda$, $N \prec (H(\mu), \in)$, $W \in N$, and for some $a \in W$, $N \cap \omega_1 \subseteq a \subseteq N$ *then* W is semi-stationary [if not, some $M = (\lambda, \ldots, F_n, \ldots)$ exemplify W is not semi-stationary, so some such M belongs to N, hence $N \cap \lambda$ is a submodel of M, contradiction].

1.3. CLAIM. A forcing notion P is semi-proper iff

$$W_P = \{a \in \mathbf{S}_{<\aleph_1}(P \cup {}^P(\omega_1 + 1)) \text{: for every } p \in P \cap a \text{ there is } q, \text{ such that } p \leqq q \in P \text{ and } q \text{ is } (a, P)\text{-semi-generic}\}$$

contains a club of $\mathbf{S}_{<\aleph_1}(P \cup {}^P(\omega_1 + 1))$ where $h : P \to (\omega_1 + 1)$ is interpreted as a P-name $\underset{\sim}{\alpha}_h$ by:

$$\underset{\sim}{\alpha}_h^0[G] = \operatorname{Min}\{h(r) \colon r \in G\},$$

$\underset{\sim}{\alpha}_h[G]$ is $\underset{\sim}{\alpha}_h^0[G]$ if $\underset{\sim}{\alpha}_h^0[G] < \omega_1$ and zero otherwise.

1.4. CLAIM. The following are equivalent for a forcing notion P:

(1) P is semi-proper.

(2) P preserves semi-stationarity.

(3) P preserves semi-stationarity of subsets of $\mathbf{S}_{<\aleph_1}(2^{|P|})$.

PROOF. (1)$\Rightarrow$(2). Let $\omega_1 \subseteq A$, $W \subseteq \mathbf{S}_{<\aleph_1}(A)$ be semi-stationary. Suppose $p \in P$, $p \Vdash_P$ "W is not semi-stationary". So there are P-names of functions $\underset{\sim}{F}_n$ $(n < \omega)$ from A to A, $\underset{\sim}{F}_n$ n-place, and $p \Vdash$ "if $a \subseteq A$ is countable closed under $\underset{\sim}{F}_n$ $(n < \omega)$ *then* $\neg(\exists b)[a \cap \omega_1 \subseteq b \subseteq a \wedge b \in W]$".

Let λ be regular large enough. Let $N \prec (H(\lambda), \in)$ be countable, let A, $(\underset{\sim}{F}_n : n < \omega)$, p, P belong to N, and there is $b \in W$ such that $N \cap \omega_1 \subseteq b \subseteq N$ (which is possible as W is semi-stationary). Let q be (N, P)-semi-generic, $p \leqq q \in P$. So $q \Vdash_P$ "$N[\underset{\sim}{G}] \cap \omega_1 = N \cap \omega_1$ and $N \subseteq N[\underset{\sim}{G}]$" hence for the b above

$$q \Vdash_P \text{“}N[\underset{\sim}{G}] \cap \omega_1 \subseteq b \subseteq N[\underset{\sim}{G}]\text{”}.$$

Also $q \Vdash_P$ "$N[\underset{\sim}{G}] \cap A$ is closed under the $\underset{\sim}{F}_n$'s" (as $N[G] \prec (H(\lambda)[G], \in)$, see Notation (9) in §0), contradictory to the choice of the F_n's.

(2)$\Rightarrow$(3). Trivial.

$\neg$(1)$\Rightarrow$$\neg$(3). Let $W = \mathbf{S}_{<\aleph_1}(P \cup {}^P(\omega_1 + 1)) - W_P$ (W_P from 1.3). As $\neg$(1), W_P is stationary; for each $a \in W$ choose $p_a \in P \cap a$ which exemplifies $a \notin W_p$. By normality for some $p(*) \in P$, $W_1 = \{a \in W : p_a = p(*)\}$ is stationary.

Hence W_1 is semi-stationary. But easily $p(*) \Vdash$ "W_1 is not semi-stationary" so (3) fails.

1.5. DEFINITION.

(1) $\mathrm{Rss}(\kappa, \lambda)$ (reflection for semi-stationarity) is the assertion that for every semi-stationary $W \subseteq \mathbf{S}_{<\aleph_1}(\lambda)$ there is $A \subseteq \lambda$, $\omega_1 \subseteq A$, $|A| < \kappa$ such that $W \cap \mathbf{S}_{<\aleph_1}(A)$ is semi-stationary.

(2) $\mathrm{Rss}(\kappa)$ is $\mathrm{Rss}(\kappa, \lambda)$ for every $\lambda \geqq \kappa$.

(3) $\mathrm{Rss}^+(\kappa, \lambda)$ means that for every semi-proper P of cardinality $<\kappa$, $\Vdash_P$ "$\mathrm{Rss}(\kappa, \lambda)$".

(4) $\mathrm{Rss}^+(\kappa)$ is $\mathrm{Rss}^+(\kappa, \lambda)$ for every $\lambda \geqq \kappa$.

1.6. CLAIM.

(1) In Definition 1.5(1) we can replace λ by B, when $|B| = \lambda$, $\omega_1 \subseteq B$.

(2) If $\kappa \leqq \kappa_1 \leqq \lambda_1 \leqq \lambda$ and $\mathrm{Rss}(\kappa, \lambda)$ then $\mathrm{Rss}(\kappa_1, \lambda_1)$; and if $\kappa \leqq \lambda_1 \leqq \lambda$, $\mathrm{Rss}^+(\kappa, \lambda)$ then $\mathrm{Rss}^+(\kappa, \lambda_1)$; lastly if $\mathrm{Rss}^+(\kappa_i, \lambda)$ $(i < \alpha)$ then $\mathrm{Rss}^+(\sup_{i<\alpha} \kappa_i, \lambda)$.

(3) If κ is a compact cardinal then $\mathrm{Rss}(\kappa)$.

(4) If κ is a compact cardinal then $\mathrm{Rss}^+(\kappa)$.

(5) If κ is measurable, $W_i \subseteq \mathbf{S}_{<\kappa}(A)$ and $\bigcup_{i<\kappa} W_i$ is semi-stationary then for some $\alpha < \kappa$, $\bigcup_{i<\alpha} W_i$ is semi-stationary.

(6) If κ is a limit of compact cardinals, then $\mathrm{Rss}^+(\kappa)$.

PROOF. (1) Trivial.

(2) Use 1.2(2).

(3) Let $\kappa \subseteq A$, $W \subseteq \mathbf{S}_{<\aleph_1}(A)$, $W \subseteq \mathbf{S}_{<\aleph_1}(A)$, $W \cap \mathbf{S}_{<\aleph_1}(B)$ not semi-stationary for $B \subseteq A$, $|B| < \kappa$.

Define the set of sentences Γ:

$$\Gamma = \Gamma^a \cup \Gamma^b \cup \Gamma^c$$

where

$$\Gamma^a = \{c_1 \neq c_2 : c_1, c_2 \text{ are distinct members of } A\},$$

$$\Gamma^b = \{R(c_0, c_1, \ldots, c_n, \ldots)_{l<\omega} : c_l \in A,\ \{c_l : l < \omega\} \in W\},$$

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$$\Gamma^c = \Big\{ (\forall x_0, x_1, \dots, x_n, \dots)_{n<\omega}$$

$$\Big[\text{if } \{x_0, x_1, \dots\} \text{ is closed under } F_n (n<\omega) \textit{ then}$$

$$\neg(\exists y_0, y_1, \dots) \Big(R(y_0, \dots, y_n, \dots) \wedge \Big\{ x_l : l<\omega, \bigvee_{i<\omega_1} x_l = i \Big\}$$

$$\subseteq \{y_l : l<\omega\} \subseteq \{x_m : m<\omega\} \Big) \Big] \Big\}.$$

Every subset of Γ of power $<\kappa$ has a model (if it mentions only $c \in B$, $|B| < \kappa$, then use a model witnessing "$W \cap S_{<\aleph_1}(B)$ is not semi-stationary"). A model M of Γ exemplifies W is not semi-stationary (in $|M|$ hence in A by 1.2(2)).

(4) As forcing notions of cardinality $<\kappa$ preserve the compactness of κ.

(5) Let Γ^a, Γ^c be as in the proof of 1.6(4),

$$\Gamma^b_i = \{R(c_0, c_1, \dots) : c_l \in A, \{c_l : l<\omega\} \in W_i\}.$$

Now $\Gamma^a \cup \Gamma^c \cup \bigcup_{i<\kappa} \Gamma^b_i$ has no model, hence (using the Los theorem) for some $\alpha<\kappa$, $\Gamma^a \cup \Gamma^c \bigcup_{i<\alpha} \Gamma^b_i$ has no model.

(6) Easy.

1.7. Claim.

(1) If $\mathrm{Rss}(\kappa, 2^{|P|})$, P not semi-proper, *then* P destroys the semi-stationarity of some $W \subseteq S_{<\aleph_1}(A)$, $|A| < \kappa$ [use (1)$\Leftrightarrow$(3) from 1.4, then 1.5(1), 1.2(2)].

(2) If P destroys the semi-stationarity of $W \subseteq S_{<\aleph_1}(A)$, $|A| = \aleph_1$, *then* P destroys the stationarity of $S_W \subseteq \omega_1$ [S_W defined in 1.2(4), which says that it is stationary in V but not in V^P].

(3) If $\mathrm{Rss}(\aleph_2, 2^{|P|})$ and P does not destroy stationarity of subsets of ω_1, *then* P is semi-proper [by parts (1), (2) above].

(4) If $W \subseteq S_{<\aleph_1}(A)$ exemplifies the failure of $\mathrm{Rss}(\aleph_2, |A|)$, *then* there is a forcing notion P of power $|A|^{\aleph_0}$, not semi-proper but not destroying stationary subsets of $\aleph_1$.

(5) $\mathrm{Rss}(\aleph_2)$ is equivalent to: every forcing notion preserving stationarity of subsets of ω_1 is semi-proper.

1.8. Definition. $\langle P_i, \tilde{Q}_j : i \leqq \alpha, j<\alpha \rangle$ is a semi-proper iteration if:

(1) it is an RCS iteration [see [Sh 1, Ch. X, §1];

(2) if $i<j\leqq\alpha$ are non-limit, then $\Vdash_{P_i}$ "P_j/P_i is semi-proper";
(3) for every i, $\Vdash_{P_{i+2}}$ "$(2^{\aleph_1})^{V^{P_i}}$ is collapsed to $\aleph_1$" (we can use another variant instead).

1.9. THEOREM. *Suppose λ is measurable, $\langle P_i, \tilde{Q}_i : i<\lambda\rangle$ is a semi-proper iteration, $|P_i|<\lambda$ for $i<\lambda$, and $\{i<\lambda : Q_i$ semi-proper$\}$ belong to some normal ultrafilter D on λ. Then in V^{P_λ}, Player II wins* $\mathrm{Gm}=\mathrm{Gm}(\{\aleph_1\},\omega,\aleph_2)$ (see below).

1.9A. REMARK as in [Sh 1, Ch. XII, Def. 1.1].
(1) The game last ω moves; in the nth move Player I chose $f_n:\aleph_2\rightarrow\omega_1$ and Player II chose $\xi_n<\omega_1$. In the end Player II wins if $A\stackrel{\mathrm{def}}{=}\{i<\aleph_2 : \bigwedge_n\bigvee_m f_n(i)<\xi_m\}$ is unbounded in $\aleph_2$.
(2) We can modify the game by demand $A\neq\emptyset \bmod E$ for a filter E on ω_2. We then denote the game by $\mathrm{Gm}(\{\aleph_1\},\omega,E)$. The result is true for $E=D$.
(3) If Player II wins $\mathrm{Gm}(\{\aleph_1\},\omega,\aleph_2)$, $\lambda>2^{\aleph_2}$, N a countable elementary submodel of $(H(\lambda),\in,<^*_\lambda)$, *then* for arbitrarily large $i<\omega_2$, there is $N'\prec(H(\lambda),\in,<^*_\lambda)$, N' countable, $N\subseteq N'$, $i\in N_1$ and $N\cap\omega_1=N'\cap\omega_1$ (hence $N<_{\omega_2}N'$, see Basic Fact (10) in §0).
If Player II wins $\mathrm{Gm}(\{\aleph_1\},\omega,E)$ (E a filter on ω_2) then the set of such i is $\neq\emptyset \bmod E$.
(4) Can we demand in (3) $N'\cap i=N\cap i$? If $\{\delta<\omega_2 : \mathrm{cf}\,\delta=\aleph_0\}\in E$ the answer is No. If $\{\delta<\omega_2 : \mathrm{cf}\,\delta=\aleph_1\}\in E$ the answer is Yes if Player I is also allowed to choose regressive function $F_n:\aleph_2\rightarrow\aleph_2$, and Player II also $\xi'_n<\omega_2$, and in the end Player II wins if $S=\{\delta<\aleph_2 :$ for $n<\omega$, $\delta\geqq\xi_n$, $f_n(\delta)<\bigcup_m\xi_m$, $F'_n(\delta)<\bigcup_m\xi'_m\}\neq\emptyset \bmod E$ (or just S is non-empty).
(5) If in the theorem $\Vdash_P$ "$\{\delta<\aleph_2 : \mathrm{cf}\,\delta=\aleph_1\}\neq\emptyset \bmod D$" *then* Player II wins also in this variant (from (4) above), hence we can demand in (3) that $N'\cap i=N\cap i$.
(6) We can replace $\aleph_1$ by any regular ϑ, $\aleph_0<\vartheta<\lambda$, and use the game $\mathrm{Gm}(\{\vartheta\},\mu,E)$, E a normal filter on λ, $\langle P_i, Q_i : i<\lambda\rangle$ is a $(<\vartheta)$-revised support iteration, such that, i: in V^{P_i} in $\tilde{P}G^\mu(p,P_\lambda/P_i,\vartheta)$, the second player has a winning strategy.

PROOF OF 1.9. Let D be a normal ultrafilter on λ (in V), $A\in D$ a set of (strongly) inaccessible cardinals, such that: $(\forall\kappa\in A)\,[(\forall i<\kappa)\ |P_i|<\kappa)\wedge Q_\kappa$ is semi-proper].

For each $\kappa \in A$, P_λ / P_κ (in V^{P_κ}) is a semi-proper forcing, hence in the following game, $PG^\omega(p, P_\lambda/P_\kappa, \aleph_1)$ Player II has a winning strategy which we call $F(P_\lambda/P_\kappa)(\in V^{P_\kappa})$: [by [Sh 1], Ch. XII, 2.7(3), p. 403, Definition 2.4, p. 401] a play of the game lasts ω-moves, in the nth move Player I chooses a P_λ/P_κ-name $\underset{\sim}{\zeta}_n$ of a countable ordinal and Player II chooses a countable ordinal ξ_n.

Player II wins a play if $(\exists q)\ (p \leqq q \in P_\lambda/P_\kappa \wedge q \Vdash$ "$\bigwedge_n [\underset{\sim}{\zeta}_n < \bigcup_{m<\omega} \xi_m]$"); without loss of generality the ξ_n are strictly increasing.

Let us describe a winning strategy for Player II in $\mathrm{Gm}(\{\aleph_1\}, \omega_1, \aleph_2)$ in $V[G_\lambda]$, $G_\lambda \subseteq P_\lambda$ generic over V.

In the nth move Player I chooses $f_n : \omega_2 \to \omega_1$, Player II, in addition to choosing $\xi_n < \omega_1$, chooses $A_n, \underset{\sim}{f}_n, \alpha_n$ such that:

(0) $\alpha_n < \alpha_{n+1} < \lambda$; in stage n Player II works in $V[G_{\alpha_n}]$, so D is still an ultrafilter;

(1) $A_n \in D$, $A_{n+1} \subseteq A_n \subseteq A$ (and $A_n \in V$);
$(\forall \delta \in A_n)\ (\alpha_n < \delta)$;

(2) $\Vdash_P$ "$\underset{\sim}{f}_n : \omega_2 \to \omega_1$";

(3) $\underset{\sim}{f}_n[G_\lambda] = f_n$; $\underset{\sim}{f}_n$ is the first such name;

(4) for $\kappa \in A_n$, $\langle \underset{\sim}{f}_l(\kappa), \xi_n \rangle : l \leqq n$ is (a P_κ-name of) an initial segment of a play of $PG^\omega(\varnothing_{P_\lambda}, P_\lambda/G_\kappa, \aleph_1)$ in which Player II uses his winning strategy $F(P_\lambda/G_\kappa)$ and some condition in G_{α_n} forces this. (Remember $P_\kappa \vDash \kappa$-c.c., so some $p \in G_{\alpha_n}$ force this.)

