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THE SPECTRUM OF CHARACTERS OF ULTRAFILTERS ON ω

BY

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Abstract. We show the consistency of the statement: "the set of regular cardinals which are the characters of ultrafilters on ω is not convex". We also deal with the set of π-characters of ultrafilters on ω.

0. Introduction. Some cardinal invariants of the continuum are actually the minimum of a natural set of cardinals $\leq 2^{\aleph_0}$ which can be called the spectrum of the invariant. Such a case is Sp_χ, the set of characters $\chi(D)$ of non-principal ultrafilters D on ω (the minimal number of generators). On the history see [BnSh:642]; there this spectrum and others were investigated and it was asked if Sp_χ can be non-convex (formally 0.1(2) below).

The main result here is 1.1, it solves the problem (starting with a measurable). This was presented at a conference in honor of I. Juhász, quite fitting as he had started the investigation of consistency on $\chi(D)$. In §2 we note what we can say on the strict π-character of ultrafilters.

The investigation is continued in [Sh:915] trying to get more 'disorderly" behaviours in smaller cardinals and in particular answering negatively the original question, 0.2(2).

Recall

0.1. Definition.

(1) $\mathrm{Sp}_\chi = \mathrm{Sp}(\chi)$ is the set of cardinals θ such that $\theta = \chi(D)$ for some non-principal ultrafilter D on ω where
(2) For D an ultrafilter on ω let $\theta = \chi(D)$ be the minimal cardinality θ such that D is generated by some family of θ members, i.e. $\mathrm{Min}\{|\mathscr{A}| : \mathscr{A} \subseteq D$ and $(\forall B \in D)(\exists A \in \mathscr{A})[A \subseteq^* B]\}$; it does not matter if we use "$A \subseteq B$".

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Now, Brendle and Shelah [BnSh:642, Problem 5] asked the question formulated in 0.2(2) below, but it seems to me, at least now, that the question is really 0.2(1)+(3).

0.2. PROBLEM.

(1) Can $\mathrm{Sp}(\chi)\cap\mathrm{Reg}$ have gaps, i.e., can it be that $\theta < \mu < \lambda$ are regular, $\theta \in \mathrm{Sp}(\chi)$, $\mu \notin \mathrm{Sp}(\chi)$, $\lambda \in \mathrm{Sp}(\chi)$?
(2) In particular, does $\aleph_1, \aleph_3 \in \mathrm{Sp}(\chi)$ imply $\aleph_2 \in \mathrm{Sp}(\chi)$?
(3) Are there any restrictions on $\mathrm{Sp}(\chi) \cap \mathrm{Reg}$?

We thank the referee for helpful comments and in particular 2.5(1).

DISCUSSION. This relies on [Sh:700, §4]; there is no point to repeat it but we try to give a description. Let $\aleph_0 < \kappa < \mu < \lambda$ be regular cardinals and κ be a measurable cardinal.

Let $S = \{\alpha < \lambda : \mathrm{cf}(\alpha) \neq \kappa\}$ or any unbounded subset of it. We define ([Sh:700, 4.3]) the class $\mathfrak{K} = \mathfrak{K}_{\lambda,S}$ of objects $\mathfrak{t}$ approximating our final forcing. Each $\mathfrak{t} \in K$ consists mainly of a finite support iteration $\langle \mathbb{P}^{\mathfrak{t}}_i, \mathbb{Q}^{\mathfrak{t}}_i : i < \mu\rangle$ of c.c.c. forcing of cardinality $\leq \lambda$ with limit $\mathbb{P}^*_{\mathfrak{t}} = \mathbb{P}^{\mathfrak{t}} = \mathbb{P}^{\mathfrak{t}}_\mu$, but also $\mathbb{Q}^{\mathfrak{t}}_i$-names $\underset{\sim}{\tau}^{\mathfrak{t}}_i$ $(i < \mu)$, so it is a $\mathbb{P}^t_{i+1}$ satisfying a strong version of the c.c.c. and for $i \in S$, also $\underset{\sim}{D}^{\mathfrak{t}}_i$, a $\mathbb{P}^t_i$-name of a non-principal ultrafilter on ω from which $\mathbb{Q}^{\mathfrak{t}}_i$ is nicely defined, and $\underset{\sim}{A}^{\mathfrak{t}}_i$, a $\mathbb{Q}^{\mathfrak{t}}_i$-name (so $\mathbb{P}^t_{i+1}$-name) of a pseudo-intersection (and $\mathbb{Q}_i$, $i \in S$, nicely defined) of $\underset{\sim}{D}^t_i$ such that $i < j \in S \Rightarrow \underset{\sim}{A}^{\mathfrak{t}}_i \in D^{\mathfrak{t}}_j$. So $\{\underset{\sim}{A}_i : i \in S\}$ witness $\mathfrak{u} \leq \mu$ in $\mathbf{V}^{\mathbb{P}_{\mathfrak{t}}}$; we do not necessarily have to use nicely defined $\mathbb{Q}_i$, though for $i \in S$ we do.

