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BAIRE PROPERTY AND AXIOM OF CHOICE*

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ABSTRACT

We show that without using inaccessible cardinals it is possible to get models of "ZF + all sets of reals have the Baire property + DC(ω_1)" and "ZFC + all projective sets have the Baire property + the union of less than ω_2 many meager sets is meager", answering two well-known open questions of Woodin and Judah, respectively.

1. Introduction

In 1979 Shelah proved that in order to obtain a model in which every set of reals has the Baire property, a large cardinal assumption is not necessary, thus finding a deep asymmetry in the study of measure and category on the real line. Shelah

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started from **L** and by a method called **amalgamation** he built a forcing notion $\mathbb{P}$ satisfying

(i) $\mathbf{HOD}(\mathbf{L}^{\mathbb{P}}) \models$"every set of reals has the Baire property",

(ii) $\mathbf{L}^{\mathbb{P}}$ and **L** have the same cofinalities; moreover $\mathbb{P} \models$ccc,

(iii) $\mathbf{L}^{\mathbb{P}} \models \mathbf{CH}$.

From (ii) it is possible to conclude that in $\mathbf{HOD}(\mathbf{L}^{\mathbb{P}})$ there are uncountable well-ordered sets of reals (namely, the constructible reals!). From this evidence it was natural to ask the following question:

> **Woodin:** Can we get a model where every projective set of reals has the Baire property and $\mathrm{DC}(\omega_1)$ holds?

Recall here that $\mathrm{DC}(\omega_1)$ is the following sentence:

> if $\mathcal{R}$ is a relation such that $(\forall X)(\exists Y)(\mathcal{R}(X,Y))$ then there is a sequence $\langle Z_\alpha : \alpha < \omega_1\rangle$ such that
>
> $$(\forall \alpha < \omega_1)(\mathcal{R}(\langle Z_\beta : \beta < \alpha\rangle,\ Z_\alpha)).$$

Note that $\mathrm{DC}(\omega_1)$ implies the following version of choice:

> if $\mathcal{R} \subseteq \omega_1 \times \mathbb{R}$, $\mathrm{dom}(\mathcal{R}) = \omega_1$
> then there exists a choice function $f: \omega_1 \longrightarrow \mathbb{R}$ such that $\mathcal{R}(\alpha, f(\alpha))$
> for each $\alpha < \omega_1$.

It is interesting to remark that in the same work Shelah proved that the existence of an uncountable well-ordered set of reals provides non-measurable sets.

In [JS1] we studied the consistency strength of "ZFC + variants of MA + suitable sets of reals have the Baire property". We showed that Baire property for Σ^1_3-sets of reals plus MA(σ-centered) implies that ω_1 is a Mahlo cardinal in L. Since MA(σ-centered) implies that the union of less than continuum meager sets is meager, the following question arises naturally at this point.

> **Judah:** Do we need large cardinals to construct a model in which all projective sets of reals have Baire property and the union of any ω_1 meager sets is meager?

Note that if unions of ω_1 many null sets are null, then every Σ^1_2-set of reals is Lebesgue measurable. Consequently if each projective sets of reals has the

Baire property and any union of ω_1 null sets is null, then ω_1 is inaccessible in $\mathbf{L}$ (cf [Rai]). It was also asked if it is possible to build a model, starting from $\mathbf{L}$, satisfying (i) and (ii) above in which the continuum is large. This question was answered in [JuR] where we proved that we can keep (i) and (ii) adding Cohen reals to $\mathbf{L}^{\mathbb{P}}$. We suggest that the reader looks at this work for a better understanding of the method of amalgamation.

The aim of the present paper is to prove the following two Theorems:

THEOREM 1.1: *If ZF is consistent then the following theory is consistent:*

ZF + $DC(\omega_1)$ + "Every set of reals has Baire property."

THEOREM 1.2: *If ZF is consistent then the following theory is consistent:*

ZFC + "Every projective set of reals has Baire property" + "Any union of ω_1 meager sets is meager."

The method presented in this paper allows one to prove stronger results. One can show that ω_1 in the above Theorems may be replaced by any resonable cardinal κ, by using $\diamondsuit$ and club filters in $\mathcal{P}_\omega(\kappa)$ when $\kappa > \omega_2$.

Our notation is standard and essentially derived from [Jec]. Since we work with Boolean algebras we keep the convention that $p \leq q$ means that the condition p is stronger than q. For a partial order $\mathbb{P}$, $\mathrm{BA}(\mathbb{P})$ stands for the complete Boolean algebra determined by the order $\mathbb{P}$. $\mathbf{1}$ denotes the largest element of a forcing notion or just the unit in the Boolean algebra.

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2. Basic definitions and facts

In this section we recall some definitions and results from [She]. They will be applied in the next section.

The basic tool in the construction of models in which definable sets have Baire property is the amalgamation. To define this operation we need the following definition.

Recall that $\mathbb{P} \lessdot \mathbb{P}'$ means $\mathbb{P} \subseteq \mathbb{P}'$ and each maximal antichain in $\mathbb{P}$ is a maximal antichain in $\mathbb{P}'$. Note that for complete Boolean algebras $\mathcal{B}_1 \subseteq \mathcal{B}_2$, $\mathcal{B}_1 \lessdot \mathcal{B}_2$ means $\sup_{\mathcal{B}_1} A = \sup_{\mathcal{B}_2} A$ for any $A \subseteq \mathcal{B}_1$.

For a forcing notion $\mathbb{P}$ let $\Gamma_{\mathbb{P}}$ be a $\mathbb{P}$-name for the generic subset of $\mathbb{P}$.

Definition 2.1: Suppose that $\mathbb{P} \lessdot \mathrm{BA}(\mathbb{Q})$. Then $(\mathbb{Q} : \mathbb{P})$ is the **$\mathbb{P}$-name** of a forcing notion which is a subset of $\mathbb{Q}$,

$$(\mathbb{Q} : \mathbb{P}) = \{q \in \mathbb{Q} : q \text{ is compatible with every } p \in \Gamma_{\mathbb{P}}\}.$$

Thus $p \Vdash q \in (\mathbb{Q} : \mathbb{P})$ if and only if every $p' \in \mathbb{P}$, $p' \leq p$ is compatible with q. Recall that if $\mathbb{P} \lessdot \mathrm{BA}(\mathbb{Q})$, then forcing notions $\mathbb{Q}$ and $\mathbb{P} * (\mathbb{Q} : \mathbb{P})$ are equivalent.