Why can Player II carry this strategy? Suppose he arrives at stage n and Player I has chosen $f_n \in V^{P_\lambda}$, $f_n : \lambda \to \omega_1$. He chooses $\underset{\sim}{f}_n \in V$, the first P_λ-name $\underset{\sim}{f}$ such that $\underset{\sim}{f}_n[G_\lambda] = f_n$. Now for every $\kappa \in A_n$, working in $V[G_\kappa]$ he continues the play $\langle \underset{\sim}{f}_l(\kappa), \xi_l^0 \rangle : l < n$ of $PG^\omega(\varnothing_P, P_\lambda/G_\kappa, \aleph_1)$, letting the first player play $\underset{\sim}{f}_n(\kappa)$, and let $\xi_n^0(\kappa)$ be the choice of the second player according to the strategy $F(P_\lambda/G_\kappa)$. Really $\xi_n^0(\kappa)$ is a P_κ-name. Now for every $p \in P_\lambda$ and $\kappa \in A_n$ there is $q_\kappa \in P_\kappa$ compatible with p and forcing a value to $\xi_n^0(\kappa)$, hence for some $\xi < \omega_1$, $A_p^{n+1} \in D$, $A_p^{n+1} \subseteq A_n$ and q, $(\forall \kappa \in A_p^{n+1})$ $[q_\kappa = q$ and $q \Vdash_{P_\kappa}$ "$\underset{\sim}{\xi}_n^0(\kappa) = \xi$"]. So there are such $q \in G_\lambda$, and ξ (which we call ξ_n^0) and a set which we call A_{n+1}. It is easy to choose α_n.

Still we should prove that this is a winning strategy. We shall consider one play and work in V, so everything is a P_λ-name (as we are using RCS, no problems arise).

Now P_λ satisfies the λ-c.c., so we have a bound $\alpha(*) < \lambda$ on the α_n's (forced by $\varnothing$). Work in $V[G_{\alpha(*)}]$. D is (essentially) an ultrafilter in $V[G_{\alpha(*)}]$. Each A_n has a P_λ-name so really there are $< \lambda$ candidates so we have $A_\omega \subseteq \bigcap_n A_n$, $A_\omega \subseteq A_n$, $A_\omega \in D$ (really we can compute $\bigcap A_n$ in $V[G_{\alpha(*)}]$). Now for $\kappa \in A_\omega$, $\kappa > \alpha(*)$ the sequence $\langle \underset{\sim}{f_l}(\kappa), \xi_l \rangle : l < \omega$ is a play of $PG^{\omega}(\emptyset_P, P_\kappa/P_\lambda, \aleph_1)$ where Player II uses his winning strategy (this is a P_κ-name, but fortunately $\langle \xi_l : l < \omega \rangle \in V[G_{\alpha(*)}]$). So there is $q_\kappa \in P_\lambda/P_\kappa$ a P_κ-name, so that

$$q_\kappa \Vdash_{P_\lambda/P_\kappa} \text{“} \bigwedge_l \underset{\sim}{f_l}(\kappa) < \bigcup_n \xi_m \text{”}$$

(more exactly:

$$q_\kappa \Vdash_{(P_\lambda/G_{\alpha(*)})/(P_\kappa/G_{\alpha(*)})} \text{“} \bigwedge_l \underset{\sim}{f_l}(\kappa) < \bigcup_n \xi_m \text{”}$$

q_κ a $P_\kappa/G_{\alpha(*)}$-name of a P_λ/P_κ-condition).

We can consider q_κ as a P_λ-condition with Dom $q_\kappa \subseteq [\kappa, \lambda)$, because we use RCS-iteration. Now easily $\langle q_\kappa : \kappa \in A_\omega \rangle \in V[G_{\alpha(*)}]$,

$$\Vdash_{P_\lambda/G_{\alpha(*)}} \text{“}\{\kappa \in A : \underset{\sim}{q_\kappa} \in \underset{\sim}{G_\lambda}\} \quad \text{is unbounded in } \lambda\text{”}$$

as every $r \in P_\lambda/G_{\alpha(*)}$ has domain bounded in λ, so q_κ, for κ large enough, is possible, i.e., compatible with it.

1.10. Claim. Suppose κ is measurable, $\bar{Q}$ a semi-proper iteraction, $\lg(\bar{Q}) = \kappa$, $|P_i| < \kappa$ for $i < \kappa$ and $\{i : Q_i \text{ semi-proper}\}$ belong to some normal ultrafilter on κ. Then:

(1) $\mathrm{Rss}^+(\kappa, \lambda)$ implies $\Vdash_{P_\kappa} \mathrm{Rss}(\kappa, \lambda)$.

(2) If $\underset{\sim}{Q}$ is a P_κ-name of a semi-proper forcing notion, $\Vdash_{P_{i+1}}$ “$(P_\kappa/P_{i+1} * \underset{\sim}{Q})$ is semi-proper for $i < \kappa$” then $\Vdash_{P_\kappa}$ “$\underset{\sim}{Q}$ is semi-proper”.

(3) We can replace measurability of κ by: κ strongly inaccessible $\Vdash_{P_\kappa}$ “Player II wins $\mathrm{Gm}(\{\aleph_1\}, \omega_1, \aleph_2)$” and P_κ satisfies the κ-c.c.

Proof. (1) Let $\underset{\sim}{W}$ be a P_κ-name, $p \in P_\kappa$, $p \Vdash_{P_\kappa}$ “$\underset{\sim}{W} \subseteq \mathrm{S}_{<\aleph_1}(\lambda)$ is semi-stationary”.

Let for $i < \kappa$, $\underset{\sim}{W}_i = \{a : a \in V^{P_i},\ a \in \mathrm{S}_{<\aleph_1}(\lambda)$, and for some $q \in \underset{\sim}{G}_{P_i}$, $q \Vdash_{P_\kappa}$ “$a \in \underset{\sim}{W}$”$\}$. So $\underset{\sim}{W}_i$ is a P_i-name.

Let μ be regular and large enough, $<^*_\mu$ a well ordering of $H(\mu)^V$.

Let $p \in G \subseteq P_\kappa$, G generic over V and $G_i = G \cap P_i$ for $i < \kappa$. In $V[G_\kappa]$, as $\underset{\sim}{W}[G_\kappa]$ is semi-stationary, there is a countable $(N, G_\kappa \cap N) \prec (H(\mu)^V, \in, <^*_\mu, G_\kappa)$, such that for some $a \in \underset{\sim}{W}[G_\kappa]$, $N \cap \omega_1 \subseteq a \subseteq N \cap \lambda$, and p, $\underset{\sim}{W}$, λ, κ, $\bar{Q}$ belongs to N.

So there are $q \in G_\kappa$ and P_κ-names $\underset{\sim}{N}$, $\underset{\sim}{a}$ such that $q \Vdash_{P_\kappa}$ “$\underset{\sim}{N}$, $\underset{\sim}{a}$ are as above”, and without loss of generality $p \leqq q$. As N, a are countable subsets of $H(\mu)^V$, λ respectively and $P_\kappa = \bigcup_{i<\kappa} P_i$ satisfies the κ-c.c. (by [Sh 1] 5.3 (3), p. 336), for some $i<\kappa$, $\underset{\sim}{N}$, $\underset{\sim}{q}$ are P_i-names, and $q \in P_i$. Now by 1.9, in V^{P_κ}, for arbitrarily large $\theta<\kappa$, $N^{[\theta]} \cap \omega_1 = N \cap \omega_1$, and Q_θ is semi-proper, where we let:

$$N^{[\theta]} \stackrel{\text{def}}{=} \text{Skolem Hull } (N \cup \{\theta\})$$
$$(\text{in } (H(\mu)^V, \in, <^*_\mu, G_\kappa) \text{ working in the universe } V[G_\kappa]).$$

Choose such a $\theta > i$. Now as $\theta \in N^{[\theta]}$, $(N^{[\theta]}, G_\theta) \prec (H(\mu)^V, \in, <^*_\theta, G_\theta)$. Clearly $\underset{\sim}{W}_\theta[G_\theta] \in N^{[\theta]}$, $\underset{\sim}{q}[G_\theta] \in \underset{\sim}{W}_\theta[G_\theta]$, $\omega_1 \cap N^{[\theta]} \subseteq \underset{\sim}{q}[G_\theta] \subseteq N^{[\theta]}$, hence by 1.2(6), $V[G_\theta] \models$ "$\underset{\sim}{W}^\theta[G_\theta]$ is a semi-stationary subset of $S_{<\aleph_1}(\lambda)$".

As $\mathrm{Rss}^+(\kappa,\lambda)$ clearly $V[G_\theta] \models \mathrm{Rss}(\kappa,\lambda)$, hence in $V[G_\theta]$ for some $A \subseteq \lambda$, $|A|<\kappa$, $\underset{\sim}{W}_\theta[G_\theta] \cap S_{<\aleph_1}(A)$ is semi-stationary, clearly $V[G_\kappa] \models$ "$A \in V[G_\kappa]$" and as P_κ/P_θ is semi-proper (see the choice of θ) it preserves the semi-stationary of A, hence $V[G_\theta] \models$ "A is semi-stationary".

(2) Similar: suppose $p \Vdash_{P_\kappa}$ "$\underset{\sim}{N} \prec (H(\mu)^V, \in, <^*_\mu, \underset{\sim}{G}_{P_\kappa})$ and $\underset{\sim}{q} \in \underset{\sim}{\theta} \cap \underset{\sim}{N}$ are counterexample to semi-properness of $\underset{\sim}{Q}$".

Let $G_\kappa \subseteq P_\kappa$ be generic over V, $p \in G_\kappa$. Let $\theta<\kappa$, $\theta > \sup(\underset{\sim}{N}[G] \cap \kappa)$ be such that $\underset{\sim}{N}[G_\kappa^{[\theta]} \cap \omega_1 = \underset{\sim}{N}[G_\kappa] \cap \omega_1$ (and $\underset{\sim}{N}$ is a P_θ-name). Now work in $V[G_\kappa \cap P_{\theta+1}]$ and use $\Vdash_{P_{\theta+1}} (P_\kappa/P_{\theta+1}) * \underset{\sim}{Q}$ is semi-proper.

(3) In the proof of (2) we use this only. In the proof of (1) choose θ a successor ordinal (so $\underset{\sim}{Q}_i$ is semi-proper). It preserves the semi-stationarity of A, hence $V[G_\theta] \models$ "A is semi-stationary".

1.11. Claim. Suppose $\mathrm{Rss}(\kappa, 2^\kappa)$, κ regular and: $\kappa=\aleph_2$ or $(\forall \mu<\kappa)$ $\mu^{\aleph_0}<\kappa$, *then* for $\lambda \geqq 2^\kappa$ for every countable $N \prec (H(\lambda), \in, <^*_\lambda)$ to which κ belongs, for arbitrarily large $i<\kappa$, letting $N^{[i]} =$ Skolem Hull $(N \cup \{i\})$, $N <_{\omega_2} N^{[i]}$ (see below), $N \cap \kappa \neq N' \cap \kappa$.

1.12. Remark. (1) The "$\kappa=\aleph_2 \cdots$" can be omitted if we replace "for arbitrarily large i" by "for some $i<\kappa$, $i>\sup(N\cap\kappa)$".

(2) We can replace "$\kappa=\aleph_2$, or ..." by "if $\alpha<\kappa$, then there is $C \in S_{<\aleph_1}(\alpha)$ of power $<\kappa$" (see the proof). It even suffices to assume "for every stationary $W \subseteq S_{<\aleph_1}(\alpha)$, $(\alpha<\kappa)$ there is a semi-stationary $W' \subseteq W$ of cardinality $<\kappa$".

(3) If we want in the conclusion to get $N <_\kappa N^{[i]}$ we have to replace in the definition of semi-stationary, "$N_1 \cap \omega_1 = N_2 \cap \omega_2$" by $N_1 <_\kappa N_2$.

Proof. Let

$$W = \{|N| : N \prec (H(\kappa^+), \in, <^*_{\kappa^+}), N \text{ countable and}$$
$$\text{for some } i_N<\kappa \text{ for no } i \in [i_N, \kappa), N <_\kappa N^{[i]}\}.$$

Assume W is a stationary subset of $H(\kappa^+)$. So, as $\mathrm{Rss}(\kappa, 2^\kappa)$ holds (and $|H(\kappa^+)| = 2^\kappa$) there is $A \subseteq H(\kappa^+)$, $\omega_1 \subseteq A$, $|A| < \kappa$ such that: $W_A = \{a \in W : a \subseteq A\}$ is a semi-stationary subset of A. Without loss of generality (see 1.2(2))

$$M \stackrel{\text{def}}{=} (A, \in \restriction A, <^*_{\kappa^+} \restriction A) \prec (H(\kappa^+), \in, <^*_{\kappa^+}).$$

As for $N_1 \subseteq N_2$ countable elementary submodels of $(H(\kappa^+), \in, <^*_{\kappa^+})$, $N_1 \in W_A$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$ implies $N_2 \in W_A$, clearly W_A is stationary. We know by assumption that for some closed unbounded $C \subseteq \mathbf{S}_{<\aleph_1}(A)$, C has cardinality $< \kappa$. So

$$\zeta \stackrel{\text{def}}{=} \sup\{i_N : |N| \in C \cap W\} < \kappa.$$

Now for some club $C_1 \subseteq C$, for every $a \in C_1$, $a^{[\zeta]} =$ Skolem Hull of $a \cup \{\zeta\}$, satisfies $a^{[\zeta]} \cap A = a$, hence $a <_\kappa a^{[\zeta]}$, but some $a \in C_1 \cap W_A$, contradiction.

So W is not stationary and let $C^* \subseteq \mathbf{S}_{<\aleph_1}(H(\kappa^+))$ be a club disjoint to W.

Let $\lambda > 2^\kappa$, so $H(\kappa^+)$, $W \in H(\lambda)$, and let $\kappa \in N \prec (H(\lambda), \in, <^*_\lambda)$ be countable. So $H(\kappa^+) \in N$ hence $W \in N$ and without loss of generality $C^* \in N$. Hence $N \cap H(\kappa^+) \in C^*$, and for arbitrarily large $i < \kappa$ there is $N_1^i \prec (H(\kappa^+), \in, <^*_{\kappa^+})$, countable, $i \in N_1^i$, $N \cap H(\kappa^+) <_{\omega_2} N_1^i$. Let N^i be the Skolem Hull of $N \cup (N_1 \cap \kappa)$. We can easily check that $N^i \cap \kappa = N_1^i \cap \kappa$, so N^i is as required.

1.13. Definition. A forcing notion P satisfies the $(\mathbf{S}, S)$-condition ($\mathbf{S}$ a set of regular cardinals, $S \subseteq \omega_1$ stationary) if there is a function F (domain implicitly defined in (c)) so that:

Suppose

(a) T is an $\mathbf{S}$-tree, $f : T \to P$, $g : T \to \omega_1$.

(b) $\nu \lhd \eta$ in T implies $f(\nu) \leqq f(\eta)$ in P, and $g(\nu) < g(\eta)$ $(<\omega_1)$.

(c) There are fronts J_n $(n < \omega)$ of T such that every member of J_{n+1} has a proper initial segment from J_n and $\eta \in J_n$ implies (η is splitting node of T and)

$$\langle \mathrm{Suc}_T(\eta), \langle\langle f(\nu), g(\nu)\rangle : \nu \in \mathrm{suc}_T(\eta)\rangle\rangle = F(\eta, w[\eta], \langle\langle f(\nu), g(\nu)\rangle : \nu \lhd \eta\rangle)$$

where $w[\eta] = \{k : \eta \restriction k \in \bigcup_{l<\omega} J_l\}$.

Then for every T', $T \leqq^* T'$ there is $p \in P$ such that $p \Vdash_P$ "there is $\eta \in \lim T'$ such that if $\sup\{g(\eta \restriction k) : k < \omega\} \in S$ then $\{f(\eta \restriction k) : k < \omega\} \subseteq G_P$".