The order $\leq_{\mathfrak{K}}$ is the natural order; we prove the existence of the so-called canonical limit.

Now a major point of [Sh:700] is: for $\mathfrak{s} \in \mathfrak{K}$, letting $\mathscr{D}$ be a uniform κ-complete ultrafilter on κ (or just κ_1-complete $\aleph_0 < \theta < \kappa$), we can consider $\mathfrak{t} = \mathfrak{s}^\kappa/\mathscr{D}$; by the Łoś theorem, more exactly by Hanf's Ph.D. thesis, (the parallel of) the Łoś theorem for $\mathbb{L}_{\kappa,\kappa}$ applies; it gives that $\mathfrak{t} \in \mathfrak{K}$, well if $\lambda = \lambda^\kappa/\mathscr{D}$; and moreover $\mathfrak{s} \leq_{\mathfrak{K}} \mathfrak{t}$ under the canonical embedding.

The effect is that, e.g., being "a linear order having cofinality $\theta \neq \kappa$" is preserved, even by the same witness, whereas having cardinality $\theta < \lambda$ is not necessarily preserved, and sets of cardinality $\geq \kappa$ are increased. As $\mathfrak{d}$ is the cofinality (not of a linear order, but) of a partial order, there are complications; anyhow, as $\mathfrak{d}$ is defined by cofinality whereas $\mathfrak{a}$ by cardinality of sets, this helps in [Sh:700], noting that as we deal with c.c.c. forcing, names of reals are represented by ω-sequences of conditions, the relevant things are preserved. So we use a $\leq_{\mathfrak{K}}$-increasing sequence $\langle \mathfrak{t}_\alpha : \alpha \leq \lambda\rangle$ such that for unboundedly many $\alpha < \lambda$, $\mathfrak{t}_{\alpha+1}$ is essentially $(\mathfrak{t}^\alpha_\alpha)^\kappa/\mathscr{D}$.

What does "nice" $\mathbb{Q} = \mathbb{Q}(D)$ mean, for D a non-principal ultrafilter over ω? We need that

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(α) $\mathbb{Q}$ satisfies a strong version of the c.c.c.,

(β) the definition commutes with the ultrapower used,

(γ) if $\mathbb{P}$ is a forcing notion then we can extend D to an ultrafilter D^+ for every (or at least some) $\mathbb{P}$-name of an ultrafilter $\underset{\sim}{D}$ extending D, and we have $\mathbb{Q}(D) \lessdot \mathbb{P} * \mathbb{Q}(\underset{\sim}{D}^+)$ (used for the existence of canonical limit).

Such a forcing is combining Laver forcing and Mathias forcing for an ultrafilter D on ω, that is: $p \in D$ iff p is a subtree of ω with trunk $\mathrm{tr}(p) \in p$ such that for $\eta \in p$ we have $\lg(\eta) < \lg(\mathrm{tr}(p)) \Rightarrow (\exists! n)(\eta\hat{}\langle n\rangle \in p)$ and $\lg(\eta) \geq \lg(\mathrm{tr}(p)) \Rightarrow \{n : \eta\hat{}\langle n\rangle \in p\} \in D$.

1. Using measurables and FS iterations with non-transitive memory. We use [Sh:700] in 1.1 heavily. We use measurables (we could have used extenders to get more). The question on $\aleph_1, \aleph_2, \aleph_3$, i.e. Problem 0.2(2) remains open.

