[M2] Megibben, C., "Crawley's problem on the unique ω-elongation of p-groups is undecidable," *Pacific J. Math.*, **107**(1983), 205-212.

[S] Shoenfield, J., "Martin's axiom," *Amer. Math. Monthly* **82**(1975), 610-617.

When κ-Free Implies Strongly κ-Free

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Abstract: For many cardinals κ that there are κ-free not strongly κ-free groups of cardinality κ. In particular, in L κ-free implies strongly κ-free iff κ is a limit cardinal or the successor of a cardinal of uncountable weakly compact cofinality. However it is consistent (assuming the consistency of the existence of a large cardinal) that there is some κ such that $cf(\kappa) = \omega$, κ^+-free implies strongly κ^+-free, but κ^+-free does not imply κ^{++}-free. It is also shown if every subgroup of cardinality κ is strongly κ-free then the group itself is strongly κ-free.

0. INTRODUCTION

One afternoon in the summer of 1983, Mekler asked Shelah some questions about almost free abelian groups. This paper records Shelah's answers together with some results due to Mekler. We would like to thank Paul Eklof for preserving notes about this conversation and for prodding Mekler to

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write this paper.

Henceforth "group" will mean "abelian group" and κ and μ will denote infinite cardinals. For uncountable κ, a group is κ-free if every subgroup of cardinality $< \kappa$ is free. Amongst the κ-free groups an important subclass are those which are strongly κ-free. Here a κ-free group A is said to be strongly κ-free if for all $X \subseteq A$ with $|X| < \kappa$, there is $X \subseteq B \subseteq A$ such that $|B| < \kappa$ and A/B is κ-free. Much of the study of κ-free groups has concentrated on strongly κ-free groups. The question was "for which κ, does κ-free imply strongly κ-free". We summarise some of the previously known information in the following theorem.

Theorem 0.1

(1) [M] If A is κ-free and $\mu < \kappa$, then A is strongly μ-free.

(2) [M] If κ is a limit cardinal, then κ-free implies strongly κ-free.

(3) [S1] If κ is a singular cardinal, then κ-free implies κ^+-free.

(4) [M] If κ is weakly compact, then κ-free implies κ^+-free.

(5) [M] Suppose $\kappa = \mu^+$, μ is regular and μ-free implies μ^+-free. Then κ-free implies strongly κ-free.

(In section 2, we will improve on (5))

(6) ([E1])(V=L). If κ is regular and not weakly compact, then strongly κ-free does not imply κ^+-free.

(Here as elsewhere we shall state theorems as consequences of the assumption that (V=L) without detailing which combinatorial principles are used.)

A related question to the one considered in this paper is: "if there is a κ-free group which is not κ^+-free,

then is there a strongly κ-free which is not κ^+-free? The answer to this question is yes ([S2]).

1. CONSTRUCTING κ-FREE GROUPS WHICH ARE NOT STRONGLY κ-FREE

Our approach follows that of [M]. We define a constructio[n] principle which in the presence of a suitable set theoreti[c] hypotheses enables us to construct the desired groups. Th[en] we will investigate when the set theoretic hypothesis hold[s].

Definition. Let $F(\kappa)$ abbreviate the following statement:

> there is a free group F on κ generators and $K \subseteq F$ so that F/K is not free but if K' is any direct summand of K generated by a set of size $< \kappa$, then F/K' is free.

(This is essentially definition 2.6 of [M].)

Lemma 1.1 (cf. [M] or [E1]) 1. $F(\omega)$ is true.

2. If κ-free does not imply κ^+-free, then $F(\kappa)$ is tru[e].

Corollary 1.2 (V=L) If κ is regular and not weakly compact, then $F(\kappa)$ is true.

Definition. Let $*(\kappa,\mu)$ denote the following statement:

There exists $\{S_i: i < \kappa^+\}$ a family of subsets of [κ] each of cardinality μ such that for each $I \subseteq \kappa^+$ with $|I| = \kappa$ there is $\{S_i^*: i \in I\}$ where $S_i^* \subseteq S_i$, $|S_i \setminus S_i^*| < \mu$ and $S_i^* \cap S_j^* = 0$ if $i \neq j$.