1.14. Notation. We say "P is $(*, S)$-complete" if f satisfies the $(\{\lambda : \lambda$ regular $> \aleph_1\}, S)$-condition.

1.15. THEOREM. *The natural generalization of the theorems from* [Sh 1, ch. X1] (and Gitik–Shelah [GSh]) *holds for the* (S, S)*-condition*.

1.16. REMARK. We use 1.13, 1.14, 1.15 only in 2.16; we can alternatively use pseudo-(S, S)-completeness (see [Sh 1, X]) but use Nm(D), or use 2.16 A(3).

§2

2.1. DEFINITION. We say $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j, \mathrm{t}_j : i \leqq \alpha,\ j < \alpha \rangle$ is S-suitable ($S \subseteq \omega_1$ stationary) if:

(A) it is an RCS iteration;

(B) we denote $|\bigcup_{j<i} P_{j+1}| = \kappa_i = \kappa_i^{\bar{Q}}$ so $\kappa_0 = 1$, κ_i increasing continuous. We demand that κ_i is strictly increasing;

(C) for i successor κ_i is strongly inaccessible;

(D) for $i < j < \alpha$ non-limit, P_j / P_i is semi-proper;

(E) $\underset{\sim}{Q}_i$ satisfies the κ_{i+1}-c.c., $\aleph_2^{V^{P_{i+1}}} = \kappa_{i+1}$;

(F) if $\mathrm{t}_i = 1$, $i < j \leqq \alpha$, j successor, then $\mathfrak{B}^{P_i} \restriction S \lessdot \mathfrak{B}^{P_j} \restriction S$.

We may allow t_α to be defined. We may but do not use t_β which are names.

NOTATION. $\alpha^{\bar{Q}} = \alpha$, $P_i^{\bar{Q}} = P_i$, $\underset{\sim}{Q}_j^{\bar{Q}} \mathrm{t}_j^{\bar{Q}} = \mathrm{t}_j$.

2.2. CLAIM.

(1) Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j, \mathrm{t}_j : i \leqq \alpha, j < \alpha \rangle$ is a semi-proper iteration (see 1.8 for definition). Then:

(a) If $i < \alpha$ is non-limit or $\underset{\sim}{Q}_i$ is semi-proper or $\underset{\sim}{Q}_i$ preserve stationarity of subsets of ω_1 from V^{P_i} *then* every stationary subset of ω_1 in V^{P_i} is stationary in V^{P_α} too (i.e., $\mathfrak{B}[P_i]$ is a subalgebra of $\mathfrak{B}[P_\alpha]$).

(b) $\aleph_1^V = \aleph_1^{V^{P_\alpha}}$.

(c) If α is strongly inaccessible ($> \omega$), and $|P_i| < \alpha$ for $i < \alpha$, *then* P_α satisfies the α-c.c. and so

$$\mathscr{P}(\omega_1)^{V^{P_\alpha}} = \bigcup_{i<\alpha} \mathscr{P}(\omega_1)^{V^{P_i}}, \quad V^{P_\alpha} \models \text{“}2^{\aleph_1} = \aleph_2\text{”}.$$

(d) If $\omega_1 - S$ is stationary, each $\underset{\sim}{Q}_i$ is $(\omega_1 - S)$-complete [Sh 1, ch. V], *then* so is P_α, hence forcing by P_α does not add ω-sequences of ordinals (hence $V^{P_\alpha} \models \mathrm{CH}$).

(e) If $\bar{Q} \in N_1 \prec N_2 \prec (H(\lambda), \in)$, N_2 countable, $N_1 <_\alpha N_2$, α inaccessible, $\alpha > |P_i|$ for $i < \alpha$ and q is (N_1, P_α)-semi-generic, then q is

(N_2, P_i)-semi-generic where $i = \mathrm{Min}(\alpha \cap N_2 - N_1)$ is strongly inaccessible.

(2) Any S-suitable iteration $\bar{Q}$ is a semi-proper iteration and $[\mathbf{t}_i = 1 \Rightarrow \mathfrak{B}[P_i] \restriction S \lessdot \mathfrak{B}[P_j] \restriction S]$ when: $j \geqq i$, i successor, j successor or strongly inaccessible.

(3) If (in (1)) $\kappa < \alpha$ is strongly inaccessible, $|P_i| < \kappa$ for $i < \kappa$, and $\Vdash_{P_\kappa}$ "Rss($\aleph_2$)" then $\underset{\sim}{Q}_\kappa$ (and P_j/P_κ when $\kappa \leqq j \leqq \alpha$) are semi-proper.

Proof. Left to the reader. For instance:

(1)(c) If $I \in N_2$ is a maximal antichain of P_κ, *then* by [Sh 1] X 5.3(3) for some $j < i$, $I \subseteq P_j$, hence there is such j in N_2, hence $j \in N_1$ and also the rest is easy.

(3) By 1.7(3) it is enough to prove that forcing with $\underset{\sim}{Q}_\kappa$ does not destroy the stationarity of any $A \subseteq \omega_1$, $A \in V^{P_\kappa}$. However, by 2.2(1)(c) (and 2.2(2)) for some $\beta < \alpha$, $A \in V^{P_\beta}$. Clearly $A \in V^{P_\beta}$ and is a stationary subset of ω_1 in $V^{P_{\beta+1}}$. As $P_{\kappa+1}/P_{\beta+1}$ is semi-proper, A is stationary also in $(V^{P_{\beta+1}})^{P_{\kappa+1}/P_{\beta+1}} = V^{P_{\kappa+1}} = (V^{P_\kappa})^{\underset{\sim}{Q}_\kappa}$ as required.

2.2A. Remark. So if κ is strongly inaccessible, and $|P_i| < \kappa$ for $i < \kappa$, *then* if A is a stationary subset of ω_1 in V^{P_κ} *then* A is a stationary subset of ω_1 in V^{P_α} for every large enough $\alpha < \kappa$.

2.3. Claim. Suppose $\bar{Q} = \langle P_j, \underset{\sim}{Q}_i : j \leqq \alpha, i < \alpha \rangle$ is an RCS iteraction, α a limit ordinal.

(1) If $\bar{Q} \restriction \beta$ is S-suitable for $\beta < \alpha$, *then* $\bar{Q}$ is S-suitable.

(2) If for $\beta < \alpha$, $\bar{Q} \restriction \beta$ is a semi-proper iteration, then $\bar{Q}$ is a semi-proper iteration.

(3) In (2) if $i < \alpha$, $\underset{\sim}{\mathscr{A}}$ a P_i-name then: $\Vdash_{P_\alpha}$ "$\underset{\sim}{\mathscr{A}} \lessdot \mathfrak{B}^{\bar{Q}}$" if and only if $\alpha = \sup\{j < \alpha : \Vdash_{P_{j+1}}$ "$\underset{\sim}{\mathscr{A}} \lessdot \mathfrak{B}^{P_j+1}$"$\}$ if and only if for arbitrarily large $j < \alpha$ $\Vdash_{P_{j+1}}$ "$\underset{\sim}{\mathscr{A}} \lessdot \mathfrak{B}^{P_j+1} \restriction \mathfrak{B}^{P_j}$".

(4) In (2) if $\alpha > |P_i|$ for $i < \alpha$, and α is strongly inaccessible, then $\mathfrak{B}^{\bar{Q}} = \mathfrak{B}^{P_\alpha}$.

Proof.

(1) For (D) use the semi-proper iteration lemma. The others are obvious too.

(2), (3), (4) Easy, too.

2.4. Definition. Let $\bar{\mathfrak{A}} = \langle \mathfrak{A}_\zeta : \zeta < \xi \rangle$ be a sequence of subalgebras of $\mathfrak{B}(= \mathfrak{B}^V)$, $S \subseteq \omega_1$ stationary.

(1) $\mathrm{Sm}(\bar{\mathfrak{A}}, S) = \{A \subseteq S$: for some $\zeta < \xi$, $\{x \in \mathfrak{A}_\zeta : \mathfrak{A}_\zeta \models x \neq 0, x \cap A = \varnothing$

$\bmod D_{\omega_1}\}$ is predense in $\mathfrak{A}_\zeta\}$ (we should have written $x/D_{\omega_1} \in \mathfrak{A}_\zeta$, $x \subseteq \omega_1\}$.

(2) We define the sealing forcing Seal $(\bar{\mathfrak{A}}, S) = \{\bar{c} : \bar{c}$ a partial function from $\mathrm{Sm}(\bar{\mathfrak{A}}, S)$, with countable domain, and for $A \in \mathrm{Sm}(\bar{\mathfrak{A}}, S)$, *if* $A = \emptyset$ then $\bar{c}_A$ is a function from some $\alpha < \omega_1$ to $2^{\aleph_1}$, and if $A \neq \emptyset$ then $\bar{c}_A$ is a continuously inceasing function from some countable $\gamma + 1$ to $\omega_1 - A\}$, the ordering is defined by:

$$\bar{c}^1 \leqq \bar{c}^2 \quad \textit{if:} \quad A \in \mathrm{Dom}\, \bar{c}^1 \text{ implies } A \in \mathrm{Dom}\, \bar{c}^2 \text{ and } \bar{c}^1_A \subseteq \bar{c}^2_A.$$

(3) If $\bar{\mathfrak{A}} = \langle \mathfrak{B} \rangle$ we write in (1), (2) above $\mathfrak{B}$ instead of $\bar{\mathfrak{A}}$.

(4) We define, for κ strongly inaccessible ($> \aleph_0$), the strong sealing forcing $\mathrm{SSeal}(\bar{\mathfrak{A}}, S, \kappa)$ as P_κ, where $\langle P_i, \underset{\sim}{Q}_j : i \leqq \kappa, j < \kappa \rangle$ is an RCS iteration, with $Q_j = \mathrm{Seal}(\bar{\mathfrak{A}}, S)^{\tilde{V}^{P_j}}$.

(5) For $I \subseteq \mathfrak{B}^V$ let $\mathrm{seal}(I) = \{\langle a_i : i \leqq \alpha \rangle : a_i \in S_{<\aleph_1}(H((2^{\aleph_1})^+))$, a_i $(i \leqq \alpha)$ is increasing continuous and $a_i \cap \omega_1$ is an ordinal which belongs to $\bigcup_{A \in I \cap a_i} A\}$ and is order by the inverse of being an initial segment.

(6) We call $I \subseteq \mathfrak{B}^V$ semi-proper *if* seal(I) is a semi-proper forcing notion.

(7) WSeal(S) is the product, with countable support, of seal(I), I semi-proper, $\omega_1 - S \in I$.

(8) We define, for κ not strongly inaccessible, but

(*) $(\forall \mu < \kappa)[\mu^{\aleph_0} < \kappa]$, $\kappa = \mathrm{cf}\, \kappa$, $\kappa^{|\mathscr{A}_\zeta|} = \kappa$ for $\zeta < \xi$, and $\xi \leqq \kappa$, $\kappa > \aleph_1$

the strong sealing forcing $\mathrm{SSeal}(\bar{\mathscr{A}}, S, \kappa)$ as P_κ where $\langle P_i, \underset{\sim}{Q}_j : i \leqq \kappa j < \kappa \rangle$ is an RCS iteration; $\underset{\sim}{Q}_j = \mathrm{seal}(\underset{\sim}{I}_j, S)^{V^{P_j}}$, $\underset{\sim}{I}_j$ is a maximal antichain of $\mathfrak{A}_{\zeta(j)}$ for some $\zeta(j) < \xi$ (in V^{P_j}) and every maximal antichain $\underset{\sim}{I}$ of some $\mathfrak{A}_\zeta$ from V^{P_κ} is $\underset{\sim}{I}_j$ for some $j < \kappa$.

We call κ $\bar{\mathfrak{A}}$-inaccessible if it satisfies (*) above, and call it $\aleph_0$-inaccessible if $(\forall \mu < \kappa)(\mu^{\aleph_0} < \kappa = \mathrm{cf}\, \kappa)$.

(9) If $\mathbf{I} \subseteq \{I : I \subseteq \mathfrak{B}\}$ then seal$(\mathbf{I})$ is the product, with countable support of seal(I) for $I \in \mathbf{I}$.

2.5. Remarks.

(1) We could have used CS iteration for SSeal.

(2) If every maximal antichain of $\mathfrak{B}^V$ is semi-proper, the difference between WSeal$(S) * \mathrm{Levy}(\aleph_1, 2^{\aleph_1})$ and Seal$(\mathfrak{B}^V, S)$ (defined in 2.4 (7), (2), respectively) is nominal.

(3) If $\mathfrak{A}_\zeta \restriction S \lessdot \mathfrak{B}^V \restriction S$ for $\zeta < \xi$, then Seal$(\bar{\mathfrak{A}}, S)$ is equivalent to the Levy collapse of $2^{\aleph_1}$ to $\aleph_1$ by countable conditions.

2.6. NOTATION. We omit κ in SSeal($\bar{\mathfrak{A}}, S, \kappa$) *when* it is the first strongly inaccessible. We omit S when $S = \omega_1$. We write $\mathfrak{A}$ instead of $\langle \mathfrak{A} \rangle$.

2.7. CLAIM. If in V,

$$\bar{\mathfrak{A}}^l = \langle \mathfrak{A}^l_\zeta : \zeta < \xi_l \rangle \text{ for } l = 1, 2 \text{ and}$$

$$(\forall \zeta_1 < \xi_1)(\exists \zeta_2 < \xi_2)[\mathfrak{A}^1_{\zeta_1} \lessdot \mathfrak{A}^2_{\zeta_2}]$$

$$(\forall \zeta_2 < \xi_2)(\exists \zeta_1 < \xi_1)[\mathfrak{A}^2_{\zeta_2} \lessdot \mathfrak{A}^1_{\zeta_1}]$$

then Seal($\bar{\mathfrak{A}}^1, S, \kappa$) = Seal($\bar{\mathfrak{A}}^2, S, \kappa$), Sm($\bar{\mathfrak{A}}^1, S$) = Sm($\bar{\mathfrak{A}}^2, S$), and SSeal($\bar{\mathfrak{A}}^1, S, \kappa$) = SSeal($\bar{\mathfrak{A}}^2, S, \kappa$).

PROOF. Easy.

2.8. CLAIM.

(1) Let $I \subseteq \mathfrak{B}^V$ be predense. Then I is semi-proper *iff* for λ regular large enough, $N \prec (H(\lambda), \in)$ countable $I \in N$, *there is* $N', N \prec N' \prec (H(\lambda), \in)$, N' countable, $N \cap \omega_1 = N' \cap \omega_1 \in \bigcup_{A \in I \cap N'} A$.

(2) $\Vdash_{\text{seal}(I)}$ "$I \subseteq \mathfrak{B}[\text{seal}(I)]$ is predense".

(3) WSeal(S) is semi-proper and $\Vdash_{\text{WSeal}(S)}$ "*if* $I \in V$ is semi-proper in $\mathfrak{B}^V$, $(\omega_1 - S) \in I$, *then* I is predense in $\mathfrak{B}[\text{WSeal}(S)]$".

(4) seal(I) is A-complete for $A \in I$; so WSeal(S) is $(\omega_1 - S)$-complete.

(5) If I is predense in $\mathfrak{B}(V)$, then seal(I) preserves stationarity of subsets of ω_1.

(6) Seal($\bar{\mathscr{A}}, S$) is $(\omega_1 - S)$-complete; S Seal($\bar{\mathscr{A}}, S, \kappa$) is $(\omega_1 - S)$-complete and if $\kappa > \aleph_0$ is $\bar{\mathscr{A}}$-inaccessible it satisfies the κ-c.c.

PROOF. Check.

2.9. CLAIM. Suppose seal(I) is semi-proper for every maximal antichain of $\mathfrak{B}^V$ to which $\omega_1 - S$ belongs, and $\kappa > \aleph_0$ is $\mathfrak{B}^V$-inaccessible.

Then $P \overset{\text{def}}{=}$ SSeal($\mathfrak{B}^V, S, \kappa$) is semi-proper and $(\omega_1 - S)$-complete.

PROOF. The $(\omega_1 - S)$-completeness is trivial by the definition of P and [Sh 1, Ch. V, Def. 1.1, p. 154].

