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A strong failure of $\aleph_0$-stability for atomic classes

Michael C. Laskowski[1] · **Saharon Shelah**[2,3]

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Abstract We study classes of atomic models $\mathbf{At}_T$ of a countable, complete first-order theory T. We prove that if $\mathbf{At}_T$ is not pcl-small, i.e., there is an atomic model N that realizes uncountably many types over $\text{pcl}_N(\bar{a})$ for some finite $\bar{a}$ from N, then there are $2^{\aleph_1}$ non-isomorphic atomic models of T, each of size $\aleph_1$.

Keywords Atomic models · Pseudo-algebraic · Non-structure

Mathematics Subject Classification Primary 03C50; Secondary 03C35

1 Introduction

In a series of papers [3–5], Baldwin and the authors have begun to develop a model theory for complete sentences of $L_{\omega_1,\omega}$ that have fewer than $2^{\aleph_1}$ non-isomorphic models of size $\aleph_1$. By well known reductions, see e.g., Sect. 6.1 of [1], one can replace the reference to infinitary sentences by restricting to the class of *atomic*[1] models of a

[1] A model M is *atomic* if, for every finite tuple $\bar{a}$ from M, $\text{tp}_M(\bar{a})$ is *principal* i.e., is uniquely determined by a single formula $\varphi(\bar{x}) \in \text{tp}_M(\bar{a})$.

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✉ Michael C. Laskowski
mcl@math.umd.edu

1 Department of Mathematics, University of Maryland, College Park, USA

2 Hebrew University, Jerusalem, Israel

3 Rutgers University, New Brunswick, USA

countable, complete first-order theory. Specifically, for every complete sentence Φ of $L_{\omega_1,\omega}$, there is a complete first-order theory T in a countable vocabulary containing the vocabulary of Φ such that the models of Φ are precisely the reducts of the class $\mathbf{At}_T$ of atomic models of T to the smaller vocabulary.

The whole of this paper concerns complete theories T in a countable language such that the class $\mathbf{At}_T$ of atomic models of T has at least one uncountable element. By theorems of Vaught, these restrictions on T are well understood. A countable, complete T has an atomic model if and only if every consistent formula can be extended to a complete formula. Furthermore, any two countable, atomic models of T are isomorphic, and a model is prime if and only if it is countable and atomic. Using a well-known union of chains argument, T has an atomic model of size $\aleph_1$ if and only if the countable atomic model is not minimal, i.e., it has a proper elementary substructure.

To date, the analysis of uncountable atomic models in $\mathbf{At}_T$ has followed the first-order setting. Recall that if T is a complete theory in a countable language that is not $\aleph_0$-stable, then there are $2^{\aleph_1}$ non-isomorphic models of size $\aleph_1$, see e.g., VIII Conclusion 1.7(2) of [8]. The proof of this splits into two cases. One first establishes the result for unsuperstable theories, and then invokes a separate argument for theories that are superstable, but not $\aleph_0$-stable.

Superstability itself does not make a good dividing line for atomic models. This can be seen by considering a two-sorted structure $M = (U, V)$, where U denotes an infinite set with no structure, and V consists of a single copy of $(\mathbb{Z}, \leq)$. Even though $T = Th(M)$ is unstable, $\mathbf{At}_T$ is κ-categorical for every infinite cardinal κ – the point being that the $(\mathbb{Z}, \leq)$ sort cannot be increased in any atomic model.

To adjust for this, in [3], Baldwin and the authors defined the notion of *pseudo-algebraicity*, which was introduced in [3], that is the correct analog of algebraicity in the context of atomic models. Suppose M is an atomic model, and $b, \bar{a}$ are from M. We say $b \in \mathrm{pcl}_M(\bar{a})$ if $b \in N$ for every elementary submodel $N \preceq M$ that contains $\bar{a}$.

By analogy to weak minimality, call a formula $\varphi(x, \bar{a})$ is *pseudo-minimal* if it is not pseudo-algebraic, yet pseudo-algebraic closure pcl_M satisfies the exchange axiom on the set of solutions $\varphi(M, \bar{a})$. [Weakly minimal formulas can be characterized as the non-algebraic formulas for which the relation of algebraic closure satisfies exchange.] In the first order context, if T is superstable, then every non-algebraic formula extends to a weakly minimal formula. By analogy, in [3] we prove that if $\mathbf{At}_T$ is an atomic class and there is some non-pseudo-algebraic formula that cannot be extended to a pseudo-minimal formula, then there are $2^{\aleph_1}$ non-isomorphic atomic models of size $\aleph_1$.

This paper seeks an atomic model analogue of the superstable, non-$\aleph_0$-stable many-models result in first order. To begin, it is natural to restrict our attention to types that can be realized in an atomic model. Suppose M is atomic and $A \subseteq M$. We let $S_{at}(A)$ denote the set of complete types p over A for which Ab is an atomic set for some (equivalently, for every) realization b of p. It is easily checked that when A is countable, $S_{at}(A)$ is a G_δ subset of the Stone space $S(A)$, hence $S_{at}(A)$ is Polish with respect to the induced topology. By analogy with the first order case, we call an atomic class $\mathbf{At}_T$ $\aleph_0$-*stable* if $S_{at}(M)$ is countable, where M denotes the unique countable model in $\mathbf{At}_T$.

The grail, which remains open, would be to prove that non-$\aleph_0$-stability of an atomic class $\mathbf{At}_T$ implies many atomic models in $\aleph_1$. Here, we content ourselves with some-what less. We repeatedly use the fact that any countable, atomic set A is contained in a countable, atomic model M. However, unlike the first-order case, some types in $S_{at}(A)$ need not extend to types in $S_{at}(M)$. Indeed, there are examples where the space $S_{at}(A)$ is uncountable (hence contains a perfect set) while $S_{at}(M)$ is countable. Thus, for analyzing types over countable, atomic sets $A \subseteq M$, we are led to consider

$$S^+_{at}(A, M) := \{p|A : p \in S_{at}(M)\}.$$

Equivalently, $S^+_{at}(A, M)$ is the set of $q \in S_{at}(A)$ that can be extended to a type $q^* \in S_{at}(M)$.

We repeatedly use the following observations. Suppose $\bar{a} \subseteq M \preceq M'$ and $f : M \to M'$ is an isomorphism fixing $\bar{a}$ pointwise. Then $\mathrm{pcl}_M(\bar{a}) = \mathrm{pcl}_{M'}(\bar{a})$. Moreover, f induces an elementary permutation of $D = \mathrm{pcl}_M(\bar{a})$, which in turn induces a bijection between the spaces of types $S^+_{at}(D, M)$ and $S_{at}(D, M')$.

We now give the major new definition of this paper.

Definition 1.1 An atomic class $\mathbf{At}_T$ with an uncountable model is *pcl-small* if, for every atomic model N and for every finite $\bar{a}$ from N, N realizes only countably many complete types over $\mathrm{pcl}_N(\bar{a})$.

The name of this notion is by analogy with the first-order case – A complete, first-order theory T is small if and only if for every model N and every finite $\bar{a}$ from N, N realizes only countably many complete types over $\bar{a}$. The following proposition relates pcl-smallness with the spaces of types $S^+_{at}(D, M)$.

Proposition 1.2 *The atomic class* $\mathbf{At}_T$ *is* pcl-*small if and only if the space of types* $S^+_{at}(\mathrm{pcl}_M(\bar{a}), M)$ *is countable for every countable, atomic model* M *and every finite* $\bar{a}$ *from* M.

Proof First, assume that some atomic model N and finite sequence $\bar{a}$ from N witness that $\mathbf{At}_T$ is not pcl-small. Choose $\{c_i : i \in \omega_1\} \subseteq N$ realizing distinct complete types over $D = \mathrm{pcl}_N(\bar{a})$. Also, choose a countable $M \preceq N$ that contains $\bar{a}$, and hence D. Then $\{\mathrm{tp}(c_i/D) : i \in \omega_1\}$ witness that $S^+_{at}(D, M)$ is uncountable.

For the converse, choose a countable, atomic model M and $\bar{a}$ from M such that $S^+_{at}(D, M)$ is uncountable, where $D = \mathrm{pcl}_M(\bar{a})$. We will inductively construct a continuous, increasing elementary chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable, atomic models with $M = M_0$ and, for each ordinal α, there is an element $c_\alpha \in M_{\alpha+1}$ such that $\mathrm{tp}(c_\alpha/D)$ is not realized in M_α. Given such a sequence, it is evident that $N = \bigcup_{\alpha<\omega_1} M_\alpha$ and $\bar{a}$ witness that $\mathbf{At}_T$ is not pcl-small. To construct such a sequence, we have defined M_0 to be M and take unions at limit ordinals. For the successor step, assume M_α has been defined. As M and M_α are each countable atomic models that contain $\bar{a}$, choose an isomorphism $f : M \to M_\alpha$ fixing $\bar{a}$ pointwise. As noted above, f fixes D setwise. As M_α is countable, so is the set $\{\mathrm{tp}(c/D) : c \in M_\alpha\}$. As $S^+_{at}(D, M)$ is uncountable, choose an atomic type $p \in S_{at}(M)$, whose restriction to D is distinct from $\{f^{-1}(\mathrm{tp}(c/D)) : c \in M_\alpha\}$. Now choose c_α to realize $f(p)$. Then, as $M_\alpha c_\alpha$ is a countable atomic set, choose a countable elementary extension $M_{\alpha+1} \succeq M_\alpha$ containing c_α. □

Recall that an atomic class $\mathbf{At}_T$ is $\aleph_0$-stable[2] if $S_{at}(M)$ is countable for all (equivalently, for some) countable atomic models M. As $S^+_{at}(A, M)$ is a set of projections of types in $S_{at}(M)$, it will be countable whenever $S_{at}(M)$ is. This observation makes the following corollary to Proposition 1.2 immediate:

Corollary 1.3 *If an atomic class* $\mathbf{At}_T$ *is* $\aleph_0$*-stable, then* $\mathbf{At}_T$ *is* pcl*-small.*

The converse to Corollary 1.3 fails. For example, the theory $T = REF(bin)$ of countably many, binary splitting equivalence relations is not $\aleph_0$-stable, yet $\mathrm{pcl}_M(\bar{a}) = \bar{a}$ for every model M and $\bar{a}$ from M. Thus, $S_{at}(\mathrm{pcl}_M(\bar{a}))$ and hence $S^+_{at}(\mathrm{pcl}(\bar{a}), M)$ is countable for every finite tuple $\bar{a}$ inside any atomic model M. The main theorem of this paper is:

Theorem 1.4 *Let T be a countable, complete theory T with an uncountable atomic model. If the atomic class* $\mathbf{At}_T$ *is not* pcl*-small, then there are* $2^{\aleph_1}$ *non-isomorphic models in* $\mathbf{At}_T$*, each of size* $\aleph_1$*.*

Section 2 sets the stage for the proof. It describes the spaces of types $S^+_{at}(A, M)$, states a transfer theorem for sentences of $L_{\omega_1,\omega}(Q)$, and details a non-structural configuration arising from non-pcl-smallness. In Sect. 3, the non-structural configuration is exploited to give a family of $2^{\aleph_0}$ non-isomorphic structures $(N, \bar{b}^*)$, where each of the reducts N is in $\mathbf{At}_T$ and has size $\aleph_1$. Theorem 1.4 is finally proved in Sect. 4. It is remarkable that whereas it is a ZFC theorem, the proof is non-uniform depending on the relative sizes of the cardinals $2^{\aleph_0}$ and $2^{\aleph_1}$.

2 Preliminaries

In this section, we develop some general tools that will be used in the proof of Theorem 1.4.

2.1 On $S^+_{at}(A, M)$

In this subsection we explore the space of types

$$S^+_{at}(A, M) = \{p|A : p \in S_{at}(M)\}$$

where A is a subset of a countable, atomic model M.