Definition 2.2: Let $\mathbb{P}^0, \mathbb{P}^1$ and $\mathbb{P}^2$ be forcing notions. Suppose that $f_1 \colon \mathbb{P}^0 \xrightarrow{1-1} \mathrm{BA}(\mathbb{P}^1)$, $f_2 \colon \mathbb{P}^0 \xrightarrow{1-1} \mathrm{BA}(\mathbb{P}^2)$ are complete embeddings (i.e. they preserve order and $f_i[\mathbb{P}^0] \lessdot \mathrm{BA}(\mathbb{P}^i)$). We define the **amalgamation** of $\mathbb{P}^1$ and $\mathbb{P}^2$ over f_1, f_2 by

$$\mathbb{P}^1 \times_{f_1,f_2} \mathbb{P}^2 =$$
$$\{(p_1,p_2) \in \mathbb{P}^1 \times \mathbb{P}^2 \colon (\exists p \in \mathbb{P}^0)(p \Vdash \text{“}p_1 \in (\mathbb{P}^1 : f_1[\mathbb{P}^0]) \;\&\; p_2 \in (\mathbb{P}^2 : f_2[\mathbb{P}^0])\text{”})\}.$$

$\mathbb{P}^1 \times_{f_1,f_2} \mathbb{P}^2$ is ordered in the natural way: $(p_1,p_2) \geq (p_1',p_2')$ if and only if $p_1 \geq p_1'$, $p_2 \geq p_2'$.

The amalgamation $\mathbb{P}^1 \times_{f_1,f_2} \mathbb{P}^2$ is equivalent to the iteration $\mathbb{P}^0 * ((\mathbb{P}^1 : f_1[\mathbb{P}^0]) \times (\mathbb{P}^2 : f_2[\mathbb{P}^0]))$.

Note that $\mathbb{P}^1$, $\mathbb{P}^2$ can be completely embedded into the amalgamation $\mathbb{P}^1 \times_{f_1,f_2} \mathbb{P}^2$ by $p_1 \in \mathbb{P}^1 \mapsto (p_1, \mathbf{1})$ and $p_2 \in \mathbb{P}^2 \mapsto (\mathbf{1}, p_2)$. Thus we think of $\mathbb{P}^1 \times_{f_1,f_2} \mathbb{P}^2$ as a forcing notion extending both $\mathbb{P}^1$ and $\mathbb{P}^2$.

The amalgamation is applied in the construction of Boolean algebras admitting a lot of automorphisms. If $\mathbb{P}^1 = \mathbb{P}^2 = \mathbb{P}$ then the mapping

$$f_2 \circ f_1^{-1} \colon f_1[\mathbb{P}^0] \longrightarrow \mathbb{P}$$

can be naturally extended to an embedding

$$\phi \colon \mathbb{P} \longrightarrow \mathbb{P} \times_{f_1,f_2} \mathbb{P} \colon (p, \mathbf{1}) \mapsto (\mathbf{1}, p),$$

where we identify $p \in \mathbb{P}$ with $(p, \mathbf{1}) \in \mathbb{P} \times_{f_1,f_2} \mathbb{P}$.

Now suppose that $\mathcal{B}$ is a complete Boolean algebra such that for sufficiently many pairs $(\mathbb{P}^1, \mathbb{P}^2)$ of complete suborders of $\mathcal{B}$ and for complete embeddings $f_i \colon \mathbb{P}^0 \longrightarrow \mathbb{P}^i$ $(i = 1,2)$ the algebra $\mathcal{B}$ contains the amalgamation $\mathbb{P}^1 \times_{f_1,f_2} \mathbb{P}^2$ (as a suborder extending both $\mathbb{P}^i$, $i = 1,2$). Then many partial isomorphisms of $\mathcal{B}$ can be extended to automorphisms of $\mathcal{B}$.

Definition 2.3: A complete Boolean algebra $\mathcal{B}$ is **strongly Cohen-homogeneous** if for every $\mathcal{B}$-name τ for an ω_1-sequence of ordinals there exists a complete subalgebra $\mathcal{B}'$ of the algebra $\mathcal{B}$ such that

- τ is a $\mathcal{B}'$-name,
- if $\mathcal{B}' \lessdot \mathcal{B}'' \lessdot \mathcal{B}$, $\mathcal{B}' \Vdash$"$(\mathcal{B}'' : \mathcal{B}')$ is the Cohen algebra" and $f: \mathcal{B}'' \longrightarrow \mathcal{B}$ is a complete embedding such that $f|\mathcal{B}' = \mathrm{id}_{\mathcal{B}'}$, then there exists an automorphism $\phi: \mathcal{B} \xrightarrow{onto} \mathcal{B}$ extending f.

For more details on extending homomorphisms see [JuR].

Solovay showed the connection between the strong homogeneity of the algebra $\mathcal{B}$ and the fact that in generic extensions via $\mathcal{B}$ all projective sets of reals have the Baire property. Let $\mathbf{S}_1$ be the class of all ω_1-sequences of ordinal numbers.

THEOREM 2.4 (Solovay): *Let $\mathcal{B}$ be a strongly Cohen homogeneous complete Boolean algebra satisfying ccc. Suppose that for any $\mathcal{B}$-name τ for an ω_1-sequence of ordinals*

$\mathcal{B} \Vdash$"the union of all meager Borel sets coded in $\mathbf{V}[\tau]$ is meager".
Then $\mathcal{B} \Vdash$"any set of reals definable over $\mathbf{S}_1$ has Baire property".

Proof: See [Sol] or for a detailed argument [JuR]. ∎

The class $\mathbf{HOD}(\mathbf{S}_1)$ consists of all sets hereditarily ordinal definable over $\mathbf{S}_1$.

THEOREM 2.5 (Solovay, McAloon): *Assume that every set of reals ordinal definable over $\mathbf{S}_1$ has Baire property. Then $\mathbf{HOD}(\mathbf{S}_1) \models$" $\mathbf{ZF} + \mathbf{DC}(\omega_1) +$ every set of reals has Baire property".*

Proof: This is a standard modification of arguments given in [Jec], pp. 545–548 (Lemma 42.8, Corollary 2). ∎

In the next section we will built a model in which there exists an algebra $\mathcal{B}$ satisfying the assumptions of Theorem 2.4 and such that

$\mathcal{B} \Vdash$"the union of ω_1 meager sets is meager".

To be sure that the algebra $\mathcal{B}$ satisfies ccc we will use the following notion.

Definition 2.6: A triple $(\mathbb{P}, \mathcal{D}, \{E_n\}_{n\in\omega})$ is a model of **sweetness** if

1. $\mathbb{P}$ is a notion of forcing and $\mathcal{D}$ is a dense subset of $\mathbb{P}$,
2. E_n are equivalence relations on $\mathcal{D}$ such that
   - each E_n has countably many equivalence classes (the equivalence class of the element $p \in \mathcal{D}$ in the relation E_n will be denoted by $[p]_n$),
   - $E_{n+1} \subseteq E_n$,
   - equivalence classes of all relations E_n are downward directed,
   - if $\{p_i : i \leq \omega\} \subseteq \mathcal{D}$, $p_i \in [p_\omega]_i$ for all i, then for every $n < \omega$ there exists $q \in [p_\omega]_n$ which is stronger than all p_i for $i \geq n$,
   - if $p, q \in \mathcal{D}$, $p \geq q$ and $n \in \omega$ then there exists $k \in \omega$ such that

$$(\forall p' \in [p]_k)(\exists q' \in [q]_n)(p' \geq q').$$

Note that if $(\mathbb{P}, \mathcal{D}, \{E_n\}_{n\in\omega})$ is a model of sweetness then $\mathbb{P}$ is σ-centered.

