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A transversal of full outer measure

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ABSTRACT

We show that for every partition of a set of reals into countable sets there is a transversal of the same outer measure.

1. Introduction

Our aim is to prove the following.

Theorem 1.1. *Suppose $\langle X_\alpha : \alpha \in S\rangle$ is a partition of $X \subseteq [0,1]$ into countable sets. Then there exists $Y \subseteq X$ such that $|Y \cap X_\alpha| = 1$ for each $\alpha \in S$ and $\mu^\star(Y) = \mu^\star(X)$.*

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Here $\mu^\star$ denotes Lebesgue outer measure on $\mathbb{R}$. For partitions into finite sets, this follows from an old result of Lusin [9] which says that any set of reals can be partitioned into two sets of full outer measure (see Lemma 2.2). Another special case of the above theorem was established in [8]: Every set of reals has a subset of full outer measure that avoids rational distances. The proof given there relied on a theorem of Gitik and Shelah [4–6] which says that forcing with a sigma ideal cannot be isomorphic to a product of random and Cohen forcing (we give another proof of this in Theorem A.1). As a byproduct of our proof, we get a generalization of this theorem to a larger class of forcings (see Lemma 6.1) – For example, an ω-length finite support iteration of random forcing. For background on forcing and generic ultrapowers, we refer the reader to [2,7].

On notation: For a set of reals X, by $\text{env}(X)$ (envelope of X), we mean a G_δ set G containing X such that $G \setminus X$ has zero inner measure. All relations involving envelopes are supposed to hold modulo null sets. A subset Y of X has full outer measure in X if $\text{env}(X) = \text{env}(Y)$. If $Y \subseteq X$ and $\text{env}(X) \neq \text{env}(X \setminus Y)$ we say that Y has positive inner measure in X; otherwise, we say that Y has zero inner measure in X. For $T \subseteq {}^{<\omega}2$, define $[T] = \{x \in 2^\omega : (\forall n < \omega)(x \restriction n \in T)\}$. For $\sigma \in {}^{<\omega}2$, define $[\sigma] = \{x \in 2^\omega : \sigma \preceq x\}$. In forcing, we use the convention that a larger condition is the stronger one – So $p \leq q$ means q extends p. If $\mathbb{Q}, \mathbb{P}$ are forcing notions, we write $\mathbb{Q} \lessdot \mathbb{P}$ if $\mathbb{Q} \subseteq \mathbb{P}$ and every maximal antichain in $\mathbb{Q}$ is also a maximal antichain in $\mathbb{P}$. For an ideal $\mathcal{I}$ over a set X, define the following.

- $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$;
- $\mathsf{add}(\mathcal{I})$ is the least cardinal κ satisfying: there exists $\mathcal{F} \subseteq \mathcal{I}$, $|\mathcal{F}| \leq \kappa$ and $\bigcup \mathcal{F} \notin \mathcal{I}$;
- For $Y \in \mathcal{I}^+$, $\mathcal{I} \restriction Y = \{W \subseteq Y : W \in \mathcal{I}\}$ is the restriction of $\mathcal{I}$ to Y.

2. A sufficient condition

Without loss of generality, $(\forall \alpha \in S)(|X_\alpha| = \aleph_0)$. For each $\alpha \in S$, let $X_\alpha = \{x_{\alpha,n} : n < \omega\}$. Put $Y_n = \{x_{\alpha,n} : \alpha \in S\}$. For $W \subseteq S$, write $Y_n \restriction W = \{x_{\alpha,n} : \alpha \in W\}$. Note that Y_n depends on the specific enumeration of X_α we fixed.

Claim 2.1. *It is enough to show the following.*

($\star$): *For every $X \subseteq [0,1]$, for every partition $\langle X_\alpha : \alpha \in S\rangle$ of X into $\aleph_0$-sized subsets, for every enumeration $X_\alpha = \{x_{\alpha,n} : n < \omega\}$ (so we can speak of Y_n's w.r.t. this enumeration), there is a subset W of S such that either*

(a) *Y_0 is null or*
(b) *$Y_0 \restriction W$ has positive outer measure and for all $n \geq 1$, $Y_n \restriction W$ has zero inner measure in Y_n.*

Proof of Claim 2.1. Assume ($\star$). It is enough to show that we can strengthen "$Y_0 \restriction W$ has positive outer measure" to "$\mu^\star(Y_0 \restriction W) \geq 0.5(\mu^\star(Y_0))$" in ($\star$) above. For then we can inductively construct a sequence $\langle (W_i, n_i) : i < \omega\rangle$ such that

- the W_i's are pairwise disjoint subsets of S,
- each $n < \omega$ equals n_i for infinitely many i's,
- if $m = n_i = n_j$, $i < j$, then $\mathrm{env}(Y_m \restriction W_i) \cap \mathrm{env}(Y_m \restriction W_j) = \emptyset$,
- for each i, letting $m = n_i$ and $S_i = S \setminus \bigcup\{W_j : j < i\}$, we have $\mu^\star(Y_m \restriction W_i) \geq 0.5\mu^\star[(Y_m \restriction S_i) \setminus B_i]$, where $B_i = \mathrm{env}(\bigcup\{Y_m \restriction W_j : j < i, n_j = m\})$,
- if $n \neq n_i$, then $\mu^\star(Y_n \restriction (S_i \setminus W_i)) = \mu^\star(Y_n \restriction S_i)$.

Claim 2.1 will immediately follow by taking $Y = \bigcup\{Y_{n_i} \restriction W_i : i < \omega\}$.