Fix a countable, atomic model M and an arbitrary subset $A \subseteq M$. Let $\mathcal{P}$ denote the space of complete types in one free variable over finite subsets of M. As M is atomic, $\mathcal{P}$ can be identified with the set of complete formulas $\varphi(x, m)$ over M. Implication gives a natural partial order on $\mathcal{P}$, namely $p \leq q$ if and only if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $q \vdash p$. One should think of elements of $\mathcal{P}$ as 'finite approximations' of types in $S^+_{at}(A, M)$. We describe two conditions on $p \in \mathcal{P}$ that identify extreme behaviors in this regard.

[2] Sadly, this usage of '$\aleph_0$-stability' is analogous, but distinct from, the familiar first-order notion.

Definition 2.1 We say a type $p^* \in S^+_{at}(A, M)$ *lies above* $p \in \mathcal{P}$ if there is some $\bar{p} \in S_{at}(M)$ extending $p \cup p^*$. As every $p \in \mathcal{P}$ extends to a type in $S_{at}(M)$, it follows that at least one $p^* \in S^+_{at}(A, M)$ lies above p.

- An element $p \in \mathcal{P}$ *determines a type in* $S^+_{at}(A, M)$ if exactly one $p^* \in S^+_{at}(A, M)$ lies above p.
- An element $p \in \mathcal{P}$ is A-*large* if $\{p^* \in S^+_{at}(A, M) : p^*$ lies above $p\}$ is uncountable.

To understand these extreme behaviors, we define a rank function $\mathrm{rk}_A : \mathcal{P} \to (\omega_1 + 1)$ as follows:

- $\mathrm{rk}_A(p) \geq 0$ for all $p \in \mathcal{P}$;
- For $\alpha \leq \omega_1$, $\mathrm{rk}_A(p) \geq \alpha$ if and only if for every $\beta < \alpha$ and for all finite F, $\mathrm{dom}(p) \subseteq F \subseteq M$, there is $q \in S_{at}(F)$ with $q \geq p$ that β-A *splits*, where:
  - A type $q \in S_{at}(F)$ A-*splits* if, for some $\varphi(x, \bar{a})$ with $\bar{a}$ from A, there are $q_1, q_2 \geq q$ with $q \cup \varphi(x, \bar{a}) \subseteq q_1$ and $q \cup \neg\varphi(x, \bar{a}) \subseteq q_2$; and $q \in S_{at}(F)$ β-A *splits* if, in addition, $\mathrm{rk}_A(q_1), \mathrm{rk}_A(q_2) \geq \beta$.
- For $\alpha < \omega_1$, we say $\mathrm{rk}_A(p) = \alpha$ if $\mathrm{rk}_A(p) \geq \alpha$, but $\mathrm{rk}_A(p) \not\geq \alpha + 1$.

Proposition 2.2 *If* $p \in \mathcal{P}$ *and* $\mathrm{rk}_A(p) = \alpha < \omega_1$, *then some* $r \geq p$ *determines a type in* $S^+_{at}(A, M)$.

Proof We prove this by induction on α. We begin with $\alpha = 0$. Suppose $\mathrm{rk}_A(p) = 0$. As $\mathrm{rk}_A(p) \not\geq 1$, there is a finite F, $\mathrm{dom}(p) \subseteq F \subseteq M$ for which there is no $q \in S_{at}(F)$ and $\varphi(x, \bar{a})$ with $\bar{a}$ from A for which $q \geq p$ and both $q \cup \{\varphi(x, \bar{a})\}$ and $q \cup \{\neg\varphi(x, \bar{a})\}$ are consistent. So fix any $r \in S_{at}(F)$ with $r \geq p$. Any such r determines a type in $S^+_{at}(A, M)$.

Next, choose $0 < \alpha < \omega_1$ and assume the Proposition holds for all $\beta < \alpha$. Choose $p \in S_{at}(E)$ with $\mathrm{rk}_A(p) = \alpha$. As $\mathrm{rk}_A(p) \geq \alpha$, while $\mathrm{rk}_A(p) \not\geq \alpha + 1$, there is a finite F, $E \subseteq F \subseteq M$ for which there is no $q \in S_{at}(F)$ that both extends p and α-A splits. So choose any $q \in S_{at}(F)$ with $q \geq p$. If q determines a type in $S^+_{at}(A, M)$, then we finish, so assume otherwise. Thus, there is some $\varphi(x, \bar{a})$ with $\bar{a}$ from A such that both $q \cup \{\varphi(x, \bar{a})\}$ and $q \cup \{\neg\varphi(x, \bar{a})\}$ are consistent. Choose complete types $q_1, q_2 \in S_{at}(F\bar{a})$ extending these partial types. Clearly, both $q_1, q_2 \geq q$, but since q does not α-A split, at least one of them has $\mathrm{rk}_A(q_\ell) < \alpha$. But then by our inductive hypothesis, there is $r \geq q_\ell$ that determines a type in $S^+_{at}(A, M)$ and we finish. □

Next, we turn our attention to A-large types and types of rank at least ω_1 and see that these coincide. We begin with two lemmas, the first involving types of rank at least ω_1 and the second involving A-large types.

Lemma 2.3 *Assume that* $E \subseteq M$ *is finite and* $p \in S_{at}(E)$ *has* $\mathrm{rk}_A(p) \geq \omega_1$. *Then:*

1. *For every finite* F, $E \subseteq F \subseteq M$, *there is* $q \in S_{at}(F)$, $q \geq p$, *with* $\mathrm{rk}_A(q) \geq \omega_1$; *and*
2. *There is some formula* $\varphi(x, \bar{a})$ *with* $\bar{a}$ *from* A *and* $q_1, q_2 \in \mathcal{P}$ *with* $p \cup \{\varphi(x, \bar{a})\} \subseteq q_1$, $p \cup \{\neg\varphi(x, \bar{a})\} \subseteq q_2$, *and both* $\mathrm{rk}_A(q_1), \mathrm{rk}_A(q_2) \geq \omega_1$.

Proof (1) Fix a finite F satisfying $E \subseteq F \subseteq M$. As $\text{rk}_A(p) \geq \omega_1$, for every $\beta < \omega_1$ there is some $q \geq p$ with $q \in S_{at}(F)$ for which certain extensions of q have rank at least β. It follows that $\text{rk}_A(q) \geq \beta$ for any such witness. However, as $S_{at}(F)$ is countable, there is some $q \in S_{at}(F)$ which serves as a witness for uncountably many β. Thus, $\text{rk}_A(q) \geq \omega_1$ for any such $q \geq p$.

(2) Assume that there were no such formula $\varphi(x, \bar{a})$. Then, for any formula $\varphi(x, \bar{a})$, since $\mathcal{P}$ is countable, there would be an ordinal $\beta^* < \omega_1$ such that **either** every $q \in \mathcal{P}$ extending $p \cup \{\varphi(x, \bar{a})\}$, $\text{rk}_A(q) < \beta^*$ **or** every $q \in \mathcal{P}$ extending $p \cup \{\neg\varphi(x, \bar{a})\}$ has $\text{rk}_A(q) < \beta^*$. Continuing, as there are only countably many formulas $\varphi(x, \bar{a})$, there would be an ordinal $\beta^{**} < \omega_1$ that works for all formulas $\varphi(x, \bar{a})$. Restating this, p does not β^{**}-A split, so no extension of p could β^{**}-A split either. This contradicts $\text{rk}_A(p) \geq \beta^{**} + 1$. □

Lemma 2.4 *Suppose $q \in S_{at}(F)$ is A-large. Then:*

1. *For every finite F', $F \subseteq F' \subseteq M$, there is some A-large $r \in S_{at}(F')$ with $r \geq q$; and*
2. *For some $\varphi(x, \bar{a})$ with parameters from A, there are A-large extensions $r_1 \supseteq q \cup \{\varphi(x, \bar{a})\}$ and $r_2 \supseteq q \cup \{\neg\varphi(x, \bar{a})\}$.*

Proof Fix such a q and let $\mathcal{S} = \{p^* \in S^+_{at}(A, M) : p^* \text{ lies above } q\}$. (1) is immediate, since $\mathcal{S}$ is uncountable, while $S_{at}(F')$ is countable.

For (2), first note that if there is no such $\varphi(x, \bar{a})$, then there is at most one $p^* \in \mathcal{S}$ with the property that:

For any formula $\varphi(x, \bar{a})$ with parameters from A, $\varphi(x, \bar{a}) \in p^*$ if and only if there is an A-large $r \in S_{at}(F\bar{a})$ extending $q \cup \{\varphi(x, \bar{a})\}$.

It follows that for any $q^* \in \mathcal{S} - \{p^*\}$, q^* lies over some $r \geq q$ that is not A-large. That is, using the fact that there are only countably many $r \geq q$, $\mathcal{S} - \{p^*\}$ is contained in the union of countably many countable sets. But this contradicts q being A-large. □

Proposition 2.5 *For $p \in \mathcal{P}$, $\text{rk}_A(p) \geq \omega_1$ if and only if p is A-large.*

Proof First, assume that $\text{rk}_A(p) \geq \omega_1$. Fix an enumeration $\{c_n : n \in \omega\}$ of M. Using Clauses (1) and (2) of Lemma 2.3, we inductively construct a tree $\{p_\nu : \nu \in 2^{<\omega}\}$ of elements of $\mathcal{P}$ satisfying:

1. $\text{rk}_A(p_\nu) \geq \omega_1$ for all $\nu \in 2^{<\omega}$;
2. If $\lg(\nu) = n$, then $\{c_i : i < n\} \subseteq \text{dom}(p_\nu)$;
3. $p_{\langle\rangle} = p$;
4. For $\nu \trianglelefteq \mu$, $p_\nu \leq p_\mu$;
5. For each ν there is a formula $\varphi(x, \bar{a})$ with $\bar{a}$ from A such that $\varphi(x, \bar{a}) \in p_{\nu 0}$ and $\neg\varphi(x, \bar{a}) \in p_{\nu 1}$.

Given such a tree, for each $\eta \in 2^\omega$, let $\bar{p}_\eta := \bigcup\{p_{\eta|n} : n \in \omega\}$ and let $p^*_\eta := \bar{p}_\eta | A$. By Clauses (2) and (4), each $\bar{p}_\eta \in S_{at}(M)$, so each $p^*_\eta \in S^+_{at}(A, M)$. By Clause (5), $p^*_\eta \neq p^*_{\eta'}$ for distinct $\eta, \eta' \in 2^\omega$. Finally, each of these types lies over p by Clause (3). Thus, p is A-large.

Conversely, we argue by induction on $\alpha < \omega_1$ that:

$(*)_\alpha$: If $p \in \mathcal{P}$ is A-large, then $\mathrm{rk}_A(p) \geq \alpha$.

Establishing $(*)_0$ is trivial, and for limit $\alpha < \omega_1$, it is easy to establish $(*)_\alpha$ given that $(*)_\beta$ holds for all $\beta < \alpha$. So assume $(*)_\alpha$ holds and we will establish $(*)_{\alpha+1}$. Choose any A-large $p \in \mathcal{P}$. Towards showing $\mathrm{rk}_A(p) \geq \alpha + 1$, choose any finite F, $\mathrm{dom}(p) \subseteq F \subseteq M$. As $S_{at}(F)$ is countable and uncountably many types in $S^+_{at}(A, M)$ lie above p, there is some A-large $q \in S_{at}(F)$ with $q \geq p$.

Next, by Lemma 2.4 choose a formula $\varphi(x, \bar{a})$ with $\bar{a}$ from A such that there are A-large extensions $r_1 \supseteq q \cup \{\varphi(x, \bar{a})\}$ and $r_2 \supseteq q \cup \{\neg\varphi(x, \bar{a})\}$. Applying $(*)_\alpha$ to both r_1, r_2 gives $\mathrm{rk}_A(r_1), \mathrm{rk}_A(r_2) \geq \alpha$. Thus, q α-A splits. Thus, by definition of the rank, $\mathrm{rk}_A(p) \geq \alpha + 1$. □

We obtain the following Corollary, which is analogous to the statement 'If T is small, then the isolated types are dense' from the first-order context.