1.1. THEOREM. *There is a c.c.c. forcing notion $\mathbb{P}$ of cardinality λ such that in $\mathbf{V}^{\mathbb{P}}$ we have $\mathfrak{a} = \lambda$, $\mathfrak{b} = \mathfrak{d} = \mu$, $\mathfrak{u} = \mu$, $\{\mu, \lambda\} \subseteq \mathrm{Sp}_\chi$ but $\kappa_2 \notin \mathrm{Sp}(\chi)$ if*

⊛ *κ_1, κ_2 are measurable and $\kappa_1 < \mu = \mathrm{cf}(\mu) < \kappa_2 < \lambda = \lambda^\mu = \lambda^{\kappa_2} = \mathrm{cf}(\lambda)$.*

Proof. Let $\mathscr{D}_l$ be a normal ultrafilter on κ_l for $l = 1, 2$. Repeat [Sh:700, §4] with (κ_1, μ, λ) here standing for (κ, μ, λ) there, getting $\mathfrak{t}_\alpha \in \mathfrak{K}$ for $\alpha \leq \lambda$ which is $\leq_{\mathfrak{K}}$-increasing. Letting $\mathbb{P}^\alpha_i = \mathbb{P}^{\mathfrak{t}_\alpha}_i$ we see that $\overline{\mathbb{Q}}^\alpha = \langle \mathbb{P}^\alpha_\varepsilon : \varepsilon < \mu\rangle$ is a $\lessdot$-increasing continuous sequence of c.c.c. forcing notions, $\mathbb{P}^\alpha_\mu = \mathbb{P}^\alpha = \mathbb{P}_{\mathfrak{t}_\alpha} := \mathrm{Lim}(\overline{\mathbb{Q}}^\alpha) = \bigcup\{\mathbb{P}^\alpha_\varepsilon : \varepsilon < \mu\}$; in fact $\langle \mathbb{P}^\alpha_\varepsilon, \mathbb{Q}^\alpha_\varepsilon : \varepsilon < \mu\rangle$ is an FS iterated forcing etc., but we add the demand that for unboundedly many $\alpha < \lambda$,

$\boxtimes^1_\alpha$ $\mathbb{P}^{\alpha+1}$ is isomorphic to the ultrapower $(\mathbb{P}^\alpha)^{\kappa_2}/\mathscr{D}_2$, by an isomorphism extending the canonical embedding.

More explicitly, we choose $\mathfrak{t}_\alpha$ by induction on $\alpha \leq \lambda$ such that

⊛$_1$ (a) $\mathfrak{t}_\alpha \in \mathfrak{K}$ (see [Sh:700, Definition 4.3]), so the forcing notion $\mathbb{P}^{\mathfrak{t}_\alpha}_i$ for $i \leq \mu$ is well defined and is $\lessdot$-increasing with i,

(b) $\langle \mathfrak{t}_\beta : \beta \leq \alpha\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous, which means that:

(α) $\gamma \leq \beta \leq \alpha \Rightarrow \mathfrak{t}_\gamma \leq_{\mathfrak{K}} \mathfrak{t}_\beta$ (see [Sh:700, Definition 4.6(1)]), so $\mathbb{P}^{\mathfrak{t}_\gamma}_i \lessdot \mathbb{P}^{\mathfrak{t}_\beta}_i$ for $i \leq \mu$,

(β) if α is a limit ordinal then $\mathfrak{t}_\alpha$ is a canonical $\leq_{\mathfrak{K}}$-u.b. of $\langle \mathfrak{t}_\beta : \beta < \alpha\rangle$ (see [Sh:700, Definition 4.6(2)]),

(c) if $\alpha = \beta + 1$ and $\mathrm{cf}(\beta) \neq \kappa_2$ then $\mathfrak{t}_\alpha$ is essentially $\mathfrak{t}^{\kappa_1}_\beta/\mathscr{D}_1$ (i.e. we have to identify $\mathbb{P}^{\mathfrak{t}_\beta}_\varepsilon$ with its image under the canonical embedding of it into $(\mathbb{P}^{\mathfrak{t}_\beta}_\varepsilon)^{\kappa_1}/\mathscr{D}_1$, in particular this holds for $\varepsilon = \mu$, see [Sh:700, Subclaim 4.9]),

(d) if $\alpha = \beta + 1$ and $\mathrm{cf}(\beta) = \kappa_2$ then $\mathfrak{t}_\alpha$ is essentially $\mathfrak{t}^{\kappa_2}_\beta/\mathscr{D}_2$.