Call a sequence $\langle W_i : i < \delta\rangle$ (where $\delta \leq \omega$) of pairwise disjoint subsets of S greedy, if for each $i < \delta$ the following "greediness condition" holds: $\mu^\star(Y_0 \restriction W_i) \geq 0.5(\sup\{\mu^\star(Y_0 \restriction W) : W \subseteq S_i, \mathrm{env}(Y_0 \restriction W) \cap \mathrm{env}(Y_0 \restriction S \setminus S_i) = 0, (\forall n \geq 1)(Y_n \restriction W$ has zero inner measure in $Y_n \restriction S_i)\})$ where $S_i = S \setminus \bigcup\{W_j : j < i\}$ and $Y_n \restriction W_i$ has zero inner measure in $Y_n \restriction S_i$ for every $n \geq 1$. Clearly such sequences exist by $(\star)$ so fix a maximal one and let $W = \bigcup\{W_i : i < \delta\}$. If possible, suppose $Y_0 \restriction W$ does not have full outer measure in Y_0. So $\delta = \omega$. Using $(\star)$, pick $W' \subseteq S \setminus W$ such that $Y_0 \restriction W'$ is non null, its envelope is disjoint with the envelope of $Y_0 \restriction W$ and for every $n \geq 1$, $\mu^\star(Y_n \restriction S \setminus (W \cup W')) = \mu^\star(Y_n \restriction S \setminus W)$. Since $\mu^\star(Y_0 \restriction W_i)$ goes to zero as $i \to \infty$, we conclude that at some stage $i < \omega$, our sequence ceased to be greedy: A contradiction. Hence $Y_0 \restriction W$ has full outer measure in Y_0 and by taking the union of sufficiently many W_i's we get the required strengthening of $(\star)$ above. □

Lemma 2.2. *Given $n \geq 1$, $A \subseteq [0, 1]$ and a partition $\langle\{x_{\alpha,k} : k < n\} : \alpha \in T\rangle$ of A into ordered sets of size n, there exists a partition $\langle T_k : k < n\rangle$ of T such that for every $k < n$, $\{x_{\alpha,k} : \alpha \in T_k\}$ has full outer measure in $\{x_{\alpha,k} : \alpha \in T\}$.*

Proof of Lemma 2.2. Use induction on n. If $n = 1$, there is nothing to show. So assume that the result holds for $n \geq 1$. Arguing as in the beginning of the proof of Claim 2.1, it is enough to show the following:

For every $A \subseteq [0, 1]$, for every partition $\langle\{x_{\alpha,k} : k < n+1\} : \alpha \in T\rangle$ of A into ordered sets of size $n + 1$, writing $Z_k = \{x_{\alpha,k} : \alpha \in T\}$, there exists $W \subseteq T$ such that

- $\mu^\star(Z_0 \restriction W) \geq 0.5\mu^\star(Z_0)$ (here $Z_k \restriction W = \{x_{\alpha,k} : \alpha \in W\}$),
- $Z_k \restriction W$ has zero inner measure in Z_k for every $1 \leq k \leq n$.

By inductive hypothesis, we can choose a partition $\langle T_k : 1 \leq k \leq n\rangle$ of T such that $Z_k \restriction T_k$ has full outer measure in Z_k. Using Lusin's result [9], for each $1 \leq k \leq n$ choose a partition $T_k = T_{k,0} \sqcup T_{k,1}$ such that each one of $Z_k \restriction T_{k,0}$ and $Z_k \restriction T_{k,1}$ has full outer measure in $Z_k \restriction T_k$. Define $W = \bigcup\{T_{k,i(k)} : 1 \leq k \leq n\}$ where $i(k) \in \{0, 1\}$ is such that $\mu^\star(Z_0 \restriction T_{k,i(k)}) \geq 0.5\mu^\star(Z_0 \restriction T_k)$. It is easily checked that W is as required. □

3. Forcing

Assume $(\star)$ fails. Fix a witnessing $X \subseteq [0,1]$, a partition $\langle X_\alpha : \alpha \in S\rangle$, enumerations $X_\alpha = \{x_{\alpha,n} : n \in \omega\}$ and the corresponding Y_n's.

For each $n \geq 1$, let $A_n \subseteq S$ be such that $Y_0 \restriction A_n$ is null and for every $A \subseteq S \setminus A_n$, if $Y_0 \restriction A$ is null, then $Y_n \restriction A \subseteq \mathrm{env}(Y_n \restriction A_n)$. Let $W_0 = \bigcup\{A_n : n \geq 1\}$. Let $C_n = \mathrm{env}(Y_n \restriction W_0)$, $B_n = \mathrm{env}(Y_n) \setminus C_n$. We can assume that $\mu(B_n) > 0$ for infinitely many n – Otherwise we can use Lemma 2.2 to get a contradiction to the failure of $(\star)$. By ignoring the Y_n's for which B_n is null, we can also assume that $\mu(B_n) > 0$ for every $n \geq 1$.

Replace S by $S \setminus W_0$. By modifying the Y_n's, we can assume that $\mathrm{env}(Y_n) = B_n$. This ensures the following: If $W \subseteq S$ is such that $Y_0 \restriction W$ is null, then for every $n \geq 1$, $Y_n \restriction W$ has zero inner measure in Y_n.

For $n \geq 1$, let $W_n \subseteq S$ be such that $Y_n \restriction W_n$ is null and if $W \subseteq S \setminus W_n$ is such that $Y_n \restriction W$ is null, then for all $m \geq n+1$, $Y_m \restriction W$ has zero inner measure in Y_m.

Let $\mathcal{I} = \{W \subseteq S : \mu(Y_0 \restriction W) = 0\}$. Then $\mathcal{I}$ is a sigma ideal on S. Let $\mathbb{P} = \mathcal{P}(S)/\mathcal{I}$ be forcing with $\mathcal{I}$. Observe that $Z \in \mathbb{P}$ iff $(\exists n \geq 1)(Y_n \restriction Z$ has positive inner measure in $Y_n)$. In fact, by Lemma 2.2, we can assume that $Z \in \mathbb{P}$ iff $(\exists^\infty n)(Y_n \restriction Z$ has positive inner measure in $Y_n)$. For $n \geq 1$, let $\mathbb{Q}_n = \{Z \subseteq S : (\exists B \subseteq B_n)$ (B is Borel positive and $Z = \{\alpha \in S \setminus W_n : x_{\alpha,n} \in B \cap Y_n\})\}$. Note that if $Z \in \mathbb{Q}_n$, then $Y_n \restriction Z$ has positive inner measure in Y_n and hence $Y_0 \restriction Z$ is non null. So $\mathbb{Q}_n \subseteq \mathbb{P}$ for every $n \geq 1$. Let $\mathbb{P}_n = \{Z \in \mathbb{P} : Z \cap W_n = 0\}$.