Corollary 2.6 *If $S^+_{at}(A, M)$ is countable, then every $p \in \mathcal{P}$ has an extension $q \geq p$ that determines a type in $S^+_{at}(A, M)$.*

Proof If $S^+_{at}(A, M)$ is countable, then no $p \in \mathcal{P}$ is A-large. Thus, every $p \in \mathcal{P}$ has $\mathrm{rk}_A(p) < \omega_1$ by Proposition 2.5, so has an extension determining a type in $S^+_{at}(A, M)$ by Proposition 2.2. □

We close with a complementary result about extensions of A-large types.

Definition 2.7 A type $r \in S_{at}(M)$ is A-*perfect* if $r{\restriction}_A$ is omitted in M and for every finite $\overline{m}$ from M, the restriction $r{\restriction}_{\overline{m}}$ is A-large.

The name *perfect* is chosen because, relative to the usual topology on $S_{at}(M)$, there are a perfect set of A-perfect types extending any A-large $p \in \mathcal{P}$. However, for what follows, all we need to establish is that there are uncountably many, which is notationally simpler to prove.

Proposition 2.8 *Suppose $p \in \mathcal{P}$ is A-large. Then there are uncountably many A-perfect $r \in S_{at}(M)$ extending p.*

Proof Fix an A-large $p \in \mathcal{P}$. Choose a set $R \subseteq S_{at}(M)$ of representatives for $\{p^* \in S^+_{at}(A, M) : p^*$ lies above $p\}$, i.e., for every such p^*, there is exactly one $\bar{p} \in R$ whose restriction $\bar{p}{\restriction}_A = p^*$. As p is A-large, R is uncountable. Now, for each finite $\overline{m}$ from M, there are only countably many complete $q \in S_{at}(\overline{m})$, and if some $q \in S_{at}(\overline{m})$ is A-small, then only countably many $\bar{p} \in R$ extend q. As M is countable, there are only countably many $\overline{m}$, hence all but countably many $\bar{p} \in R$ satisfy $\bar{p}{\restriction}_{\overline{m}}$ A-large for every $\overline{m}$. Further, again since M is countable, at most countably many $\bar{p} \in R$ have restrictions to A that are realized in M. Thus, all but countably many $\bar{p} \in R$ are A-perfect. □

2.2 A transfer result

In this brief subsection we state a transfer result that follows immediately by Keisler's completeness theorem for the logic $L_{\omega_1,\omega}(Q)$, given in [7]. Recall that $L_{\omega_1,\omega}(Q)$ is the logic obtained by taking the (usual) set of atomic L formulas and closing under boolean combinations, existential quantification, the 'Q-quantifier,' i.e., if $\theta(y, \overline{x})$ is a formula, then so is $Qy\theta(y, \overline{x})$; and countable conjunctions of formulas involving a finite set of free variables, i.e., if $\{\psi_i(\overline{x}) : i \in \omega\}$ is a set of formulas, then so is $\bigwedge_{i\in\omega} \psi_i(\overline{x})$. We are only interested in *standard interpretations* of these formulas, i.e., $M \models \bigwedge_{i\in\omega} \psi_i(\bar{a})$ if and only if $M \models \psi_i(\bar{a})$ for every $i \in \omega$; and $M \models Qy\theta(y, \bar{a})$ if and only if the solution set $\theta(M, \bar{a})$ is uncountable.

Throughout the discussion let ZFC^* denote a sufficiently large, finite subset of the ZFC axioms. In the notation of [10], Proposition 2.9 states that sentences of $L_{\omega_1,\omega}(Q)$ are *grounded.*

Proposition 2.9 *There is a sufficiently large, finite subset ZFC^* of ZFC such that whenever a countable language L and a sentence $\Phi \in L_{\omega_1,\omega}(Q)$ are given, IF there is a countable, transitive model $(\mathcal{B}, \epsilon) \models ZFC^*$ with $L, \Phi \in \mathcal{B}$ and*

$$(\mathcal{B}, \epsilon) \models \text{'There is } M \models \Phi \text{ and } |M| = \aleph_1\text{'}$$

THEN (in V!) there is $N \models \Phi$ and $|N| = \aleph_1$.

Proof This follows immediately from Keiser's completeness theorem for $L_{\omega_1,\omega}$, given that provability is absolute between transitive models of set theory. More modern, 'constructive' proofs can be found in [2] and [3]. These use the existence $\mathcal{B}$-normal ultrafilters. Given an arbitrary language $L^* \in \mathcal{B}$ and any countable L^*-structure $(\mathcal{B}, E, \dots)$ where the reduct $(\mathcal{B}, E)$ is an ω-model of ZFC^*, for any $\mathcal{B}$-normal ultrafilter $\mathcal{U}$, the ultrapower $Ult(\mathcal{B}, \mathcal{U})$ is a countable ω-model that is an L^*-elementary extension of $(\mathcal{B}, E, \dots)$. It has the additional property that for any L^*-definable subset D, $D^{Ult(\mathcal{B},\mathcal{U})}$ properly extends $D^{\mathcal{B}}$ if and only if $(\mathcal{B}, E, \dots) \models$ 'D is uncountable'.

Using this, one constructs (in V!) a continuous, L^*-elementary ω_1-sequence $\langle \mathcal{B}_\alpha : \alpha < \omega_1\rangle$ of ω-models, where each $\mathcal{B}_{\alpha+1} = Ult(\mathcal{B}_\alpha, \mathcal{U}_\alpha)$. Then the interpretation $M^{\mathcal{C}}$ where $\mathcal{C} = \bigcup_{\alpha\in\omega_1} \mathcal{B}_\alpha$ will be a suitable choice of N. More details of this construction are given in [2] or [3]. □

2.3 A configuration arising from non-pcl-smallness

The goal of this subsection is to prove the following Proposition, the data from which will be used throughout Sect. 3.

Proposition 2.10 *Assume T is a countable, complete theory for which $\mathbf{At}_T$ has an uncountable atomic model, but is not* pcl*-small. Then there are a countable, atomic $M^* \in \mathbf{At}_T$, finite sequences $\bar{a}^* \subseteq \bar{b}^* \subseteq M^*$, and complete 1-types $\{r_j(x, \bar{b}^*) : j \in \omega\}$ such that, letting $D^* = \mathrm{pcl}_{M^*}(\bar{a}^*)$, $A_n = \bigcup\{r_j(M^*, \bar{b}^*) : j < n\}$ and $A^* = \bigcup\{A_n : n \in \omega\}$ we have:*

1. *$A^* \subseteq D^*$;*
2. *$S_{at}^+(A_n, M^*)$ is countable for every $n \in \omega$; but*
3. *$S_{at}^+(A^*, M^*)$ is uncountable.*

Proof Fix any countable, atomic $M^* \in \mathbf{At}_T$. Using Proposition 1.2 and the non-pcl-smallness of $\mathbf{At}_T$, choose a finite tuple $\bar{a}^* \subseteq M^*$ such that $S^+_{at}(D^*, M^*)$ is uncountable, where $D^* = \mathrm{pcl}_{M^*}(\bar{a}^*) \subseteq M^*$.

Fix any finite tuple $\bar{b} \supseteq \bar{a}^*$ from M^* and look at the complete 1-types $\mathcal{Q}_{\bar{b}} := \{r \in S_{at}(\bar{b})$ such that $r(M^*) \subseteq D^*\}$. These types visibly induce a partition of D^*, and it is easily seen that if $\bar{b}' \supseteq \bar{b}$, the partition induced by $\bar{b}'$ refines the partition induced by $\bar{b}$. Let $\mathcal{Q} := \bigcup\{\mathcal{Q}_{\bar{b}} : \bar{a}^* \subseteq \bar{b} \subseteq M^*\}$.

Define a rank function $\mathrm{rk} : \mathcal{Q} \to ON \cup \{\infty\}$ as follows:

- $\mathrm{rk}(c/\bar{b}) \geq 0$ if and only if $\mathrm{tp}(c/\bar{b}) \in \mathcal{Q}$;
- $\mathrm{rk}(c/\bar{b}) \geq 1$ if and only if $\mathrm{tp}(c/\bar{b}) \in \mathcal{Q}$ and there are infinitely many $c' \in D^*$ realizing $\mathrm{tp}(c/\bar{b})$; and
- for an ordinal $\alpha \geq 2$, $\mathrm{rk}(c/\bar{b}) \geq \alpha$ if and only if for every $\beta < \alpha$ and every $\bar{b}'$ from M^*, there is $c' \in D^*$ realizing $\mathrm{tp}(c/\bar{b})$ such that $\mathrm{rk}(c'/\bar{b}\bar{b}') \geq \beta$.
- $\mathrm{rk}(c/\bar{b}) = \alpha$ if and only if $\mathrm{rk}(c/\bar{b}) \geq \alpha$ but $\mathrm{rk}(c/\bar{b}) \not\geq \alpha + 1$.

Claim 1. For every $r \in \mathcal{Q}$, $\mathrm{rk}(r)$ is a countable ordinal.

Proof Assume by way of contradiction that $\mathrm{rk}(c/\bar{b}) \geq \omega_1$ for some type $c/\bar{b}$. Then, for any $\bar{b}'$ from M, as D^* is countable, there is an element $c' \in D^*$ such that $\mathrm{rk}(c'/\bar{b}\bar{b}') \geq \beta$ for uncountably many β's, hence $\mathrm{rk}(c'/\bar{b}\bar{b}') \geq \omega_1$ as well. Using this idea, if we let $\langle \bar{b}_n : n \in \omega \rangle$ be an increasing sequence of finite sequences from M^* whose union is all of M^*, then we can find a sequence $\langle c_n : n \in \omega \rangle$ of elements from D^* such that, for each n, $\mathrm{rk}(c_n/\bar{b}_n) \geq \omega_1$ and $\mathrm{tp}(c_n/\bar{b}_n) \subseteq \mathrm{tp}(c_{n+1}/\bar{b}_{n+1})$. The union of these 1-types yields a complete, atomic 1-type $q \in S_{at}(M^*)$ all of whose realizations are in $\mathrm{pcl}_{M^*}(\bar{a})$. However, since the type asserting that '$x = c$' has rank 0 for each $c \in D^*$, q is omitted in M^*. To obtain a contradiction, choose a realization e of q and, as M^*e is a countable, atomic set, construct a countable, atomic elementary extension $M' \succeq M^*$ with $e \in M'$. But now, q implies that $e \in \mathrm{pcl}_{M'}(\bar{a})$, yet this is contradicted by the fact that M^* contains $\bar{a}$ but not e. □

As notation, for a subset $\mathcal{S} \subseteq \mathcal{Q}_{\bar{b}}$, let $A_{\mathcal{S}} = \bigcup\{r(M^*) : r \in \mathcal{S}\}$, which is always a subset of D^*. Define the set of 'candidates' as

$$\mathcal{C} = \{(\mathcal{S}, \bar{b}) : \bar{b} \supseteq \bar{a}^*, \mathcal{S} \subseteq \mathcal{Q}_{\bar{b}}, \text{ and } S^+_{at}(A_{\mathcal{S}}, M^*) \text{ uncountable}\}$$

Note that $\mathcal{C}$ is non-empty as $(\mathcal{S}_0, \bar{a}^*) \in \mathcal{C}$, where $\mathcal{S}_0$ is an enumeration of all the complete, pseudo-algebraic types over $\bar{a}^*$. Among all candidates, choose $(\mathcal{S}^*, \bar{b}^*) \in \mathcal{C}$ such that

$$\alpha^* := \sup\{rk(r) + 1 : r \in \mathcal{S}^*\}$$

is as small as possible. Enumerate $\mathcal{S}^* = \{r_j : j \in \omega\}$ and put $A^* := A_{\mathcal{S}^*}$ and $A_n := \bigcup\{r_j(M^*, \bar{b}^*) : j < n\}$ for each $n \in \omega$. As Clauses (1) and (3) are immediate, it suffices to prove the following Claim:

Claim 2. For each $n \in \omega$, $S^+_{at}(A_n, M^*)$ is countable.