Now let λ be regular and large enough, and $N \prec (H(\lambda), \in)$ countable, $P \in N$, $p \in P \cap N$. Applying repeatedly 2.8(1), there is $N', N \prec N' \prec (H(\lambda), \in)$, $N \cap \omega_1 = N' \cap \omega_1$, N' countable, and for every maximal antichain $I \subseteq \mathfrak{B}$ (or just predense $I \subseteq \mathfrak{B}^V$):

$$I \in N', \quad N \cap \omega_1 \in S \Rightarrow N \cap \omega_1 = N' \cap \omega_1 \in \bigcup_{A \in I \cap N'} A.$$

Then we proceed as in the proof of 2.10 below (using N' instead of N and the choice of N' instead of (**)), i.e., using 2.10A.

2.10. Claim. If $\bar{\mathfrak{A}} = \langle \mathfrak{A}_\zeta : \zeta < \xi \rangle$, $\mathfrak{A}_\zeta \lessdot \mathfrak{B}^V$ for $\zeta < \xi$, each $\mathfrak{A}_\zeta$ satisfies the $\aleph_2$-c.c. (or just has power $\leqq \aleph_1$) and $\kappa > \aleph_0$ is strongly inaccessible, *then*

(1) $P_\kappa \stackrel{\text{def}}{=} \mathrm{SSeal}(\bar{\mathfrak{A}}, S, \kappa)$ is proper;

(2) $\Vdash_{P_\kappa}$ "$\mathfrak{A}_\zeta \restriction S \lessdot \mathfrak{B}^{P_\kappa} \restriction S$ for $\zeta < \xi$";

(3) in fact, P_κ is $(\omega_1 - S)$-complete and strongly proper satisfying the κ-c.c.;

(4) if $\omega_1 - S$ is stationary, P_κ does not add ω-sequences of ordinals.

Proving 2.10(1), we really prove the following, which in fact is used several times (the only difference is that (**) becomes an assumption).

2.10A. Claim. Suppose $\bar{\mathfrak{A}} = \langle \mathfrak{A}_\zeta : \zeta < \xi \rangle$, $\kappa > \aleph_0$ is strongly inaccessible, $\bar{\mathfrak{A}}, \kappa \in N \prec (H(\lambda), \in)$, N countable, $P = \mathrm{SSeal}(\bar{\mathfrak{A}}, S, \kappa)$ and

$\oplus$ if $I \in N$ is a predense subset of $\mathfrak{A}_\zeta$, $\omega_1 - S \in I$, then $N \cap \omega_1 \in \bigcup_{A \in I \cap N} A$.

Then for every $p \in P \cap N$ *there is* $q \in P$, (N, P)-generic, $p \leqq q$.

Proof. (1) Let λ be regular large enough and $N \prec (H(\lambda), \in)$ countable, $\underset{\sim}{Q} \in N$ (hence $P_\kappa \in N$) and $p \in P_\kappa \cap N$. We have to find q, $p \leqq q \in P$, which is (N, P)-generic. Now

(*) if $\zeta, \underset{\sim}{I} \in N$ are P-names, $\Vdash_P$ "$\underset{\sim}{I}$ a predense subset of $\mathfrak{A}_\zeta$", $p \in N \cap P$, *then* for some p^2, $p \leqq p^2 \in N \cap P$, $p^2 \Vdash$ "for some $\underset{\sim}{A} \in \underset{\sim}{I} \cap N \cap \underset{\sim}{\mathfrak{A}}_\zeta$ (so P forces that $\underset{\sim}{A} \in V$), $N \cap \omega_1 \in A$".

Proof of (*). We can find p^0, $p \leqq p^0 \in N \cap P$, and ζ, $p^0 \Vdash$ "$\underset{\sim}{\zeta} = \zeta$" (so necessarily $\zeta \in N$). Next define

$$J = \{A \in \mathfrak{A}_\zeta : \text{for some } p^1,\ p \leqq p^1 \in P,\ p^1 \Vdash \text{“}A \in \underset{\sim}{I}\text{”}\}.$$

Clearly $J \in N$, $J \in V$, and J is a predense subset of $\mathfrak{A}_\zeta$, $\zeta \in N$. We now have:

(**) if $\zeta \in N \cap \xi$, $J \subseteq \mathfrak{A}_\zeta$ is predense, $(J \in V)$ $J \in N$ *then* for some $A \in J \cap N$, $N \cap \omega_1 \in A$.

[Proof of (**). As $\| \mathfrak{A}_\zeta \| \leqq \aleph_1$, or $\mathfrak{A}_\zeta \models \aleph_2$-c.c., clearly without loss of generality $|J| \leqq \aleph_1$, so let $J = \{A_i : i < \omega_1\}$ (as $J \neq \emptyset$ this is possible). Since $\mathfrak{A}_\zeta \lessdot \mathfrak{B}^V$, clearly J is predense in $\mathfrak{B}^V$, hence we know $\{\delta : \delta \in \bigcup_{i<\delta} A_i\} \in D_{\omega_1}$ (otherwise the complement contradicts the predensity), so there is a closed unbounded $C \subseteq \omega_1$, $C \subseteq \{\delta : \delta \in \bigcup_{i<\delta} A_i\}$. As $J \in N$ without loss of generality $\langle A_i : i < \omega_1 \rangle \in N$ and without loss of generality $C \in N$. As $N \prec (H(\lambda), \in)$ clearly $C \cap N$ is unbounded in $N \cap \omega_1$, hence $N \cap \omega_1 = \sup(C \cap N \cap \omega_1) \in C$, so $N \cap \omega_1 \in \bigcup \{A_i : i \in N \cap \omega_1\}$, so for some $j \in N \cap \omega_1$, $N \cap \omega_1 \in A_j$. But $\langle A_i : i < \omega_1 \rangle \in N$ so $A_j \in N$, as required.]

Continuation of Proof of (*). By (**) there is $A \in J \cap N$, $N \cap \omega_1 \in A$. By the definition of J there is p^2, $p^0 \leqq p^2 \in P$, $p^2 \Vdash$ "$A \in \underset{\sim}{I}$". As $p^0, A, \underset{\sim}{I}$ are all in N, we can choose such p^2 in N, thus finishing the proof of (*).

Now we prove 2.10(1). We define p_n for $n < \omega$ such that:

(a) $p_0 = p$, $p_{n+1} \geqq p_n$;

(b) $p_n \in P_\kappa \cap N$;

(c) *for every* dense subset J of P_κ which belongs to N for some n, $p_{n+1} \in J$;

(d) *if* $j \in \kappa \cap N$ and $\underset{\sim}{I}, \underset{\sim}{\zeta}$ are P_j-names from N, $\Vdash_{P_j}$ "$\underset{\sim}{\zeta} < \xi, \underset{\sim}{I} \subseteq \mathfrak{A}_{\underset{\sim}{\zeta}}$ predense" *then for some* $n < \omega$ and $B \in \mathfrak{B}^V \cap N$,

$$p_{n+1} \restriction j \Vdash_{P_j} \text{“} B \in \underset{\sim}{I}, \quad N \cap \omega_1 \in B \text{”}.$$

This clearly suffices, as (using the notation of Definition 2.4(4)): $(\bigcup_{n<\omega} p_n)(j)$ is in Q_j by (d), and $\bigcup_{n<\omega} p_n$ is (N, P)-generic by (c). So we can assign the tasks, and for satisfying (b) and (c) there is no problem. For (d) use (*).

(2) If $A \in \mathscr{P}(\omega_1)^{V^{P_\kappa}}$ then as P_κ satisfies the κ-c.c. (see 2.2(1)(c)) for some $\alpha < \kappa$, $A \in \mathscr{P}(\omega_1)^{V^{P_\alpha}}$, and so by the definition of Seal$(\bar{\mathfrak{A}}, S, \kappa)$, if A/D_{ω_1} is disjoint to a dense subset of $x \in \mathfrak{A}_\zeta$, $A \subseteq S$, $\zeta < \xi$ then we "shoot" a club through its completion in the $(\beta+1)$-th iterand in the iteration defining SSeal$(\bar{\mathfrak{A}}, S, \kappa)$ for $\beta \in (\alpha, \kappa)$ large enough. Why? As $V^{P_1} \models |\mathfrak{A}_\zeta| \leqq \aleph_1$ (or P_1 collapse $2^{\aleph_1}$) there is β, $\alpha < \beta < \kappa$ such that for every $x \in \mathfrak{A}_\zeta$, if $x \cap A$ is not stationary in V^{P_κ}, then it is not stationary in V^{P_β}.

(3), (4) Easy.

2.11. Claim. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : i \leqq \alpha+1,\ j < \alpha+1 \rangle$ is an RCS iteration, $\bar{Q} \restriction \alpha$ is S-suitable, $\kappa > |P_\alpha|$ strongly inaccessible.

(1) If $\mathbf{t}_\alpha = 0$, $\underset{\sim}{Q}_\alpha = \text{SSeal}(\langle \mathfrak{B}[P_j] : j < \alpha, \mathbf{t}_j = 1 \rangle, S, \kappa)$ then $\bar{Q}$ is S-suitable.

(2) If $\bar{\mathfrak{A}} = \langle \mathfrak{A}_\zeta : \zeta < \xi \rangle$, α limit and for every $\zeta < \xi$ $\Vdash_{P_\alpha}$ "$\{i < \alpha : \mathfrak{A}_\zeta \restriction S \lessdot \mathfrak{B}[P_i] \restriction S\}$ is unbounded below α", $\mathbf{t}_\alpha = 0$, $\underset{\sim}{Q}_\alpha = \text{SSeal}(\bar{\mathfrak{A}}, S, \kappa)$ *then* $\bar{Q}$ is S-suitable.

Proof. (1) For α non-limit this holds by Claim 2.10(1) for Definition 2.1(D) and 2.10(2) for Definition 2.1(F) (the other parts of Definition 2.1 hold trivially). If α is limit, the proof is similar, using 2.13; see proof of 2.11(2).

(2) By 2.13(1) below it suffices to show: if $\underset{\sim}{I}$ is a P_α-name of a maximal antichain of $\mathfrak{A}_\zeta$ ($\zeta < \xi$) *then* $I \in \mathbf{I}$ ($\mathbf{I}$ of 2.13(1)). For this we apply 2.12: (a) is the desired conclusion so it suffices to verify (b). W.l.o.g. $\mathfrak{A}_\zeta \subseteq \mathfrak{B}^{P_i}$. Now (b) is proved (with $N^1 = N$) as in the proof of (**) in 2.10(1).

2.12. Claim. Let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j, \mathrm{t}_j : i \leqq \kappa, j < \alpha \rangle$ be a semi-proper iteration, α limit.

Suppose $\Vdash_{P_\alpha}$ "$\underset{\sim}{I} \subseteq \mathfrak{B}^Q$ is predense"; the following are equivalent:

(a) $(P_\alpha / P_i) * \mathrm{seal}(\underset{\sim}{I})$ is semi-proper (in V^{P_i}) for non-limit $i < \alpha$,

(b) If λ is regular large enough, $\bar{Q} \in N \prec (H(\lambda), \in, <^*_\lambda)$, N countable, $\underset{\sim}{I} \in N$, $p \in N \cap P_\alpha$, $i \in N \cap \alpha$, i non-limit, $q \in P_i$ is (N, P_i)-semi-generic, $p \restriction i \leqq q$ *then* there are N^1, p^1, q^1, $\underset{\sim}{A}$ and j such that:

(i) $N \prec N^1 \prec (H(\lambda), \in)$,

(ii) N^1 is countable, $N^1 \cap \omega_1 = N \cap \omega_1$,

(iii) $p \leqq p^1 \in N^1 \cap P$,

(iv) $i < j < \alpha$, j non-limit,

(v) $j \in N^1$,

(vi) $q \leqq q^1 \in P_j$,

(vii) q^1 is (N^1, P_j)-semi-generic,

(viii) $p^1 \restriction j \leqq q^1$,

(ix) $\underset{\sim}{A}$ is a P_j-name,

(x) $q^1 \Vdash$ "$\underset{\sim}{A} \in \underset{\sim}{I}$ and $N^1 \cap \omega_1 \in \underset{\sim}{A}$".

Proof. Easy.

2.13. Claim. Let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : j \leqq \alpha,\ i < \alpha \rangle$ be a semi-proper iteraction and α is a limit ordinal.

(1) If $\mathbf{I} = \{ I \in V^{P_\alpha} : I$ a maximal antichain of $\mathfrak{B}^Q$, and for every $i < \alpha$, $(P_\alpha / P_{i+1}) * \mathrm{seal}(I)$ is semi-proper$\}$ *then* $(P_\alpha / P_{i+1}) * \mathrm{seal}(\mathbf{I})$ is semi-proper for every $i < \alpha$.

(2) If (*)$(P_\alpha / P_{i+1}) * \mathrm{seal}(\underset{\sim}{I})$ is semi-proper for every $i < \alpha$ and maximal antichain antichain $\underset{\sim}{I}$ of $\mathfrak{B}^Q$ (from V^{P_α}) to which $\omega_1 - S$ belongs, *then* $(P_\alpha / P_{i+1}) * \mathrm{Seal}(\mathfrak{B}^Q, S)$ is semi-proper for every $i < \alpha$, as is $(P_\alpha / P_{i+1}) * \mathrm{SSeal}(\mathfrak{B}^Q, S, \kappa)$ for $\kappa > |P_\alpha|$ strongly inaccessible.

(3) The hypothesis (*) of (2) holds if for arbitrarily large $i < \alpha$:

$$\underset{\sim}{Q}_i \text{ is semi-proper and } \Vdash_{P_i} \text{"Rss}(\aleph_2)\text{"}.$$

Proof. (1) Use Claim 2.12.

(2) Use 2.13(1), and for the SSeal case, also 2.10A.

(3) By 1.7(5) $\mathrm{Rss}(\aleph_2)$ implies that semi-properness and not preserving stationarity of subsets of ω_1 are equivalent. Suppose $i<\alpha$, Q_i is semi-proper and $\Vdash_{P_i}$ "$\mathrm{Rss}(\aleph_2)$". As (by 2.8(5)) $\mathrm{seal}(\underset{\sim}{I})$ ($I \subseteq \mathfrak{B}^{Q}$ a maximal antichain) do not destroy stationarity of subsets of ω_1 from V^{P_i} (and this property is preserved by composition (though not by iteration)) and $P_\alpha/P_i = Q_i * (P_\alpha/P_{i+1})$ is semi-proper, we get that $(P_\alpha/P_i) * \mathrm{seal}(\underset{\sim}{I})$ is semi-proper (in V^{P_i} of course). This holds for arbitrarily large $i<\alpha$, hence (by the composition of semi-properness) for every non-limit i which is the assumption of (2).

2.14. Claim. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leqq \kappa,\ j<\kappa\rangle$ is semi-proper, κ strongly inaccessible and $\kappa > |P_i|$ for $i<\kappa$, $S \subseteq \omega_1$ stationary.

If

(*) (a) for $i<\kappa$, in V^{P_i}, Player II wins $\mathrm{Gm}(\{\aleph_1\}, \omega, D_\kappa + E_i^+)$ where $E_i^+ = \{\delta<\kappa : \delta > i, \delta$ strongly inaccessible and $\Vdash_{P_\delta/P_i}$ "Q_δ is semi-proper"$\}$, see 1.9A(2).

(b) $E^* = \{i<\kappa : \Vdash_{P_i}$ "$\mathrm{Rss}(\aleph_2)$, $\underset{\sim}{Q}_i$ semi-proper"$\}$ is unbounded,

then $R_{i+1} \stackrel{\mathrm{def}}{=} (P_\kappa/P_{i+1}) * \mathrm{Nm}' * \mathrm{SSeal}(\mathfrak{B}[P_\kappa], S)$ *is semi-proper for every* $i<\kappa$.

2.14A. Remark. (1) $\mathrm{Nm}' = \{T : T \subseteq {}^{\omega>}\aleph_2$ is closed under initial segments, is non-empty, and for every $\eta \in T\ |\{\nu : \eta \leqq \nu \in T\}| = \aleph_2\}$.

(2) We can use $\mathrm{Nm}'(D)$ instead of Nm' and even Nm, $\mathrm{Nm}(D)$.