Lemma 3.1. *The following hold:*

(1) *For each $n \geq 1$, $\mathbb{Q}_n \lessdot \mathbb{P}_n$;*
(2) *$\bigcup\{\mathbb{Q}_n : n \geq 1\}$ is dense in $\mathbb{P}$;*
(3) *Each $\mathbb{Q}_n$ is isomorphic to random forcing;*
(4) *$\mathbb{P}$ adds a Cohen real.*

Proof of Lemma 3.1. (1) Suppose $\{Z_k : k < \theta \leq \omega\}$ is a maximal antichain in $\mathbb{Q}_n$ but not in $\mathbb{P}_n$. Let $Z \in \mathbb{P}_n$ be incompatible with Z_k for every $k < \theta$. So $Z \subseteq S \setminus W_n$, $Y_n \restriction Z$ is null and $Y_0 \restriction Z$ is non null. Choose $m > n$ such that $Y_m \restriction Z$ has positive inner measure in Y_m. But now $Z \subseteq S \setminus W_n$, $Y_n \restriction Z$ is null and $Y_m \restriction Z$ has positive inner measure in Y_m which is impossible by our choice of W_n.

(2) Let $Z \in \mathbb{P}$. Choose $n \geq 1$ such that $Y_n \restriction Z$ has positive inner measure in Y_n. Let $B \subseteq B_n$ be Borel positive such that $B \cap (Y_n \restriction (S \setminus Z)) = \emptyset$. Let $Z' = \{\alpha \in (Z \setminus W_n) : x_{\alpha,n} \in B\}$. Then $Z' \in \mathbb{Q}_n$ and $Z' \subseteq Z$.

(3) Should be clear.

(4) Construct a tree $\langle A_\sigma : \sigma \in {}^{<\omega}2\rangle$ of subsets of S such that the following hold.

(a) $A_\emptyset = S$, and for every $\sigma \in {}^{<\omega}2$,
(b) A_σ is a disjoint union of $A_{\sigma 0}$ and $A_{\sigma 1}$,
(c) for every $0 \leq k \leq |\sigma| + 1$, $\mu^\star(Y_k \restriction A_{\sigma 0}) = \mu^\star(Y_k \restriction A_{\sigma 1}) = \mu^\star(Y_k \restriction A_\sigma)$.

For Clause (c), we make use of Lemma 2.2 and Lusin's result [9]. We claim that if G is $\mathbb{P}$-generic over V, then in $V[G]$, the real $x = \bigcup\{\sigma \in {}^{<\omega}2 : A_\sigma \in G\}$ is Cohen over V. To see this, suppose that $D \subseteq {}^{<\omega}2$ is dense and $x \notin \bigcup\{[\sigma] : \sigma \in D\}$. Then $S \setminus \bigcup\{A_\sigma : \sigma \in D\} = Z \in G$ so that $Y_0 \upharpoonright Z$ is non null. But then, for some $n \geq 1$, $Y_n \upharpoonright Z$ has positive inner measure in Y_n. So Z meets A_σ for every σ extending some σ_0, $|\sigma_0| = n$, and hence also meets some condition in $\{A_\sigma : \sigma \in D\}$: A contradiction. □

Note that Lemma 3.1 implies that $\mathbb{P}$ satisfies ccc.

4. A dichotomy

Claim 4.1. *Let $\mathbb{Q} \lessdot \mathbb{P}$ be atomless. Then forcing with $\mathbb{Q}$ adds a new real.*

Proof of Claim 4.1. Since $\mathbb{Q}$ satisfies ccc (as $\mathbb{Q} \lessdot \mathbb{P}$ and $\mathbb{P}$ satisfies ccc), it is enough to show that every generic extension contains a new ω-sequence of members of V. Towards a contradiction, suppose $p \in \mathbb{Q}$ forces that no such sequence appears in the extension. Let α be the least ordinal such that for some $\nu \in V^{\mathbb{Q}}$ and $q \in \mathbb{Q}$, $q \geq p$ and $q \Vdash_{\mathbb{Q}} \nu : \alpha \to V \wedge \nu \notin V$. It is clear that α is regular uncountable. Choose $q \geq p$, $\nu \in V^{\mathbb{Q}}$ such that $q \Vdash_{\mathbb{Q}} \nu : \alpha \to V \wedge \nu \notin V$. For each $\beta < \alpha$, choose $\langle(q_{\beta,n}, \nu_{\beta,n}) : n < \omega\rangle$ such that $A_\beta = \{q_{\beta,n} : n < \omega\}$ is a maximal antichain in $\mathbb{Q}$ above q, $\nu_{\beta,n} \in V$ and $q_{\beta,n} \Vdash_{\mathbb{Q}} \nu \upharpoonright \beta = \nu_{\beta,n}$. Choose $N < \omega$ such that $W = \{\beta < \alpha : A_\beta \cap \mathbb{Q}_N \neq \emptyset\}$ has size α. For each $\beta \in W$, let $n_\beta < \omega$ be such that $q_{\beta,n_\beta} \in \mathbb{Q}_N$. Since $\mathbb{Q}_N$ is isomorphic to random forcing we can find $W' \in [W]^\alpha$ such that $\{q_{\beta,n_\beta} : \beta \in W'\}$ consists of pairwise compatible conditions. Put $\nu_\star = \bigcup\{\nu_{\beta,n_\beta} : \beta \in W'\}$. Since $\mathbb{Q}$ satisfies ccc, there exists $q' \geq q$ such that $q' \Vdash_{\mathbb{Q}} |G_{\mathbb{Q}} \cap \{q_{\beta,n_\beta} : \beta \in W'\}| = \alpha$. But now $q' \Vdash \nu = \nu_\star \in V$: A contradiction. □