Proof Fix any $n \in \omega$. First, note that if $\mathrm{rk}(r_j) = 0$ for every $j < n$, then A_n would be finite, which would imply $S_{at}(A_n)$ is countable. As $S_{at}(A_n)$ contains $S^+_{at}(A_n, M^*)$, the result follows.

Now assume $rk(r_j) > 0$ for at least one $j < n$. Let $\beta := \max\{rk(r_j) : j < n\}$ and let $F = \{j < n : rk(r_j) = \beta\}$. Clearly, $\beta < \alpha^*$. For each $j \in F$, as $\beta > 0$ but $rk(r_j) \not\geq \beta + 1$, there is a finite tuple $\bar{b}_j$ such that $\mathrm{rk}(c/\bar{b}^*\bar{b}_j) < \beta$ for all $c \in r_j(M^*)$.

Let $\bar{b}'$ be the concatenation of $\bar{b}^*$ with each $\bar{b}_j$ for $j \in F$ and let

$$\mathcal{S}' := \{r' \in \mathcal{Q}_{\bar{b}'} : r' \text{ extends some } r_j \text{ with } j < n\}$$

Subclaim. $\mathrm{rk}(r') < \beta$ for every $r' \in \mathcal{S}'$.

Proof Fix $r' \in \mathcal{S}'$ and choose $c \in r'(M^*, \bar{b}')$. There are two cases. On one hand, if r' extends some r_j with $j \in F$, then $\mathrm{rk}(c/\bar{b}') \leq \mathrm{rk}(c/\bar{b}^*\bar{b}_j) < \beta$. On the other hand, if r' extends some r_j with $r_j \notin F$, then as $\mathrm{rk}(r_j) < \beta$, $\mathrm{rk}(c/\bar{b}') \leq \mathrm{rk}(c/\bar{b}^*) < \beta$. □

Clearly $A_{\mathcal{S}'} = A_n$, so $S^+_{at}(A_n, M^*) = S^+_{at}(A_{\mathcal{S}'}, M^*)$. Thus, if $S^+_{at}(A_n, M^*)$ were uncountable, then $(\mathcal{S}', \bar{b}')$ would be a candidate, i.e., an element of $\mathcal{C}$. But, as $\beta < \alpha^*$, this is impossible by the Subclaim and the minimality of α^*. □

3 A family of $2^{\aleph_0}$ atomic models of size $\aleph_1$

Throughout the whole of this section, we assume that T is a complete theory in a countable language for which $\mathbf{At}_T$ has an uncountable atomic model, but is not pcl-small. Appealing to Proposition 2.10,

> **Fix, for the whole of this section, a countable atomic model $\mathbf{M}^*$, tuples $\bar{\mathbf{a}}^* \subseteq \bar{\mathbf{b}}^* \subseteq \mathbf{M}^*$ and sets A^* and A_n for each $n \in \omega$ as in Proposition 2.10.**

We work with this fixed configuration for the whole of this section and, in Sect. 3.3 eventually prove:

Proposition 3.1 *There is a family $\{(N_\eta, \bar{b}^*) : \eta \in 2^\omega\}$ of atomic models of T, each of size $\aleph_1$, that are pairwise non-isomorphic over $\bar{b}^*$.*

3.1 Colorings of models realizing many types over A^*

Definition 3.2 Call a structure $(N, \bar{b}^*)$ *rich* if $N \in \mathbf{At}_T$ has size $\aleph_1$, $M^* \preceq N$, and N realizes uncountably many 1-types over A^*.

Lemma 3.3 *For each $n \in \omega$, a rich $(N, \bar{b}^*)$ realizes only countably many distinct 1-types over A_n.*

Proof Fix any $(N, \bar{b}^*)$ and $n < \omega$ as above. If $\{c_i : i \in \omega_1\}$ realize distinct types over A_n, then the types $\{\mathrm{tp}_N(c_i/M^*) : i \in \omega_1\}$ would be distinct, contradicting $S^+_{at}(A_n, M^*)$ countable. □

How can we tell whether rich structures are non-isomorphic? We introduce the notion of $\mathcal{U}$-colorings and Corollary 3.6 gives a sufficient condition.

Definition 3.4 Fix a subset $\mathcal{U} \subseteq \omega$ and a rich $(N, \bar{b}^*)$.

- For elements $d, d' \in N$, define the *splitting number* $\mathrm{spl}(d, d') \in (\omega + 1)$ to be the least $k < \omega$ such that $\mathrm{tp}(d/A_k) \neq \mathrm{tp}(d'/A_k)$ if such exists; and $\mathrm{spl}(d, d') = \omega$ if $\mathrm{tp}(d/A^*) = \mathrm{tp}(d'/A^*)$.
- A *$\mathcal{U}$-coloring of a rich* $(N, \bar{b}^*)$ is a function

$$c : N \to \omega$$

  such that for all pairs $d, d' \in N$, at least one of the following hold:
  1. $\mathrm{tp}(d/A^*) = \mathrm{tp}(d'/A^*)$; or
  2. $c(d) \neq c(d')$; or
  3. $\mathrm{spl}(d, d') \in \mathcal{U}$.
- The *color filter* $\mathcal{F}(N, \bar{b}^*) := \{\mathcal{U} \subseteq \omega : \text{a } \mathcal{U}\text{-coloring of } (N, \bar{b}^*) \text{ exists}\}$.

Lemma 3.5 *Fix a rich* $(N, \bar{b}^*)$. *Then:*

1. $\mathcal{F}(N, \bar{b}^*)$ *is a filter;*
2. $\mathcal{F}(N, \bar{b}^*)$ *contains the cofinite subsets of* ω*; but*
3. *No finite* $\mathcal{U} \subseteq \omega$ *is in* $\mathcal{F}(N, \bar{b}^*)$.

Proof (1) First, note that if $\mathcal{U} \subseteq \mathcal{U}' \subseteq \omega$, then every $\mathcal{U}$-coloring c is also a $\mathcal{U}'$-coloring. Thus, $\mathcal{F}(N, \bar{b}^*)$ is upward closed. Next, suppose $\mathcal{U}_1 \in \mathcal{F}(N, \bar{b}^*)$ via the coloring $c_1 : N \to \omega$ and $\mathcal{U}_2 \in \mathcal{F}(N, \bar{b}^*)$ via the coloring $c_2 : N \to \omega$. Fix any bijection $t : \omega \times \omega \to \omega$. It is easily checked that $c^* : N \to \omega$ defined by $c^*(d) = t(c_1(d), c_2(d))$ is a $\mathcal{U}_1 \cap \mathcal{U}_2$-coloring of $(N, \bar{b}^*)$. Thus, $\mathcal{U}_1 \cap \mathcal{U}_2 \in \mathcal{F}(N, \bar{b}^*)$. So $\mathcal{F}(N, \bar{b}^*)$ is a filter.

(2) As $\mathcal{F}(N, \bar{b}^*)$ is a filter, it suffices to show that for every $n \in \omega$, $B_n \in \mathcal{F}(N, \bar{b}^*)$, where $B_n = (\omega - \{0, \ldots, n-1\})$. Fix such an n. By Lemma 3.3, N realizes at most countably many types over A_n. Thus, we can produce a map $c : N \to \omega$ such that $c(d) = c(d')$ if and only if $\mathrm{tp}(d/A_n) = \mathrm{tp}(d'/A_n)$. As any such c is a B_n-coloring, $B_n \in \mathcal{F}(N, \bar{b}^*)$.

(3) It suffices to show that no $n = \{0, \ldots, n-1\}$ is in $\mathcal{F}(N, \bar{b}^*)$. To see this, let $c : N \to \omega$ be an arbitrary map. We will show that c is not an $\{0, \ldots, n-1\}$-coloring. As N realizes $\aleph_1$ distinct types over A^*, there is some $m^* \in \omega$ and an uncountable subset $\{d_\alpha : \alpha < \omega_1\} \subseteq N$ that realize distinct types over A^*, yet $c(d_\alpha) = m^*$ for each α. However, as N realizes only countably many types over A_n, there are $\alpha \neq \beta$ such that $n \leq \mathrm{spl}(d_\alpha, d_\beta) < \omega$. Thus, c is not an $\{0, \ldots, n-1\}$-coloring. □

We close with a sufficient condition for non-isomorphism of rich models.

Corollary 3.6 *Suppose that for* $\ell = 1, 2$, $(N_\ell, \bar{b}^*)$ *is a* $\mathcal{U}_\ell$*-colored rich model, and* $\mathcal{U}_1 \cap \mathcal{U}_2$ *is finite. Then there is no isomorphism* $f : N_1 \to N_2$ *fixing* $\bar{b}^*$ *pointwise.*

Proof If there were such an isomorphism, then $(N_2, \bar{b}^*)$ would be both $\mathcal{U}_1$-colored and $\mathcal{U}_2$-colored. Thus, both $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{F}(N_2, \bar{b}^*)$, which contradicts Lemma 3.5. □

3.2 Constructing a colored rich model via forcing

Arguing as in the proof of Proposition 1.2, from the data of Proposition 2.10 we can construct a rich $(N, \bar{b}^*)$ as the union of a continuous, elementary chain $\langle M_\alpha : \alpha \in \omega_1\rangle$ of countable, atomic models with $M_0 = M^*$ such that, for each $\alpha \in \omega_1$ there is a distinguished $b_\alpha \in M_{\alpha+1}$ such that $\mathrm{tp}(b_\alpha/A^*)$ is omitted in M_α.

Our goal is to construct a sufficiently generic rich $(N, \bar{b}^*)$, along with a coloring $c : N \to (\omega + 1)$ via forcing. Our forcing $(\mathbb{Q}, \leq_{\mathbb{Q}})$ encodes finite approximations of such an $(N, \bar{b}^*)$ and c. A fundamental building block is the notion of a *striated type* over a finite subset $\bar{a}$ satisfying $\bar{b}^* \subseteq \bar{a} \subseteq M^*$. As an atomic type over a finite subset is generated by a complete formula, we use the terms interchangeably.

Definition 3.7 Choose a finite tuple $\bar{a}$ with $\bar{b}^* \subseteq \bar{a} \subseteq M^*$. A *striated type over* $\bar{a}$ is a complete formula $\theta(\bar{x}) \in S_{at}(\bar{a})$ whose variables are partitioned as $\bar{x} = \langle \bar{x}_j : j < \ell\rangle$ where, for each j, $\bar{x}_j = \langle x_{j,n} : n < n(j)\rangle$ is an $n(j)$-tuple of variable symbols such that $\theta(\bar{x})$ implies $\mathrm{tp}(x_{j,0}/\bar{a} \cup \{\bar{x}_i : i < j\})$ is A^*-large. The integer ℓ is the *length* of the striated type.

A *simple realization* of a striated type $\theta(\bar{x})$ of length ℓ is a sequence $\bar{b} = \langle \bar{b}_j : j < \ell\rangle$ of tuples from M^* such that $M^* \models \theta(\bar{b})$. A *perfect chain realization* of $\theta(\bar{x})$ is a pair $(\overline{M}, \bar{b})$, consisting of a chain $M_0 \preceq M_1 \preceq M_{\ell-1} \preceq M^*$ of ℓ elementary submodels of M^* and a simple realization $\bar{b} = \langle \bar{b}_j : j < \ell\rangle$ from M^* that satisfy: For each $j < \ell$,

1. $\bar{a} \cup \{\bar{b}_i : i < j\} \subseteq M_j$; and
2. $\mathrm{tp}(b_{j,0}/M_j)$ is A^*-perfect (see Definition 2.7).

Lemma 3.8 *Every striated type $\theta(\bar{x}) \in S_{at}(\bar{a})$ has a perfect chain realization.*

Proof We argue by induction on ℓ, the length of the striation. For striations of length zero there is nothing to prove, so assume the Lemma holds for striated types of length ℓ and choose an $(\ell + 1)$-striation $\theta(\bar{x}) \in S_{at}(\bar{a})$. Let $\theta\restriction_\ell$ be the truncation of θ to the variables $\bar{x}\restriction_\ell = \langle \bar{x}_j : j < \ell\rangle$. As $\theta\restriction_\ell$ is clearly an ℓ-striation, it has a perfect chain realization, i.e., a chain $M_0 \preceq M_1 \preceq M_{\ell-1} \preceq M^*$ and a tuple $\bar{b} = \langle \bar{b}_j : j < \ell\rangle$ from M^* realizing $\theta\restriction_\ell$ such that $\bar{a} \cup \{\bar{b}_i : i < j\} \subseteq M_j$ and $\mathrm{tp}(b_{j,0}/M_j)$ is A^*-perfect for each $j < \ell$.