Theorem 1.2. Suppose $*(\kappa,\mu)$ and $F(\mu)$ hold. Then the[re] is a κ^+-free group which is not strongly κ^+-free.

Proof. (We will just sketch the proof. For more details [the] reader can consult Theorem 3.6 of [M].) Let $\{S_i: i < \kappa^+\}$ witness $*(\kappa,\mu)$ and enumerate each S_i as $\{a_{i,\alpha}: \alpha < \mu\}$. Choose $F_i \supseteq K_i$ $(i < \kappa^+)$ witnessing $F(\mu)$. We can assum[e] the F_i's are disjoint and $\{k_{i,\alpha}: \alpha < \mu\}$ freely genera[te]

K_i. Then the desired group, A, is $(\bigoplus_{i<\kappa^+} F_i)/N$, where N is the subgroup generated by $\{k_{i,\alpha} - k_{j,\beta} : i,j < \kappa^+, \alpha,\beta < \mu$ and $a_{i,\alpha} = a_{j,\beta}\}$. It is not hard to see A is κ^+-free.

To see that A is not strongly κ^+-free, let $B = (\bigoplus_{i<\kappa^+} K_i)/N$. Then $|B| = \kappa$ and $A/B \cong \bigoplus_{i<\kappa^+}(F_i/K_i)$.

Note: there are groups which are ω_1-free but not strongly ω_1-free which cannot be constructed as above. Perhaps the most interesting of these are what are called Shelah groups in [E2]. These are ω_1-free not strongly ω_1-free groups, A, with the following property: If $B \subseteq A$ and $|B| \leq \omega$ there is $C \supseteq B$ such that $|C| = \omega$ and whenever $D \cap C = B$, D/B is ω_1-free.

It remains to discover when $*(\kappa,\mu)$ holds.

Lemma 1.3. (1) If κ is regular then $*(\kappa,\kappa)$ holds.
(2) (Litman, Shelah [BD]). If κ is singular and $\square_\kappa$ holds (in particular if V=L), then $*(\kappa, cf(\kappa))$ holds. (Here $cf(\kappa)$ denotes the cofinality of κ.)

Proof (1). First note that it suffices to find $\{S_i : i < \kappa^+\}$ subsets of κ each of cardinality κ such that for all $i \neq j$ $|S_i \setminus S_j| < \kappa$. Suppose $\{S_i : i < \kappa^+\}$ is as above and $I \subseteq \kappa^+$, where $|I| = \kappa$. Let $\prec$ order I with order type κ. Then set $S_i^* = S_i \setminus \bigcup_{j \prec i} S_j$.

It is easy to define by induction on $i < \kappa^+$ functions $f_i : \kappa \to \kappa$ so that for $j < i$ there is an ordinal $\alpha < \kappa$ such that for $\nu > \alpha$, $f_i(\nu) > f_j(\nu)$. If we view each f_i as a subset of $\kappa\times\kappa$, then for $i\neq j$ $|f_i \cap f_j| < \kappa$. Since $|\kappa\times\kappa| = \kappa$, we are done.

Corollary 1.4 (V=L) If κ-free implies strongly κ-free, then either κ is a limit cardinal or κ is the successor of a cardinal of uncountable weakly compact cofinality.

2. WHEN κ-FREE IMPLIES STRONGLY κ-FREE.

In this section we will improve on Theorem 0.1.5. First we recall a result from [M].

Lemma 2.1. ([M] claim from Theorem 1.13). Suppose μ is a regular cardinal and μ-free implies μ^+-free. If $F \supseteq K = \bigcup_{\alpha<\mu} K_\alpha$, F is free and F/K_α is free for all α, then F/K is free. Here we assume $\beta < \alpha$ implies $K_\beta \subseteq K_\alpha$, but we do not assume the chain is smooth. In [M] this claim is stated with the hypothesis that μ is weakly compact. However in the proof all that is used is that μ is regular and μ-free implies μ^+-free.)

Theorem 2.2. Suppose $\kappa=\mu^+$ and $cf(\mu)$-free implies $cf(\mu)^+$-free. Then κ-free implies strongly κ-free.