Definition 2.7: We say that a model of sweetness $(\mathbb{P}^2, \mathcal{D}^2, \{E^2_n\}_{n\in\omega})$ **extends** a model $(\mathbb{P}^1, \mathcal{D}^1, \{E^1_n\}_{n\in\omega})$ (we write $(\mathbb{P}^1, \mathcal{D}^1, \{E^1_n\}_{n\in\omega}) \geq (\mathbb{P}^2, \mathcal{D}^2, \{E^2_n\}_{n\in\omega})$) whenever

1. $\mathbb{P}^1 \lessdot \mathbb{P}^2$, $\mathcal{D}^1 \subseteq \mathcal{D}^2$ and $E^1_n = E^2_n | \mathcal{D}^1$ for each $n \in \omega$,
2. if $p \in \mathcal{D}^1$, $n \in \omega$ then $[p]^2_n \subseteq \mathcal{D}^1$,
3. if $p \geq q$, $p \in \mathcal{D}^2$, $q \in \mathcal{D}^1$ then $p \in \mathcal{D}^1$.

LEMMA 2.8:

(a) *The relation $\geq$ is transitive on models of sweetness.*

(b) *Suppose that $(\mathbb{P}^i, \mathcal{D}^i, \{E^i_n\}_{n\in\omega})$ are models of sweetness such that*

$$(\mathbb{P}^i, \mathcal{D}^i, \{E^i_n\}_{n\in\omega}) \geq (\mathbb{P}^j, \mathcal{D}^j, \{E^j_n\}_{n\in\omega})$$

for all $i < j < \xi$ $(\xi < \omega_1)$. Then

$$\lim_{i<\xi}(\mathbb{P}^i, \mathcal{D}^i, \{E^i_n\}_{n\in\omega}) = (\bigcup_{i<\xi} \mathbb{P}^i, \bigcup_{i<\xi} \mathcal{D}^i, \{\bigcup_{i<\xi} E^i_n\}_{n\in\omega})$$

is a model of sweetness extending all models $(\mathbb{P}^i, \mathcal{D}^i, \{E^i_n\}_{n\in\omega})$.

Proof: See [She]. ∎

Sweetness may be preserved by amalgamation.

LEMMA 2.9: *Suppose that $(\mathbb{P}^i, \mathcal{D}^i, \{E^i_n\}_{n\in\omega})$ for $i = 1,2$ are models of sweetness and f_i: $\mathbb{P}^0 \longrightarrow BA(\mathbb{P}^i)$ are complete embeddings. Then there exists a model of sweetness $(\mathbb{P}^1 \times_{f_1,f_2} \mathbb{P}^2, \mathcal{D}^*, \{E^*_n\}_{n\in\omega})$ based on the amalgamation $\mathbb{P}^1 \times_{f_1,f_2} \mathbb{P}^2$ and extending both $(\mathbb{P}^1, \mathcal{D}^1, \{E^1_n\}_{n\in\omega})$ and $(\mathbb{P}^2, \mathcal{D}^2, \{E^2_n\}_{n\in\omega})$.*

Proof: See Lemmas 7.5, 7.12 of [She]. ∎

To ensure that our algebra satisfies

$\mathcal{B} \Vdash$"the union of ω_1 meager sets is meager"

we will use the Hechler order $\mathbb{D}$. Recall that $\mathbb{D}$ consists of all pairs (n, f) such that $n \in \omega$, $f \in \omega^\omega$. It is ordered by

$(n, f) \geq (n', f')$ if and only if
$n \leq n'$, $f|n = f'|n$ and $(\forall k \in \omega)(f(k) \leq f'(k))$.

The forcing with $\mathbb{D}$ adds both a dominating real and a Cohen real. Consequently

$\mathbb{D}*\dot{\mathbb{D}} \Vdash$"the union of all Borel meager sets coded in the ground model is meager".

The composition with $\mathbb{D}$ preserves sweetness.

LEMMA 2.10: *Let $(\mathbb{P}, \mathcal{D}, \{E_n\}_{n\in\omega})$ be a model of sweetness and let $\dot{\mathbb{D}}$ be a $\mathbb{P}$-name for Hechler forcing. Then there exists a model of sweetness $(\mathbb{P} * \dot{\mathbb{D}}, \mathcal{D}^*, \{E^*_n\}_{n\in\omega})$ based on $\mathbb{P} * \dot{\mathbb{D}}$ and extending the model $(\mathbb{P}, \mathcal{D}, \{E_n\}_{n\in\omega})$.*

Proof: Similar to the proof of Lemmas 7.6, 7.11 of [She]; see [JuR] for the details. ∎

In [She] the Amoeba Forcing for Category $\mathcal{U}$ was applied to add a comeager set of Cohen reals. The same notion of forcing can be used in our construction instead of $\mathcal{D}$.

3. The proof of the main result

In this section we present proofs of Theorems 1.2 and 1.1.

Definition 3.1: Let $\mathcal{K}$ be the class consisting of all sequences $\bar{\mathbb{P}} = \langle (P^i, M^i) : i < \omega_1 \rangle$ such that

1. M^i is a model of sweetness based on P^i,
2. if $i < j < \omega_1$ then $P^i \lessdot P^j$.
3. $P^{\omega_1} = \bigcup_{i<\omega_1} P^i$ satisfies ccc.

Let us remark that if $\bar{\mathbb{P}} \in \mathcal{K}$ then P^i extend each other only as forcing notions, not as models of sweetness. Moreover no continuity is assumed. Note that if $\bar{\mathbb{P}} \in \mathcal{K}$ then each P^i is σ-centered.

We define the relation $\leq$ on $\mathcal{K}$.

Definition 3.2: Let $\bar{\mathbb{P}}_1, \bar{\mathbb{P}}_2 \in \mathcal{K}$. We say $\bar{\mathbb{P}}_1 \geq \bar{\mathbb{P}}_2$ if $P_1^{\omega_1} \lessdot P_2^{\omega_1}$ and there exists a closed unbounded subset C of ω_1 such that

(!) if $i \in C$ then $M_1^i \geq M_2^i$,

(!!) if $i \in C$, $q \in P_1^{\omega_1}$, $p \in P_1^i$ and $p \Vdash_{P_1^i} q \in (P_1^{\omega_1} : P_1^i)$ then $p \Vdash_{P_2^i} q \in (P_2^{\omega_1} : P_2^i)$.

Remark: We do not keep a strict formalism, but we would like to alert the reader to a notational problem here. In Definition 3.1 the relation $P^i \lessdot P^j$ assumes a literal inclusion of the underlying sets. The relation $P_1^{\omega_1} \lessdot P_2^{\omega_1}$ in Definition 3.2 says that there is a (canonical) complete embedding of $P_1^{\omega_1}$ into $P_2^{\omega_1}$ satisfying the suitable conditions.

Condition (!!) can be written in the language of projections. It says that if $\pi_i^k \colon \mathrm{BA}(P_k^{\omega_1}) \xrightarrow{\text{onto}} \mathrm{BA}(P_k^i)$ are the projections (for $k = 1, 2$), $i \in C$, $q \in P_1^{\omega_1} \subseteq P_2^{\omega_1}$ then $\pi_i^1(q) = \pi_i^2(q)$.

