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On Groups A Such That $A \oplus \mathbf{Z}^n \cong A$

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Abstract. For any $m \geq 2$ we construct a locally free group A of cardinality $2^{\aleph_0}$ such that $A \oplus \mathbf{Z}^m \cong A$ but $A \oplus \mathbf{Z}^n \not\cong A$ if $1 \leq n < m$. We show that it is not necessarily the case that there exists such an A of cardinality $\aleph_1$.

In 1983, Gabriel Sabbagh pointed out to the first author the following question: if A is an abelian group such that $A \oplus \mathbf{Z}^2 \cong A$, is it the case that $A \oplus \mathbf{Z} \cong A$? (cf.[4;p.222]). The question may be re-formulated and generalized as follows. Given A, let $S(A) = \{n \varepsilon \mathbf{Z}_+ : A \oplus \mathbf{Z}^n \cong A\}$ (where $\mathbf{Z}_+$ is the set of positive integers). It is not hard to see that if $S(A) \neq \phi$, then $S(A) = m\mathbf{Z}_+ (=\{mk : k \varepsilon \mathbf{Z}_+\})$ for some m. The question then becomes: is there for each $m \varepsilon \mathbf{Z}_+$ a group A such that $S(A) = m\mathbf{Z}_+$?

M. Dugas observed in 1983 that Stein's Lemma [1; Cor. 19.3] implies that for any countable A, $S(A) \neq \phi$ implies $A \oplus \mathbf{Z}^{(\omega)} \cong A$, so $S(A) = \mathbf{Z}_+$ (see also the Corollary below). In the spring of 1985, while both authors were visiting Simon

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Fraser University, Shelah found a construction which answered Sabbagh's question, as follows:

MAIN THEOREM. For every $m \in \mathbf{Z}_+$ there is a locally free group A of cardinality $2^{\aleph_0}$ such that for all $n \in \mathbf{Z}_+$, $A \oplus \mathbf{Z}^n \cong A$ if and only if n is a multiple of m.

Here "locally free" means that A is $\aleph_1$-free and separable: in fact, A will be a pure subgroup of $\mathbf{Z}^\omega$.

Before proving this theorem, let us observe that it is not possible to find such examples of cardinality less than the continuum. (For MA, see for example [2]).

THEOREM. Assume MA. If A is a group of cardinality $< 2^{\aleph_0}$, such that $A \oplus \mathbf{Z}^m \cong A$ for some $m \in \mathbf{Z}_+$ then $A \oplus \mathbf{Z}^{(\omega)} \cong A$ (and hence $S(A) = \mathbf{Z}_+$).

Proof: Suppose there exists a monomorphism $\varphi: A \to A$ such that $A = \varphi(A) \oplus \bigoplus_{i=1}^{m} \mathbf{Z}x_i$, for some $x_i \in A$. Then $A = \varphi^2(A) \oplus \bigoplus_{i=1}^{m} (\mathbf{Z}x_i \oplus \mathbf{Z}\varphi(x_i))$. More generally, for each $k \geq 1$, if we define $x_{mk-m+i} = \varphi^k(x_i)$ for $i = 1,\cdots,m$, then $A = \varphi^k(A) \oplus F_k$, where F_k is the free subgroup of A with basis $\{x_1,\cdots,x_{mk}\}$. Let π_k be the projection of A on F_k along $\varphi^k(A)$. Let F be the free subgroup of A with basis $\{x_i : i \in \mathbf{Z}_+\}$.

We shall employ MA to show that there is a homomorphism $g: A \to F$ whose range is not finitely-generated. This will clearly suffice, for then the range of g is isomorphic to $\mathbf{Z}^{(\omega)}$, and $A \cong A' \oplus \mathbf{Z}^{(\omega)}$, where $A' = \ker g$.

To this end, we let $\mathbf{P}$ consist of all homomorphisms $f: B \to F_k$ (for some $k \in \mathbf{Z}_+$) such that: B is a finitely-generated subgroup of A; and $f = h \circ (\pi_k \restriction B)$ where h is an endomorphism of F_k. We partially order $\mathbf{P}$ by extension, i.e., $f_1 \leq f_2$ iff f_1 is an extenson of f_2.