Lemma 4.2. *Suppose ν is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \nu \in 2^\omega \setminus V$. Then for every $p \in \mathbb{P}$ there exist $q \geq p$ and a Borel function $B : 2^\omega \to \omega^\omega$ such that either $q \Vdash_{\mathbb{P}} B(\nu)$ is an infinitely often equal real over V or $q \Vdash_{\mathbb{P}} V[\nu]$ is a random real extension of V.*

Proof of Lemma 4.2. For simplicity, assume that p is the empty condition. For each $n \geq 1$, let $\mathring{T}_n \in V^{\mathbb{Q}_n}$ be such that for every $\mathbb{Q}_n$-generic G_n, $\mathring{T}_n[G_n] = \{\sigma \in {}^{<\omega}2 : (\forall p \in G_n)(\exists q \in \mathbb{P})(q \geq p \text{ and } q \Vdash_{\mathbb{P}} \sigma \prec \nu)\}$. Clearly, $\Vdash_{\mathbb{Q}_n} \mathring{T}_n$ is a leafless subtree of ${}^{<\omega}2$. We break our proof into two cases.

Claim 4.3. *Suppose for some $n \geq 1$ and $q \in \mathbb{Q}_n$, $q \Vdash_{\mathbb{Q}_n} \mathring{T}_n$ is not a perfect tree. Then there exists $p \in \mathbb{P}$, $p \geq q$ such that $p \Vdash_{\mathbb{P}} V[\nu]$ is a random real extension of V.*

Proof of Claim 4.3. Choose $q_1 \in \mathbb{Q}_n$, $q_1 \geq q$ and $\sigma \in {}^{<\omega}2$ such that $q_1 \Vdash_{\mathbb{Q}_n} \sigma \in \mathring{T}_n$ and $\mathring{T}_n$ has a unique branch above σ. Choose $p \in \mathbb{P}$, $p \geq q_1$ such that $p \Vdash_{\mathbb{P}} \sigma \prec \nu$. Let $G_{\mathbb{P}}$ be $\mathbb{P}$-generic over V with $p \in G_{\mathbb{P}}$. Put $G_{\mathbb{Q}_n} = G_{\mathbb{P}} \cap \mathbb{Q}_n$. Then, $\nu[G_{\mathbb{P}}] \in V[G_{\mathbb{Q}_n}]$ since it is the unique branch through $\mathring{T}_n[G_{\mathbb{Q}_n}]$ above σ. Since intermediate models in a random real extension are also random real extensions (as a complete subalgebra of a measure algebra is also a measure algebra), the claim follows. □

Claim 4.4. *Suppose for every $n \geq 1$, $\Vdash_{\mathbb{Q}_n} \mathring{T}_n$ is a perfect tree. Then for some Borel function B, $\Vdash_{\mathbb{P}} (\forall x \in {}^{\omega}\omega \cap V)(\exists^{\infty} l)(B(\nu)(l) = x(l))$.*

Proof of Claim 4.4. We will think of conditions in $\mathbb{Q}_n$ as the set of branches through a perfect subtree $S \subseteq {}^{<\omega}2$ such that for each $\sigma \in S$, $\mu([S] \cap [\sigma]) > 0$ – This is a dense set of conditions in $\mathbb{Q}_n$. Let us call such trees fat. Let $\langle (p_{n,k}, S_{n,k}, g_{n,k}, G_{n,k}, f_{n,k}) : k < \omega \rangle$ satisfy the following.

- $\{p_{n,k} : k < \omega\}$ is a maximal antichain in $\mathbb{Q}_n$;
- $S_{n,k}$ is a fat perfect subtree of ${}^{<\omega}2$ and $p_{n,k} = [S_{n,k}]$;
- $g_{n,k}$ is a function with domain $S_{n,k}$;
- For each $\sigma \in S_{n,k}$, $g_{n,k}(\sigma)$ is a finite subtree of ${}^{\leq h}2$ for some $h < \omega$ such that each terminal node of $g_{n,k}(\sigma)$ has length $h = h_{n,k}(|\sigma|)$ – So $h_{n,k}$ depends only on $|\sigma|$;
- Whenever $\sigma \preceq \tau$ are from $S_{n,k}$, $g_{n,k}(\sigma)$ is a subtree of $g_{n,k}(\tau)$;
- $G_{n,k}$ is a function from $p_{n,k}$ to perfect subtrees of ${}^{<\omega}2$;
- For every $x \in p_{n,k}$, the height of $g_{n,k}(x \restriction m)$ goes to infinity as m goes to infinity and the limiting tree is $G_{n,k}(x)$;
- $p_{n,k} \Vdash_{\mathbb{Q}_n}$ "$G_{n,k}(\mathring{r}_n) = \mathring{T}_n$ is a perfect tree" where $\mathring{r}_n$ is the $\mathbb{Q}_n$-name for the random real added by $\mathbb{Q}_n$;
- $f_{n,k} : \omega \to \omega$ is increasing such that if $\sigma \preceq \tau$ are from $S_{n,k}$ and $|\tau| \geq f_{n,k}(|\sigma|)$, then for every maximal node $\nu_1 \in g_{n,k}(\sigma)$ there are $\geq 2^{|\sigma|+3} + h_{n,k}(|\sigma|)$ maximal nodes $\nu_2 \succeq \nu_1$ in $g_{n,k}(\tau)$ – $f_{n,k}$ is well defined because for every $x \in [S_{n,k}]$, $G_{n,k}(x)$ is a perfect tree.

The existence of such objects follows from the following well known fact about random forcing.