So we need

$\circledast_2$ [Sh:700, Subclaim 4.9] also applies to the ultrapower $\mathfrak{t}^{\kappa_2}_\beta/D$.
[Why? The same proof applies as $\mu^{\kappa_2}/\mathscr{D}_2 = \mu$, i.e., the canonical embedding of μ into $\mu^{\kappa_2}/\mathscr{D}_2$ is one-to-one and onto (and $\lambda^{\kappa_1}/\mathscr{D}_1 = \lambda^{\kappa_2}/\mathscr{D}_2 = \lambda$, of course).]

Let $\mathbb{P}^\alpha_\varepsilon = \mathbb{P}^{\mathfrak{t}_\alpha}_\varepsilon$ for $\varepsilon \le \mu$ so $\mathbb{P}^\alpha = \bigcup\{\mathbb{P}^\alpha_\varepsilon : \varepsilon < \mu\}$ and $\mathbb{P} = \mathbb{P}^\lambda$. It is proved in [Sh:700, 4.10] that in $\mathbf{V}^{\mathbb{P}}$, by construction,

$$\mu \in \mathrm{Sp}(\chi), \quad \mathfrak{a} \le \lambda, \quad \mathfrak{u} = \mu, \quad 2^{\aleph_0} = \lambda.$$

By [Sh:700, 4.11] we have $\mathfrak{a} \ge \lambda$, hence $\mathfrak{a} = \lambda$, and always $2^{\aleph_0} \in \mathrm{Sp}(\chi)$, hence $\lambda = 2^{\aleph_0} \in \mathrm{Sp}(\chi)$. So what is left to prove is $\kappa_2 \notin \mathrm{Sp}(\chi)$. Assume toward a contradiction that $p^* \Vdash$ "$\underset{\sim}{D}$ is a non-principal ultrafilter on ω and $\chi(\underset{\sim}{D}) = \kappa_2$, and let it be exemplified by $\langle \underset{\sim}{A}_\varepsilon : \varepsilon < \kappa_2\rangle$".

Without loss of generality $p^* \Vdash_{\mathbb{P}}$ "for each $\varepsilon < \kappa_2$, $\underset{\sim}{A}_\varepsilon \in \underset{\sim}{D}$ does not belong to the filter on ω generated by $\{\underset{\sim}{A}_\zeta : \zeta < \varepsilon\} \cup \{\omega \setminus n : n < \omega\}$, and trivially also $\omega \setminus \underset{\sim}{A}_\varepsilon$ does not belong to this filter".

As λ is regular $> \kappa_2$ and the forcing notion $\mathbb{P}^\lambda$ satisfies the c.c.c., clearly for some $\alpha < \lambda$ we have $p^* \in \mathbb{P}^\alpha$ and $\varepsilon < \kappa_2 \Rightarrow \underset{\sim}{A}_\varepsilon$ is equivalently a $\mathbb{P}^\alpha$-name. So for every $\beta \in [\alpha, \lambda)$ we have

$\boxtimes^2_\beta$ $p^* \Vdash_{\mathbb{P}^\beta}$ "for each $i < \kappa_2$ the set $\underset{\sim}{A}_i \in [\omega]^{\aleph_0}$ is not in the filter on ω generated by $\{\underset{\sim}{A}_j : j < i\} \cup \{\omega \setminus n : n < \omega\}$, and also the complement of $\underset{\sim}{A}_i$ is not in this filter (as $\underset{\sim}{D}$ exemplifies)".

But for some such β, the statement $\boxtimes^1_\beta$ holds, i.e. $\circledast_1$(d) applies, so in $\mathbb{P}^{\beta+1}$ which is essentially a $(\mathbb{P}^\beta)^{\kappa_2}/\mathscr{D}_2$ we get a contradiction. That is, let $\mathbf{j}_\beta$ be an isomorphism from $\mathbb{P}^{\beta+1}$ onto $(\mathbb{P}^\beta)^{\kappa_2}/\mathscr{D}_2$ which extends the canonical embedding of $\mathbb{P}^\beta$ into $(\mathbb{P}^\beta)^{\kappa_2}/\mathscr{D}_2$. Now $\mathbf{j}_\beta$ induces a map $\hat{\mathbf{j}}_\beta$ from the set of $\mathbb{P}^{\beta+1}$-names of subsets of ω into the set of $(\mathbb{P}^\beta)^{\kappa_2}/\mathscr{D}_2$-names of subsets of ω, and let