Clearly the relation $\leq$ is transitive and reflexive.

LEMMA 3.3: *Suppose that $\bar{\mathbb{P}}_m \in \mathcal{K}$ for $m < \omega$ are such that $m_1 < m_2 < \omega$ implies $\bar{\mathbb{P}}_{m_1} \geq \bar{\mathbb{P}}_{m_2}$ (and let C_{m_1,m_2} witness it). Let $C = \bigcap_{m_1<m_2<\omega} C_{m_1,m_2}$. Put*

$$P_\omega^i = \bigcup_{m<\omega} P_m^{\cap(C\setminus i)}, \quad M_\omega^i = \lim_{m<\omega} M_m^{\cap(C\setminus i)}.$$

Then $\bar{\mathbb{P}}_\omega = <(P_\omega^i, M_\omega^i) : i < \omega_1> \in \mathcal{K}$ and $\bar{\mathbb{P}}_m \geq \bar{\mathbb{P}}_\omega$ for each $m < \omega$.

Proof: First note that C is a closed unbounded subset of ω_1. Since $C \subseteq \bigcap_{m<\omega} C_{m,m+1}$ we may apply Lemma 2.8 (b) to conclude that each M_ω^i is a model of sweetness based on P_ω^i.

CLAIM: If $i < j < \omega_1$ then $P^i_\omega \lessdot P^j_\omega$.

Indeed, let $i < j$. We may assume that $i, j \in C$ (recall that $P^i_\omega = P^{\cap(C\setminus i)}_\omega$). Note that $P^i_m \lessdot P^i_\omega$ and $P^i_m \lessdot P^j_m$ for each $m \in \omega$. Let $\mathcal{A} \subseteq P^i_\omega$ be a maximal antichain. Clearly it is an antichain in P^j_ω but we have to prove that it is maximal. Let $q \in P^j_\omega$. Then $q \in P^j_m$ for some $m < \omega$. Let

$$Z = \{r \in P^i_m \colon (\exists p_r \in \mathcal{A})(r \Vdash_{P^i_m} p_r \in (P^i_\omega : P^i_m))\}.$$

Clearly Z is dense in P^i_m. Hence we find $r \in Z$ such that $r \Vdash_{P^i_m} q \in (P^j_m : P^i_m)$. Let $p_r \in \mathcal{A}$ witness $r \in Z$. Take k such that $p_r \in P^i_k$, $m < k < \omega$. Consider $\bar{\mathbb{P}}_m$ and $\bar{\mathbb{P}}_k$. Since $i, j \in C \subseteq C_{m,k}$ we may apply condition (!!) to conclude that

$$r \Vdash_{P^i_k} q \in (P^j_k : P^i_k).$$

By the choice of p_r we have

$$r \Vdash_{P^i_m} p_r \in (P^i_k : P^i_m).$$

Thus p_r and r are compatible and any $p' \in P^i_k$, $p' \geq r, p_r$ is compatible with q. Consequently q and p_r are compatible. The claim is proved.

It follows from the above claim that $\bar{\mathbb{P}}_\omega \in \mathcal{K}$.

CLAIM: The club C witnesses that $\bar{\mathbb{P}}_m \geq \bar{\mathbb{P}}_\omega$ for each $m < \omega$.

Indeed, first note that

$$P^{\omega_1}_\omega = \bigcup_{i<\omega_1} P^i_\omega = \bigcup_{i<\omega_1} \bigcup_{m<\omega} P^{\cap(C\setminus i)}_m = \bigcup_{m<\omega} P^{\omega_1}_m.$$

Since $P^{\omega_1}_{m_1} \lessdot P^{\omega_1}_{m_2}$ for each $m_1 < m_2$ we see that $P^{\omega_1}_m \lessdot P^{\omega_1}_\omega$. It follows from the definition of M^i_ω and Lemma 2.8 that if $i \in C$ then $M^i_m < M^i_\omega$. Thus we have to check condition (!!) only. Suppose $i \in C$, $q \in P^{\omega_1}_m$, $p \in P^i_m$ and $p \Vdash_{P^i_m} q \in (P^{\omega_1}_m : P^i_m)$. Assume $p \nVdash_{P^i_\omega} q \in (P^{\omega_1}_\omega : P^i_\omega)$. Then we find $r \in P^i_\omega$ such that $r \geq p$ and r is incompatible with q. Let $k > m$ be such that $r \in P^i_k$. Since $i \in C_{m.k}$ we have $p \Vdash_{P^i_k} q \in (P^{\omega_1}_k : P^i_k)$ (by condition (!!) for $\bar{\mathbb{P}}_m, \bar{\mathbb{P}}_k$). But $r \Vdash_{P^i_k} q \notin (P^{\omega_1}_k : P^i_k)$—a contradiction. ∎

LEMMA 3.4: *Assume that*

- $\bar{\mathbb{P}}_\xi \in \mathcal{K}$ *for* $\xi < \omega_1$,
- *if* $\xi < \zeta < \omega_1$ *then* $\bar{\mathbb{P}}_\xi \geq \bar{\mathbb{P}}_\zeta$ *is witnessed by the club* $C_{\xi,\zeta} \subseteq \omega_1$,

- *if $\delta < \omega_1$ is a limit ordinal and $i \in \bigcap_{\xi<\zeta<\delta} C_{\xi,\zeta}$ then $M^i_\delta = \lim_{\xi<\delta} M^i_\xi$.*

Let

$$C = \{\delta < \omega_1 \colon \delta \text{ is limit } \& \ (\forall \xi < \zeta < \delta)(\delta \in C_{\xi,\zeta})\}$$

and let $C(i) = \cap(C \backslash i)$ for $i < \omega_1$. Put $P^i_{\omega_1} = P^{C(i)}_{C(i)}$, $M^i_{\omega_1} = M^{C(i)}_{C(i)}$. Then $\bar{\mathbb{P}}_{\omega_1} \in \mathcal{K}$ and $(\forall \xi < \omega_1)(\bar{\mathbb{P}}_\xi \geq \bar{\mathbb{P}}_{\omega_1})$.

Proof: First note that the set $\{\delta < \omega_1 \colon (\forall \xi < \zeta < \delta)(\delta \in C_{\xi,\zeta})\}$ is the diagonal intersection of clubs $\bigcap_{\xi<\zeta} C_{\xi,\zeta}$ (for $\zeta < \omega_1$). Hence C is closed and unbounded and $\bar{\mathbb{P}}_{\omega_1}$ is well defined.

CLAIM: If $i < j < \omega_1$ then $P^i_{\omega_1} \lessdot P^j_{\omega_1}$.

Indeed, suppose $i < j < \omega_1$. Then $P^i_{\omega_1} = P^{C(i)}_{C(i)}$, $P^j_{\omega_1} = P^{C(j)}_{C(j)}$ and we may assume that $C(i) < C(j)$. By 3.1 (2) we have that $P^{C(i)}_{C(i)} \lessdot P^{C(j)}_{C(i)}$. Since C consists of limit ordinals only and $C(j) \in \bigcap_{\xi<\zeta<C(j)} C_{\xi,\zeta}$ we get $P^{C(j)}_{C(j)} = \bigcup_{\xi<C(j)} P^{C(j)}_\xi$ (and it is a direct limit). Since $C(i) < C(j)$ we conclude $P^{C(j)}_{C(i)} \lessdot P^{C(j)}_{C(j)}$ and consequently $P^{C(i)}_{C(i)} \lessdot P^{C(j)}_{C(j)}$. The claim is proved.