Proof. Suppose A is κ-free and $X \subseteq A$ with $|X| = \mu$. Let $\gamma = cf(\mu)$ and write $X = \bigcup_{\alpha<\gamma} X_\alpha$ where $|X_\alpha| < \mu$. We can now choose B_α $(\alpha < \gamma)$ so that for all $\alpha,\beta < \gamma$: if $\beta < \alpha$, $B_\beta \subseteq B_\alpha$; $X_\alpha \subseteq B_\alpha$; $|B_\alpha| < \mu$; and if $B_\alpha \subseteq F \subseteq A$ and F is free, then F/B_α is free. [cf. [M], Lemma 1.9]. Let $B = \bigcup B_\alpha$. By Lemma 2.1, if $B \subseteq F \subseteq A$ and F is free, then F/B is free. Since A is κ-free, A/B is κ-free.

Corollary 2.3 (V=L) An uncountable cardinal κ is a limit cardinal or the successor of a cardinal of weakly compact cofinality iff κ-free implies strongly κ-free.

If there are no inaccessible cardinals in L, then κ-free implies strongly κ-free iff κ is singular iff κ-free implies κ^+-free. These considerations are meant to justify the use of large cardinals in the next section.

3. WHEN κ-FREE MAY IMPLY STRONGLY κ-FREE.

In this section we show, assuming the consistency of certain large cardinals, that: there may exist a cardinal κ so that κ^+-free implies κ^{++}-free but κ^{++}-free does not imply κ^{+++}-free; there may exist a cardinal κ so that $cf(\kappa) = \omega$, κ^+-free implies strongly κ^+-free but κ^+-free does not imply κ^{++}-free. We first recall some definitions and results.

Definition. Suppose κ is a regular cardinal. Then $E(\kappa)$ holds if there is a stationary set $E \subseteq \kappa$ such that if $\nu \in E$ then $cf(\nu) = \omega$ and for all $\nu < \kappa$ $\nu \cap E$ is not stationary in ν.

Theorem 3.1 (1) ([El]) Suppose κ is regular and $E(\kappa)$ holds. Then there is a strongly κ-free group of cardinality κ which is not free.
(2) ([M]) Suppose κ is strongly compact, then κ-free implies free.

If we have a regular cardinal κ and we wish to arrange for $E(\kappa)$ to hold in generic extension we can force with the following poset $\mathbf{P}$: $\mathbf{P} = \{S$: there is $\alpha < \kappa$ such that $S: \alpha \to 2$, if $S(\nu) = 0$ then $cf(\nu) = \omega$, and $\{\nu : S(\nu) = 0\}$ is not stationary in any $\gamma \leq \alpha\}$ and $\mathbf{P}$ is ordered by inclusion. [cf [J] p. 255 Exercise 24.13). Suppose G is $\mathbf{P}$-generic. It is not hard to show: in $V[G]$ there are no new subsets of κ of cardinality $< \kappa$; and $\bigcup G$ witnesses $E(\kappa)$ in $V[G]$

Theorem 3.2. Assume ZFC and there exists a strongly compact cardinal is consistent. Then it is consistent that for some cardinal κ, κ^+-free implies κ^{++}-free but κ^{++}-free does not imply κ^{+++}-free. Also κ^{++}-free implies strongly κ^{++}-free.

Proof The last statement is a consequence of Theorem 2.2. Now assume μ is a strongly compact cardinal and $\kappa > \mu$. Let $\mathbf{P}$ be as above for κ^{++} and assume G is $\mathbf{P}$-generic. Then in $V[G]$, there is a κ^{++}-free group which is not κ^{+++}-free. However since there are no new (up to isomorphism) groups of cardinality κ^+ added to V, κ^+-free implies κ^{++}-free.

If we wish to deal with successors of singular cardinals, we need a more complicated argument.

Theorem 3.3. Assume it is consistent with ZFC that there exists a supercompact cardinal. Then it is consistent that there is a cardinal κ so that $cf(\kappa) = \omega$, κ^+-free implies strongly κ^+-free, but κ^+-free does not imply κ^{++}-free.