$$\underset{\sim}{A}^* = \hat{\mathbf{j}}^{-1}_\beta(\langle \underset{\sim}{A}_i : i < \kappa_2\rangle/\mathscr{D}_2),$$

so $p^* \Vdash_{\mathbb{P}^{\beta+1}}$ "$\underset{\sim}{A}^* \in [\omega]^{\aleph_0}$ and the sets $\underset{\sim}{A}^*, \omega \setminus \underset{\sim}{A}^*$ do not include any finite intersection of some members of $\{\underset{\sim}{A}_\varepsilon : \varepsilon < \kappa_2\} \cup \{\omega \setminus n : n < \omega\}$". So $p^* \Vdash_{\mathbb{P}^{\beta+1}}$ "$\{\underset{\sim}{A}_\varepsilon : \varepsilon < \kappa_2\}$ does not generate an ultrafilter on ω", but $\mathbb{P}^{\beta+1} \lessdot \mathbb{P}$, a contradiction. ■

1.2. Remark. (1) As the referee pointed out, if we waive "$\mathfrak{u} < \mathfrak{a}$" in 1.1, we can forget κ_1 (and $\mathscr{D}_1$) so not take ultrapowers by $\mathscr{D}_1$ so $\mu = \aleph_0$ is allowed, but we have to start with $\mathfrak{t}_0$ such that $\mathbb{P}^{\mathfrak{t}_0}_0$ is adding κ_2-Cohen.

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(2) Moreover, in this case we can demand that $\mathbb{Q}^{\mathfrak{t}}_\alpha = \mathbb{Q}(\underset{\sim}{D}^{\mathfrak{t}}_\alpha)$ and so we do not need the $\underset{\sim}{\tau}^{\mathfrak{t}}_\alpha$. Still this way was taken in [Sh:915, §1]. But this gain in simplicity has a price in lack of flexibility in choosing the $\mathfrak{t}$. We use this mildly in §2, only for $\mathbb{P}_1$. See more in [Sh:915, §§2, 3].

2. Remarks on π-bases

2.1. DEFINITION.

(1) $\mathscr{A}$ is a *π-base* if:

(a) $\mathscr{A} \subseteq [\omega]^{\aleph_0}$,
(b) for some ultrafilter D on ω, $\mathscr{A}$ is a π-base of D (see below; note that D is necessarily non-principal).

(A) We say $\mathscr{A}$ is a *π-base of D* if $(\forall B \in D)(\exists A \in \mathscr{A})(A \subseteq^* B)$.
(B) $\pi\chi(D) = \mathrm{Min}\{|\mathscr{A}| : \mathscr{A}$ is a π-base of $D\}$.

(2) $\mathscr{A}$ is a *strict π-base* if:

(a) $\mathscr{A}$ is a π-base of some D,
(b) no subset of $\mathscr{A}$ of cardinality $< |\mathscr{A}|$ is a π-base.

(3) *D has a strict π-base* when D has a π-base $\mathscr{A}$ which is a strict π-base.
(4) $\mathrm{Sp}^*_{\pi\chi} = \{|\mathscr{A}|$: there is a non-principal ultrafilter D on ω such that $\mathscr{A}$ is a strict π-base of $D\}$.

2.2. DEFINITION. For $\mathscr{A} \subseteq [\omega]^{\aleph_0}$ let $\mathrm{Id}_{\mathscr{A}} = \{B \subseteq \omega$: for some $n < \omega$ and partition $\langle B_l : l < n\rangle$ of B, for no $A \in \mathscr{A}$ and $l < n$ do we have $A \subseteq^* B_l\}$.

2.3. OBSERVATION. *For $\mathscr{A} \subseteq [\omega]^{\aleph_0}$ we have:*

(a) *$\mathrm{Id}_{\mathscr{A}}$ is an ideal on $\mathscr{P}(\omega)$ including the finite sets, though it may be equal to $\mathscr{P}(\omega)$,*
(b) *if $B \subseteq \omega$ then: $B \in [\omega]^{\aleph_0} \backslash \mathrm{Id}_{\mathscr{A}}$ iff there is a (non-principal) ultrafilter D on ω to which B belongs and $\mathscr{A}$ is a π-base of D,*
(c) *$\mathscr{A}$ is a π-base iff $\omega \notin \mathrm{Id}_{\mathscr{A}}$.*

Proof. (a) Obvious.