Since each $M^i_{\omega_1}$ is a model of sweetness based on $P^i_{\omega_1}$ we have proved that $\bar{\mathbb{P}}_{\omega_1} \in \mathcal{K}$. Let $\xi < \omega_1$.

CLAIM: $P^{\omega_1}_\xi \lessdot P^{\omega_1}_{\omega_1}$

First note that

$$P^{\omega_1}_{\omega_1} = \bigcup_{i<\omega_1} P^i_{\omega_1} = \bigcup_{i<\omega_1} P^{C(i)}_{C(i)} = \bigcup_{\zeta, i<\omega_1} P^i_\zeta = \bigcup_{\zeta<\omega_1} P^{\omega_1}_\zeta.$$

Since $\zeta_1 < \zeta_2 < \omega_1$ implies $\bar{\mathbb{P}}_{\zeta_1} \geq \bar{\mathbb{P}}_{\zeta_2}$ we have $P^{\omega_1}_{\zeta_1} \lessdot P^{\omega_1}_{\zeta_2}$ for $\zeta_1 < \zeta_2 < \omega_1$. Consequently $P^{\omega_1}_\xi \lessdot P^{\omega_1}_{\omega_1}$.

CLAIM: If $i \in C \backslash (\xi + 1)$ then $M^i_\xi \geq M^i_{\omega_1}$.

If $i \in C\backslash(\xi+1)$ then $C(i) = i > \xi$. Moreover it follows from our assumptions that $M^i_i = \lim_{\zeta<i} M^i_\zeta$. By Lemma 2.8 we get $M^i_\xi \geq M^i_i = M^{C(i)}_{C(i)} = M^i_{\omega_1}$.

CLAIM: Suppose $i \in C\backslash(\xi + 1)$, $q \in P^{\omega_1}_\xi$, $p \in P^i_\xi$ and $p \Vdash_{P^i_\xi} q \in (P^{\omega_1}_\xi : P^i_\xi)$. Then $p \Vdash_{P^i_{\omega_1}} q \in (P^{\omega_1}_{\omega_1} : P^i_{\omega_1})$.

Assume not. Then we have $r \in P^i_{\omega_1} = P^i_i$, $r \leq p$ such that r and q are incompatible. There is $\zeta \in (\xi, i)$ such that $r \in P^i_\zeta$. Thus $p \not\Vdash_{P^i_\zeta} q \in (P^{\omega_1}_\zeta : P^i_\zeta)$. Since $i \in C_{\xi,\zeta}$ we get a contradiction with condition (!!) for $\bar{\mathbb{P}}_\xi \geq \bar{\mathbb{P}}_\zeta$.

We have proved that the club $C\setminus(\xi+1)$ witnesses $\bar{\mathbb{P}}_\xi \geq \bar{\mathbb{P}}_{\omega_1}$. ∎

Suppose $\bar{\mathbb{P}} = <(P^i, M^i) : i < \omega_1> \in \mathcal{K}$. Let

$$P^i_D = \{(p,\tau) \in P^{\omega_1} * \dot{\mathbb{D}} : p \in P^i \ \&\ \tau \text{ is a } P^i\text{-name }\} = P^i * \dot{\mathbb{D}}.$$

Note that though it is possible that P^{i+1} is isomorphic to $P^i * \dot{\mathbb{D}}$, we think of algebra $\dot{\mathbb{D}}$ (in P^i_D) being iterated "on a new coordinate" above $\sup(\mathbb{P}^{\omega_1})$. In other words, while $P^i \subseteq P^j$ for $i<j$ we do not treat P^i also as a subset of $P^i * \dot{\mathbb{D}}$, it is only embedded into the iteration.

Let M^i_D be the canonical model of sweetness based on P^i_D and extending the model M^i (see Lemma 2.10). Let

$$\bar{\mathbb{P}}_D = \langle (P^i_D, M^i_D) : i < \omega_1 \rangle.$$

LEMMA 3.5: *$\bar{\mathbb{P}}_D \in \mathcal{K}$, $\bar{\mathbb{P}} \geq \bar{\mathbb{P}}_D$ and $P^{\omega_1}_D = P^{\omega_1} * \dot{\mathbb{D}}$.*

Proof: The last assertion is a consequence of the fact that P^{ω_1} is a ccc notion of forcing. Since for $i<j$, $\mathbb{D}^{\mathbf{V}^{P_i}}$ is a complete suborder of $\mathbb{D}^{\mathbf{V}^{P_j}}$ (cf [JS2]) we have that $P^i_D \lessdot P^j_D$ provided $i<j$. Consequently $\bar{\mathbb{P}}_D \in \mathcal{K}$. To show $\bar{\mathbb{P}} \geq \bar{\mathbb{P}}_D$ note that $M^i \geq M^i_D$ for all $i<\omega_1$ and $P^{\omega_1} \lessdot P^{\omega_1}_D$. Suppose now that $i<\omega_1$, $p \in P^i$, $q \in P^{\omega_1}$ and $p \Vdash_{P^i} q \in (P^{\omega_1} : P^i)$. Assume that $p \not\Vdash_{P^i_D} q \in (P^{\omega_1}_D : P^i_D)$. Then we find a condition $r = (r_0, \tau) \in P^i_D$ below p which is inconsistent with q. We consider q as an element of P^{ω_1}, while r is an element of $P^{\omega_1} * \dot{\mathbb{D}}$. Consequently incompatibility of q and r means that q and r_0 are not compatible. But $r_0 \in P^i$ lies below p—a contradiction. ∎

LEMMA 3.6: *Suppose that $\mathcal{B}, \mathbb{C}, \mathcal{D}, \mathbb{C}_0$ are complete Boolean algebras such that*

(1) $\mathcal{B} \lessdot \mathcal{D} \lessdot \mathbb{C}$, $\mathbb{C}_0 \lessdot \mathbb{C}$

Let $\mathcal{B}_0 = \mathcal{B} \cap \mathbb{C}_0$, $\mathcal{D}_0 = \mathcal{D} \cap \mathbb{C}_0$ (note that $\mathcal{B}_0 \lessdot \mathcal{D}_0 \lessdot \mathbb{C}_0$). We assume that

(2) *$\mathcal{B} \Vdash$ "$(\mathcal{D} : \mathcal{B})$ is a subset of $(\mathcal{D}_0 : \mathcal{B})$" (i.e. every element of $(\mathcal{D} : \mathcal{B})$ is (modulo $\Gamma_{\mathcal{B}}$) equivalent to an element of $(\mathcal{D}_0 : \mathcal{B})$);*

(3) *if $b \in \mathcal{B}, b_0 \in \mathcal{B}_0$ and $b_0 \Vdash_{\mathcal{B}_0} b \in (\mathcal{B} : \mathcal{B}_0)$ then $b_0 \Vdash_{\mathbb{C}_0} b \in (\mathbb{C} : \mathbb{C}_0)$.*

Then

(3*) *if $d \in \mathcal{D}, d_0 \in \mathcal{D}_0$ and $d_0 \Vdash_{\mathcal{D}_0} d \in (\mathcal{D} : \mathcal{D}_0)$ then $d_0 \Vdash_{\mathbb{C}_0} d \in (\mathbb{C} : \mathbb{C}_0)$.*

Remark: Note that (3) is equivalent to: $\pi^{\mathbb{C}}_{\mathbb{C}_0}(b) = \pi^{\mathcal{B}}_{\mathcal{B}_0}(b)$, where $\pi^A_{A_0}$ is the projection from the algebra A onto its complete subalgebra A_0.