(3) We can replace Nm' by any forcing notion satisfying, e.g., the **I**-condition or is S-complete (see [Sh 1, X, XI]) where $\mathbf{I} \in V$ is a family of κ-complete normal ideals.

(4) Instead of (*)(b) we can have "largeness" demands on κ. We need it to make $(P_\kappa/P_j) * \mathrm{seal}(\underset{\sim}{I})$ semi-proper for $j \in E_i^+$, $\underset{\sim}{I}$ a maximal antichain of $\mathfrak{B}^*$ from V^{P_κ}.

Proof. We work in $V^{P_{i+1}}$. Let λ be regular and large enough, $N \prec (H(\lambda), \in, <^*_\lambda)$ countable, $i \in N$, $\kappa \in N$, $\bar{Q} \in N$ and $(p^a, p^b, p^c) \in R_{i+1} \cap N$.

We now define by induction on n, T_n, N_η, q_η ($\eta \in T_n$), such that:

(A) $T_n \subseteq {}^n\kappa$.

(B) $T_0 = \{\langle\ \rangle\}$.

(C) $(\forall \nu \in T_{n+1})[\nu \restriction n \in T_n]$.

(D) $(\forall \eta \in T_n)[\{i : \eta^\wedge\langle i\rangle \in T_{n+1}\}$ has power $\kappa]$.

(E) $N <_{\omega_2} N_{\langle\ \rangle}$; $p^a \leqq q_{\langle\ \rangle}$.

(F) For $\eta \in T_{n+1}$ the model $N_\eta \prec (H(\lambda), \in, <^*_\lambda)$ is countable, extend $N_{\eta\restriction n}$, and $N_\eta \cap \omega_1 = N \cap \omega_1$; moreover $N_{\eta \restriction n} <_\kappa N_\eta$.

(G) $\eta \in N_\eta$.

(H) $q_\eta \in P_\kappa/P_{i+1}$ is $(N_\eta, P_\kappa/P_{i+1})$-semi-generic.

(I) For $\eta \in T_{n+1}$, $q_\eta \restriction \mathrm{Min}(N_\eta \cap \kappa - N_{\eta\restriction n}) = q_{\eta \restriction n}$.

(J) If $\underset{\sim}{I}$ is a P_κ/P_{i+1}-name of a dense subset of $\mathfrak{B}(P_\kappa)$, $\underset{\sim}{I} \in N_\eta$, $\eta \in T_n$, *then* for some natural number $k = k(\underset{\sim}{I}, \eta)$ for every ν: if $\eta \triangleleft'' \nu \in T_{n+k}$ then:

$$q_\nu \Vdash (\exists \underset{\sim}{A} \in N_\nu)[\underset{\sim}{A} \in \underset{\sim}{I} \wedge \underset{\sim}{A} \text{ a } P_\kappa/P_{i-1}\text{-name} \wedge N \cap \omega_1 \in \underset{\sim}{A}].$$

(K) E^0_η has cardinality κ, where

$$E^0_\eta \overset{\text{def}}{=} \{j < \kappa : N_\eta <_\kappa N_{\eta,j} \text{ where } N_{\eta,j} \text{ is the Skolem Hull of } N_\eta \cup \{j\} \text{ in } (H(\lambda), \in, <^*_\lambda) \text{ and } j = \mathrm{Min}(N_{\eta,j} \cap \kappa - N_\eta) \text{ and } j \text{ is strongly inaccessible and } (\forall i < j)[|P_i| < j] \text{ and } \Vdash_{P_j/P_i} \text{“}\underset{\sim}{Q}_j \text{ is semi-proper”}\}.$$

Now in carrying out the definition, (J) involves standard bookkeeping.

For $n = 0$ our main problem is satisfying (K). For $j < \kappa$ let N_j be the Skolem Hull of N in $(H(\lambda), \in, <^*_\lambda)$. By (*)(a)

$$E^1 = \{j < \kappa : N <_{\omega_2} N_j,\ \mathrm{cf}\, j > \aleph_0,\ \Vdash_{P/P_i} \text{“}\underset{\sim}{Q}_j \text{ is semi-proper”}\}$$

is a stationary subset of κ. So by the Fodor lemma [as $\delta \in E^1 \Rightarrow \mathrm{cf}\,\delta > \aleph_0$, and $\mu < \kappa \Rightarrow \mu^{\aleph_0} < \kappa$] for some stationary $E^2 \subseteq E^1$, $\langle N_j : j \in E^2\rangle$ form a Δ-system, and let $\cap \{N_j : j \in E^2\}$ be called $N_{\langle\ \rangle}$. So $N_{\langle\ \rangle} \prec (H(\lambda), \in, <^*_\lambda)$, and let $q_{\langle\ \rangle} \in P_\kappa/P_{i+1}$ be $(N_{\langle\ \rangle}, P_\kappa/P_{i+1})$-semigeneric, $p^a \leqq q_{\langle\ \rangle}$.

For $n > 0$ assume N_η, q_η are defined. By (K), E^0_η has power κ, where for $j \in E^0_\eta$ $\mathrm{Min}(\kappa \cap N_{\eta,j} - N_\eta) \in E^+_i$, $|E^0_\eta| = \kappa$ and we let

$$T_{\lg(\eta)+1} \cap \{\nu : \eta \triangleleft \nu \in {}^{(n+1)}\kappa\} = \{\eta^\wedge\langle j\rangle : j \in E^0_\eta\}.$$

So $T_{\lg(\eta)+1}$ is really constructed as required.

For $\gamma \in E^0_\eta$ let $N_{\eta,\gamma}$ be the Skolem Hull (in $(H(\lambda), \in, <^*_\lambda)$) of $N_\eta \cup \{\gamma\}$. By 2.2(1)(e) q_η is $(N_{\eta,\gamma}, P_\gamma)$-semi-generic and $\gamma = \mathrm{Min}(\kappa \cap N_{\eta,\gamma} - N_\eta)$. Let our bookkeeping give us $\underset{\sim}{I}_\delta \in N_\eta (\in N_{\eta,\gamma})$, a P_κ-name of a predense subset of $\mathfrak{B}[P_\kappa]$. Let $i(0) < \kappa$. We can find $\gamma^* \in E^*$, $\gamma < \gamma^* < \kappa$, so

$$\Vdash_{P_{\gamma^*}} \text{“Rss}(\aleph_2^{V^{P_\gamma}}),\ \underset{\sim}{Q}_{\gamma^*} \text{ semi-proper”}.$$

Now $(P_\kappa/P_{\gamma^*}) * \mathrm{seal}(\underset{\sim}{I})$ does not destroy stationary subsets ω_1 (as P_κ/P_{γ^*} is semi-proper, $\underset{\sim}{I}$ predense hence $\mathrm{seal}(\underset{\sim}{I})$ preserves stationary subsets of ω_1) because $\gamma^* \in E^*$ is semi-proper. As $\gamma \in E^0_\eta$, P_{γ^*}/P_γ is semi-proper. Hence $(P_\kappa/P_\gamma) * \mathrm{seal}(\underset{\sim}{I})$ is semi-proper. Now by 2.12 applied in V^{P_γ}, there is a model

$N'_{\eta,\gamma}$, $N'_{\eta,\gamma} \prec (H(\lambda), \in, <^*_\lambda)$ countable, $N_{\eta,\gamma} <_\gamma N'_{\eta,\gamma}$ and $q'_{\eta,\gamma} \in P_\kappa$ and $j_{\eta,\gamma} < \kappa$, successor such that:

$$q'_{\eta,\gamma} \in P_{j_{\eta,\gamma}}, \quad \gamma < j_{\eta,\gamma} \in N'_{\eta,\gamma},$$

$$q'_{\eta,\gamma} \restriction \mathrm{Min}(N'_{\eta,\gamma} \cap \kappa - N_\eta) = q'_{\eta,\gamma} \restriction \mathrm{Min}(N_{\eta,\gamma} \cap \kappa - N_\eta) = q_\eta,$$

$q'_{\eta,\gamma}$ is $(N'_{\eta,\gamma}, P_{j_{\eta,\gamma}})$-semi-generic, and

$$q'_{\eta,\gamma} \Vdash_{P_{j_{\eta,\gamma}}} \text{"for some } \underset{\sim}{A} \in N'_{\eta,\gamma}, \underset{\sim}{A} \in I \text{ and } N \cap \omega_1 \in \underset{\sim}{A}\text{"}.$$

As in the case $n = 0$, there is $N_{\eta\char`^\langle\gamma\rangle}$ such that (for $i \in E^D_{\eta\char`^\langle\gamma\rangle}$) $N'_{\eta,\gamma} \leqq_\gamma N_{\eta\char`^\langle\gamma\rangle} \prec (H(\lambda), \in, <^*_\lambda)$, and $N_{\eta\char`^\langle\gamma\rangle}$ satisfies condition (K); now we can define $q_{\eta\char`^\langle\gamma\rangle}$ as required.

Let $G \subseteq P_\kappa$ be generic over V, $q_{\langle\ \rangle} \in G$. Let $T'_n \overset{\text{def}}{=} \{\eta \in T_n : q_\eta \in G\}$. We work in $V[G]$. We now define by induction on n, for every $\eta \in T'_n$, a condition p^b_η such that:

(a) $p^b_\eta \in N_\eta[G]$.

(b) $p^b_\eta \in \mathrm{Nm}'$, and $p^b_\eta \cap ({}^{\lg(\eta)}\kappa)$ is a singleton.

(c) $p^b_{\eta|l} \leqq p^b_\eta$; and if $p^b_{\eta\restriction l}$ has a stem of length m, $\lg(\eta) \leqq m$, then $p^b_\eta = p^b_{\eta\restriction l}$.

(d) If $\eta \in T'_n$, $\underset{\sim}{\alpha}$ is a Nm′-name of a countable ordinal, $\underset{\sim}{\alpha} \in N_\eta[G]$, *then* for some $k = k^1(\underset{\sim}{\alpha}, \eta)$, for every $\nu \in T'_{n+k}$,

$$[\eta \trianglelefteq' \nu \Rightarrow p^b_\nu \Vdash_{\mathrm{Nm}'} \text{"}\underset{\sim}{\alpha} < N \cap \omega_1\text{"}].$$

(e) If $\eta \in T'_n$, $\underset{\sim}{I}$ is a Nm′-name of a predense subset of $\mathfrak{B}(P_\kappa)$, $\underset{\sim}{I} \in N_\eta[G]$ *then* for some $k = k^1(\underset{\sim}{I}, \eta)$, for every $\rho \in T'_{n+k}$, $\eta \trianglelefteq' \rho$, for some $m = k^2(\underset{\sim}{I}, \eta, \rho)$, for every $\nu \in T'_{n+k+m}$,

$$[\rho \trianglelefteq' \nu \Rightarrow p^b_\nu \Vdash_{\mathrm{Nm}'} \text{"for some } A \in N_\nu[G], A \in \underset{\sim}{I}, N \cap \omega_1 \in A\text{"}].$$

(f) If p^b_η has a stem of length $\lg(\eta)$, call it ν_η, let h_η be a one-to-one function from κ onto $\{j < \kappa : \nu_\eta\char`^\langle j\rangle \in p^b_\eta\}$, $h_\eta \in N_\eta[G]$ and then

$$(\forall \rho \in p^b_{\eta\char`^\langle i\rangle})[\lg(\rho) > \lg(\eta) \Rightarrow \rho(\lg(\eta)) = h_\eta(i)] \text{ for } \eta\char`^\langle i\rangle \in \textstyle\bigcup_n T'_n.$$

There is no problem to do this. [For (d), when we come to deal with $\underset{\sim}{I}$ (say at ρ), we let

$$I' = \{A : (\exists\, p)(p^b_\nu \leqq p \in \mathrm{Nm}' \wedge p \Vdash_{\mathrm{Nm}'} \text{"}A \in \underset{\sim}{I}\text{"})\},$$

so $I' \in N_\rho[G]$ is a predense subset of $\mathfrak{B}(P_\kappa)$, and by (J) we can define $m = k(\underset{\sim}{I}', \rho)$, let $p^b_\nu = p^b_\rho$ if $\nu \triangleleft \rho \in T'_{\lg(\rho)}$, $\lg(\rho) \leqq \lg(\nu) + m$.]

Now let (in V^{P_κ})
$q^b = \{\rho \in {}^{\omega>}\kappa : \rho \in p^b$, and for some $\eta \in \bigcup_n T'_n$, ρ belongs to the stem of $p^b_\eta\}$.

Let $q^a = q_{\langle\ \rangle}$ and assume $q^a \in G \subseteq P_\kappa$, G generic over V.

We can easily see that $q^a \leqq q^b \in \mathrm{Nm}'$. Also (in $V[G]$) q^b is $(N[G], \mathrm{Nm}')$-semi-generic and

$$q^b \Vdash_{\mathrm{Nm}'} \text{“}N[G][\underset{\sim}{G}_{\mathrm{Nm}'}] \supseteq \bigcup_{l<\omega} N_{\eta \restriction l}[G]\text{”},$$

where $\underset{\sim}{G}_{\mathrm{Nm}'}$ is the (canonical name of the) generic subset of Nm', and η is the ω-sequence in ${}^\omega\kappa$ which it defines naturally. [Remember that if $N_1, N_2 < (H(\lambda), \in)$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$, and $i \in N_1 \cap N_2$, $i < \aleph_2$, then $N_1 \cap i = N_2 \cap i$.] Now clearly by the above and (e)

$$q^b \Vdash_{\mathrm{Nm}'} \text{“for every predense subset } I \text{ of } \mathfrak{B}[P_\kappa] \text{ in } N[G][\underset{\sim}{G}_{\mathrm{Nm}'}],$$
$$N \cap \omega_1 \in \bigcup_{A \in I} \{A : A \in I \cap N[G][\underset{\sim}{G}_{\mathrm{Nm}'}]\}\text{”}.$$

So we can apply the proof of Claim 2.10(1) (i.e., 2.10A) to get q^c, which is $(N[G][G_{\mathrm{Nm}'}], \mathrm{SSeal}(\mathfrak{B}[P_\kappa], S))$-semi-generic. Now $(q^a, \underset{\sim}{q}^b, \underset{\sim}{q}^c)$ is as required (i.e., (R_{i+1}, N)-semi-generic).

2.15. Claim. Suppose $Q = \langle P_i, \underset{\sim}{Q}_j : i \leqq \kappa,\ j < \kappa \rangle$ is a semi-proper iteration, $\kappa > |P_i|$ for $i < \kappa$, $S \subseteq \omega_1$ is stationary: If

(∗) (a) for $i < \kappa$, in V^{P_i}, Player II wins $\mathrm{Gm}(\{\aleph_1\}, \omega, D_\kappa + E_i^+)$ where $E_i^+ = \{\delta < \kappa : \delta > i,\ \delta$ strongly inaccessible $\Vdash_{P_\delta/P_i}$ “$\underset{\sim}{Q}_\delta$ is semi-proper”$\}$.

(b) $E = \{i < \kappa : \Vdash_{P_i}$ “$\mathrm{Rss}(\aleph_2)$ and $\underset{\sim}{Q}_i$-semi-proper”$\}$ is unbounded and in V^{P_κ},

(c) $\mathrm{W} \subseteq \{\delta < \kappa : V^{P_\kappa} \models \mathrm{cf}\,\delta = \aleph_0\}$ is stationary ($\underset{\sim}{W}$ a P_κ-name),

then $(P_\kappa/P_{i+1}) * \mathrm{club}(\underset{\sim}{W}) * \mathrm{SSeal}(\mathfrak{B}(P_\kappa), S)$ is semi-proper for $i < \kappa$ where $\mathrm{club}(W) = \{f$: for some non-limit $\gamma < \omega_1$, f is an increasing continuous function from γ into $W\}$.

Proof. Like the previous claim, only after defining N_η, q_η, for a set $G \subseteq P_\kappa$ generic over V, $q_{\langle\ \rangle} \in G$, in $V[G]$ there is $\eta \in {}^\omega\kappa$, $\bigwedge_n(\eta \restriction n \in T_n)$ such that $\eta(l) > \sup(N_{\eta \restriction l} \cap \kappa)$ and $\sup\{\eta(l) : l < \omega\}$ belong to ω, then in $V[G]$ continue with $\bigcup_l N_{\eta \restriction l}[G]$.