Now $\mathbf{P}$ is easily seen to be c.c.c. since if X is an uncountable subset of $\mathbf{P}$ there must exist $f_1 \neq f_2$ in X such that $f_i = h \circ (\pi_k \restriction B_i)$ for some $k \in \mathbf{Z}_+$ and $h \in \mathrm{End}(F_k)$. But then clearly f_1 and f_2 are compatible, i.e., they have a common extension (to $B_1 + B_2$).

For each $a \in A$, define $D_a = \{f \in \mathbf{P} : a \in \mathrm{dom}\, f\}$. It is easy to see that each D_a is a dense subset of $\mathbf{P}$, since for any $a \in A$, and $f = h \circ (\pi_k \restriction B)$ in $\mathbf{P}$, f extends to $h \circ (\pi_k \restriction B + \langle a \rangle)$.

For each $n \in \mathbf{Z}_+$, let $E_n = \{f \in \mathbf{P} :$ the range of f has rank $\geq n\}$. Suppose, for the moment, that each E_n is dense in $\mathbf{P}$. Then, by MA, since $|A| < 2^{\aleph_0}$, there exists a filter G in $\mathbf{P}$ such that for each

$a \in A$, $G \cap D_a \neq \phi$, and for each $n \in \mathbf{Z}_+$, $G \cap E_n \neq \phi$. Then $g = \cup G$ is the desired homomorphism: $A \to F$.

So it remains to prove that each E_n is dense. Let $f \in \mathbf{P}$; say $f = h \circ (\pi_k \upharpoonright B)$ where $B = \langle b_1, \cdots, b_r \rangle$. It suffices to prove that there is an extension f' of f whose range has rank $\geq \text{rank}(\text{rge}(f)) + 1$. Choose k' such that $m(k'-k) > r$, and let $B' = B + F_{k'}$. We claim that there is an endomorphism h' of $F_{k'}$ such that $h'(\pi_k(b_j)) = h(\pi_k(b_j))$ for $j = 1, \cdots, r$; and such that $h'(x_\ell) \in \langle x_{mk+1}, \cdots, x_{mk'} \rangle - \{0\}$ for some $\ell \in \{mk+1, \cdots, mk'\}$. If so then we can define f' to be $h' \circ (\pi_{k'} | B')$.

So it remains to prove the claim. We have

$$F_{k'} = F_k \oplus C$$

where $C = \langle x_{mk+1}, \cdots, x_{mk'} \rangle$. For each $j = 1, \cdots, r$, $\pi_{k'}(b_j) = \pi_k(b_j) + c_j$ for some $c_j \in C$. Now $\langle c_1, \cdots, c_r \rangle_*$ has rank $\leq r < rk(C) = m(k'-k)$, so $C = \langle c_1, \cdots, c_r \rangle_* \oplus C'$ for some non-zero C'. Then define h' by: $h' \upharpoonright F_k = h$; $h' \upharpoonright \langle c_1, \cdots, c_r \rangle_* = 0$; $h' \upharpoonright C' = id_{C'}$. ◻

COROLLARY. If A is a countable group such that $A \oplus \mathbf{Z}^m \cong A$ for some $m \in \mathbf{Z}_+$, then $A \oplus \mathbf{Z}^{(\omega)} \cong A$.

Proof: This follows from the proof of the theorem, and the fact that $MA(\aleph_0)$ is a theorem of ZFC.

REMARK. In fact, the assertion that (for $m \geq 2$) there is an A of cardinality $\aleph_1$ with $S(A) = m\mathbf{Z}_+$ is independent of ZFC + $\neg$CH. The above theorem shows that this assertion is not provable in ZFC + $\neg$CH. On the other hand, the assertion can be shown to be true in a model of ZFC obtained by adding $\aleph_2$ Cohen generic reals, which is a model of ZFC + $\neg$CH

Proof of the Main Theorem: Fix $m \geq 2$. For each $j \in \mathbf{Z}$, define $\varphi^j : \mathbf{Z}^\omega \to \mathbf{Z}^\omega$ by: $(\varphi^j(a))(n) = a(n-mj)$ for each $n \in \omega$. (If $a \in \mathbf{Z}^\omega$, $a(k)$ denotes its kth coordinate if $k \in \omega$; and $a(k) = 0$ if $k < 0$.) Let $\varphi = \varphi^1$.