Proof:

CLAIM: Suppose $c \in \mathbb{C}_0, d_0 \in \mathcal{D}_0$ and $d_0 \Vdash_{\mathcal{D}_0} c \in (\mathbb{C}_0 : \mathcal{D}_0)$. Then $d_0 \Vdash_{\mathcal{D}} c \in (\mathbb{C} : \mathcal{D})$.

We have to prove that each $d \leq d_0, d \in \mathcal{D}$ is compatible with c. Let $d \leq d_0, d \in \mathcal{D}$. Let $b' \in \mathcal{B}$ be such that $b' \Vdash_{\mathcal{B}}$ "$d \in (\mathcal{D} : \mathcal{B})$". By (2) we have $b' \Vdash_{\mathcal{B}}$ "$d \in (\mathcal{D}_0 : \mathcal{B})$". Thus we find $b \in \mathcal{B}$ and $d_1 \in \mathcal{D}_0$ such that

$$b \Vdash_{\mathcal{B}} \text{“}d \in (\mathcal{D} : \mathcal{B}) \ \&\ d \equiv_{(\mathcal{D}:\mathcal{B})} d_1\text{”}$$

(the last means that $b{\cdot}d = b{\cdot}d_1$). Thus $b{\cdot}d_1{\cdot}d_0 = b{\cdot}d{\cdot}d_0 = b{\cdot}d \neq \mathbf{0}$. We find $b_0 \in \mathcal{B}_0$ such that $b_0 \Vdash_{\mathcal{B}_0} b \in (\mathcal{B} : \mathcal{B}_0)$ and $b_0{\cdot}d_1{\cdot}d_0 \neq \mathbf{0}$ (it is enough to take b_0 such that $b_0 \Vdash_{\mathcal{B}_0} b{\cdot}d_1{\cdot}d_0 \in (\mathcal{D} : \mathcal{B}_0)$). Note that then $b_0 \Vdash_{\mathbb{C}_0} b \in (\mathbb{C} : \mathbb{C}_0)$ (by (3)). Since $b_0{\cdot}d_1{\cdot}d_0 \in \mathcal{D}_0$ and it is stronger than d_0 we get $b_0{\cdot}d_1{\cdot}d_0{\cdot}c \neq \mathbf{0}$. The last condition is stronger than b_0 and belongs to $\mathbb{C}_0$. Hence $b{\cdot}b_0{\cdot}d_1{\cdot}d_0{\cdot}c \neq \mathbf{0}$. Finally note that $b{\cdot}b_0{\cdot}d_1{\cdot}d_0{\cdot}c \leq b{\cdot}d_1 = b{\cdot}d \leq d$ so d and c are compatible. The claim is proved.

Now suppose that $d \in \mathcal{D}, d_0 \in \mathcal{D}_0$ and $d_0 \Vdash_{\mathcal{D}_0} d \in (\mathcal{D} : \mathcal{D}_0)$. Let $c \in \mathbb{C}_0, c \leq d_0$. Take $d^* \in \mathcal{D}_0$ such that $d^* \leq d_0$ and $d^* \Vdash_{\mathcal{D}_0} c \in (\mathbb{C}_0 : \mathcal{D}_0)$. By the claim we have $d^* \Vdash_{\mathcal{D}} c \in (\mathbb{C} : \mathcal{D})$. Since $d^* \leq d_0$ we have $d^*{\cdot}d \neq \mathbf{0}, d^*{\cdot}d \in \mathcal{D}$ and consequently $d^*{\cdot}d{\cdot}c \neq \mathbf{0}$. Hence d and c are compatible and we are done. ∎

Suppose that $\bar{\mathbb{P}}_0, \bar{\mathbb{P}}_1, \bar{\mathbb{P}}_2, \bar{\mathbb{P}}_3 \in \mathcal{K}$ and the club $C \subseteq \omega_1$ witnesses that both $\bar{\mathbb{P}}_0 \geq \bar{\mathbb{P}}_1$ and $\bar{\mathbb{P}}_2 \geq \bar{\mathbb{P}}_3$. Assume that $\mathbb{Q}_0, \mathbb{Q}_2$ are complete Boolean algebras such that for some $i_0 < \omega_1$

- $\mathrm{BA}(P_0^{\omega_1}) \lessdot \mathbb{Q}_0 \lessdot \mathrm{BA}(P_1^{\omega_1})$, $\mathrm{BA}(P_2^{\omega_1}) \lessdot \mathbb{Q}_2 \lessdot \mathrm{BA}(P_3^{\omega_1})$,
- $\mathrm{BA}(P_0^{\omega_1}) \Vdash (\mathbb{Q}_0 : \mathrm{BA}(P_0^{\omega_1})) \subseteq ((\mathbb{Q}_0 \cap \mathrm{BA}(P_1^{i_0})) : \mathrm{BA}(P_0^{\omega_1}))$
  $\mathrm{BA}(P_2^{\omega_1}) \Vdash (\mathbb{Q}_2 : \mathrm{BA}(P_2^{\omega_1})) \subseteq ((\mathbb{Q}_2 \cap \mathrm{BA}(P_3^{i_0})) : \mathrm{BA}(P_2^{\omega_1}))$.

Let $f\colon \mathbb{Q}_0 \longrightarrow \mathbb{Q}_2$ be an isomorphism such that $f[\mathbb{Q}_0 \cap \mathrm{BA}(P_1^i)] = \mathbb{Q}_2 \cap \mathrm{BA}(P_3^i)$ for all $i \in C \backslash i_0$. For $i \in C \backslash i_0$ put

$$P^i = \{(p_1, p_2) \in P_1^{\omega_1} \times_{\mathrm{id}, f} P_3^{\omega_1} : p_1 \in P_1^i \ \&\ p_2 \in P_3^i\},$$

where id stands for the identity on $\mathbb{Q}_0$. It follows from Lemma 3.6 that P^i is isomorphic to $P_1^i \times_{f_1, f_3} P_3^i$, where $f_3 = f | \mathbb{Q}_0 \cap \mathrm{BA}(P_1^i)$ and f_1 is the identity on $\mathbb{Q}_0 \cap \mathrm{BA}(P_1^i)$. Therefore we have the canonical model of sweetness M^i based on P^i and extending both models M_1^i and M_2^i (compare Lemma 2.9). At the moment P^i is defined for i from the club $C \backslash i_0$ only. For $i \notin C \backslash i_0$ we put $P^i = P^{i^*}$, where i^* is the first element of $C \backslash i_0$ greater than i.