Proof. By [L] we can assume μ is a supercompact cardinal (and so also strongly compact) and μ remains supercompact in any forcing extension by a μ-directed closed poset. Choose $\kappa > \mu$ such that $cf(\kappa) = \omega$. Let $\mathbf{P}$ be the poset which ensures $E(\kappa^+)$ holds and suppose G is $\mathbf{P}$-generic. As above in $V[G]$ κ^+-free does not imply κ^{++}-free.

Suppose now that $A \in V[G]$ and A is κ^+-free but not strongly κ^+-free. Then A remains κ^+-free but not free in any extension of $V[G]$ where no new subsets of κ^+ of cardinality $\leq \kappa$ are added. To see this suppose $B \subseteq A$, $|B| = \kappa$ and A/B is not κ^+-free. Then there is $B \subseteq C \subseteq A$ so that $|C| = \kappa$ and C/B is not free. But in any extension which adds no new subsets of κ^+ of size $\leq \kappa$ C/B

remains non-free.

In $V[G]$ let Q be the poset for making $\cup G$ non-stationary. I.E. the elements of Q are closed bounded subsets of κ^+ which are disjoint from $\cup G$ and Q is ordered by end extension. Let $\tilde{Q}$ be a $\mathbf{P}$-name for Q and consider $\mathbf{P} * \tilde{Q}$. It is easy to see the following poset $\mathbf{R}$, is isomorphic to a cofinal subset of $\mathbf{P} * \tilde{Q}$;

$\mathbf{R} = \{(s,C)\colon s \in \mathbf{P}$, C is a closed subset of κ^+, $\operatorname{dom} s = \sup C$ and for all $\alpha \in C$, $s(\alpha) \neq 0\}$; $(s_0, C_0) \leq (s_1, C_1)$ iff $s_0 \subseteq s_1$ and C_1 is an end extension of C_0. So any extension by a generic subset of $\mathbf{P} * \tilde{Q}$ is also an extension by a generic subset of $\mathbf{R}$. But $\mathbf{R}$ is κ^+-directed closed so if H is $\mathbf{R}$-generic in $V[H]$ μ is supercompact and there are no new subsets of κ^+ of cardinality $\leq \kappa$. Suppose now that K is Q-generic and consider $V[G][K]$. ($=V[H]$ for some $\mathbf{R}$-generic H). Since in $V[G][K]$ there are no subsets of κ^+ of cardinality $\leq \kappa$ not in $V[G]$, A is not free. But since $\mu < \kappa^+$, μ is supercompact (in $V[G][K]$) and A is κ^+-free, then A is free.

There was nothing special about cofinality ω in the above proof, we chose to state the theorem as we did for simplicity.

Remark: Recall that $\square_\kappa$ implies $E(\kappa^+)$. In $V[G]$, although $E(\kappa^+)$ holds, $\square_\kappa$ does not. Otherwise $\square_\kappa$ would hold in $V[G][K]$. The independence of $\square_\kappa$ from $E(\kappa^+)$, given the consistency of a Mahlo cardinal was previously known.

4. κ-FREE GROUPS IN LARGER CARDINALITIES.

If we wish to study κ-free groups of cardinality greater than κ, then there is an a priori conceivable class of groups which might cause problems, namely those groups which are not strongly κ-free but every subgroup of cardinality κ is strongly κ-free. We will show no such group exists. As a consequence we will know that κ-free implies strongly κ-free iff κ-free implies strongly κ-free for groups of cardinality κ. If $\kappa^{<\kappa} = \kappa$ (in particular if κ is regular and GCH holds), then it is easy given a group A which is not strongly κ-free to construct a subgroup B of cardinality κ which is not strongly κ-free. Choose $X \subseteq A$ so that X witnesses that A is not strongly κ-free. Then we can build $X \subseteq B \subseteq A$ so that $B = \bigcup_{\alpha<\kappa} B_\alpha$ (increasing); for $\alpha < \kappa$, $|B_\alpha| < \kappa$; and if $X \subseteq C \subseteq B$ and $|C| < \kappa$ then for some α B_α/C is not free. This is easy to arrange by a straightforward enumeration argument (given that $\kappa^{<\kappa} = \kappa$).

To prove the result without any cardinal restrictions we first need a criterion for a group not to be strongly κ-free.

