

Article

# No Uncountable Polish Group Can be a Right-Angled Artin Group

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**Abstract:** We prove that if  $G$  is a Polish group and  $A$  a group admitting a system of generators whose associated length function satisfies: (i) if  $0 < k < \omega$ , then  $lg(x) \leq lg(x^k)$ ; (ii) if  $lg(y) < k < \omega$  and  $x^k = y$ , then  $x = e$ , then there exists a subgroup  $G^*$  of  $G$  of size  $\mathfrak{b}$  (the bounding number) such that  $G^*$  is not embeddable in  $A$ . In particular, we prove that the automorphism group of a countable structure cannot be an uncountable right-angled Artin group. This generalizes analogous results for free and free abelian uncountable groups.

**Keywords:** descriptive set theory; polish group topologies; right-angled Artin groups

In a meeting in Durham in 1997, Evans asked if an uncountable free group can be realized as the group of automorphisms of a countable structure. This was settled in the negative by Shelah [1]. Independently, in the context of descriptive set theory, Becher and Kechris [2] asked if an uncountable Polish group can be free. This was also answered negatively by Shelah [3], generalizing the techniques of [1]. Inspired by the question of Becher and Kechris, Solecki [4] proved that no uncountable Polish group can be free abelian. In this paper, we give a general framework for these results, proving that no uncountable Polish group can be a right-angled Artin group (see Definition 1). We actually prove more:

**Theorem 1.** *Let  $G = (G, d)$  be an uncountable Polish group and  $A$  a group admitting a system of generators whose associated length function satisfies the following conditions:*

- (i) *if  $0 < k < \omega$ , then  $lg(x) \leq lg(x^k)$ ;*
- (ii) *if  $lg(y) < k < \omega$  and  $x^k = y$ , then  $x = e$ .*

*Then  $G$  is not isomorphic to  $A$ ; in fact, there exists a subgroup  $G^*$  of  $G$  of size  $\mathfrak{b}$  (the bounding number) such that  $G^*$  is not embeddable in  $A$ .*

After the authors proved Theorem 1, they discovered that the impossibility to endow groups  $A$  as in Theorem 1 with a Polish group topology follows from an old important result of Dudley [5]. In fact, Dudley's work implies more strongly that we cannot even find a homomorphism from a Polish group  $G$  into  $A$ . Apart from the fact that the claim about  $G^*$  in Theorem 1 is of independent interest and not subsumed by Dudley's work, our focus here is on techniques; i.e., the crucial use of the Compactness Lemma of [3]. This powerful result has a broad scope of applications, and is used by the authors in a work in preparation [6] to deal with classes of groups not covered by Theorem 1 or Dudley's work, most notably the class of right-angled Coxeter groups (see Definition 1).

**Proof of Theorem 1.** Let  $\zeta = (\zeta_n)_{n < \omega} \in \mathbb{R}^\omega$  be such that  $\zeta_n < 2^{-n}$ , for every  $n < \omega$ , and  $\bar{g} = (g_n)_{n < \omega} \in G^\omega$  such that  $g_n \neq e$  and  $d(g_n, e) < \zeta_n$ , for every  $n < \omega$ . Let  $\Lambda$  be a set of power

$\mathfrak{b}$  of increasing functions  $\eta \in \omega^\omega$  which is unbounded with respect to the partial order of eventual domination. For transparency, we also assume that for every  $\eta \in \Lambda$  we have  $\eta(0) > 0$ . For  $\eta \in \Lambda$ , define the following set of equations:

$$\Gamma_\eta = \{x_{n+1}^{\eta(n)} = x_n g_n : n < \omega\}.$$

By (3.1, [3]), for every  $\eta \in \Lambda$ ,  $\Gamma_\eta$  is solvable in  $G$ . Let  $\bar{b}_\eta = (b_{\eta,n})_{n < \omega}$  witness it; i.e.,

$$\bar{b}_\eta \in G^\omega \quad \text{and} \quad \bigwedge_{n < \omega} b_{\eta,n+1}^{\eta(n)} = b_{\eta,n} g_n.$$

Let  $G^*$  be the subgroup of  $G$  generated by  $\{g_n : n < \omega\} \cup \{b_{\eta,n} : \eta \in \Lambda, n < \omega\}$ . Towards contradiction, suppose that  $\pi$  is an embedding of  $G^*$  into  $A$ , and let  $S$  be a system of generators for  $A$  whose associated length function  $lg_S = lg$  satisfies conditions (i) and (ii) of the statement of the theorem. For  $\eta \in \Lambda$  and  $n < \omega$ , let:

$$\pi(g_n) = g'_n, \quad \pi(b_{\eta,n}) = c_{\eta,n} \quad \text{and} \quad m_*(\eta) = lg(c_{\eta,0}).$$

Now,  $m_*$  is a function from  $\Lambda$  to  $\omega$  and so there exists unbounded  $\Lambda_1 \subseteq \Lambda$  such that for every  $\eta \in \Lambda_1$  the value  $m_*(\eta)$  is a constant  $m_*$ . Fix such a  $\Lambda_1$  and  $m_*$ , and let  $f_1, f_2 \in \omega^\omega$  increasing satisfying the following:

- (1)  $f_1(n) > lg(g'_n)$ ;
- (2)  $f_2(n) = (m_* + 1) + \sum_{\ell < n} f_1(\ell)$ .