Now, since $\mathrm{tp}(x_{\ell,0}/\bar{a}\bar{b})$ is A^*-large, by applying Proposition 2.8 there is an A^*-perfect type $\bar{p} \in S_{at}(M^*)$ (in a single variable $x_{\ell,0}$) extending $\mathrm{tp}(x_{\ell,0}/\bar{a}\bar{b})$. Choose a countable, atomic $N \succeq M^*$ and $e \in N$ realizing $\bar{p}$. As N and M^* are both countable and atomic, choose an isomorphism $f : N \to M^*$ that fixes $\bar{a}\bar{b}$ pointwise. Then $f(M_0) \preceq f(M_1) \preceq \ldots f(M_{\ell-1}) \preceq f(M^*) \preceq M^*$ is a chain. Let $b_{\ell,0} := f(e)$ and choose $\langle b_{\ell,1} \ldots, b_{\ell,n(\ell)-1}\rangle$ arbitrarily from M^* so that, letting $\bar{b}_\ell = \langle \bar{b}_{\ell,n} : n < n(\ell)\rangle$, $\bar{b} \frown \bar{b}_\ell$ realizes $\theta(\bar{x})$. This chain and this sequence form a perfect chain realization of θ. □

The following Lemma is immediate, and indicates the advantage of working with A^*-perfect types.

Lemma 3.9 *Let $(\overline{M}, \bar{b})$ be any perfect chain realization of a striated type $\theta(\bar{x}) \in S_{at}(\bar{a})$. Then for every $\bar{c} \subseteq M_0$, $\mathrm{tp}(\bar{b}/\bar{a}\bar{c}) \in S_{at}(\bar{a}\bar{c})$ is a striated type extending $\theta(\bar{x})$, and $(\overline{M}, \bar{b})$ is a perfect chain realization of it.*

The Lemma below, whose proof simply amounts to unpacking definitions, demonstrate that striated types are rather malleable.

Lemma 3.10 *1. If* $\text{tp}(\bar{c}/\bar{a})$ *is a striated type of length* k *and* $\text{tp}(\bar{d}/\bar{a}\bar{c})$ *is a striated type of length* ℓ*, then* $\text{tp}(\bar{c}\bar{d}/\bar{a})$ *is a striated type of length* $k+\ell$*.*

2. *Suppose* $\text{tp}(\bar{b}/\bar{a})$ *is a striated type of length* ℓ *and* $k<\ell$*. Let* $\bar{b}_{<k}$ *and* $\bar{b}_{\geq k}$ *be the induced partition of* $\bar{b}$*. Then* $\text{tp}(\bar{b}_{<k}/\bar{a})$ *is a striated type of length* ℓ *and* $\text{tp}(\bar{b}_{\geq k}/\bar{a}\bar{b}_{<k})$ *is a striated type of length* $(\ell-k)$*. Moreover, if* $(\overline{M},\bar{b})$ *is a perfect chain realization of* $\text{tp}(\bar{b}/\bar{a})$*, then* $(\overline{M}_{<k},\bar{b}_{<k})$ *is a perfect chain realization of* $\text{tp}(\bar{b}_{<k}/\bar{a})$ *and* $(\overline{M}_{\geq k},\bar{b}_{\geq k})$ *is a perfect chain realization of* $\text{tp}(\bar{b}_{\geq k}/\bar{a}\bar{b}_{<k})$*.*

We begin by defining a partial order $(\mathbb{Q}_0,\leq_{\mathbb{Q}_0})$ of 'preconditions'. Then our forcing $(\mathbb{Q},\leq_{\mathbb{Q}})$ will be a dense suborder of these preconditions.

Definition 3.11 $\mathbb{Q}_0$ is the set of all $\mathbf{p}=(\bar{\mathbf{a}}_\mathbf{p},u_\mathbf{p},\bar{n}_\mathbf{p},\theta_\mathbf{p}(\bar{x}_\mathbf{p}),k_\mathbf{p},\mathcal{U}_\mathbf{p},c_\mathbf{p})$, where

1. $\bar{\mathbf{a}}_\mathbf{p}$ is a finite subset of M^* containing $\bar{b}^*$;
2. $u_\mathbf{p}$ is a finite subset of ω_1;
3. $\bar{n}_\mathbf{p}=\langle n_t : t\in u_\mathbf{p}\rangle$ is a sequence of positive integers;
4. $\bar{x}_\mathbf{p}=\langle \bar{x}_{t,\mathbf{p}} : t\in u_\mathbf{p}\rangle$, where each $\bar{x}_{t,\mathbf{p}}=\langle x_{t,n} : n<\bar{n}_t\rangle$ is a finite sequence from the set $X=\{x_{t,n} : t\in\omega_1, n\in\omega\}$ of variable symbols;
5. $\theta_\mathbf{p}(\bar{x}_\mathbf{p})\in S_{at}(\bar{\mathbf{a}}_\mathbf{p})$ is a striated type of length $|u_\mathbf{p}|$ (see Definition 3.7);
6. $k_\mathbf{p}\in\omega$;
7. $\mathcal{U}_\mathbf{p}\subseteq k_\mathbf{p}=\{0,\ldots,k_\mathbf{p}-1\}$;
8. $c_\mathbf{p}:\bar{x}_\mathbf{p}\to\omega$ is a function such that for all pairs $x_{t,n},x_{s,m}$ from $\bar{x}_\mathbf{p}$ with $c_\mathbf{p}(x_{t,n})=c_\mathbf{p}(x_{s,m})$
   (a) **either** $\text{spl}(b_{t,n},b_{s,m})\geq k_\mathbf{p}$ for all perfect chain realizations $(\overline{M},\bar{b})$ of $\theta_\mathbf{p}(\bar{x}_\mathbf{p})$;
   (b) **or** there is some $k\in\mathcal{U}_\mathbf{p}$ such that $\text{spl}(b_{t,n},b_{s,m})=k$ for all perfect chain realizations $(\overline{M},\bar{b})$ of $\theta_\mathbf{p}(\bar{x}_\mathbf{p})$.

We order elements of $\mathbb{Q}_0$ by: $\mathbf{p}\leq_{\mathbb{Q}_0}\mathbf{q}$ if and only if

- $\bar{\mathbf{a}}_\mathbf{p}\subseteq\bar{\mathbf{a}}_\mathbf{q}$;
- $u_\mathbf{p}\subseteq u_\mathbf{q}$ and $n_{t,\mathbf{p}}\leq n_{t,\mathbf{q}}$ for all $t\in u_\mathbf{p}$, hence $\bar{x}_{t,\mathbf{p}}$ is a subsequence of $\bar{x}_{t,\mathbf{q}}$;
- $\theta_\mathbf{q}(\bar{x}_\mathbf{q})\vdash\theta_\mathbf{p}(\bar{x}_\mathbf{p})$;
- $k_\mathbf{p}\leq k_\mathbf{q}$;
- $\mathcal{U}_\mathbf{p}=\mathcal{U}_\mathbf{q}\cap k_\mathbf{p}$ (hence, for $j<k_\mathbf{p}$, $j\in\mathcal{U}_\mathbf{p}$ if and only if $j\in\mathcal{U}_\mathbf{q}$);
- $c_\mathbf{p}=c_\mathbf{q}\restriction_{\bar{x}_\mathbf{p}}$.

Visibly, $(\mathbb{Q}_0,\leq_{\mathbb{Q}_0})$ is a partial order. As notation, for $\mathbf{p}\in\mathbb{Q}_0$ and $x_{t,n}\in\bar{x}_\mathbf{p}$, let $p(x_{t,n})\in S_{at}(\bar{a}_\mathbf{p})$ be $\text{tp}(e_{t,n}/\bar{a}_\mathbf{p})$ for any realization $\bar{e}_\mathbf{p}$ in M^* of $\theta_\mathbf{p}(\bar{x}_\mathbf{p})$. Call a precondition $\mathbf{p}\in\mathbb{Q}_0$ *unarily decided* if, for every $x_{t,n}\in\bar{x}_\mathbf{p}$, $p(x_{t,n})$ determines a type in $S^+_{at}(A_{k_\mathbf{p}},M^*)$ (see Definition 2.1). That the unarily decided preconditions are dense follows easily from the fact that $S^+_{at}(A_{k_\mathbf{p}},M^*)$ is countable.

Lemma 3.12 *The set* $\{\mathbf{p}\in\mathbb{Q}_0 : \mathbf{p}$ *is unarily decided*$\}$ *is dense in* $(\mathbb{Q}_0,\leq_{\mathbb{Q}_0})$*. Moreover, given any* $\mathbf{p}\in\mathbb{Q}_0$*, there is a unarily decided* $\mathbf{q}\geq_{\mathbb{Q}_0}\mathbf{p}$ *with* $\bar{x}_\mathbf{q}=\bar{x}_\mathbf{p}$ *and* $k_\mathbf{q}=k_\mathbf{p}$ *(hence* $\mathcal{U}_\mathbf{q}=\mathcal{U}_\mathbf{p}$*).*

Proof Fix $\mathbf{p} \in \mathbb{Q}_0$ and let $k := k_\mathbf{p}$. Arguing by induction on the size of the finite set $\overline{x}_\mathbf{p}$, it is enough to strengthen $p(x_{t,n})$ individually for each $x_{t,n} \in \overline{x}_\mathbf{p}$. So fix $x_{t,n} \in \overline{x}_\mathbf{p}$. By Corollary 2.6 there is an $\bar{a}' \supseteq \bar{a}_\mathbf{p}$ and a 1-type $q_1(x_{t,n}) \in S_{at}(\bar{a}')$ extending $p(x_{t,n})$ that determines a type in $S^+_{at}(A_{k_\mathbf{p}}, M^*)$. Then, using Lemma 3.9 we can choose a striated type $\theta'(\overline{x}_\mathbf{p}) \in S_{at}(\bar{a}')$ extending $\theta_\mathbf{p}(\overline{x}_\mathbf{p}) \cup q_1$.

We iterate the above procedure for each of the (finitely many) elements of $\overline{x}_\mathbf{p}$, thereby getting a unarily decided precondition $\mathbf{p}' \geq_{\mathbb{Q}_0} \mathbf{p}$ whose type $\theta_{\mathbf{p}'}(\overline{x}_\mathbf{p})$ still has the same free variables, and each of $k_\mathbf{p}, \mathcal{U}_\mathbf{p}, c_\mathbf{p}$ are unchanged. □

Next, call a precondition $\mathbf{p} \in \mathbb{Q}_0$ *fully decided* if it is unarily decided and for each pair $x_{t,n}, x_{s,m}$ from $\overline{x}_\mathbf{p}$ with $c_\mathbf{p}(x_{t,n}) = c_\mathbf{p}(x_{s,m})$, if $\mathrm{spl}(b_{t,n}, b_{s,m}) \geq k_\mathbf{p}$ for some perfect chain realization $(\overline{M}, \bar{b})$, then $\mathrm{tp}(b_{t,n}/A^*) = \mathrm{tp}(b_{s,m}/A^*)$ for all perfect chain realizations $(\overline{M}, \bar{b})$ of $\theta_\mathbf{p}(\overline{x}_\mathbf{p})$.

Lemma 3.13 *The set* $\{\mathbf{p} \in \mathbb{Q}_0 : \mathbf{p}$ *is fully decided*$\}$ *is dense in* $(\mathbb{Q}_0, \leq_{\mathbb{Q}_0})$. *Moreover, given any* $\mathbf{p} \in \mathbb{Q}_0$, *there is a fully decided* $\mathbf{q} \geq_{\mathbb{Q}_0} \mathbf{p}$ *with* $\overline{x}_\mathbf{q} = \overline{x}_\mathbf{p}$.