2.15A. Remark. Really 2.15 is just a case of 2.14(A)(3).

2.16. Theorem. *Suppose $\{\mu < \kappa : \mu$ supercompact$\}$ is unbounded below κ and κ is 2-Mahlo.*

Let $S \subseteq \omega_1$ be stationary, then for some semi-proper $(*, (\omega_1 - S))$-complete *forcing notion* P (see 1.4), *satisfying the* κ-c.c., $\Vdash_P$ "$\mathfrak{B}[P_\kappa] \restriction S$ *has a dense subset which is* (*up to isomorphism*) Levy$(\aleph_0, <\aleph_2)$".

2.16A. Remark. (1) So really (see introduction) from one supercompact κ we get, e.g., P such that: $\Vdash_P$ "$ZFC + \mathfrak{B} \restriction S_1$ has a dense subset isomorphic to Levy$(\aleph_0, <\aleph_2) + \mathrm{MA}_{\omega_1}$ $(\omega_1 - S_1 \cup S_2)$-complete semi-proper)" ($S_l \subseteq \omega_1$ stationary).

For this use 2.18A(3).

We should, while defining the iteration $\bar{Q}$, ensure that: if over V^{P_κ} we force by club$_\kappa(w^*)$, the relevant variant of MA still holds. Really we can ensure something like Laver's indestructibility of supercompactness holds.

(2) An alternative way to iterate is to let κ be limit of supercompacts, and: if $|P_i| < \lambda$ for $i < \lambda$, $\lambda < \kappa$, λ supercompact or limit of supercompacts, ensure in the iteration that

$$\mathfrak{B}^{Q \restriction i} \lessdot \mathfrak{B}[P_j] \qquad \text{when } i < j < \kappa.$$

In this way we get rid of $(*, S)$-completeness.

Proof. We define by induction in i, P_i, $\underset{\sim}{Q}_i$, $\mathbf{t}_i$ such that

(A) $\bar{Q}^\alpha = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : i \leqq \alpha, j < \alpha \rangle$ is S-suitable;

(B) there is no strongly inaccessible Mahlo λ, $i < \lambda \leqq |P_i|$;

(C) if i is a singular ordinal or $(\exists j < i)[|P_j| > i]$ then $\mathbf{t}_i$ is 0, $\underset{\sim}{Q}_i =$ SSeal$(\langle \mathfrak{B}[P_j] : j \leqq i, \mathbf{t}_j = 1 \rangle, S)$;

(D) if i is supercompact, $\mathbf{t}_i = 1$, $\underset{\sim}{Q}_i =$ SSeal$(\mathfrak{B}[P_i], S)$;

(E) if $(\forall j < i)[|P_j| < i]$, i limit of supercompacts and i is inaccessible but not Mahlo, we let $\mathbf{t}_i = 1$, $\underset{\sim}{Q}_i = \mathrm{Nm}' * $ SSeal$(\mathfrak{B}[P_i])$;

(F) if $(\forall j < i)[|P_j| < i]$, i Mahlo and limit of supercompacts then

$$W_i = \{\delta < i : \delta = \mathrm{cf}\,\delta \text{ limit of supercompacts and } (\forall j < \delta)[|P_j| < \delta]\}$$

is a stationary subset of i, and we let:

$$\mathbf{t}_i = 1, \qquad \underset{\sim}{Q}_i = \mathrm{club}(W_i) * \mathrm{SSeal}(\mathfrak{B}[P_i]).$$

Why is $\bar{Q}$ S-suitable?

Note that the use of SSeal guarantees (F) of Definition 2.1, as well as (E) (see 2.10(3), 2.10(2)). So it suffices to show by induction on i that $\bar{Q} \restriction i$ is a semi-proper iteration.

We shall show below that every Q_i is smi-proper (the only problematic cases are i inaccessible limit of supercompacts, but then $\mathrm{Rss}^+(i)$ (by 1.6(2), 1.6(4)), so in V^{P_i}, every forcing notion not destroying stationary sets is semi-proper).

For $i=0$: trivially.

For i limit: by Claim 2.3(2).

For $i+1$, apply (C) to i: by Claim 2.11.

For $i+1$, apply (D) to i: by Claim 2.9, 2.2(3), 1.6(4), $\mathrm{SSeal}(\mathfrak{B}[P_\kappa], S)$ is semi-proper in V^{P_κ}.

For $i+1$, apply (E) to i: by 2.14 (*)(a) holds as $E_j^+ = \kappa$, and as said above, $\mathrm{Rss}^+(i)$ (see 1.11).

For $i+1$, apply (F) to i: by Claim 2.15 (and remember (5) of the Notation).

Also each Q_i is $(*, (\omega_1 - S))$-complete, hence P_κ is $(*, (\omega_1 - S))$-complete so when S is costationary

$$\Vdash_{P_\kappa} \text{“}2^{\aleph_0} = \aleph_1,\ 2^{\aleph_1} = \aleph_2\text{”}.$$

Let $\mathfrak{B}_i = \mathfrak{B}[P_i]$, so $\mathrm{t}_i = 1 \Rightarrow \mathfrak{B}_i \restriction S \lessdot \mathfrak{B}[P_\kappa] \restriction S$. Let $w^* = \{i < \kappa : \mathfrak{B}_i \restriction S \lessdot \mathfrak{B}[P_\kappa] \restriction S\}$. So in V^{P_κ} (as case F occurs stationarily often): $\{\delta \in w^* : \mathrm{cf}\,\delta = \aleph_1,\ w^* \text{ contains a club of } \delta\}$ is stationary. Hence it is well-known that in V^{P_κ}, $\mathrm{club}_\kappa(w^*) = \{h : h \text{ an increasing continuous function from some } \alpha + 1 < \kappa \text{ to } w^*\}$ does not add bounded subsets to $\kappa\ (=\aleph_2)$. So forcing will give us a universe as required.

2.17. Claim. Suppose $\bar{Q} = \langle P_i, Q_j : i \leqq \alpha, j < \alpha \rangle$ is a semi-proper iteration, $\mu < \alpha$ ($\mu = 0$ is alright), and $\Vdash_{P_\mu}$ “$\mathrm{Rss}(\aleph_2[V^{P_\mu}])$” (e.g., if μ is supercompact, $[i < \mu \Rightarrow |P_i| < \mu]$ note that $\Vdash_{P_\mu}$ “$\mu = \aleph_2$” if $\{\kappa < \mu : \underset{\sim}{Q}_\kappa$ semi-proper$\}$ belong to some normal ultrafilter on μ).

Let $\underset{\sim}{A}$ be a P_α-name of a subset of S and $\underset{\sim}{B}$ a P_α-name of a member of $\mathfrak{B}[P_\mu]$ such that:

$$\Vdash_{P_\alpha} (\forall x \in \mathfrak{B}[P_\mu])[0 < x \leqq \underset{\sim}{B} \rightarrow x \cap \underset{\sim}{A} \neq 0 \text{ (in } \mathfrak{B}[P_\alpha])].$$

Then

(*) *if* λ is regular and large enough, $N \prec (H(\lambda), \in, <^*_\lambda)$ is countable, and $\bar{Q}$, $\lambda, p, \underset{\sim}{A}, \underset{\sim}{B}$ and μ belong to N, $p \in P_\alpha \cap N$ and $q \in P_\mu$ is (N, P_μ)-generic, $p \restriction \mu \leqq q$ and $q \Vdash_{P_\mu}$ “$N \cap \omega_1 \in \underset{\sim}{B}$”, *then* there is a countable N', $N \leqq_\mu N' \prec (H(\lambda), \in, <^*_\lambda)$, $N \cap \omega_1 = N' \cap \omega_1$ and $q' \in P_\alpha$, $q' \restriction \mu = q$ such that $q' \Vdash$ “$N \cap \omega_1 \in \underset{\sim}{A}$”.

2.17A. Remark. (1) If $\bar{Q}$ is S-suitable, $\mathrm{t}_\mu = 1$, and $\underset{\sim}{A} \neq \emptyset \bmod D_{\omega_1}$, $\underset{\sim}{A}$ is a P_β-name for some $\beta < \alpha$, then we know that such $\underset{\sim}{B}$ exists as $\mathrm{t}_\mu = 1$. If α is strongly inaccessible, $\alpha > |P_i|$ for $i < \alpha$, such B will exist.

(2) Now, e.g., for suitable $\bar{Q}$, $\lg(\bar{Q}) = \alpha = \bigcup_{n<\omega} \alpha_n$, $\alpha_n < \alpha_{n+1}$, $\mathrm{t}_{\alpha_n} = 1$, we can use $Q_\alpha = \mathrm{SSeal}(\mathfrak{B}^{\bar{Q}}, S)$ by new reasons.

PROOF. As we can increase p, without loss of generality p forces $\underset{\sim}{B}$ to be equal to some P_μ-name, so without loss of generality $\underset{\sim}{B}$ is a P_μ-name.

Let us fix $p, \underset{\sim}{A}, \underset{\sim}{B}, \mu$ and work in $V[G_\mu]$, $G_\mu \subseteq P_\mu$ generic over V, $q \in G_\mu$. Let

$$\begin{aligned} W = \{N \prec (H(\lambda), \in) : \ & N \cap \omega_1 \in \underset{\sim}{B}[G_\mu], \\ & \text{but there is no } r \in P_\alpha/G_\mu \text{ such that:} \\ & r \text{ is } (N, P_\alpha/G_\mu)\text{-semi-generic}, p \restriction [\mu, \alpha) \leqq r \\ & \text{and } r \Vdash_{P_\alpha/G_\mu} \text{“} N \cap \omega_1 \in \underset{\sim}{A}\text{”}\}. \end{aligned}$$

If $W = \emptyset \bmod D_{<\aleph_1}(H(\lambda))$, we can easily get the desired result (as in the proof of 1.11).

So (in $V[G_\mu]$) W is a stationary subset of $S_{<\aleph_1}(H(\lambda))$. Hence there is $u \subseteq H(\lambda)$, $\omega_1 \subseteq u$, $|u| < \aleph_2$ (in $V[G_\mu]$) and $W \cap S_{<\aleph_1}(u)$ semi-stationary; now without loss of generality $(u, \in, <^*_\lambda \restriction u) \prec (H(\lambda), \in, <^*_\lambda)$. Let

$$u = \bigcup_{\zeta < \omega_1} u_\zeta, \qquad (u_\zeta, \in, <^*_\lambda \restriction u_\zeta) \prec (u, \in, <^*_\lambda \restriction u_\zeta),$$

u_ζ countable increasing continuous. So

$$B_1 = \{\zeta < \omega_1 : (\exists N \in W)(\omega_1 \cap u_\zeta \subseteq N \subseteq u_\zeta)\}$$

is stationary, it is a stationary subset of ω_1, it belongs to $\mathfrak{B}[P_\mu]$, and obviously:

(*) $\qquad p \Vdash$ “$\underset{\sim}{A} \cap B_1$ is not stationary”.

[Let for $\zeta \in B_1$, $\omega_1 \cap u_\zeta \subseteq N_\zeta \subseteq u_\zeta$, $N_\zeta \in W$; let for $\xi < \omega_1$, N'_ξ be the Skolem Hull in $(H(\lambda), \in, <^*_\lambda)$ of $\{\zeta : \zeta < \xi\} \cup \{p, \langle u_\zeta, N_\zeta : \zeta \in B_1\rangle\}$, and

$$\underset{\sim}{C} = \{\xi < \omega_1 : N'_\xi[G_{P_\kappa}] \cap \omega_1 = \xi\}.$$

$\underset{\sim}{C}$ is a P_α/G_μ-name of a club of ω_1. clearly $\underset{\sim}{C} \cap \underset{\sim}{A}$ is necessarily disjoint to B_1 by the definition of W [if $\zeta < \omega_1$, $q \in P_\alpha/G_\mu$, $q \Vdash_{P_\alpha/G_\mu}$ “$\zeta \in \underset{\sim}{C} \cap \underset{\sim}{A} \cap \underset{\sim}{B}_1$”, then $N_\zeta \in W$ is defined, q_α is $(N_\zeta, P_\alpha/G_\mu)$-semi-generic, $q_\alpha \Vdash$ “$N_\zeta \cap \omega_1 \in \underset{\sim}{A}$”, contradicting “$N_\zeta \in W$”]. Also

$$p \Vdash \text{“}B_1[\underset{\sim}{G}_\alpha] \subset B[\underset{\sim}{G}_\alpha]\text{”}$$

[by the clause “$N \cap \omega_1 \notin \underset{\sim}{B}[G_\mu]$” in the definition of W]. So (*) holds.]

Of course $B_1 \in V^{P_\alpha}$ and we get a contradiction to an assumption on $\underset{\sim}{A}, \underset{\sim}{B}$.

2.18. CLAIM. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leqq \alpha,\ j < \alpha\rangle$ is a semi-proper iteration, $\langle \mu_\zeta : \zeta < \xi\rangle$ is an increasing sequence of strongly inaccessible cardinals,

$$\bigwedge_{\zeta<\xi} [(\forall i<\mu_\zeta)(|P_i|<\mu_\zeta) \text{ and } \Vdash_{P_{\mu_\zeta}} \mathrm{Rss}(\mu_\zeta) \text{ and } \Vdash_{P_{\mu_\zeta+1}} \text{“cf}\,\mu_\zeta=\aleph_1\text{”}].$$

Suppose further $\underset{\sim}{B}$ is a P_{μ_0}-name, $\underset{\sim}{A}_\zeta$ a $P_{\mu_\zeta+1}$-name of subsets of S. Suppose further $p\in P$ and:

$p\restriction\mu_0\Vdash_{P_{\mu_0}}$ “$\underset{\sim}{B}$ is stationary”,

$p\restriction\mu_{\zeta+1}\Vdash_{P_{\mu_\zeta+1}}$ “for every $x\in\mathfrak{B}[P_{\mu_\zeta}]-\{0\}$ if $x\subseteq\underset{\sim}{B}$ *then* $\underset{\sim}{A}_\zeta\cap x$ is stationary”.

Then $p\Vdash_{P_\alpha}$ “the intersection of any countable subset of $\{A_\zeta:\zeta<\xi\}$ is stationary”.

Proof. Let $\underset{\sim}{W}$ be a P_α-name of a countable subset of ξ. So without loss of generality $\underset{\sim}{W}=\{\zeta(n):n<\omega\}$, $\Vdash_{P_\alpha}$ “$\underset{\sim}{\zeta}_n\in\xi$”.

We now prove by induction on $j\leqq\alpha$,

(*) *if* $\mu_0<i<j$, i non-limit, λ regular, and large enough, $N\prec(H(\lambda),\in)$ countable, $\underset{\sim}{B}\in N$, $\langle\mu_\zeta,\underset{\sim}{A}_\zeta:\zeta<\xi\rangle\in N$ and i,j, $\bar{Q}\in N$, $p\leqq p'\in N\cap P_\alpha$ and $q\in P_i$ is (N,P_i)-semi-generic, $p'\restriction i\leqq q$, and

$$q\Vdash_{P_i}\text{“}N\cap\omega_1\in\underset{\sim}{B}\text{ and for } n<\omega\ [\mu_{\zeta(n)}<i\Rightarrow N\cap\omega_1\in A_{\zeta(n)}]\text{”}$$

then there is $q'\in P_j$, (N,P_j)-semi-generic, $p'\restriction j\leqq q'$, $q'\restriction i=q$ and $q'\Vdash_{P_j}$ “$N\cap\omega_1\in\underset{\sim}{B}$ and for $n<\omega$ $[\mu_{\zeta(n)}<j\Rightarrow N\cap\omega_1\in A_{\zeta(n)}]$”.

Clearly this is enough (apply it with $p'=p$, $i=\mu_0$, $j=\alpha$, and those suitable N,q possible as B is a $P_{\mu_{\zeta_0}}$-name of a stationary subset of $S\subseteq\omega_1$).

Case 1. $j\leqq\mu_0$. Trivial.

Case 2. j limit. As in the proof of the iteration lemma for semi-properness.

Case 3. $j=(j-1)+1$, $j-1\notin\{\mu_\zeta:\zeta<\xi\}$. Trivial (use iteration of semi-properness of length 2).