Lemma 4.1. Suppose κ is a regular cardinal and $H = \bigcup_{\alpha<\kappa} H_\alpha$ where for all α $|H_\alpha| < \kappa$ and for $\alpha < \beta$, $H_\alpha \subseteq H_\beta$. Further suppose that for all α there is $G_\alpha \subseteq H_0$ so that H_α/G_α is free but $H_{\alpha+1}/G_\alpha$ is not free. Then H is not strongly κ-free.

Proof. Suppose H is strongly κ-free. Choose $H \supseteq K \supseteq H_0$ so that $|K| < \kappa$ and H/K is κ-free. Choose α such that $H_\alpha \supseteq K$. Since $K/G_\alpha \subseteq H_\alpha/G_\alpha$, K/G_α is free. Also by the choice of K, $H_{\alpha+1}/K$ is free. Hence $H_{\alpha+1}/G_\alpha$ is free. This is a contradiction.

Theorem 4.2. Suppose every subgroup of a group A of cardinality κ is strongly κ-free. Then A is strongly κ-free.

Proof. We can assume κ is regular. (Otherwise as A is κ-free, A would automatically be strongly κ-free.) Suppose A is not strongly κ-free and choose a subgroup $K_0 \subseteq A$ which witnesses this. We define K_α for $\alpha < \kappa$ by induction. We have already defined K_0. If δ is a limit ordinal then let $K_\delta = \bigcup_{\alpha < \delta} K_\alpha$. Suppose now that K_α has been defined. To choose $K_{\alpha+1}$ we attempt to choose an auxiliary sequence $H_{\alpha\nu}, G_{\alpha\nu}$ for $\nu < \kappa$. Let $H_{\alpha 0} = K_\alpha$. If δ is a limit ordinal and $H_{\alpha\nu}$ has been chosen for all $\nu < \delta$ let $H_{\alpha\delta} = \bigcup_{\nu<\delta} H_{\alpha\nu}$. Suppose $H_{\alpha\nu}$ has been chosen. Then choose $K_0 \subseteq G_{\alpha\nu} \subseteq K_\alpha$ so that $H_{\alpha\nu}/G_{\alpha\nu}$ is free (if such a $G_{\alpha\nu}$ exists). If $H_{\alpha\nu}$ and $G_{\alpha\nu}$ have been chosen then choose $H_{\alpha\nu+1} \supseteq H_{\alpha\nu}$ so that $|H_{\alpha\nu+1}| < \kappa$ and $H_{\alpha\nu+1}/G_{\alpha\nu}$ is not free. Note that such an $H_{\alpha\nu+1}$ must exist. (It is possible to think of this sequence as a game between two players I and II, where I chooses the $H_{\alpha\nu}$'s and II the $G_{\alpha\nu}$'s.)

Is it possible that for all $\nu < \kappa$, both $H_{\alpha\nu}$ and $G_{\alpha\nu}$ exist? Clearly the answer is no. Otherwise $H = \bigcup_{\nu<\kappa} H_{\alpha\nu}$ would by Lemma 4.1, be a non-strongly κ-free subgroup of A with cardinality κ. So for some $\nu < \kappa$ given $H_{\alpha\nu}$ we cannot choose $G_{\alpha\nu}$. Let $K_{\alpha+1} = H_{\alpha\nu}$. To sum up we have chosen $K_{\alpha+1}$ so that: $|K_{\alpha+1}| < \kappa$; $K_\alpha \subseteq K_{\alpha+1}$; and if $K_0 \subseteq G \subseteq K_\alpha$ then $K_{\alpha+1}/G$ is not free.

Let $K = \bigcup_{\alpha < \kappa} K_\alpha$. Suppose there is $K_0 \subseteq G \subseteq K$ so that K/G is κ-free and $|G| < \kappa$. Then there is $\alpha < \kappa$ so that $G \subseteq K_\alpha$. So $K_{\alpha+1}/G$ is free contradicting the choice of $K_{\alpha+1}$.

Corollary 4.3 If for groups of cardinality κ, κ-free implies strongly κ-free, then κ-free implies strongly κ-free.

Remark. Theorem 4.2 can be thought of as a companion to the fact that subgroups of strongly κ-free groups are themselves strongly κ-free.