Proof It suffices to handle each pair $x_{t,n}, x_{s,m}$ from $\overline{x}_\mathbf{p}$ with $c(x_{t,n}) = c(x_{s,m})$ separately. Given such a pair, suppose there is some perfect chain realization $(\overline{M}, \bar{b})$ of $\theta(\overline{x}_\mathbf{p}) \in S_{at}(\bar{\mathbf{a}}_\mathbf{p})$ with $k_\mathbf{p} \leq \mathrm{spl}(b_{t,n}, b_{s,m}) < \omega$. Among all such perfect chain realizations, choose one that minimizes $k^* = \mathrm{spl}(b_{t,n}, b_{s,m})$. Choose a formula $\varphi(x, \bar{c})$ with $\bar{c}$ from A_{k^*+1} witnessing that $\mathrm{tp}(b_{t,n}/A_{k^*+1}) \neq \mathrm{tp}(b_{s,m}/A_{k^*+1})$. As $A_{k^*+1} \subseteq M_0$, by applying Lemma 3.9, let $\theta^*(\overline{x}_\mathbf{p})$ be a complete formula over $\bar{\mathbf{a}}_\mathbf{p}\bar{c}$ isolating $\mathrm{tp}(\bar{b}/\bar{\mathbf{a}}_\mathbf{p}\bar{c})$. Form the precondition $\mathbf{p}' \in \mathbb{Q}_0$ by putting $\bar{\mathbf{a}}_{\mathbf{p}'} = \bar{\mathbf{a}}_\mathbf{p}\bar{c}$; $\theta_{\mathbf{p}'} = \theta^*$; $k_{\mathbf{p}'} = k^* + 1$; and $\mathcal{U}_{\mathbf{p}'} = \mathcal{U}_\mathbf{p} \cup \{k^*\}$; while leaving $\overline{x}_\mathbf{p}$ and $c_\mathbf{p}$ unchanged. It is evident that $\mathrm{spl}(b'_{t,n}, b'_{s,m}) = k^* \in \mathcal{U}_{\mathbf{p}'}$ for all perfect chain realizations $(\overline{M}, \bar{b}')$ of $\theta_{\mathbf{p}'}$. Continuing this process for each of the (finitely many) relevant pairs gives us a fully decided extension of $\mathbf{p}$. □

Definition 3.14 The forcing $(\mathbb{Q}, \leq_\mathbb{Q})$ is the set of fully decided $\mathbf{p} \in \mathbb{Q}_0$ with the inherited order.

Lemma 3.15 *The forcing* $(\mathbb{Q}, \leq_\mathbb{Q})$ *has the countable chain condition (c.c.c.).*

Proof Suppose $\{\mathbf{p}_i : i \in \omega_1\}$ is an uncountable subset of $\mathbb{Q}$. In light of Lemma 3.13, it suffices to find $i \neq j$ for which there is some precondition $\mathbf{q} \in \mathbb{Q}_0$ satisfying $\mathbf{p}_i \leq_{\mathbb{Q}_0} \mathbf{q}$ and $\mathbf{p}_j \leq_{\mathbb{Q}_0} \mathbf{q}$. First, by the Δ-system lemma applied to the finite sets $\{u_{\mathbf{p}_i}\}$, we may assume that $|u_{\mathbf{p}_i}|$ is constant and there is some fixed u^* that is an initial segment of each $u_{\mathbf{p}_i}$ and, moreover, whenever $i < j$, every element of $(u_{\mathbf{p}_i} \setminus u^*)$ is less than every element of $(u_{\mathbf{p}_j} \setminus u^*)$. By further trimming, but preserving uncountability, we may assume that the integer $k_\mathbf{p}$, the subset $\mathcal{U}_\mathbf{p} \subseteq k_\mathbf{p}$, and the parameter $\bar{\mathbf{a}}_\mathbf{p}$ remain constant. As notation, for $i < j$, let $f : u_{\mathbf{p}_i} \to u_{\mathbf{p}_j}$ be the unique order-preserving bijection. We may additionally assume that $n_{\mathbf{p}_i}(t) = n_{\mathbf{p}_j}(f(t))$, hence f has a natural extension (also called f): $\overline{x}_{\mathbf{p}_i} \to \overline{x}_{\mathbf{p}_j}$ given by $f(x_{t,n}) = x_{f(t),n}$. With this identification, we may assume $\theta_{\mathbf{p}_i}(\overline{x}_{\mathbf{p}_i}) = \theta_{\mathbf{p}_j}(f(\overline{x}_{\mathbf{p}_i}))$. As well, we may also assume $\mathrm{tp}(x_{t,n}/A_{k_\mathbf{p}}) = \mathrm{tp}(x_{f(t),n}/A_{k_\mathbf{p}})$ for every $x_{t,n} \in \overline{x}_{\mathbf{p}_i}$. As well, the colorings match up as well, i.e., $c(x_{t,n}) = x_{f(t),n}$.

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Now fix $i < j$. Define $\mathbf{q}$ by $k_{\mathbf{q}} := k_{\mathbf{p}}$; $\mathcal{U}_{\mathbf{q}} := \mathcal{U}_{\mathbf{p}}$; and $\bar{\mathbf{a}}_{\mathbf{q}} := \bar{\mathbf{a}}_{\mathbf{p}}$ (the common values). Let $u_{\mathbf{q}} := u_{\mathbf{p}_i} \cup u_{\mathbf{p}_j}$, and, for $t \in u_{\mathbf{p}_i}$, $n_{t,\mathbf{q}} = n_{t,\mathbf{p}_i}$ while $n_{t,\mathbf{q}} = n_{t,\mathbf{p}_j}$ for $t \in u_{\mathbf{p}_j}$. To produce the striated type $\theta_{\mathbf{q}} \in S_{at}(\bar{\mathbf{a}}_{\mathbf{q}})$, first choose a perfect chain realization $(\overline{M}, \bar{b})$ of $\theta_{\mathbf{p}_i}(\overline{x}_{\mathbf{p}_i})$. Say $|u_{\mathbf{p}_i}| = \ell = |u_{\mathbf{p}_j}|$, while $|u^*| = k < \ell$. By Lemma 3.10(2), $\mathrm{tp}(\bar{b}_{<k}/\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length k and $(\overline{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of the striated type $\mathrm{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$ of length $(\ell - k)$. Choose $\bar{d}$ from M_k such that $\mathrm{tp}(\bar{d}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}) = \mathrm{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$. Then by Lemma 3.9 (with M_k playing the role of M_0 there), $(\overline{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of the striated type $\mathrm{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}\bar{d})$. So, by Lemma 3.10(1), $\mathrm{tp}(\bar{d}\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$ is a striated type of length $2(\ell - k)$. Thus, a second application of Lemma 3.10(1) implies that $\mathrm{tp}(\bar{b}_{<k}\bar{d}\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length $2\ell - k$. Let $\theta_{\mathbf{q}}$ be a complete formula over $\bar{\mathbf{a}}_{\mathbf{p}}$ generating this type.

In order to show that $\mathbf{q}$ is a precondition (i.e., an element of $\mathbb{Q}_0$) only Clause (8) requires an argument. Fix any $x_{t,n}, x_{s,m}$ in $\overline{x}_{\mathbf{q}}$ with $c_{\mathbf{q}}(x_{t,n}) = c_{\mathbf{q}}(x_{s,m})$. As both $\mathbf{p}_i, \mathbf{p}_j \in \mathbb{Q}_0$, the verification is immediate if $\{t, s\}$ is a subset of either $u_{\mathbf{p}_i}$ or $u_{\mathbf{p}_j}$, so assume otherwise. By symmetry, assume $t \in u_{\mathbf{p}_i} - u^*$ and $s \in u_{\mathbf{p}_j} - u^*$. The point is that by our trimming, $x_{f(t),n} \in \overline{x}_{\mathbf{p}_j}$, $c_{\mathbf{p}_j}(x_{f(t),n}) = c_{\mathbf{p}_i}(x_{t,n})$, and $\mathrm{tp}(x_{t,n}/A_{k_{\mathbf{p}}}) = \mathrm{tp}(x_{f(t),n}/A_{k_{\mathbf{p}}})$. There are now two cases: First, if $\mathrm{tp}(x_{f(t),n}/A^*) = \mathrm{tp}(x_{s,m}/A^*)$, then it follows that $\mathrm{tp}(x_{t,n}/A_{k_{\mathbf{p}}}) = \mathrm{tp}(x_{s,m}/A_{k_{\mathbf{p}}})$, hence $\mathrm{spl}(e_{t,n}, e_{s,m}) \geq k_{\mathbf{p}}$ for any perfect chain realization $(\overline{N}, \bar{e})$ of $\theta_{\mathbf{q}}$. On the other hand, if $\theta_{\mathbf{p}_j}$ 'says' $\mathrm{spl}(x_{f(t),n}, x_{s,m}) = k \in \mathcal{U}_{\mathbf{p}}$, then $\theta_{\mathbf{q}}$ 'says' $\mathrm{spl}(x_{t,n}, x_{s,m}) = k \in \mathcal{U}_{\mathbf{q}}$ as well. Thus, $\mathbf{q} \in \mathbb{Q}_0$, which suffices by Lemma 3.13. □

Lemma 3.16 *Each of the following sets are dense and open in* $(\mathbb{Q}, \leq_{\mathbb{Q}})$.

1. *For every* $t \in \omega_1$, $D_t = \{\mathbf{p} \in \mathbb{Q} : t \in u_{\mathbf{p}}\}$;
2. *For every* $(t, n) \in \omega_1 \times \omega$, $D_{t,n} = \{\mathbf{p} \in \mathbb{Q} : x_{t,n} \in \overline{x}_{\mathbf{p}}\}$; *and*
3. **Henkin witnesses:** *For all* $t \in \omega_1$, *all* $\langle x_{s_i,n_i} : i < m\rangle$ *with each* $s_i \leq t$ *and all* $\varphi(y, v_i : i < m)$, *the set* $\{\mathbf{p} \in \mathbb{Q} :$ **either** $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}}) \vdash \forall y \neg\varphi(y, x_{s_i,n_i} : i < m)$ **or** *for some* n^*, $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}}) \vdash \varphi(x_{t,n^*}, x_{s_i,n_i} : i < m)\}$.
4. *For all* $e \in M^*$, $D_e = \{\mathbf{p} \in \mathbb{Q} : e \in \bar{\mathbf{a}}_{\mathbf{p}}$ *and* $\theta(\overline{x}_{\mathbf{p}}) \vdash x_{0,n} = e$ *for some* $n \in \omega\}$.

Proof That each of these sets is open is immediate. As for density, in all four clauses we will show that given some $\mathbf{p} \in \mathbb{Q}$, we will find an extension $\mathbf{q} \geq_{\mathbb{Q}} \mathbf{p}$ with $\overline{x}_{\mathbf{q}}$ a one-point extension of $\overline{x}_{\mathbf{p}}$. In all cases, we will put $k_{\mathbf{q}} := k_{\mathbf{p}}$, $\mathcal{U}_{\mathbf{q}} = \mathcal{U}_{\mathbf{p}}$ and since $\overline{x}_{\mathbf{p}}$ is finite, we can choose the color $c_{\mathbf{q}}$ of the 'new element' to be distinct from the other colors. Because of that, Clause (8) for $\mathbf{q}$ follows immediately from the fact $\mathbf{p} \in \mathbb{Q}$. Thus, for all four clauses, all of the work is in finding a striated type $\theta_{\mathbf{q}}$ extending $\theta_{\mathbf{p}}$.