Let

$$\bar{\mathbb{P}}_1 \times_f \bar{\mathbb{P}}_3 = \langle (P^i, M^i) : i < \omega_1 \rangle.$$

Note that $\bigcup_{i<\omega_1} P^i = P_1^{\omega_1} \times_{\mathrm{id},f} P_3^{\omega_1}$.

LEMMA 3.7: *$\bar{\mathbb{P}}_1 \times_f \bar{\mathbb{P}}_3 \in \mathcal{K}$ and $\bar{\mathbb{P}}_1, \bar{\mathbb{P}}_3 \geq \bar{\mathbb{P}}_1 \times_f \bar{\mathbb{P}}_3$.*

Proof: To prove $\bar{\mathbb{P}}_1 \times_f \bar{\mathbb{P}}_3 \in \mathcal{K}$ we have to show the following

CLAIM: *$P^i \lessdot P^j$ for each $i < j < \omega_1$, $i, j \in C \setminus i_0$.*

Let $\mathcal{A} \subseteq P^i$ be a maximal antichain and let $(p_1, p_2) \in P^j$. Let $q \in \mathbb{Q}_0$ be such that

$$q \Vdash \text{“} p_1 \in (P_1^{\omega_1} : \mathbb{Q}_0) \ \&\ p_2 \in (P_3^{\omega_1} : f[\mathbb{Q}_0]) \text{”}.$$

Take $r_1 \in P_1^i$ such that $r_1 \Vdash_{P_1^i}$ “$p_1, q \in (P_1^{\omega_1} : P_1^i)$” (note that q and p_1 are compatible). Next find $q' \in \mathbb{Q}_0$ such that $q' \leq q$ and $q' \Vdash r_1 \in (P_1^{\omega_1} : \mathbb{Q}_0)$ (recall that r_1 and q are compatible). Since p_2 and $f(q')$ are compatible we find $r_2 \in P_3^i$ such that $r_2 \Vdash_{P_3^i}$ “$p_2, f(q') \in (P_3^{\omega_1} : P_3^i)$”. Consider the pair (r_1, r_2). There is $q'' \in \mathbb{Q}_0, q'' \leq q'$ such that $q'' \Vdash r_2 \in (P_3^{\omega_1} : f[\mathbb{Q}_0])$. Then

$$q'' \Vdash \text{“} r_1 \in (P_1^{\omega_1} : \mathbb{Q}_0) \ \&\ r_2 \in (P_3^{\omega_1} : f[\mathbb{Q}_0]) \text{”}$$

and consequently $(r_1, r_2) \in P^i$. Since (r_1, r_2) has to be compatible with some element of $\mathcal{A}$ we are done.

CLAIM: *Suppose $q \in P_1^{\omega_1}$, $i \in C \setminus i_0$, $p \in P_1^i$ are such that $p \Vdash_{P_1^i} q \in (P_1^{\omega_1} : P_1^i)$. Then $p \Vdash_{P^i} q \in (P^{\omega_1} : P^i)$.*

Suppose $r \in P^i$ is stronger than p. Let $r = (r_1, r_2)$ and let $r_0 \in \mathbb{Q}_0$ witness $r \in P_1^{\omega_1} \times_{\mathrm{id},f} P_3^{\omega_1}$. We may get $r_0 \in \mathbb{Q}_0 \cap \mathrm{BA}(P_1^i)$. Remember that $P_1^{\omega_1}$ is embedded in P^{ω_1} by $s \mapsto (s, \mathbf{1})$, thus we have $p \simeq (p, \mathbf{1})$, $q \simeq (q, \mathbf{1})$. Since $r_0, r_1 \in \mathrm{BA}(P_1^i)$ are compatible and $r_1 \geq p$ we find $r_1^* \in P_1^{\omega_1}$ below r_0, r_1 and q. Then $(r_1^*, r_2) \in P^{\omega_1}$ and it is a condition stronger than both (r_1, r_2) and $(q, \mathbf{1})$. The claim is proved.

Since $M_1^i \geq M^i$ for each $i \in C \setminus i_0$ it follows from the above claim that $\bar{\mathbb{P}}_1 \geq \bar{\mathbb{P}}_1 \times_f \bar{\mathbb{P}}_3$ (and $C \setminus i_0$ is a witness for it). Similarly one can prove $\bar{\mathbb{P}}_3 \geq \bar{\mathbb{P}}_1 \times_f \bar{\mathbb{P}}_3$. ∎

LEMMA 3.8: *Suppose $\bar{\mathbb{P}}_0, \bar{\mathbb{P}}_1 \in \mathcal{K}$, $\bar{\mathbb{P}}_0 \geq \bar{\mathbb{P}}_1$. Let $\mathbb{Q}_0, \mathbb{Q}_1$ be complete Boolean algebras such that (for $k = 0, 1$):*

- $BA(P_0^{\omega_1}) \lessdot \mathbb{Q}_k \lessdot BA(P_1^{\omega_1})$,
- $BA(P_0^{\omega_1}) \Vdash$ "$(\mathbb{Q}_k : BA(P_0^{\omega_1}))$ *is the Cohen algebra".*

Let $f: \mathbb{Q}_0 \longrightarrow \mathbb{Q}_1$ be an isomorphism such that $f|BA(P_0^{\omega_1}) = id$.

Then there exist $\bar{\mathbb{P}} \in \mathcal{K}$ and an automorphism $\phi: P^{\omega_1} \xrightarrow{onto} P^{\omega_1}$ such that $\bar{\mathbb{P}}_1 \geq \bar{\mathbb{P}}$ and $f \subseteq \phi$.

Proof: We may apply Lemma 3.7 to get that $\bar{\mathbb{P}}_2 = \bar{\mathbb{P}}_1 \times_f \bar{\mathbb{P}}_1 \in \mathcal{K}$. The amalgamation over f produces an extension of f — there is $f_1: P_1^{\omega_1} \longrightarrow P_2^{\omega_1}$ such that $f \subseteq f_1$ (we identify $p \in P_1^{\omega_1}$ with $(\mathbf{1}, p) \in P_2^{\omega_1}$). Moreover $\bar{\mathbb{P}}_1, \bar{\mathbb{P}}_2, f_1$ satisfy assumptions of Lemma 3.7 and thus $\bar{\mathbb{P}}_3 = \bar{\mathbb{P}}_2 \times_{f_1} \bar{\mathbb{P}}_2 \in \mathcal{K}$. If we identify $p \in P_2^{\omega_1}$ with $(p, \mathbf{1}) \in \bar{\mathbb{P}}_3$ we get a partial isomorphism f_2 such that $f_1 \subseteq f_2$ and $\mathrm{rng}(f_2) = P_2^{\omega_1}$. Continuing in this fashion we build $\bar{\mathbb{P}}_m \in \mathcal{K}$ and partial isomorphisms f_m such that $\bar{\mathbb{P}}_m \geq \bar{\mathbb{P}}_{m+1}$, $f_m \subseteq f_{m+1}$ and either $P_m^{\omega_1} \subseteq \mathrm{dom}(f_m)$ or $P_m^{\omega_1} \subseteq \mathrm{rng}(f_m)$. Next we apply Lemma 3.3 to conclude that $\bar{\mathbb{P}}_\omega \in \mathcal{K}$ and $f_\omega = \bigcup_{m \in \omega} f_m: P_\omega^{\omega_1} \xrightarrow{onto} P_\omega^{\omega_1}$ is the desired automorphism. ∎