**Claim 1.** For every  $\eta \in \Lambda_1$ ,  $lg(c_{\eta,n}) < f_2(n)$ .

**Proof.** By induction on  $n < \omega$ . The case  $n = 0$  is clear by the choice of  $f_1$  and  $f_2$ . Let  $n = m + 1$ . Because of assumption (i) on  $A$ , the choice of  $\Lambda_1$ , and the choice of  $f_1$  and  $f_2$ , we have:

$$\begin{aligned} lg(c_{\eta,n}) &\leq lg(c_{\eta,n}^{\eta(m)}) \\ &= lg(c_{\eta,m} g'_m) \\ &\leq lg(c_{\eta,m}) + lg(g'_m) \\ &< f_2(m) + f_1(m) \\ &= f_2(n). \end{aligned}$$

□

Now, by the choice of  $\Lambda_1$ , we can find  $\eta \in \Lambda_1$  and  $n < \omega$  such that  $\eta(n) > f_2(n + 2)$ . Notice then that by the claim above and the choice of  $f_1$  and  $f_2$ , we have:

$$\eta(n) > f_2(n + 1) = f_2(n) + f_1(n) > lg(c_{\eta,n}) + lg(g'_n) \geq lg(c_{\eta,n} g'_n), \quad (1)$$

$$\eta(n) > f_2(n + 2) \geq f_1(n + 1) > lg(g'_{n+1}). \quad (2)$$

Thus, by (1) and the fact that  $c_{\eta,n+1}^{\eta(n)} = c_{\eta,n} g'_n$ , using assumption (ii), we infer that  $c_{\eta,n+1} = e$ . Hence,

$$c_{\eta,n+2}^{\eta(n+1)} = c_{\eta,n+1} g'_{n+1} = g'_{n+1}.$$

Furthermore, if  $\eta(n + 1) > lg(g'_{n+1})$ , then again by assumption (ii), we have that  $c_{\eta,n+2} = e$ , and so  $c_{\eta,n+2}^{\eta(n+1)} = g'_{n+1} = e$ , which contradicts the choice of  $(g_n)_{n < \omega}$ . Hence,  $\eta(n) < \eta(n + 1) \leq lg(g'_{n+1})$ , contradicting (2). It follows that the embedding  $\pi$  from  $G^*$  into  $A$  cannot exist. □

**Definition 1.** Given a graph  $\Gamma = (E, V)$ , the right-angled Artin group  $A(\Gamma)$  is the group with presentation:

$$\Omega(\Gamma) = \langle V \mid ab = ba : aEb \rangle.$$

If in the presentation  $\Omega(\Gamma)$ , we ask in addition that all the generators are involutions, then we speak of right-angled Coxeter groups  $C(\Gamma)$ .

Thus, for  $\Gamma$ , a graph with no edges (resp. a complete graph)  $A(\Gamma)$  is a free group (resp. a free abelian group).

**Definition 2.** Let  $A(\Gamma)$  be a right-angled Artin group and  $lg$  its associated length function. We say that an element  $g \in A(\Gamma)$  is cyclically reduced if it cannot be written as  $g = hfh^{-1}$  with  $lg(g) = lg(f) + 2$ .

**Fact 1.** Let  $A(\Gamma)$  be a right-angled Artin group,  $lg$  its associated length function, and  $g \in A(\Gamma)$ . Then:

- (1)  $g$  can be written as  $hfh^{-1}$  with  $f$  cyclically reduced and  $lg(g) = lg(f) + 2lg(h)$ ;
- (2) if  $0 < k < \omega$  and  $f$  is cyclically reduced, then  $lg(f^k) = klg(f)$ ;
- (3) if  $0 < k < \omega$  and  $g = hfh^{-1}$  is as in (1), then  $lg(hfh^{-1})^k = klg(f) + 2lg(h)$ .

**Proof.** Item (1) is proved in (Proposition on p. 38, [7]). The rest is folklore.  $\square$

**Corollary 1.** No uncountable Polish group can be a right-angled Artin group.

**Proof.** By Theorem 1 it suffices to show that for every right-angled Artin group  $A(\Gamma)$  the associated length function  $lg$  satisfies conditions (i) and (ii) of the theorem, but by Fact 1, this is clear.  $\square$

As is well known, the automorphism group of a countable structure is naturally endowed with a Polish topology which respects the group structure, hence:

**Corollary 2.** The automorphism group of a countable structure cannot be an uncountable right-angled Artin group.

As already mentioned, the situation is different for right-angled Coxeter groups; in fact, the structure  $M$  with  $\omega$  many disjoint unary predicates of size 2 is such that  $Aut(M) = (\mathbb{Z}_2)^\omega$ ; i.e.,  $Aut(M)$  is the right-angled Coxeter group on  $K_c$  (a complete graph on continuum many vertices). Notice that in this group for any  $a \neq b \in K_c$ , we have:

- (i)  $(ab)^2 = 1$ ;
- (ii)  $lg(ab) = 2 < 3$ ,  $(ab)^3 = ab$  and  $ab \neq e$ .

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