(b) "if": Let D be a non-principal ultrafilter on ω such that $B \in D$ and $\mathscr{A}$ is a π-base of D. Now for any $n < \omega$ and partition $\langle B_l : l < n\rangle$ of B, as $B \in D$ and D is an ultrafilter, clearly there is $l < n$ such that $B_l \in D$, hence by Definition 2.1(1A) there is $A \in \mathscr{A}$ such that $A \subseteq^* B_l$. By the definition of $\mathrm{Id}_{\mathscr{A}}$ it follows that $B \notin \mathrm{Id}_{\mathscr{A}}$; but $[\omega]^{<\aleph_0} \subseteq \mathrm{Id}_{\mathscr{A}}$ so we are done.

"only if": We are assuming $B \notin \mathrm{Id}_{\mathscr{A}}$, so as $\mathrm{Id}_{\mathscr{A}}$ is an ideal of $\mathscr{P}(\omega)$ there is an ultrafilter D on ω disjoint from $\mathrm{Id}_{\mathscr{A}}$ such that $B \in D$. So if $B' \in D$

then $B' \subseteq \omega \wedge B' \notin \mathrm{Id}_{\mathscr{A}}$, hence by the definition of $\mathrm{Id}_{\mathscr{A}}$ it follows that $(\exists A \in \mathscr{A})(A \subseteq^* B')$. By Definition 2.1(1A) this means that $\mathscr{A}$ is a π-base of D.

(c) Follows from clause (b). $\blacksquare_{2.3}$

2.4. OBSERVATION.

(1) *If D is an ultrafilter on ω then D has a π-base of cardinality $\pi\chi(D)$.*
(2) *$\mathscr{A}$ is a π-base iff for every $n \in [1, \omega)$ and partition $\langle B_l : l < n\rangle$ of ω into finitely many sets, for some $A \in \mathscr{A}$ and $l < n$ we have $A \subseteq^* B_l$.*
(3) $\mathrm{Min}\{\pi\chi(D) : D$ *a non-principal ultrafilter on* $\omega\} = \mathrm{Min}\{|\mathscr{A}| : \mathscr{A}$ *is a π-base*$\} = \mathrm{Min}\{|\mathscr{A}| : \mathscr{A}$ *is a strict π-base*$\}$.

Proof. (1) By the definition.

(2) For the "only if" direction, assume $\mathscr{A}$ is a π-base of D. Then $\mathrm{Id}_{\mathscr{A}} \subseteq \mathscr{P}(\omega) \setminus D$ (see the proof of 2.2) so $\omega \notin \mathrm{Id}_{\mathscr{A}}$ and we are done.

For the "if" direction, use 2.2.

(3) Easy. $\blacksquare_{2.4}$

2.5. THEOREM. *In $\mathbf{V}^{\mathbb{P}}$ as in 1.1, we have $\{\mu, \lambda\} \subseteq \mathrm{Sp}^*_{\pi\chi}$ and $\kappa_2 \notin \mathrm{Sp}^*_{\pi\chi}$.*

Proof. Similar to the proof of 1.1 but with some additions. Defining $\mathfrak{K}$ in [Sh:700, 4.1] we allow $\mathbb{Q}_0 = \mathbb{Q}_0^{\mathfrak{t}} = \mathbb{P}_1^{\mathfrak{t}}$ to be any c.c.c. forcing notion of cardinality $\leq \lambda$ (this makes no change). The main change is in the proof of $\Vdash_{\mathbb{P}}$ "$\lambda \in \mathrm{Sp}_\chi$". The main addition is that choosing $\mathfrak{t}_\alpha$ by induction on α we also define $\mathscr{A}_\alpha$ such that

$\circledast'_1$ (a), (b) as in $\circledast_1$ in the proof of 1.1,
(c) as in $\circledast_1$(c) but only if $\alpha \neq 2 \bmod \omega$ (and $\alpha = \beta + 1$),
(d) $\underset{\sim}{A}_\alpha$ is a $\mathbb{P}_0^{\mathfrak{t}_\alpha}$-name of an infinite subset of ω,
(e) if $\alpha \neq 2 \bmod \omega$ then $\Vdash_{\mathbb{P}^{\mathfrak{t}_\alpha}} \underset{\sim}{A}_\alpha = \omega$ (or do not define $\underset{\sim}{A}_\alpha$),
(f) if $\alpha < \beta$ are $= 2 \bmod \omega$ then $\Vdash_{\mathbb{P}_\mu^{\mathfrak{t}_\beta}}$ "$\underset{\sim}{A}_\beta \subseteq^* \underset{\sim}{A}_\alpha$",
(g) if $\beta = \alpha + 1$ and $\beta = 2 \bmod \omega$ and $\underset{\sim}{B}$ is a $\mathbb{P}_\mu^{\mathfrak{t}_\alpha}$-name of an infinite subset of ω then $\Vdash_{\mathbb{P}_\mu^{\mathfrak{t}_\beta}}$ "$\underset{\sim}{B} \not\subseteq^* \underset{\sim}{A}_\alpha$".