Definition 3.9: We define the following notion of forcing:

- $\mathbb{R} = \{\bar{\mathbb{P}} \in \mathcal{K} : \bar{\mathbb{P}} \in \mathcal{H}(\omega_2)\}$, where $\mathcal{H}(\omega_2)$ is the family of those sets which are hereditarily of size less than ω_2 (we choose $\bar{\mathbb{P}} \in \mathcal{H}(\omega_2)$ in order to be sure that $\mathbb{R}$ is a set).
- $\leq_{\mathbb{R}}$ is the relation $\leq$ of 3.2.

A notion of forcing $\mathbb{P}$ is $(\omega_1 + 1)$-strategically closed if the second player has a winning strategy in the following game of length $\omega_1 + 1$.

For $i = 0$ Player I gives $p_0 \in \mathbb{P}$;
Player I gives in the i-th move a dense subset D_i of $\mathbb{P}$;
Player II gives $p_{i+1} \leq p_i$, $p_{i+1} \in D_i$, for a limit i Player II gives p_i below all p_j (for $j < i$).

Player II looses if he is not able to give the respective element of $\mathbb{P}$ for some $i \leq \omega_1$.

Note that $(\omega_1 + 1)$-strategically closed notions of forcings do not add new ω_1-sequences of elements of the ground model.

PROPOSITION 3.10: *The forcing notion $\mathbb{R}$ is ω_1-closed and (ω_1+1)-strategically closed. Consequently forcing with $\mathbb{R}$ does not collapse ω_1 and ω_2.*

Proof: For the ω_1-closure use Lemma 3.3, for the ω_1+1-strategic closure apply 3.3 and 3.4. ∎

Note that $|\mathbb{R}| = 2^{\omega_1}$. Thus if we assume that $2^{\omega_1} = \omega_2$ then forcing with $\mathbb{R}$ does not collapse cardinals.

Suppose $\mathbf{V} \models$GCH.

PROPOSITION 3.11: *Let $G \subseteq \mathbb{R}$ be a generic over $\mathbf{V}$. Let $\mathbb{P} = \bigcup\{P^{\omega_1} : \bar{\mathbb{P}} \in G\}$. Then (in $\mathbf{V}[G]$)*

1. *$\mathbb{P}$ is a ccc notion of forcing.*
2. *If τ is a $\mathbb{P}$-name for an ω_1-sequence of ordinals then $\mathbb{P} \Vdash$ "the union of all Borel meager sets coded in $\mathbf{V}[\tau]$ is meager".*
3. *The Boolean algebra BA($\mathbb{P}$) is strongly Cohen-homogeneous.*
4. *$\mathbb{P} \Vdash$ "any union of ω_1 meager sets is meager".*

Proof: 1. Work in $\mathbf{V}$. Suppose that $\dot{A}$ is a $\mathbb{R}$-name for an ω_1-sequence of pairwise incompatible elements of $\mathbb{P}$. Let $\bar{\mathbb{P}} \in \mathbb{R}$. By Proposition 3.10 there is $\bar{\mathbb{P}}_1 \geq \bar{\mathbb{P}}$ which decides all values of $\dot{A}$. We may assume that all these elements belong to $P_1^{\omega_1}$. A contradiction.

2. Let τ be a $\mathbb{P}$-name for an ω_1-sequence of ordinals. Then τ is actually an ω_1-sequence of (countable) antichains in $\mathbb{P}$. Therefore $\tau \in \mathbf{V}$ and it is a $P_0^{\omega_1}$-name for some $\bar{\mathbb{P}}_0 \in G$. By density arguments we have that $(\bar{\mathbb{P}}_D)_D \in G$ for some $\bar{\mathbb{P}} \geq \bar{\mathbb{P}}_0$ (compare Lemma 2.10). Since $\mathbb{D} * \dot{\mathbb{D}}$ forces that the union of all meager sets coded in the ground model is meager we get

$$\mathbb{P} \Vdash \text{“the union of all Borel meager sets coded in } \mathbf{V}[G][\tau] \text{ is meager”.}$$

3. Work in $\mathbf{V}[G]$. Let τ be a $\mathbb{P}$-name for an ω_1-sequence of ordinals. As in 2. we find $\bar{\mathbb{P}}_0 \in G$ such that τ is a $P_0^{\omega_1}$-name. Suppose now that

- $\mathrm{BA}(P_0^{\omega_1}) \lessdot \mathcal{B} \lessdot \mathrm{BA}(\mathbb{P})$,
- $\mathrm{BA}(P_0^{\omega_1}) \Vdash$ "$(\mathcal{B} : \mathrm{BA}(P_0^{\omega_1}))$ is the Cohen algebra",
- $f\colon \mathcal{B} \longrightarrow \mathrm{BA}(\mathbb{P})$ is a complete embedding such that $f|\mathrm{BA}(P_0^{\omega_1}) = \mathrm{id}$.

Note that $\mathcal{B}$ and f are determined by countably many elements. Each element of BA($\mathbb{P}$) is a countable union of elements of $\mathbb{P}$. Consequently $\mathcal{B}, f \in \mathbf{V}$ and there is $\bar{\mathbb{P}}_1 \in G$ such that $\mathcal{B}, \mathrm{rng}(f) \subseteq \mathrm{BA}(P_1^{\omega_1})$, $\bar{\mathbb{P}}_0 \geq \bar{\mathbb{P}}_1$. By density argument and

Lemma 3.8 we find $\bar{\mathbb{P}}_2 \in G$ and f_2 such that $\bar{\mathbb{P}}_1 \geq \bar{\mathbb{P}}_2$ and f_2 is an automorphism of $\mathrm{BA}(P_2^{\omega_1})$ extending f. Similarly, if $\bar{\mathbb{P}}_4 \in G$, $\bar{\mathbb{P}}_3 \geq \bar{\mathbb{P}}_4$ and f_3 is an automorphism of $\mathrm{BA}(P_3^{\omega_1})$ then there are $\bar{\mathbb{P}}_5 \in G$, f_5 such that f_5 is an automorphism of $\mathrm{BA}(P_5^{\omega_1})$ extending f_3.

It follows from the above that, in $V[G]$, we can extend f to an automorphism of $\mathrm{BA}(\mathbb{P})$.

4. Similar arguments as in 1. and 2. ∎

Theorems 1.2 and 1.1 follow directly from the above proposition and Theorems 2.4 and 2.5.

Remark: To get a model for $2^{\omega} = \kappa > \omega_2$ we use the same method but using the club filter on $\mathcal{P}_{\omega}(\kappa)$. We leave the details to the reader.

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