Case 4. $j=(j-1)+1$, $j-1=\mu_\zeta$.

As we know $\Vdash_{P_\alpha}$ “cf $\mu_\zeta=\aleph_1$”, there is an ordinal $\gamma<\mu_\zeta$ and condition p'', $p'\leqq p''\in P_\alpha\cap N$ such that:

$$p''\Vdash\text{“for } n<\omega,\text{ if } \underset{\sim}{\mu}_{\zeta(n)}<\mu_\zeta\text{ then }\underset{\sim}{\mu}_{\zeta(n)}<\gamma\text{”}.$$

Now apply the induction hypotheses to i,γ,N,p'' and get a condition $q'\in P_\gamma$, $p''\restriction\gamma\leqq q$, q' is (N,P_γ)-semi-generic and satisfying

$$q'\Vdash_{P_\alpha}\text{“if }\mu_{\zeta(n)}<\gamma\text{ then }N\cap\omega_1\in A_{\zeta(n)}\text{”}.$$

By the choice of γ and p''

$$q' \Vdash_{P_{\mu_\zeta}} \text{"if } \mu_{\zeta(n)} < \mu_\zeta \text{ then } N \cap \omega_1 \in A_{\zeta(n)}\text{"}.$$

Now apply the previous claim (for $\bar{Q} \restriction \mu_{\zeta+1}$).

2.19. THEOREM. *Suppose $\{\mu < \kappa : \mu$ supercompact$\}$ is stationary, $S \subseteq \omega_1$ stationary. Then for some semi-proper $P = P_\kappa$ (for some S-suitable $\bar{Q}$) which is $(\omega_1 - S)$-complete (and satisfies the* κ-c.c.) *in V^{P_κ}: from any $\aleph_2$ stationary subsets of $S \subseteq \omega_1$, there are $\aleph_2$; the intersection of any countably many of them is stationary (and $\mathfrak{B}^{P_\kappa}[P_\kappa]$ is layered, of course).*

2.19A. REMARK. We really use just (i) $\{\mu < \kappa : \mu$ measurable$\}$ is stationary; (ii) $\{\mu < \kappa : \mu$ supercompact$\}$ is unbounded suffice.

PROOF. (1) We define by induction on $\alpha \leqq \kappa$,

$$\bar{Q}^\alpha = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : i \leqq \alpha, j < \alpha \rangle,$$

such that:

(A) $\bar{Q}^\alpha$ is S-suitable.

(B) Each Q_i is $(\omega_1 - S)$-complete.

(C) There is no strongly inaccessible Mahlo μ, $i < \mu \leqq |P_i|$.

(D) If i is inaccessible, $|P_j| < i$ for $j < i$ and $\Vdash_{P_i}$ "Rss($\aleph_2$)" then $\underset{\sim}{Q}_i =$ SSeal($\langle \mathfrak{B}[P_j] : j \leqq i, t_j = 1 \rangle, S$) and $t_i = 1$.

(E) If not (D) then $\mathbf{t}_i = 0$,

$$Q_i = \text{SSeal}(\langle \mathfrak{B}[P_j] : j \leqq i, \mathbf{t}_j = 1 \rangle, S).$$

We can carry out the construction and prove by induction on $\alpha\vartheta$ that $\bar{Q}^\alpha$ is suitable.

$\alpha = 0$. Trivial.

α limit. By Claim 2.3.

$\alpha = \beta + 1$, (E) applies to β. By 2.11.

$\alpha = \beta + 1$, (D) applies to β. By Claim 2.9 (and 2.10).

Now suppose $p \in P_\kappa$, $\langle \underset{\sim}{A}_i : i < \kappa \rangle$ a P_κ-name $p \Vdash$ "$\underset{\sim}{A}_i \subseteq S$ is stationary".

For each $\mu \in Y = \{\mu < \kappa : \mu$ supercompact$\}$ choose $p_\mu \in P_\kappa$ and a P_μ-name $\underset{\sim}{B}_\mu$ such that

$$p_\mu \restriction \mu \Vdash \text{"}\underset{\sim}{B}_\mu \subseteq S \text{ is stationary, } \underset{\sim}{B}_\mu \in \mathfrak{B}[P_\mu]\text{"},$$

$$p_\mu \Vdash_{P_\kappa} \text{"for every } X \in \mathfrak{B}[P_\mu] \text{ non-zero, if } X \leqq \underset{\sim}{B}_\mu \text{ then } X \cap \underset{\sim}{A}_\mu \text{ is stationary"}.$$

Now that $\underset{\sim}{B}_\mu \in H(\chi_\mu)$ for some $\chi_\mu < \mu$, also γ is stationary by a hypothesis.

By Fodor's lemma for some stationary $Y_1 \subseteq Y$, there are p and $\underset{\sim}{B}$ such that for $\mu \in Y_i$: $p_\mu \restriction \mu = p$,

$$\underset{\sim}{B}_\mu = B.$$

As each $\underset{\sim}{A}_\zeta$ is a P_{μ_ζ}-name for some $\mu_\zeta > \mu$ without loss of generality [$\mu_\zeta < \mu_\xi$ in $Y_1 \Rightarrow \underset{\sim}{A}_{\mu_1}$ is a P_{μ_2}-name]. Now define for $\mu \in Y_1 : \underset{\sim}{A}'_\mu$ is $\underset{\sim}{A}_\mu$ if $P_\mu \in \underset{\sim}{G}_{P_\kappa}$ and S otherwise.

Note that $Y_1 \in V$ and every countable subset of Y_1 is contained in a countable set from V. Now we apply the previous claim to $\underset{\sim}{B}$, $\langle \underset{\sim}{A}'_\mu : \mu \in Y_1 \rangle$.

2.20. Theorem. *Suppose* $\{\lambda : \lambda < \kappa,\ \lambda$ *supercompact*$\}$ *is a stationary subset of* κ, $S \subseteq \omega_1$ *stationary, costationary*; then *in some forcing extension of* V, ZFC $+ 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2 +$ Ulam$(D_{\omega_1} + S)$ *holds, where, for a uniform filter* D *on* λ, Ulam(D) *means*: *there are* λ λ*-complete filters extending* D, *such that every* D*-positive set belongs to at least one of them* (A *is* D*-positive* if $A \subseteq \lambda$, *and* $(\lambda - A) \notin D$).

2.20A. Remark. Of course also here we can easily (when κ is supercompact) make, e.g., $\mathrm{MA}_{\omega_1}((\omega_1 - S \cup S')$-complete semi-proper) holds after the forcing; if $S' \subseteq \omega_1 - S$ is stationary $\omega_1 - S'$ is stationary.

Proof. Before we do the forcing, we make some combinatories, which tell us what will suffice.

A. Notation. (1) $\lambda = \lambda^{<\lambda}$ is a fixed regular uncountable cardinal.

(2) W denotes a fixed class of ordinals, $0 \in W$, for every i, $i + 1 \in W$, and

$$\aleph_0 \leqq \mathrm{cf}\, i < \lambda \Rightarrow i \notin W.$$

(3) B will denote Boolean algebras.

(4) For a Boolean Algebra B: $B^+ = B - \{0\}$.

(5) $\Pr(a_1, a_2, B_1, B_2)$ means: B_1, B_2 are Boolean algebras, $B_1 \subseteq B_2$, $a_1 \in B_1^+$, $a_2 \in B_2^+$, and $(\forall x)[x \in B_1^+ \wedge x \leqq a_1 \rightarrow x \cap a_2 \neq 0]$.

If the identity of B_2 is clear (when dealing with one Boolean Algebra and its subalgebras) we omit it.

B. Observation. Note:

(a) $\Pr(1, x, B_0)$ for $x \in B_1^+$, $|B_0| = 2$;

(b) if $B_a \subseteq B_b \subseteq B$, $x \in B_a^+$, $y \in B_b^+$, $z \in B$ and $\Pr(x, y, B_a)$, $\Pr(y, z, B_b)$ then $\Pr(x, z, B_a)$;

(c) if $\Pr(x, y, B_1, B_2)$, $0 < x' \leqq x$, $x' \in B_1$, $y \leqq y' \in B_2$ *then* $\Pr(x', y', B)$ and $\Pr(x', y \cap x, B)$.

C. Notation and Definition.

(1) We call $\bar{B}$ 1-o.k. if $\bar{B} = \langle \bar{B}_i : i < \alpha \rangle$ is increasing continuous, each B_i a Boolean Algebra of cardinality $\leqq \lambda$, $[i, j \in \alpha \cap W$ and $i < j \Rightarrow B_i \lessdot B_j$, B_i is λ-complete].

(2) We call $a \subseteq W \cap \alpha$ closed if for every accumulation point $\delta < \alpha$ of a $[\delta \notin W \Rightarrow \delta + 1 \in a]$, $[\delta \in W \Rightarrow \delta \in a]$.

(3) Let $\mathrm{CSb}(\alpha) = \{w : w$ a closed subset of $W \cap \alpha$ of power $< \lambda\}$, $\mathrm{CSb}_u(\alpha) = \{w \in \mathrm{CSb}(\alpha) : w$ unbounded below $\alpha\}$.

(4) For $w \in \mathrm{CSb}(\alpha)$, $\bar{B} = \langle B_i : i < \beta \rangle$ 1-o.k., $\beta > \alpha$ let

(i) $\mathrm{Seq}_w(\bar{B}) = \{(\langle a_i : i \in w \rangle : a_i \in B_i,$ a_i is decreasing; *if* $i \in w$, $i = \delta + 1$, $\delta \notin W$, $i > \mathrm{Min}\, w$ then $a_j = {}_{j \in w \cap i}a_j$, *if* $i \in w$, $i > \mathrm{Min}\, w$ and $\neg(\exists \delta)[i = \delta + 1 \wedge \delta \notin W]$ then $a_i \in B_{\mathrm{Max}(w \cap i)+1}$, *if* $i \in w$, $i > \mathrm{Min}\, w$, $\mathrm{cf}\, i \geqq \lambda$, then $a_i = a_j \in B_j$ for some $j \in i \cap W$, and for $i < j$ in $w \cap W$, $\Pr(a_i, a_j, B_i, B_j)\}$.

(ii) Let $\mathrm{Seq}(\bar{B}) = \bigcup \{\mathrm{Seq}_w(\bar{B}) : w \in \mathrm{CSb}(\alpha)\}$
when $\alpha = \delta + 1 \leqq \lg(\bar{B})$, $\alpha = \sup w$:

$$Z_w(\bar{B}) = \left\{ \bigcap_{i \in w} a_i : \langle a_i : i \in w \rangle \in \mathrm{Seq}_w(\bar{B} \restriction (\delta + 1)) \right\},$$

$Z^\delta(\bar{B}) = \bigcup \{Z_w(\bar{B}) : w \in \mathrm{CSb}_u(\delta)\}$, if $\lg(\bar{B}) = \delta + 1$, we omit δ.

(5) We call $\bar{B}$ a 2-o.k. if for every limit $\delta < \lg(\bar{B})$, $0 \notin Z(\bar{B} \restriction (\delta + 1))$ (and it is 1-o.k.).

(6) We call $\bar{B}$ 3-o.k. if it is 2-o.k. and for limit $\delta < \lg(\bar{B})$ $Z(\bar{B} \restriction (\delta + 1))$ is a dense subset of $B_{\delta+1}$.

(7) If $\bar{B}$ is not continuous, we identify it with the obvious correction for the purpose of our definitions.

D. Fact. Suppose $\bar{B}$ is 2-o.k., $\bar{B} = \langle B_i : i \leqq \delta + 1 \rangle$, $\mathrm{cf}\, \delta < \lambda$:

(0) $\mathrm{CSb}_u(\delta) \neq 0$, and if $w \in \mathrm{CSb}_u(\delta)$, $\alpha < \delta$, then $w - \alpha \in \mathrm{CSb}_u(\delta)$.

(1) $Z_w(\bar{B})$ includes $B_{\mathrm{Min}\, w}$, hence $B_\delta \subseteq Z(\bar{B})$.

(2) If $w_1, w_2 \in \mathrm{CSb}(\delta)$ then $w_1 \cup w_2 \in \mathrm{CSb}(\delta)$; similarly for $\mathrm{CSb}_u(\delta)$.

(3) If $w_1 \subseteq w_2$ are in both in $\mathrm{CSb}(\delta)$, $\mathrm{Min}\, w_1 = \mathrm{Min}\, w_2$, $\langle a_i : i \in w_1 \rangle \in \mathrm{Seq}_{w_1}(\bar{B})$ then $\langle a_i : i \in w_2 \rangle \in \mathrm{Seq}_{w_2}(\bar{B})$ if for $i \in w_2 - w_1$ we define $a_i = a_{\mathrm{Max}(i \cap w_1)}$ exists as $i \geqq \mathrm{Min}\, w_2 = \mathrm{Min}\, w_1$ hence $i > \mathrm{Min}\, w_1$, and w_1 is closed.

(4) If $w_1 \subseteq w_2$ are both in $\mathrm{CSb}_u(\delta)$, $\mathrm{Min}\, w_1 = \mathrm{Min}\, w_2$ then $Z_{w_1}(\bar{B}) \subseteq Z_{w_2}(\bar{B})$.

(5) If $\alpha < \delta$, $w_1 \in \mathrm{CSb}_u(\delta)$ then $Z_{w_1}(\bar{B}) \subseteq Z_{w_1 - \alpha}(\bar{B})$.

(6) If $\langle a_i : i \in w_1 \rangle$, $\langle b_j : j \in w_2 \rangle$ are in $\mathrm{Seq}_{w_1}(\bar{B})$, $\mathrm{Seq}_{w_2}(\bar{B})$ resp, $(\forall i \in w_1)$ $(\exists j \in w_2)\, a_i \leqq b_j$ then $\bigcap_{i \in w_1} a_i \leqq \bigcap_{j \in w_2} b_j$.

(7) $Z(\bar{B})$ is a dense subset of the subalgebra it generates.

(8) If $\bar{B} = \langle B_i : i \leqq \alpha \rangle$ is l-o.k. $\gamma_i \leqq \alpha$ (for $i \leqq i(*)$) is increasing continuous and $[i \in W \Rightarrow \gamma_i \in W]$, *then* $\langle B_{\gamma_i} : i \leqq i(*) \rangle$ is l-o.k.

(9) $Z(\bar{B})$ is closed under non-empty finite intersections.

PROOF. Easy, e.g.,

(7) By (9) it is enough to show that if $a, b_0, \dots, b_{n-1} \in Z(\bar{B})$, $a - \bigcup_{l<n} b_l \neq 0$ then for some $c \in Z(\bar{B})$, $c \leqq (a - \bigcup_{l<n} b_l)$. Let $a = \bigcap_{i \in w} a_i$, $b_l = \bigcap_{i \in u_l} b_i^l$, where $\langle a_i : i \in w \rangle$, $\langle b_i^l : i \in u_l \rangle \in \mathrm{Seq}_{u_l}(\bar{B})$.

As $\bigcap_i a_i \nleqq \bigcup_{l<n} (\bigcap_{i \in u_l} b_i^l)$, there are $\gamma_0 \in u_0, \dots, \gamma_{n-1} \in u_{n-1}$ such that for no $i \in w$, $a_i \leqq \bigcup_{l<n} b_{\gamma_l}^l$. As $b_{\gamma_l}^l \in B_\beta$ for some $\beta \in w$, $\beta > \gamma_l$ for $l < n$. So $w - \beta \in \mathrm{CSb}_u(\delta)$. Let for $i \in w - \beta$, $c_i \overset{\mathrm{def}}{=} a_i - \bigcup_{l<n} b_{\gamma_l}^l$. So

(i) $c_i = a_i - \bigcup_l b_{\gamma_l}^l \in B_i$ [as $a_i \in B_i$, $b_{\gamma_l}^l \in B_{\gamma_l} \subseteq B_\beta \subseteq B_i$];

(ii) for $i < j$ from $w - \beta$, $c_j \leqq c_i$ [as $a_j \leqq a_i$, clearly $a_j - \bigcup_l b_{\gamma_l}^l \leqq a_i - \bigcup b_{\gamma_l}^l$];

(iii) for $i < j$ from $w - \beta$, $\Pr(c_i, c_j, B_i)$. [Let $0 < d \leqq c_i$, $d \in B_i$ then $0 < d \leqq a_i$, $d \in B_i$ hence (by $\Pr(a_i, a_j, B_i)$)$d \cap a_j \neq 0$ and

$$
\begin{aligned}
d \cap c_j &= d \cap \left(a_j - \bigcup_l b_{\gamma_l}^l \right) \\
&= d \cap a_j - d \cap \bigcup_l b_{\gamma_l}^l \\
&= d \cap a_j - 0 = d \cap a_j \left(d \cap \bigcup_l b_{\gamma_l}^l = 0 \text{ as } d \leqq c_i \right)
\end{aligned}
$$

so $d \cap c_j \neq 0$.]