This addition requires that we also prove

$\circledast_3$ if $\mathfrak{s} \in \mathfrak{K}$ and $\underset{\sim}{D}$ is a $\mathbb{P}_1^{\mathfrak{s}}$-name of a filter on ω including all co-finite subsets of ω (such that $\emptyset \notin D$) then for some $(\mathfrak{t}, \underset{\sim}{A})$ we have
(a) $\mathfrak{s} \leq_{\mathfrak{K}} \mathfrak{t}$,
(b) $\Vdash_{\mathbb{P}_1^{\mathfrak{t}}}$ "$\underset{\sim}{A}$ is an infinite subset of ω",
(c) if $\underset{\sim}{B}$ is a $\mathbb{P}^{\mathfrak{s}}$-name of an infinite subset of ω then $\Vdash_{\mathbb{P}^{\mathfrak{t}}}$ "$\underset{\sim}{B} \not\subseteq^* \underset{\sim}{A}$".

[Why $\circledast_3$ holds? Without loss of generality $\Vdash_{\mathbb{P}_1^{\mathfrak{s}}}$ "$\underset{\sim}{D}$ is an ultrafilter on ω".

We can find a pair $(\mathbb{P}', \underset{\sim}{A}')$ such that

(α) $\mathbb{P}'$ is a c.c.c. forcing notion,
(β) $\mathbb{P}_1^{\mathfrak{s}} \lessdot \mathbb{P}'$, moreover $\mathbb{P}' = \mathbb{P}_1^{\mathfrak{s}} * \mathbb{Q}(\underset{\sim}{D})$,
(γ) $|\mathbb{P}'| \leq \lambda$,
(δ) $\Vdash_{\mathbb{P}'}$ "$\underset{\sim}{A}$ is an almost intersection of $\underset{\sim}{D}$ (i.e. $\underset{\sim}{A} \in [\omega]^{\aleph_0}$ and $(\forall B \in \underset{\sim}{D})(A \subseteq^* B)$)",
(ε) $\underset{\sim}{\eta}' \in {}^\omega\omega$ is the generic of $\mathbb{Q}[\underset{\sim}{D}]$ and $\underset{\sim}{A}' = \mathrm{Rang}(\eta)$ so both are $\mathbb{P}'$-names.

Now we define $\mathfrak{t}'$: for $\mathfrak{t} \leq_{\mathfrak{K}} \mathfrak{t}'$ and $\mathbb{P}_1^{\mathfrak{t}'} = \mathbb{P}'$, we do it by defining $\mathbb{Q}_i^{\mathfrak{t}'}$ by induction on i as in the proof of [Sh:700, 4.8] and we choose $\underset{\sim}{\tau}^{\mathfrak{t}'_i}$ naturally. Let $\langle \underset{\sim}{n}_\rho : \rho \in {}^{\omega>}2\rangle$ be a $\mathbb{P}_0^{\mathfrak{t}'}$-name listing the members of $\underset{\sim}{A}$.

Now we choose $\mathfrak{t}$ such that $\mathfrak{t}' \leq_{\mathfrak{K}} \mathfrak{t}$ and for some $\mathbb{P}_0^{\mathfrak{t}}$-name $\underset{\sim}{\rho}$ of a member of ${}^\omega 2$ we have $\Vdash_{\mathbb{P}_\mathfrak{t}}$ "$\rho \neq \underset{\sim}{\nu}$" for any $\mathbb{P}_{\mathfrak{t}'}$-name (clearly exists, e.g. when $(\mathfrak{t}, \mathfrak{t}')$ is like $(\mathfrak{t}', \mathfrak{s})$ above, e.g. do as above with $\mathbb{P}'$ adding λ^+ such reals and reflect). Now $\underset{\sim}{A} := \{\underset{\sim}{n}_{\rho\restriction k} : k < \omega\}$ is forced to be an infinite subset of $\underset{\sim}{A}'$, and if it includes a member of $\mathscr{P}(\omega)^{\mathbf{V}[\mathbb{P}_\mathfrak{s}]}$ or even $\mathscr{P}(\omega)^{\mathbf{V}[\mathbb{P}_\mathfrak{t}]}$ we find that $\underset{\sim}{\rho}$ is from $({}^\omega 2)^{\mathbf{V}[\mathbb{P}_{\mathfrak{t}'}]}$, a contradiction.]