We are also interested in generalizing Theorem 4.2 to other varieties. For varieties whose free algebras satisfy Axiom I^{**} of [S2] (i.e. If A is free over B and $C \subseteq A$, then C is free over B), exactly the same proof establishes the analogue of Theorem 4.2. For example the analoguous result holds for groups (as opposed to Abelian groups). For varieties where Axiom I^{**} fails, the situation is less clear. First we must redefine being strongly κ-free in a manner appropriate for general varieties. This can be done by defining A to be strongly κ-free iff A is $L_{\infty\kappa}$-equivalent to a free algebra (in our variety). Equivalently we can define A to be strongly κ-free iff Player II has a winning strategy in the following game: Player's I and II alternately choose an increasing a sequence $B_0 \subseteq B_1 \ \ldots$ of subalgebras of A of cardinality $< \kappa$. Player II wins is for all n B_{n+1} is free and a free factor of B_{2n+3}. In [H] this game is called the κ-Shelah game. Of course the hypothesis of Theorem 4.2 must be changed to only require that "most" subsets of cardinality κ are strongly κ-free. (Otherwise we would get in trouble if not all subalgebras of a strongly κ-free algebra were strongly κ-free; e.g. in a non-Schreier variety.) If these changes are made then Theorem 4.2 can be proved assuming $\kappa^{<\kappa} = \kappa$. This proof is essentially the one which was outlined at the beginning of this section. We do not know if the result can be proved with no set theoretic hypotheses.

REFERENCES

[BD] Ben David, S., On Shelah's compactness of cardinals, Israel J. Math. 31 (1978), 34-56.

[E1] Eklof, P., On the existence of κ-free abelian groups, Proc. Amer. Math. Soc. 47 (1975), 65-72.

[E2] Eklof, P., Set theoretic methods in Homological Algebra and Abelian Groups, les Presses de l'Université de Montréal (1980)

[J] Jech, T., Set Theory, Academic Press, New York (1973).

[L] Laver, R., Making the super compactness of κ indestructible under κ-directed closed forcing, Israel J. Math <u>29</u> (1978) 385-388.

[M] Mekler, A., How to construct almost free groups, Can. J. Math <u>32</u> (1980) 1206-1228.

[S1] Shelah, S., A compactness theorem for singular cardinals, free algebra, Whitehead problem and transversals, Israel J. Math. <u>21</u> (1975), 319-349.

[S2] Shelah, S., On regular incompactness, Note Dame J. Formal Logic, 26 (1985), 195-228.

On Groups A Such That $A \oplus \mathbf{Z}^n \cong A$

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<u>Abstract</u>. For any $m \geq 2$ we construct a locally free group A of cardinality $2^{\aleph_0}$ such that $A \oplus \mathbf{Z}^m \cong A$ but $A \oplus \mathbf{Z}^n \not\cong A$ if $1 \leq n < m$. We show that it is not necessarily the case that there exists such an A of cardinality $\aleph_1$.

In 1983, Gabriel Sabbagh pointed out to the first author the following question: if A is an abelian group such that $A \oplus \mathbf{Z}^2 \cong A$, is it the case that $A \oplus \mathbf{Z} \cong A$? (cf.[4;p.222]). The question may be re-formulated and generalized as follows. Given A, let $S(A) = \{n \in \mathbf{Z}_+ : A \oplus \mathbf{Z}^n \cong A\}$ (where $\mathbf{Z}_+$ is the set of positive integers). It is not hard to see that if $S(A) \neq \phi$, then $S(A) = m\mathbf{Z}_+ (=\{mk : k \in \mathbf{Z}_+\})$ for some m. The question then becomes: is there for each $m \in \mathbf{Z}_+$ a group A such that $S(A) = m\mathbf{Z}_+$?

M. Dugas observed in 1983 that Stein's Lemma [1; Cor. 19.3] implies that for any countable A, $S(A) \neq \phi$ implies $A \oplus \mathbf{Z}^{(\omega)} \cong A$, so $S(A) = \mathbf{Z}_+$ (see also the Corollary below). In the spring of 1985, while both authors were visiting Simon

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