Fact 4.5. *Let $\mathbb{B}$ denote the random forcing whose conditions are viewed as compact positive measure subsets of 2^{ω}. Let τ be a $\mathbb{B}$-name for a real in 2^{ω}. Then for every $p \in \mathbb{B}$, there exist a fat tree $S \subseteq 2^{<\omega}$ and a (uniformly) continuous function $G : [S] \to 2^{\omega}$ such that letting $q = [S]$, $q \geq p$ and $q \Vdash_{\mathbb{B}} G(\mathring{r}_{\mathbb{B}}) = \tau$ where $\mathring{r}_{\mathbb{B}}$ is the canonical name for the random real added by $\mathbb{B}$.*

Claim 4.6. *Given $k, m < \omega$ and $n \geq 1$ there exist $m', l = l_H$ and H where $m' > m$, $l < \omega$ and $H : {}^{l}2 \to \{0,1\}$ such that $(\star\star)$ holds: For every $\sigma \in S_{n,k} \cap {}^{m}2$, the set $\{\tau \in S_{n,k} \cap {}^{m'}2 : \tau \succeq \sigma$ and $(\forall \eta \in g_{n,k}(\sigma))(range(H \restriction \{\eta' \succeq \eta : \eta' \in {}^{l}2 \cap g_{n,k}(\tau)\} = \{0,1\})\}$ has size at least $|\{\tau \in S_{n,k} \cap {}^{m'}2 : \tau \succeq \sigma\}|(1 - 2^{-2^m})$.*

Proof of Claim 4.6. Fix $n \geq 1$ and $k, m < \omega$. Let $N_0 = \max\{|g_{n,k}(\sigma) \cap {}^{h_{n,k}(m)}2| : \sigma \in S_{n,k} \cap {}^{m}2\}$. Then $N_0 \leq 2^{h_{n,k}(m)}$. Put $m' = f_{n,k}(m)$. Let $l = h_{n,k}(m')$ be the common height of $g_{n,k}(\tau)$ for $\tau \in S_{n,k} \cap {}^{m'}2$. Let $p = \min\{|\{\eta' \succeq \eta : \eta' \in {}^{l}2 \cap g_{n,k}(\tau)\}| : \tau \in S_{n,k} \cap {}^{m'}2 \wedge \eta \in g_{n,k}(\tau \restriction m) \cap {}^{h_{n,k}(m)}2\}$. Note that $p \geq 2^{m+3} + h_{n,k}(m)$.

For each $\rho \in {}^{l}2$, we choose $H(\rho) \in \{0,1\}$ with equal probability – So for $A \subseteq \{H : H : {}^{l}2 \to \{0,1\}\}$, the probability $Pr(H \in A) = |A|2^{-2^{l}}$. For each $\sigma \in S_{n,k} \cap {}^{m}2$, $\tau \in S_{n,k} \cap {}^{m'}2$ with $\tau \succeq \sigma$, say that H satisfies (σ, τ) if for every $\eta \in g_{n,k}(\sigma)$, the range of $H \restriction \{\eta' \succeq \eta : \eta' \in {}^{l}2 \cap g_{n,k}(\tau)\}$ is $\{0,1\}$. Note that $Pr(H \text{ satisfies } (\sigma,\tau)) \geq 1 - N_0 2^{-p+1}$. It follows that for each $\sigma \in S_{n,k} \cap {}^{m}2$, the probability that $\left|\left\{\tau \succeq \sigma : \tau \in S_{n,k} \cap {}^{m'}2 \wedge H \text{ satisfies } (\sigma,\tau)\right\}\right| \geq \left(1 - 2^{-2^{m}}\right)\left|\left\{\tau \succeq \sigma : \tau \in S_{n,k} \cap {}^{m'}2\right\}\right|$ is at least $1 - N_0 2^{-p+1} 2^{2^{m}}$. So for an appropriate H to exist, it suffices to have $2^{m} N_0 2^{-p+1} 2^{2^{m}} < 1$. But this is clear since $p \geq h_{n,k}(m) + 2^{m+3} > \log(N_0) + 2^{m} + m + 1$. □

For each $n \geq 1, k < \omega$, choose $\langle (m_{n,k,j}, H_{n,k,j}, s_{n,k,j}) = (m_j, H_j, s_j) : j < \omega \rangle$ satisfying the following.

- The m_j's are increasing with j;
- For each j, $(\star\star)$ of Claim 4.6 holds with $m = m_j$, $m' = m_{j+1}$ and $H = H_{j+1}$;
- $\langle s_j : j < \omega \rangle$ lists ω infinitely often.

Define a Borel function $B : 2^{\omega} \to \omega^{\omega}$ as follows: For each $\eta \in 2^{\omega}$, $l < \omega$, letting $(n,k,j) = l$ (so l codes a triplet via some bijection between ${}^{3}\omega$ and ω) if there exists some $j_1 > j$ such that $H_{n,k,j_1}(\eta \restriction l_{H_{n,k,j_1}}) = 1$, then pick the least such j_1 and define $B(\eta)(l) = s_{n,k,j_1}$. If there is no such j_1, put $B(\eta)(l) = 0$.