$(*)_1$ $\mu \in \mathrm{Sp}^*_{\pi\chi}$, in $\mathbf{V}^{\mathbb{P}}$, of course.

[Why? As there is a $\subseteq^*$-decreasing sequence $\langle B_\alpha : \alpha < \mu\rangle$ of sets which generates a (non-principle) ultrafilter. We can use B_α as the generic of $\mathbb{Q}^{\mathfrak{t}_\lambda} = \mathbb{P}^{\mathfrak{t}_{\lambda_{\alpha+1}}}/\mathbb{P}^{\mathfrak{t}_{\lambda_\alpha}}$.]

$(*)_2$ $\kappa_2 \notin \mathrm{Sp}^*_{\pi\chi}$.

[Why? Toward a contradiction assume $p^* \in \mathbb{P}$ and $p^* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{D}$ is a non-principal ultrafilter on ω and $\{\mathscr{U}_\varepsilon : \varepsilon < \kappa_2\}$ is a sequence of infinite subsets of ω which is a strict π-base of $\underset{\sim}{D}$"; so $p^* \Vdash_{\mathbb{P}}$ "$\{\mathscr{U}_\varepsilon : \varepsilon < \zeta\}$ is not a π-base of any ultrafilter on ω" for every $\zeta < \kappa_2$, hence for some $\langle \underset{\sim}{B}_{\zeta,l} : l < \underset{\sim}{n}_\zeta\rangle$ we have $p^* \Vdash$ "$\underset{\sim}{n}_l < \omega$ and $\langle \underset{\sim}{B}_{\zeta,l} : l < \underset{\sim}{n}_l\rangle$ is a partition of ω and $\varepsilon < \zeta \wedge l < \underset{\sim}{n}_\zeta \Rightarrow \mathscr{U}_\varepsilon \not\subseteq^* \underset{\sim}{B}_{\zeta,l}$". Now, as in the proof of 1.1, we choose suitable $\beta < \lambda$ and consider $\langle \underset{\sim}{B}^*_l : l < \underset{\sim}{n}\rangle = \hat{\mathbf{j}}_\beta^{-1}(\langle \underset{\sim}{B}_{\zeta,l} : l < \underset{\sim}{n}_\zeta\rangle : \zeta < \kappa_2\rangle/\mathscr{D}_2)$ so $p^* \Vdash_{\mathbb{P}^{\beta+1}}$ "$\langle \underset{\sim}{B}^*_l : l < \underset{\sim}{n}\rangle$ is a partition of ω into finitely many sets and $\varepsilon < \kappa_2 \wedge l < \underset{\sim}{n} \Rightarrow \mathscr{U}_\varepsilon \not\subseteq^* \underset{\sim}{B}^*_l$". But this contradicts $p^* \Vdash_{\mathbb{P}}$ "$\{\mathscr{U}_\varepsilon : \varepsilon < \kappa_2\}$ is a π-base".]

$(*)_3$ $\lambda \in \mathrm{Sp}^*_\pi$.

[Why? Clearly it is forced (i.e. $\Vdash_{\mathbb{P}_\lambda}$) that $\langle \underset{\sim}{A}_{\omega\alpha+2} : \alpha < \lambda\rangle$ is a $\subseteq^*$-decreasing sequence of infinite subsets of ω, hence there is an ultrafilter of D on ω including it. Now $\underset{\sim}{A}_{\omega\alpha+2}$ witness that $\mathscr{P}(\omega)^{\mathbf{V}[\mathbb{P}_{\mathfrak{t}_{\omega\alpha+2}}]}$ is not a π-base of $\underset{\sim}{D}$ (recalling clause (g) of $\circledast'_1$). As λ is regular, we are done.] $\blacksquare_{2.5}$

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