(1) Fix $t \in \omega_1$ and choose an arbitrary $\mathbf{p} \in \mathbb{Q}$. If $t \in u_{\mathbf{p}}$ then there is nothing to prove, so assume otherwise. Let $\ell = |u_{\mathbf{p}}|$ and let $k = |\{s \in u_{\mathbf{p}} : s < t\}|$. Assume that $k < \ell$, as the case of $k = \ell$ is similar, but easier. Choose a perfect chain realization $(\overline{M}, \bar{b})$ of $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}})$. By Lemma 3.10(2), $\mathrm{tp}(\bar{b}_{<k}/\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length k. By Lemma 2.4(1), choose an A^*-large type $r \in S_{at}(\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$ and choose a realization e of r in M_k. One checks immediately that $\mathrm{tp}(\bar{b}_{<k}e/\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length $(k + 1)$. Now, also by Lemma 3.10(2), $(\overline{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of $\mathrm{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$. So, by Lemma 3.9, $(\overline{M}_{\geq k}, \bar{b}_{\geq k})$ is also a perfect chain realization of $\mathrm{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}e)$. In particular, $\mathrm{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}e)$ is a striated type of length $(\ell - k)$. Thus,

by Lemma 3.10(1), $\text{tp}(\bar{b}_{<k}e\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length $(\ell+1)$. Take $\bar{\mathbf{a}}_{\mathbf{q}} := \bar{\mathbf{a}}_{\mathbf{p}}$, $\overline{x}_{\mathbf{q}} := \overline{x}_{\mathbf{p}} \cup \{x_{t,0}\}$, and take $\theta_{\mathbf{q}}(\overline{x}_{\mathbf{q}})$ to be a complete formula in $\text{tp}(\bar{b}_{<k}e\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{q}})$.

The proofs of (2) and (3) are extremely similar. We prove (2) and indicate the adjustment necessary for (3). Fix $(t,n) \in \omega_1 \times \omega$. By (1) and an inductive argument, we may assume we are given $\mathbf{p} \in \mathbb{Q}$ with $t \in u_{\mathbf{p}}$ and $x_{t,n-1} \in \overline{x}_{\mathbf{p}}$. Say $|u_{\mathbf{p}}| = \ell$ and assume t is the $(k-1)$st element of u_p in ascending order. Choose a perfect chain realization $(\overline{M}, \bar{b})$ of $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}})$. By Lemma 3.10(2), $\text{tp}(\bar{b}_{<k}/\bar{\mathbf{a}}_{\mathbf{p}})$ is striated of length k. Choose an arbitrary $e \in M_k$[3] and adjoin it to $\bar{b}_{k-1}$. More formally, let $\bar{b}^*_{<k} := \langle \bar{b}^*_j : j < k \rangle$, where $\bar{b}^*_j = \bar{b}_j$ for $j < k-2$, while $\bar{b}^*_{k-1} := \bar{b}_{k-1}e$. Note that $\text{tp}(\bar{b}^*_{<k}/\bar{\mathbf{a}}_{\mathbf{p}})$ remains a striated type of length k. By Lemma 3.10(2), $(\overline{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of $\text{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$. So, by Lemma 3.9 it is also a perfect chain realization of $\text{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}^*_{<k})$. In particular, $\text{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}^*_{<k})$ is a striated type of length $(\ell-k)$, so $\text{tp}(\bar{b}^*_{<k}\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length ℓ extending $\theta_{\mathbf{p}}(\overline{x}_{\mathbf{p}})$. Put $\overline{x}_{\mathbf{q}} := \overline{x}_{\mathbf{p}} \cup \{x_{t,n}\}$ and let $\theta_{\mathbf{q}}(\overline{x}_{\mathbf{q}})$ be a complete formula isolating this type.

(4) is also similar and is left to the reader. □

The following Proposition follows immediately from the density conditions described above.

Proposition 3.17 *Let G be a $\mathbb{Q}$-generic filter. Then, in $V[G]$, a rich, $\mathcal{U}_G$-colored atomic model of T exists, where $\mathcal{U}_G = \{k \in \omega : k \in \mathcal{U}_{\mathbf{p}}$ for some $\mathbf{p} \in G\}$.*

Proof There is a congruence $\sim_G$ defined on $X = \{x_{t,n} : t \in \omega_1, n \in \omega\}$ defined by $x_{t,n} \sim_G x_{s,m}$ if and only if $\theta_{\mathbf{p}} \vdash x_{t,n} = x_{s,m}$ for some $\mathbf{p} \in G$. Let M_G be the model of T with universe $X/\sim_G$ and relations $M_G \models \varphi(a_1,\ldots,a_k)$ if and only if there are $(x_{t_1,n_1},\ldots,x_{t_k,n_k}) \in X^k$ such that $[x_{t_i,n_i}] = a_i$ for each i and $\theta_{\mathbf{p}} \vdash \varphi(x_{t_1,n_1},\ldots,x_{t_k,n_k})$ for some $\mathbf{p} \in G$. Since $(\mathbb{Q}, \leq_{\mathbb{Q}})$ has c.c.c., M_G has size $\aleph_1$. As notation, for each $t \in \omega_1$, let $M_{\leq t}$ be the substructure of M_G with universe $\{[x_{s,m}] : s \leq t, m \in \omega\}$. Then $M^* \preceq M_0$ and $M_{\leq s} \preceq M_{\leq t} \preceq M_G$ whenever $s \leq t < \omega_1$. The definition of a striated type implies that $\text{tp}([x_{t,0}]/A^*)$ is omitted in $M_{<t}$, hence the set $\{[x_{t,0}] : t \in \omega_1\}$ witnesses that $(M_G, \bar{b}^*)$ is rich. Also, define $c_G := \bigcup\{c_{\mathbf{p}} : \mathbf{p} \in G\}$. Using the fact that each $\mathbf{p} \in \mathbb{Q}$ is fully decided, check that c_G is a $\mathcal{U}_G$-coloring of $(M_G, \bar{b}^*)$. □

Note that in the Conclusion below, such a $G \in V$ always exists, since $\mathcal{B}$ is countable. Recall the finite fragment ZFC^* of ZFC given by Proposition 2.9.

Conclusion 3.18 *Suppose $\mathcal{B}$ is a countable, transitive model of ZFC^*, with $\{M^*, T, L\} \subseteq \mathcal{B}$, and let $G \in V$, $G \subseteq \mathbb{Q}$ be any filter meeting every dense $D \subseteq \mathbb{Q}$ with $D \in \mathcal{B}$. Then: Let $\mathcal{U}_G = \{k \in \omega : k \in \mathcal{U}_{\mathbf{p}}$ for some $\mathbf{p} \in G\}$. Then:*

1. *$\mathcal{U}_G \in V$; and*
2. *In V, there is a $\mathcal{U}_G$-colored, rich atomic model $(N, \bar{b}^*)$ of T.*

Proof That $\mathcal{U}_G \in V$ is immediate, since both $\mathcal{B}$ and G are. As for (2), as G meets every dense set in $\mathcal{B}$, $\mathcal{B}[G]$ is a countable, transitive model of ZFC^*, and by applying

[3] In the proof of (3), e would be a realization of $\varphi(y, b_{s_i,n_i} : i < m)$ in M_k, if one existed.

Proposition 3.17,

$$\mathcal{B}[G] \models \text{`There is a rich, } \mathcal{U}_G\text{-colored } (M_G, \bar{b}^*) \text{ of size } \aleph_1\text{'}$$

Let $L' = L \cup \{c, R\} \cup \{c_m : m \in M^*\}$ Working in $\mathcal{B}[G]$, expand M_G to an L'-structure M', interpreting each c_m by m, interpreting the unary function $c^{M'}$ as $c_G = \bigcup\{c_{\mathbf{p}} : \mathbf{p} \in G\}$, and the unary predicate $R^{M'} = \{[x_{t,0}] : t \in \omega_1\}$.

Now, for each $d, d' \in M'$ and $k \in \omega$, the relation $\text{tp}_{M'}(d/A_k) = \text{tp}_{M'}(d'/A_k)$ is definable by an $L'_{\omega_1,\omega}$-formula. Thus, the binary function spl : $(M')^2 \to (\omega + 1)$ is also $L'_{\omega_1,\omega}$-definable, hence, using the coloring c, there is an $L'_{\omega_1,\omega}$-sentence Ψ stating that 'c induces a $\mathcal{U}_G$-coloring.' Finally, using the Q-quantifier to state that R is uncountable, there is an $L'_{\omega_1,\omega}$-sentence $\Phi \in \mathcal{B}[G]$ stating that the $L(\bar{b}^*)$-reduct of a given L'-structure is a rich, atomic model of T, that is $\mathcal{U}_G$-colored via c. We finish by applying Proposition 2.9 to M' and Φ. □

3.3 Mass production

In this subsection we define a forcing $(\mathbb{P}, \leq_{\mathbb{P}})$ such that a $\mathbb{P}$-generic filter G produces a perfect set $\{G_\eta : \eta \in 2^\omega\}$ of $\mathbb{Q}$-generic filters such that the associated subsets $\{\mathcal{U}_{G_\eta} : \eta \in 2^\omega\}$ of ω are almost disjoint. Although the application there is very different, the argument in this subsection is similar to one appearing in [9].

We begin with one easy density argument concerning the partial order $(\mathbb{Q}, \leq_{\mathbb{Q}})$. Fundamentally, it allows us to 'stall' the construction for any fixed, finite length of time.

Lemma 3.19 *For every $\mathbf{p} \in \mathbb{Q}$ and every $k^* > k_{\mathbf{p}}$, there is $\mathbf{q} \geq_{\mathbb{Q}} \mathbf{p}$ such that $\bar{x}_{\mathbf{q}} = \bar{x}_{\mathbf{p}}$, (hence $c_{\mathbf{q}} = c_{\mathbf{p}}$); but $k_{\mathbf{q}} = k^*$ and $\mathcal{U}_{\mathbf{q}} = \mathcal{U}_{\mathbf{p}}$, i.e., $\mathcal{U}_{\mathbf{q}} \cap [k_{\mathbf{p}}, k^*) = \emptyset$.*

Proof Simply define $\mathbf{q}$ as above and then verify that $\mathbf{q} \in \mathbb{Q}$. □

Definition 3.20 For $n \in \omega$, let

$$\mathbb{P}_n = \{(k, \bar{p}) : k \in \omega, \bar{p} = \langle p_\eta : \eta \in 2^n\rangle, \text{ where each } p_\nu \in \mathbb{Q} \text{ and every } k_{p_\nu} = k\}$$

As notation, for $\mathbf{p} \in \mathbb{P}_n$, we let $k(\mathbf{p})$ denote the (integer) first coordinate of $\mathbf{p}$. For each $\ell < k(\mathbf{p})$, define the *trace of* ℓ, $\text{tr}_\ell(\mathbf{p}) = \{\nu \in 2^n : \ell \in \mathcal{U}_{p_\nu}\}$.

Let $\mathbb{P} = \bigcup_{n\in\omega} \mathbb{P}_n$. As notation, for $\mathbf{p} \in \mathbb{P}$, $n(\mathbf{p})$ is the unique n for which $\mathbf{p} \in \mathbb{P}_n$.

Definition 3.21 Define an order $\leq_{\mathbb{P}}$ on $\mathbb{P}$ by $\mathbf{p} \leq_{\mathbb{P}} \mathbf{q}$ if and only if

1. $n(\mathbf{p}) \leq n(\mathbf{q})$, $k(\mathbf{p}) \leq k(\mathbf{q})$;
2. $p_\nu \leq_{\mathbb{Q}} q_\mu$ for all pairs $\nu \in 2^{n(\mathbf{p})}$, $\mu \in 2^{n(\mathbf{q})}$ satisfying $\nu \trianglelefteq \mu$; and
3. For all $\ell \in [k(\mathbf{p}), k(\mathbf{q}))$, the set $\{\mu\restriction_{n(\mathbf{p})} : \mu \in \text{tr}_\ell(\mathbf{q})\}$ is either empty or is a singleton.

It is easily checked that $(\mathbb{P}, \leq_{\mathbb{P}})$ is a partial order, hence a notion of forcing. The following Lemma describes the dense subsets of $\mathbb{P}$.