Claim 4.7. *For every $p \in \mathbb{P}$, $x \in \omega^{\omega}$, $l < \omega$, there exist $q \geq p$ and $l_1 > l$ such that $q \Vdash_{\mathbb{P}} B(\nu)(l_1) = x(l_1)$.*

Proof of Claim 4.7. Choose $n \geq 1, k < \omega$, $q \geq p$ such that $q \in \mathbb{Q}_n$ and $q, p_{n,k}$ are compatible. Let $S \subseteq {}^{<\omega}2$ be a fat perfect subtree such that $[S] \subseteq q \cap p_{n,k}$. Choose $j < \omega$ sufficiently large and $\sigma \in S$ such that, letting $m = m_{n,k,j}$, we have $\prod_{i \geq 0}(1 - 2^{-2^{m+i}}) > 0.5$, $(n,k,j) = l_1 > l$, $\sigma \in {}^{m}2$ and $[S]$ has more than 99 percent measure in $[\sigma]$. Choose $j_1 > j$ such that $s_{n,k,j_1} = x(l_1)$. Now using $(\star\star)$, we can find $\tau \in S$, $\tau \succeq \sigma$, $\tau : m_{n,k,j_1} \to 2$, $\eta \in g_{n,k}(\tau) \cap 2^{l_{H_{n,k,j_1}}}$ such that for each $j < j' < j_1$, $H_{n,k,j'}(\eta \restriction l_{H_{n,k,j'}}) = 0$ and $H_{n,k,j_1}(\eta) = 1$. Choose $q_1 \in \mathbb{P}$ such that $q_1 \geq [S] \cap [\tau]$ and $q_1 \Vdash_{\mathbb{P}} \eta \preceq \nu$. It follows that for each $j < j' < j_1$, $q_1 \Vdash_{\mathbb{P}} H_{n,k,j'}(\nu \restriction l_{H_{n,k,j'}}) = 0$ and $q_1 \Vdash_{\mathbb{P}} H_{n,k,j_1}(\nu \restriction l_{H_{n,k,j_1}}) = 1$. Hence $q_1 \Vdash_{\mathbb{P}} B(\nu)(l_1) = s_{n,k,j_1} = x(l_1)$. □

This finishes the proof of Claim 4.4 and Lemma 4.2.

5. A non meager set in $V^{\mathbb{P}}$

Recall that $\mathsf{non}(\mathsf{Meager})$ denotes the least cardinality of a non meager set of reals.

Lemma 5.1. *Suppose* $V \models \mathsf{non}(\mathsf{Meager}) \leq \kappa$. *Then* $V^{\mathbb{P}} \models \mathsf{non}(\mathsf{Meager}) \leq \kappa$.

Proof of Lemma 5.1. Let $\mathbf{A}$ be the collection of all quadruples $\mathbf{s} = (\mathbf{m}, \mathbf{n}, \mathbf{k}, \mathbf{h})$ where $\mathbf{m} = \langle m_i : i < \omega \rangle$, $\mathbf{n} = \langle n_i : i < \omega \rangle$ and $\mathbf{k} = \langle k_i : i < \omega \rangle$ are sequences in ω, $m_0 = 0$, the m_i's are strictly increasing and $\mathbf{h} = \langle h_i : i < \omega \rangle$ where $h_i : {}^{k_i}2 \to {}^{[m_i, m_{i+1})}2$. Let $\{x_\alpha : \alpha < \kappa\}$ be non meager in $\mathbf{A}$ where the topology over $\mathbf{A}$ is generated by declaring finite restrictions of members of $\mathbf{A}$ clopen. For $\alpha < \kappa$, define $\mathring{y}_\alpha \in 2^\omega \cap V^{\mathbb{P}}$ by $\mathring{y}_\alpha \upharpoonright [m_{\alpha,i}, m_{\alpha,i+1}) = h_{\alpha,i}(\mathring{r}_{n_{\alpha,i}} \upharpoonright k_{\alpha,i})$ where $m_{\alpha,i}, n_{\alpha,i}, k_{\alpha,i}, h_{\alpha,i}$ correspond to x_α and $\mathring{r}_j$ is the random real added by $\mathbb{Q}_j$. It suffices to show the following.

Claim 5.2. $\Vdash_{\mathbb{P}} \mathring{Y} = \{\mathring{y}_\alpha : \alpha < \kappa\}$ *is non meager.*

Proof of Claim 5.2. Suppose not. Then we can find $p_\star \in \mathbb{P}$, $\mathring{T} \in V^{\mathbb{P}}$, $W \subseteq \kappa$, $n_\star \geq 1$ such that, $p_\star \Vdash_{\mathbb{P}} \mathring{T}$ is a nowhere dense leafless subtree of ${}^{<\omega}2$, $\{x_\alpha : \alpha \in W\}$ is non meager and for every $\alpha \in W$, $\exists q_\alpha \geq p_\star$ such that $q_\alpha \in \mathbb{Q}_{n_\star}$ and $q_\alpha \Vdash_{\mathbb{P}} \mathring{y}_\alpha \in [\mathring{T}]$. Let $\mathring{T}_{n_\star} \in V^{\mathbb{Q}_{n_\star}}$ be such that $\mathring{T}_{n_\star}[G_{\mathbb{Q}_{n_\star}}] = \{\sigma \in {}^{<\omega}2 : (\nexists p \in \mathbb{P})(p \geq p_\star,$ p is compatible with every condition in $G_{\mathbb{Q}_{n_\star}}$ and $p \Vdash_{\mathbb{P}} \sigma \notin \mathring{T})\}$. Note that for every $q \in \mathbb{Q}_{n_\star}$, if $q \geq p_\star$, then $q \Vdash_{\mathbb{Q}_{n_\star}} [\mathring{T}_{n_\star}]$ is nowhere dense. We need the following

Claim 5.3. *There exist* $q_\star \in \mathbb{Q}_{n_\star}$, $S_\star \subseteq 2^{<\omega}$, $W_\star \subseteq W$, H *such that the following hold.*