The other conditions are easy too.

So $\langle c_j : j \in w - \beta \rangle \in \mathrm{Seq}_{w-\beta}(\bar{B})$ hence $c \overset{\mathrm{def}}{=} \bigcup_{i \in w - \beta} c_i \in Z_{w-\beta}(\bar{B}) \subseteq Z(\bar{B})$. As $\bar{B}$ is a 2-o.k., $c \neq 0$. Now $c \leqq a$, i.e., $\bigcap_{i \in w-\beta} c_i \leqq \bigcap_{i \in w} a_i$ as $c_i \leqq a_i$, and $c \cap b_l = 0$, i.e., $(\bigcap_{i \in w-\beta} c_i) \cap (\bigcap_{i \in u_l} b_i^l) = 0$ as $\bigcap_{i \in u_l} b_i^l \leqq b_{\gamma_l}^l$, $c_i \cap b_{\gamma_l}^l = 0$.

(9) Let, for $l = 1, 2$, $w_l \in \mathrm{CSb}_u(\delta)$ and $a_l \in Z_{w_l}(\bar{B})$. Choose successor $j < \delta$, $j > \mathrm{Min}\, w_l$ for $l = 1, 2$; let $w_3 = w_1 \cup \{j\}$, $w_4 = w_2 \cup \{j\}$, $w_5 = w_3 - j$, $w_6 = w_4 - j$. By D(4), $a_1 \in Z_{w_3}(\bar{B})$, by D(5), $a_1 \in Z_{w_5}(\bar{B})$. By D(4), $a_1 \in Z_{w_5 \cup w_6}(\bar{B})$, similarly $a_2 \in Z_{w_5 \cup w_6}(\bar{B})$ and we finish easily.

E. CLAIM. If $\bar{B} = \langle B_i : i \leqq \lambda^+ \rangle$ is 3-o.k., $[i < \lambda^+,\ \mathrm{cf}\, i = \lambda \Rightarrow i \in W]$ *then* $B_{\lambda^+}^+$ is the union of λ λ-complete filters.

PROOF. Note that

(∗) for every $x \in B_\alpha^+$ for some $\langle a_i : i \in w\rangle \in \mathrm{Seq}(\langle B_i : i \leqq \alpha + 1\rangle)$ $(0 <)$ $\bigcap_{i\in w} a_i \leqq x$, $0 \in w$, and w has a last element [prove this by induction on α, for $\alpha = 0$ — trivial, for $\alpha = \beta + 1$, $\beta \notin W$ use (D)(7), for $\alpha = \beta + 1$, $\beta \in W$, note that $B_\beta \gg B_\alpha$, hence for some $x_1 \in B_\beta$, $\mathrm{Pr}(x_1, x, B_\beta, B_\alpha)$. By the induction hypothesis there is $\langle a_i : i \in w\rangle \in \mathrm{Seq}(\langle B_i : i \leqq \beta + 1\rangle)$ $(0 <)$ $\bigcap_{i\in w} a_i \leqq x_1$, $0 \in w$ and has a last element (note $w \subseteq \beta + 1$). Now let $a_{\beta+1} = a_{\mathrm{Max}(w)} \cap x_1 \cap x$, $u = w \cup \{\beta + 1\}$ and $\langle a_i : i \in u\rangle$ is as required except when $\mathrm{cf}\,\beta = \lambda$, but the change is obvious.

Now $\mathrm{Seq}(\bar{B}) = \bigcup_{\alpha<\lambda^+} \mathrm{Seq}(\langle B_i : i \leqq \alpha\rangle)$.

It is well known that there is $H : \{w \subseteq \lambda^+ : |w| < \lambda\} \to \lambda$ such that: $H(w) = H(u)$, $\alpha \in w \cap u$ implies $\alpha \cap w = \alpha \cap u$; also $H(w) = H(u)$ implies w, u have the same order type.

Let F_i be a one-to-one function from B_{i+1} into λ. We say $\langle a_i^1 : i \in w_1\rangle$, $\langle a_i^2 : i \in w_2\rangle \in \mathrm{Seq}(\bar{B})$ are equivalent *if*:

(a) $H(w_1) = H(w_2)$ and

(b) *if* $\beta_1, \alpha_1 \in w_1$ and $\beta_2, \alpha_2 \in w_2$, $w_1 \cap \alpha_1$, $w_2 \cap \alpha_2$ has the same order type and $\alpha_1 = \gamma_1 + 1$, $\alpha_2 = \gamma_2 + 1$, *then*

$$F_{\gamma_1}(a_{\alpha_1}^1) = F_{\gamma_2}(a_{\alpha_2}^2).$$

It is enough to show that if $\langle a_i^\zeta : i \in w_\zeta\rangle \in \mathrm{Seq}(\bar{B})$ are equivalent for $\zeta < \zeta(*) < \lambda$, $0 \in w_\zeta$, $\mathrm{Max}\, w_\zeta \in w_\zeta$ then $\bigcap_\zeta a^\zeta_{\mathrm{Max}\, w_\zeta} \neq 0$. Note that if $\alpha \in w_{\zeta_1} \cap w_{\zeta_2}$, $\beta_l = \mathrm{Min}(w_{\zeta_l} - (\alpha + 1))$ then $a_{\beta_1}^{\zeta_1}\, a_{\beta_2}^{\zeta_2}$.

For this end we prove be induction on $\alpha \in W$, $\alpha > 0$

(∗) (1) $x_\alpha \overset{\mathrm{def}}{=} \bigcap_{\zeta<\zeta(*)} a^\zeta_{\mathrm{Max}(w_\zeta \cap (\alpha+1))}$ is not zero;

(2) if $\beta < \alpha$ $(\beta \in W)$ then $\mathrm{Pr}(x_\beta, x_\alpha, B_\beta)$.

Clearly x_α is decreasing (as a_α^ζ is decreasing in α for each ζ).

Case 1. $\alpha = 0$

Then $\mathrm{Max}(w_\zeta \cap (\alpha + 1)) = 0$ and $a_0^\zeta = a_0^+ \in B_0^+$ for every ζ. So (∗)(1) holds; and (∗)(2) is empty.

Case 2. $\alpha = \beta + 1, \beta \in W$

If $\alpha = \beta + 1 \notin w_\zeta$ then $a^\zeta_{\mathrm{Max}(w_\zeta \cap (\alpha+1))} = a^\zeta_{\mathrm{Max}(w_\zeta \cap (\beta+1))}$.

So if $\alpha \notin w_\zeta$ for every $\zeta < \zeta(*)$ then $x_\alpha = x_\beta$, so (∗)(1) holds. As for (∗)(2): for $\gamma < \beta$ use the induction hypothesis; for $\gamma = \beta$ this is easy.

If for some ζ, $\alpha \in w_\zeta$, let $v = \{\zeta < \zeta(*) : \alpha \in w_\zeta\}$. So $x_\alpha = \bigcap_{\zeta\notin v} a^\zeta_{\mathrm{Max}(w_\zeta \cap (\beta+1))} \cap \bigcap_{\zeta\in v} a_\alpha^\zeta$. By the definition of the equivalence relation for some a, $\zeta \in v \Rightarrow a_\alpha^\zeta = a$ or $\zeta \notin v \Rightarrow a_\alpha^\zeta \in B_\beta$. Clearly

$$x_\alpha = \bigcap_{\zeta \in v} a^\zeta_{\mathrm{Max}(w_\zeta \cap (\beta+1))} \cap \bigcap_{\zeta \in v} a^\zeta_\alpha = \bigcap_{\zeta} a^\zeta_{\mathrm{Max}(w_\zeta \cap (\beta+1))} \cap \bigcap_{\zeta \in v} a^\zeta_\alpha = x_\beta \cap a.$$

Now as $\beta \in W$, B_β is λ-complete, hence $x_\beta \in B_\beta$. Now $a \in B_\alpha$ and letting $\zeta(0) = \mathrm{Min}\, v$, $\gamma(0) = \mathrm{Max}(w_{\zeta(0)} \cap (\beta+1))$. Clearly $a \leqq a^{\zeta(0)}_{\gamma(0)}$ and $\Pr(a^{\zeta(0)}_{\gamma(0)}, a, B_\beta)$. As $x_\beta \in B_\beta$, and easily $a^{\zeta(0)}_{\gamma(0)} \geqq x_\beta$, clearly $x_\beta \cap a \neq 0$. So (*)(1) holds. As for (*)(2), by (B)(b) (and the induction hypothesis), without loss of generality, $\gamma = \beta$. So let $d \in B_\beta$, $0 < d \leqq x_\beta$, then $d \leqq a^{\zeta(0)}_{\gamma(0)}$ hence by $\Pr(a^{\zeta(0)}_{\gamma(0)}, a, B_\beta)$, $a \cap d \neq 0$, but $a \cap d = d \cap x_\beta \cap a = d \cap x_\alpha$, so we finish.

Case 3. $\alpha = \beta + 1, \beta \notin W$

Let $u \subseteq \beta$, $|u| < \lambda$, $\sup u = \beta$. Note that

$$a^\zeta_{\mathrm{Max}(w_\zeta \cap (\alpha+1))} = \bigcap_{\gamma<\beta} a^\zeta_{\mathrm{Max}(w_\zeta \cap (\gamma+1))}.$$

[If $\alpha \notin w_\zeta$, as $\langle a^\zeta_{\mathrm{Max}(w_\zeta \cap (\gamma+1))} : \gamma < \beta \rangle$ is eventually constant and equal to

$$a^\zeta_{\cap(\alpha+1))} = \bigcap_{\gamma<\beta} a^\zeta_{\mathrm{Max}(w_\zeta \cap (\gamma+1))}.$$

[If $\alpha \notin w_\zeta$, as $\langle a^\zeta_{\mathrm{Max}(w_\zeta \cap (\gamma+1))} : \gamma < \beta \rangle$ is eventually constant and equal to $a^\zeta_{\mathrm{Max}(w_\zeta \cap (\alpha+1))}$, and if $\alpha \in w_\zeta$, as $\langle a^\zeta_\gamma : \gamma \in w_\zeta \rangle \in \mathrm{Seq}(\bar{B})$). So

$$x_\alpha = \bigcap_{\zeta<\zeta(*)} a^\zeta_{\mathrm{Max}(w_\zeta \cap (\alpha+1))} = \bigcap_{\zeta<\zeta(*)} \bigcap_{\gamma<\beta} a^\zeta_{\mathrm{Max}(w_\zeta \cap (\gamma+1))}$$

$$= \bigcap_{\gamma<\beta} \left(\bigcap_{\zeta<\zeta(*)} a^\zeta_{\mathrm{Max}(w_\zeta \cap (\gamma+1))} \right) = \bigcap_{\gamma<\beta} x_\gamma.]$$

As $\bar{B}$ is 2-o.k. (as (*)(2) holds below β) we know $x_\alpha \neq 0$. Similarly we can check (*)(2).

Case 4. α limit

As $\alpha \in W$, necessarily $\mathrm{cf}\,\alpha = \lambda$. But then, by the definition of $\mathrm{Seq}_w(\bar{B})$, if $\alpha \in w_\zeta$ though $\mathrm{Max}(w_\zeta \cap (\alpha+1)) \neq \mathrm{Max}(w_\zeta \cap (\gamma+1))$ for $\gamma < \alpha$, still $a^\zeta_{\mathrm{Max}(w_\zeta \cap (\alpha+1))} = \bigcap_{\gamma<\beta} a^\zeta_{\mathrm{Max}(w_\zeta \cap (\gamma+1))}$ for every large enough $\gamma < \alpha$. If $\alpha \notin w_\zeta$ this holds on simpler ground. So $x_\alpha = x_\gamma$ for every large enough $\gamma < \alpha$, and we can finish easily.

Remark. The proof is written such that it will be easy to change it for $\bar{B} = \langle B_i : i < \gamma \rangle$, $\gamma < (2^\lambda)^+$, so $|B_i| = |i| + \lambda$, B_{i+1} is generated by $B_i \cup B'_i$, $|B'_i| = \lambda$; but it is not clear whether there is interest in this.

CONTINUATION OF THE PROOF OF THEOREM 2.20. We define by induction on $\alpha \leqq \kappa$

$$\bar{Q}^\alpha = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : i \leqq \alpha, j < \alpha \rangle$$

and γ_α, $\underset{\sim}{B}_i$, $\alpha[i]$ for $i < \gamma_\alpha$ and $W \cap \gamma_\alpha$ such that:

(A) $\bar{Q}^\alpha$ is a suitable iteration.

(B) Each Q_i is $(\omega_1 - S)$-complete.

(C) $|P_j| < \kappa$ for $j < \kappa$.

(D) γ_α is the ordering type of the closure of $\{j \leqq \alpha : \mathbf{t}_j = 1\}$. If $i < j$, $\underset{\sim}{t}_i = 1$, j non-limit, then $\Vdash_{P_j}$ "$\underset{\sim}{\mathfrak{B}}_{\gamma_i - 1} \lessdot \underset{\sim}{\mathfrak{B}}[V^{P_j}] \restriction S$".

(E) If i is inaccessible, $|P_j| < i$ for $j < i$ and $\Vdash_{P_i}$ "Rss($\aleph_2$)" then $\mathbf{t}_i = 1$, $Q_i = \mathrm{SSeal}(\langle \mathfrak{B}[V^{P_j}] : j \leqq i, \mathbf{t}_j = 1 \rangle, S)$ and $\underset{\sim}{\mathfrak{B}}_{\gamma_i - 1} = \underset{\sim}{\mathfrak{B}}[V^{P_i}] \gamma_i = (\bigcup_{\beta < i} \gamma_\beta) + 1$, $\gamma_i - 1 \in W$.

(F) If i is not limit, or a limit but not a limit of ordinals satisfying the assumption of (E), *then* $\gamma = \bigcup_{\beta < i} \gamma_\beta$ and $Q_i = \mathrm{SSeal}(\langle \mathfrak{B}_j : j \leqq i, \mathbf{t}_j = 1 \rangle, S)$ and $\mathbf{t}_i = 0$, except when (G) for $i - 1$ decrees $\mathbf{t}_i = 1$.

(G) If i is a limit of j's satisfying the assumption of (E), then $\gamma_i = (\bigcup_{j<i} \gamma_j) + 2$, $\alpha[\gamma_i - 2] = i$, $\gamma_i \notin w$, $\gamma_i + 1 \in w$, $\underset{\sim}{\mathfrak{B}}_{\gamma_i - 2} = \bigcup_{j<i} \mathfrak{B}[V^{P_j}]$, $\mathfrak{B}_{\gamma_i - 1} =$ the subalgebra of $\mathfrak{B}[V^{P_i}] \restriction S$ which $Z(\langle \mathfrak{B}_j : j < i, \mathbf{t}_j = 1 \rangle \hat{} \langle \mathfrak{B}[V^{P_i}] \rangle)$ generates (see Definition (B) above) and

$$Q_i = \mathrm{SSeal}(\langle \mathfrak{B}_j : j \in \gamma_i \cap W \rangle, S).$$

(H) In V^{P_α}, $\langle \underset{\sim}{\mathfrak{B}}_i : j \in \gamma_i \rangle \hat{} \langle \mathfrak{B}[V^{P_\alpha}] \rangle$ is 2-o.k. and every proper initial segment is 3-o.k.

The proof is like 2.19.

Now in order to be able to prove the Ulamness, we need to force (over V^{P_κ}) with

$R = \{f : f$ is an increasing continuous function from some $\gamma + 1 < \kappa$ to $W \cup \{\delta < \kappa : \mathrm{cf}\, \delta = \aleph_0$ (in V^{P_κ})$\}$.

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