Lemma 3.22 *1. For each n and k, $\{\mathbf{p} \in \mathbb{P} : n(\mathbf{p}) \geq n\}$ and $\{\mathbf{p} \in \mathbb{P} : k(\mathbf{p}) \geq k\}$ are dense;*

2. Suppose D is a dense, open subset of $\mathbb{Q}$. Then for every n and every $\mathbf{p} \in \mathbb{P}_n$, there is $\mathbf{q} \in \mathbb{P}_n$ such that $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$ and, for every $\nu \in 2^n$, $\mathbf{q}_\nu \in D$.

Proof Arguing by induction, it suffices to prove that for any given $\mathbf{p} \in \mathbb{P}$, there is $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$ with $n(\mathbf{q}) = n(\mathbf{p}) + 1$ and an $\mathbf{r} \geq_{\mathbb{P}} \mathbf{p}$ with $k(\mathbf{r}) > k(\mathbf{p})$. Fix $\mathbf{p} \in \mathbb{P}$. Say $\mathbf{p} \in \mathbb{P}_n$ and $\mathbf{p} = (k, \bar{p})$. To construct $\mathbf{q}$, for each $\nu \in 2^n$, define $q_{\nu 0} = q_{\nu 1} = p_\nu$. Let $\bar{q} := \langle q_\mu : \mu \in 2^{n+1}\rangle$ and $\mathbf{q} = (k, \bar{q})$. Then $\mathbf{q} \in \mathbb{P}_{n+1}$ and $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$ (note that Clause (3) in the definition of $\leq_{\mathbb{P}}$ is vacuously satisfied since $k(\mathbf{p}) = k(\mathbf{q})$).

To construct $\mathbf{r}$, simply apply Lemma 3.19 to each p_ν to produce an extension $r_\nu \geq_{\mathbb{Q}} p_\nu$ with $k_{r_\nu} = k + 1$, but $\mathcal{U}_{r_\nu} = \mathcal{U}_{p_\nu}$. Then let $\bar{r} := \langle r_\nu : \nu \in 2^n\rangle$ and $\mathbf{r} = (k + 1, \bar{r})$. Then $\mathbf{r} \geq_{\mathbb{P}} \mathbf{p}$ as required.

(2) Fix such a D and n. As we are working exclusively in $\mathbb{P}_n$ and because 2^n is a fixed finite set, it suffices to prove that for any chosen $\nu \in 2^n$,

For every $\mathbf{p} \in \mathbb{P}_n$ there is $\mathbf{q} \in \mathbb{P}_n$ with $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$ and $q_\nu \in D$.

To verify this, fix $\nu \in 2^n$ and $\mathbf{p} \in \mathbb{P}_n$. Concentrating on p_ν, as D is dense, choose $q_\nu \in D \cap \mathbb{Q}$ with $q_\nu \geq_{\mathbb{Q}} p_\nu$. Let $k^* := k_{q_\nu}$. Next, for each $\delta \in 2^n$ with $\delta \neq \nu$, apply Lemma 3.19 to p_δ, obtaining some $q_\delta \in \mathbb{Q}$ satisfying $q_\delta \geq_{\mathbb{Q}} p_\delta$, $k_{q_\delta} = k^*$, but $\mathcal{U}_{q_\delta} = \mathcal{U}_{p_\delta}$. Now, collect all of this data into a condition $\mathbf{q} \in \mathbb{P}_n$ defined by $k(\mathbf{q}) = k^*$ and $\bar{q} = \langle q_\gamma : \gamma \in 2^n\rangle$, where each q_γ is as above. To see that $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$, Clause (3) is verified by noting that for every $\ell \in [k(\mathbf{p}), k^*)$, $\mathrm{tr}_\ell(\mathbf{q})$ is either empty, or equals $\{\nu\}$, depending on whether or not $\ell \in \mathcal{U}_{q_\nu}$. ◻

Notation 3.23 Suppose $\mathcal{B} \models ZFC^*$ and let $G^* \subseteq \mathbb{P}$, $G^* \in V$ be a filter meeting every dense subset $D^* \subseteq \mathbb{P}$ with $D^* \in \mathcal{B}$. For each n and $\nu \in 2^n$, let

$$G_\nu := \{\mathbf{p} \in \mathbb{Q} : \text{ for some } \mathbf{p}^* = (k, \bar{p}) \in G^*, \mathbf{p} = \mathbf{p}^*_\nu\}$$

Then, for each $\eta \in 2^\omega$, let

$$G_\eta := \bigcup\{G_{\eta|n} : n \in \omega\} \text{ and } \mathcal{U}_\eta := \{\ell \in \omega : \ell \in \mathcal{U}_{\mathbf{q}} \text{ for some } \mathbf{q} \in G_\eta\}$$

Proposition 3.24 *In the notation of 3.23:*

1. *For every $\eta \in 2^\omega$, $G_\eta \subseteq \mathbb{Q}$ is a filter meeting every dense $D \subseteq \mathbb{Q}$ with $D \in \mathcal{B}$;*
2. *The sets $\{\mathcal{U}_\eta : \eta \in 2^\omega\}$ are an almost disjoint family of infinite subsets of ω.*

Proof (1) follows immediately from Lemma 3.22(2).

(2) Choose distinct $\eta, \eta' \in 2^\omega$. Choose n_0 such that $\eta|n \neq \eta'|n$ whenever $n \geq n_0$. By Lemma 3.22(1), choose $\mathbf{p}^* \in G^*$ with $n(\mathbf{p}^*) \geq n_0$. We show that $\mathcal{U}_\eta \cap \mathcal{U}_{\eta'}$ is finite by establishing that if $\ell \in \mathcal{U}_\eta \cap \mathcal{U}_{\eta'}$, then $\ell \leq k(\mathbf{p}^*)$.

To establish this, choose $\ell \in \mathcal{U}_\eta \cap \mathcal{U}_{\eta'}$. By unpacking the definitions, choose $\mathbf{q}^*, \mathbf{r}^* \in G^*$ such that, letting $\mu := \eta|n(\mathbf{q}^*)$ and $\mu' := \eta'|n(\mathbf{r}^*)$, we have $\ell \in \mathcal{U}_{\mathbf{q}^*_\mu} \cap \mathcal{U}_{\mathbf{r}^*_{\mu'}}$. As G^* is a filter, choose $\mathbf{s}^* \in G^*$ with $\mathbf{s}^* \geq_{\mathbb{P}} \mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*$. As notation, let $\delta := \eta|n(\mathbf{s}^*)$ and $\delta' := \eta'|n(\mathbf{s}^*)$.

Claim: $\ell \in \mathcal{U}_{\mathbf{s}^*_\delta} \cap \mathcal{U}_{\mathbf{s}^*_{\delta'}}$.

Proof As $\ell \in \mathcal{U}_{\mathbf{q}^*_\mu}$, $\ell < k(\mathbf{q}^*)$. From $\mathbf{q}^* \leq_{\mathbb{P}} \mathbf{s}^*$ we conclude $k(\mathbf{q}^*) \leq k(\mathbf{s}^*)$, so $\ell < k(\mathbf{s}^*)$ as well. From $\mathbf{q}^* \leq_{\mathbb{P}} \mathbf{s}^*$ and $\mu \trianglelefteq \delta$ we obtain $\mathbf{q}^*_\mu \leq_{\mathbb{Q}} \mathbf{s}^*_\delta$. But then, as $\ell \in \mathcal{U}_{\mathbf{q}^*_\mu}$, it follows that $\ell \in \mathcal{U}_{\mathbf{s}^*_\delta}$. That $\ell \in \mathcal{U}_{\mathbf{s}^*_{\delta'}}$ is analogous, using $\mathbf{r}^*$ in place of $\mathbf{q}^*$.

Finally, assume by way of contradiction that $\ell \geq k(\mathbf{p}^*)$. The Claim implies that $\{\delta, \delta'\} \subseteq \mathrm{tr}_\ell(\mathbf{s}^*)$. As $\ell \in [k(\mathbf{p}^*), k(\mathbf{s}^*))$, Clause (3) of $\mathbf{p}^* \leq_{\mathbb{P}} \mathbf{s}^*$ implies that $\delta|n(\mathbf{p}^*) = \delta'|n(\mathbf{p}^*)$. But, as $\eta|n(\mathbf{p}^*) = \delta|n(\mathbf{p}^*)$ and $\eta'|n(\mathbf{p}^*) = \delta'|n(\mathbf{p}^*)$, we contradict our choice of $\mathbf{p}^*$. □

We close this section with the proof of Proposition 3.1, which we restate for convenience.

Conclusion 3.25 *There is a family $\{(N_\eta, \bar{b}^*) : \eta \in 2^\omega\}$ of $2^{\aleph_0}$ rich, atomic models of T, each of size $\aleph_1$, that are pairwise non-isomorphic over $\bar{b}^*$.*

Proof Choose any countable, transitive model $\mathcal{B}$ of ZFC^* and choose any $G^* \in V$, $G^* \subseteq \mathbb{P}$, G^* meets every dense subset $D^* \in \mathcal{B}$ (as $\mathcal{B}$ is countable, such a G^* exists). For each $\eta \in 2^\omega$, choose G_η and $\mathcal{U}_\eta$ as in Proposition 3.24, and apply Conclusion 3.18 to get a rich $\mathcal{U}_\eta$-colored $(N_\eta, \bar{b}^*)$ in V. That this family is pairwise non-isomorphic over $\bar{b}^*$ follows immediately from Corollary 3.6, since the sets $\{\mathcal{U}_\eta : \eta \in 2^\omega\}$ are almost disjoint. □

4 The proof of Theorem 1.4

Assume that the class $\mathbf{At}_T$ is not pcl-small, as witnessed by an (uncountable) model N^* containing a finite tuple $\bar{a}^*$. Fix a countable, elementary substructure $M^* \preceq N^*$ that contains $\bar{a}^*$. To aid notation, let $D^* := \mathrm{pcl}_{N^*}(\bar{a}^*)$. We now split into cases, depending on the relationship between the cardinals $2^{\aleph_0}$ and $2^{\aleph_1}$.

Case 1. $2^{\aleph_0} < 2^{\aleph_1}$.

In this case, expand the language of T to $L(D^*)$, adding a new constant symbol for each $d \in D^*$. Then, the natural expansion $N^*_{D^*}$ of N^* to an $L(D^*)$-structure is a model of the infinitary $L(D^*)$-sentence Φ that entails $Th(N^*_{D^*})$ and ensures that every finite tuple is L-atomic with respect to T. As $N^*_{D^*}$ is a model of Φ that realizes uncountably many types over the empty set (after fixing D^*!), it follows from [6], Theorem 45 of Keisler that there are $2^{\aleph_1}$ pairwise non-$L(D^*)$-isomorphic models Φ, each of size $\aleph_1$. As $2^{\aleph_0} < 2^{\aleph_1}$, it follows that there is a subfamily of $2^{\aleph_1}$ pairwise non-L-isomorphic reducts to the original language L. As each of these models are L-atomic, we conclude that $\mathbf{At}_T$ has $2^{\aleph_1}$ non-isomorphic models of size $\aleph_1$.

Case 2. $2^{\aleph_0} = 2^{\aleph_1}$.

Choose $\bar{b}^*$ from M^* as in Proposition 2.10 and apply Conclusion 3.25 to get a set $\mathcal{F}^* = \{(N_\eta, \bar{b}^*) : \eta \in 2^\omega\}$ of atomic models, each of size $\aleph_1$, that are pairwise non-isomorphic over $\bar{b}^*$. Let $\mathcal{F} = \{N_\eta : \eta \in 2^\omega\}$ be the set of reducts of elements from $\mathcal{F}^*$. By our cardinal hypothesis, $\mathcal{F}$ has size $2^{\aleph_1}$. The relation of L-isomorphism is an equivalence relation on $\mathcal{F}$, and each L-isomorphism equivalence class has size at most $\aleph_1$ (since $\aleph_1^{<\omega} = \aleph_1$). As $\aleph_1 < 2^{\aleph_1}$ we conclude that $\mathcal{F}$ has a subset of size $2^{\aleph_1}$ of pairwise non-isomorphic atomic models of T, each of size $\aleph_1$. □

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