- $S_\star$ *is a fat perfect tree and* $[S_\star] = q_\star \geq p_\star$;
- H *is a function with domain* $S_\star$;
- *For each* $\sigma \in S_\star$, $H(\sigma) = (H_0(\sigma), H_1(\sigma)) = (m, t)$ *where* $m < \omega$ *and* $t \subseteq {}^{<m}2$ *is a subtree;*
- *If* $\sigma \preceq \tau$ *are from* $S_\star$, *then* $H_0(\sigma) \leq H_0(\tau)$ *and* $H_1(\sigma) = H_1(\tau) \cap {}^{<H_0(\sigma)}2$;
- *For every* $k < \omega$, *there exists* $k' < \omega$ *such that for every* $\sigma \in S_\star \cap {}^{k'}2$, $H_0(\sigma) \geq k$;
- *For every* $r \in [S_\star]$, $\bigcup\{H_1(r \upharpoonright k) : k < \omega\}$ *is nowhere dense in* ${}^{<\omega}2$;
- $q_\star \Vdash_{\mathbb{Q}_{n_\star}} \mathring{T}_{n_\star} = \bigcup\{H_1(\mathring{r}_{n_\star} \upharpoonright k) : k < \omega\}$;
- $\{x_\alpha : \alpha \in W_\star\}$ *is non meager;*
- *For every* $\alpha \in W_\star$, *there exists* $q_\alpha \in \mathbb{Q}_{n_\star}$ *such that* $q_\alpha \geq q_\star$ *and* $q_\alpha \Vdash_{\mathbb{P}} \mathring{y}_\alpha \in [\mathring{T}]$.

Proof of Claim 5.3. First using Fact 4.5, choose a maximal antichain $\{q_{i,\star} : i < \omega\}$ of conditions $q_{i,\star} \in \mathbb{Q}_{n_\star}$ such that for each $i < \omega$, $q_{i,\star} \geq p_\star$ and there are $S_{i,\star}$ and H for this $q_{i,\star}$ satisfying all but the last two clauses about $W_\star$. Then note that $W = \bigcup_{i<\omega} W_{i,\star}$ where $W_{i,\star} = \{\alpha \in W : (\exists q_\alpha \in \mathbb{Q}_{n_\star})(q_\alpha \geq q_{i,\star} \wedge q_\alpha \Vdash_{\mathbb{P}} \mathring{y}_\alpha \in [\mathring{T}])\}$. Choose $i_\star < \omega$, such that $\{x_\alpha : \alpha \in W_{i_\star,\star}\}$ is non meager and take $q_\star = q_{i_\star,\star}$, $W_\star = W_{i_\star,\star}$ and the corresponding $S_\star$ and H. $\square$

Claim 5.4. *$C = \{x \in \mathbf{A} : (\exists i < \omega)(\forall \sigma \in S_\star \cap 2^{k_{x,i}})(\forall \rho \in {}^{m_{x,i}}2)(n_{x,i} = n_\star \wedge H_0(\sigma) \geq m_{x,i+1} \wedge \rho \cup h_{x,i}(\sigma) \notin H_1(\sigma))\}$ is open dense in $\mathbf{A}$. Here $m_{x,i}, n_{x,i}, k_{x,i}, h_{x,i}$ correspond to $x \in \mathbf{A}$.*

Proof of Claim 5.4. Let $(\mathbf{m}, \mathbf{n}, \mathbf{l}, \mathbf{h})$ be the restriction of some member of $\mathbf{A}$ to some $i < \omega$. Extend $(\mathbf{m}, \mathbf{n}, \mathbf{l}, \mathbf{h})$ to a member $x \in \mathbf{A}$ as follows. Put $n_{x,i} = n_\star$. Choose $m_{x,i} > m_{i-1}$. For each $r \in [S_\star]$, let $\tau_r \in 2^{<\omega}$, $k_r < \omega$ be such that $(\forall \rho \in {}^{m_{x,i}}2)$, $[\rho^\frown \tau_r] \cap [H_1(r)] = \emptyset$ and $H_0(r \restriction k_r) > m_{x,i} + |\tau_r|$ where $H_1(r) = \bigcup\{H_1(r \restriction k) : k < \omega\}$. Choose $F \subseteq [S_\star]$ finite such that $\{[r \restriction k_r] : r \in F\}$ covers $[S_\star]$. Choose $m_{x,i+1} > \max\{H_0(r \restriction k_r) : r \in F\}$. Choose $k_{x,i}$ such that for every $r \in F$, $k_{x,i} > k_r$ and for every $\sigma \in S_\star \cap {}^{k_{x,i}}2$, $H_0(\sigma) \geq m_{x,i+1}$. For $\sigma \in S_\star \cap {}^{k_{x,i}}2$, choose $h_{x,i}(\sigma) \in {}^{[m_{x,i}, m_{x,i+1})}2$ such that for some $r \in F$, $r \restriction k_r = \sigma \restriction k_r$ and for all $t < |\tau_r|$, $h_{x,i}(\sigma)(m_{x,i} + t) = \tau_r(t)$. Extend x arbitrarily to a member of $\mathbf{A}$. It is easily checked that $x \in C$. $\square$

Now let $\alpha \in W_\star$ be such that $x_\alpha \in C$ as witnessed by $i < \omega$. Choose $q_\alpha \in \mathbb{Q}_{n_\star}$ such that $q_\alpha \geq q_\star$ and $q_\alpha \Vdash_{\mathbb{P}} \mathring{y}_\alpha \in [\mathring{T}]$. Let $\sigma \in {}^{k_{\alpha,i}}2$ be such that $q = q_\alpha \cap [\sigma]$ has positive measure. Notice that $q \Vdash_{\mathbb{P}} \mathring{y}_\alpha \restriction m_{\alpha,i+1} \notin \mathring{T}_{n_\star}$ hence for some $q_1 \in \mathbb{P}$, $q_1 \geq q \geq q_\alpha$ and $q_1 \Vdash_{\mathbb{P}} \mathring{y}_\alpha \notin [\mathring{T}]$: A contradiction. $\square$

6. Contradiction

The following lemma contradicts Lemma 3.1 and thus finishes the proof of Theorem 1.1.

Lemma 6.1. *The following is impossible.*

(1) *$\mathcal{I}$ is an ideal over S, $add(\mathcal{I} \restriction A) = \kappa > \aleph_0$ for every $A \in \mathcal{I}^+$;*
(2) *$\langle(\mathbb{Q}_n, p_n) : n < \omega\rangle$ satisfies the following: For every $n < \omega$, $p_n \in \mathbb{P}$, $\mathbb{Q}_n \lessdot \mathbb{P}_{\geq p_n}$ where $\mathbb{P}_{\geq p_n} = \{p \in \mathbb{P} : p \geq p_n\}$, $\mathbb{Q}_n$ is forcing isomorphic to random forcing and $\bigcup\{\mathbb{Q}_n : n < \omega\}$ is dense in $\mathbb{P}$;*
(3) *$\mathbb{P}$ is forcing isomorphic to $\mathcal{P}(S)/\mathcal{I}$;*
(4) *Forcing with $\mathbb{P}$ adds a Cohen real.*

Proof of Lemma 6.1. Let G be $\mathcal{P}(S)/\mathcal{I}$-generic over V and let $j : V \to N \subseteq V[G]$ be the generic ultrapower embedding with critical point κ. Let $f : S \to \kappa$ represent κ. Define $\mathcal{J} = \{A \subseteq \kappa : f^{-1}[A] \in \mathcal{I}\}$. Let $\mathbb{Q} = \mathcal{P}(\kappa)/\mathcal{J}$. Then $\mathbb{Q} \lessdot \mathbb{P}$ – So $\mathbb{Q}$ adds a new real by Claim 4.1. Let H be $\mathbb{Q}$-generic over V and let $k : V \to M \subseteq V[H]$ be the generic ultrapower embedding. Note that ${}^\kappa N \cap V[G] \subseteq N$ and ${}^\kappa M \cap V[H] \subseteq M$ (see Proposition 2.14, [2]) and we'll use this freely below. We can divide our proof into two cases.

Case 1: There is a real $r \in {}^\omega\omega \cap M$ such that r is infinitely often equal to every real in V. Let $\langle r_\alpha : \alpha < \kappa\rangle$ represent r. Then for every $x \in {}^\omega\omega \cap V$, there exists $\alpha < \kappa$ such that r_α and x agree infinitely often. It follows that there is a non meager set of size κ in V. Since $\mathbb{P}$ adds a random real, $V \cap 2^\omega$ is meager in $V[G]$. Let B be a meager F_σ-set coded in $V[G]$ that contains $V \cap 2^\omega$. Then B is also coded in N. So by elementarity of j, it follows that every set of reals in V of size $< \kappa$ is meager in V. So $V \models \text{non(Meager)} = \kappa$. Hence $N \models \text{non(Meager)} = j(\kappa) > \kappa$. By Lemma 5.1, $V[G]$ and hence N has a non meager set of size κ: A contradiction.

Case 2: No real in M is infinitely often equal to every real in V. By Lemma 4.2, for every new real $r \in M$, $V[r]$ is a random real extension of V. Using Claim 4.1, choose $r \in V[H]$ random over V. Then $r \in M$. Let $\langle r_\alpha : \alpha < \kappa\rangle$ represent r. Then $\{r_\alpha : \alpha < \kappa\}$ is a non null set in V. Since $\mathbb{P}$ adds a Cohen real, $V \cap 2^\omega$ is null in $V[G]$. Let B be a null G_δ-set coded in $V[G]$ that contains $V \cap 2^\omega$. Then B is also coded in N. So by elementarity of j, it follows that every set of reals in V of size $< \kappa$ is null in V. In particular, for every $\gamma < \kappa$, $\{r_\alpha : \alpha < \gamma\}$ is null in V. By considering $k(\langle r_\alpha : \alpha < \kappa\rangle)$, it follows that $\{r_\alpha : \alpha < \kappa\}$ is null in M. Let r' be a null Borel set coded in M witnessing this. It follows that $V[r']$ is not a random real extension of V: A contradiction. □

Appendix A

Let Random_θ (resp. Cohen_θ) denote the forcing for adding θ random (resp. Cohen) reals. Random_θ is the measure algebra on the product measure space $[0,1]^\theta$ (with Lebesgue measure on $[0,1]$) and Cohen_θ is the forcing for adding a function in ${}^{\omega\times\theta}2$ using finite approximations. Gitik and Shelah [4–6] proved that forcing with a sigma ideal cannot be isomorphic to $\mathsf{Random}_1 \times \mathsf{Cohen}_1$. In [3] (Section 546), Fremlin asked if the result generalizes to a product that adds more of these reals. We show that this is the case.

Theorem A.1. *Let θ_1, θ_2 be positive cardinals. Let $\mathcal{I}$ be a sigma ideal over X. Then forcing with $\mathcal{P}(X)/\mathcal{I}$ is not isomorphic to* $\mathbb{P} = \mathsf{Random}_{\theta_1} \times \mathsf{Cohen}_{\theta_2}$.

Proof of Theorem A.1. Suppose not. By restricting $\mathcal{I}$, we can assume $\text{add}(\mathcal{I} \restriction A) = \kappa$ for every $A \in \mathcal{I}^+$. Note that since $\mathbb{P}$ is ccc, $\kappa > \aleph_1$. Let G be $\mathbb{P}$-generic over V and let $j : V \to M \subseteq V[G]$ be the generic ultrapower embedding with critical point κ.

The following is well known [1].

Fact A.2. $V^{\mathsf{Cohen}_1} \models \mathsf{cov(Null)} = \aleph_1$. *Here,* $\mathsf{cov(Null)}$ *is the least cardinality of a family of null sets whose union covers* $\mathbb{R}$.

So $V[G]$ and therefore M (as ${}^\kappa M \cap V[G] \subseteq M$) thinks that $\mathsf{cov(Null)} = \aleph_1$. Hence also $V \models \mathsf{cov(Null)} = \aleph_1$. Let $\langle N_i : i < \omega_1\rangle$ be a sequence of null G_δ-sets witnessing this in V. By elementarity of j, $M \models \bigcup\{N_i : i < \omega_1\} = \mathbb{R}$. But this is impossible as $V[G]$ and therefore M contains a random real over V which cannot belong to any of the N_i's. □

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