Let $\overline{\mathbf{Z}^{(\omega)}}$ denote the closure in $\mathbf{Z}^\omega$ of $\mathbf{Z}^{(\omega)}$ with respect to the $\mathbf{Z}$-adic topology. Then $K \overset{\text{def}}{=} \{a \in \mathbf{Z}^\omega : a(n) \in n!\mathbf{Z}\}$ is a subgroup of $\overline{\mathbf{Z}^{(\omega)}}$.

Let $\{\psi_\nu : \nu < 2^{\aleph_0}\}$ be a list of all homomorphisms: $\mathbf{Z}^{(\omega)} \to \mathbf{Z}^\omega$. Notice that each ψ_ν has at most one extension to a homomorphism $\overline{\psi}_\nu : \overline{Z^{(\omega)}} \to Z^\omega$.

We shall choose a certain countable subgroup S of

$\overline{\mathbf{Z}^{(\omega)}}$ containing $\mathbf{Z}^{(\omega)}$ and closed under φ and φ^{-1}. We shall then select inductively elements $a_\nu (\nu < 2^{\aleph_0})$ of K and let A be the pure closure in $\mathbf{Z}^\omega$ of $S + \langle \varphi^j(a_\nu): \nu < 2^{\aleph_0}, j \in \mathbf{Z} \rangle$. Then A will be a subgroup of $\overline{Z^{(\omega)}}$ and $\varphi \restriction A: A \to A$ will be a monomorphism. Moreover since A is closed under φ^{-1}, we will have $A = \varphi(A) \oplus \bigoplus_{i=0}^{m-1} \mathbf{Z}e_i$ (where $e_i(j) = \delta_{ij}$). Since $A \cong \varphi(A)$, we will have $A \cong A \oplus \mathbf{Z}^m$.

We shall choose the a_ν in order to rule out the possibility that $A \cong A \oplus \mathbf{Z}^n$ if $n < m$. Say that an endomorphism $\Psi: A \to A$ is <u>undesirable</u> if it is one-one and $A/\psi(A) \cong \mathbf{Z}^n$ for some n, $0 < n < m$; we must choose the a_ν so that no ψ_ν extends to an undesirable endomorphism of A.

We begin by identifying a class of ψ_ν's whose extensions cannot be undesirable. Let $\mathcal{D}'$ be the set of all $\theta: \mathbf{Z}^{(\omega)} \to \mathbf{Z}^{(\omega)}$ such that $\theta = \sum_{j=-r}^{r} k_j \varphi^j$ for some $r \in \omega$, and $k_j \in \mathbf{Z}$. For each $n \in \omega$, let $U_n = \{x \in \mathbf{Z}^\omega : x(i) = 0$ for $i<n\}$. Let $\mathcal{D}$ be the set of all ψ_ν such that there exists $n \in \omega$ and $\theta \in \mathcal{D}'$ such that $\psi_\nu \restriction \mathbf{Z}^{(\omega)} \cap U_n = \theta \restriction \mathbf{Z}^{(\omega)} \cap U_n$.

Now we make three claims:

(1) For every $\theta \in \mathcal{D}'$, the kernel of $\overline{\theta}$ is countable;

(2) There is a countable subgroup S of $\overline{\mathbf{Z}^{(\omega)}}$ containing $\mathbf{Z}^{(\omega)}$ and such that: for every subgroup A of $\overline{Z^{(\omega)}}$ containing S and closed under φ and φ^{-1}, if $\psi_\nu \in \mathcal{D}$, then $\overline{\psi}_\nu | A$ is not an undesirable endomorphism of A.

(3) Given subgroups G of $\overline{\mathbf{Z}^{(\omega)}}$ and H of K, both of cardinality $< 2^{\aleph_0}$, if $\psi_\nu \notin \mathcal{D}$, then there exists $a_\nu \in K-H$ such that $\overline{\psi}_\nu(a_\nu) \notin \langle G \cup \{\theta(a_\nu): \theta \in \mathcal{D}'\}\rangle_*$

Supposing for the moment that these claims are true, we proceed to describe the construction of A, i.e., the inductive choice of the a_ν. Suppose that for some $\nu < 2^{\aleph_0}$ we have chosen $a_\mu \in K$ and a subgroup H_μ of K of cardinality $< 2^{\aleph_0}$ for each $\mu < \nu$. If $\psi_\nu \in \mathcal{D}$, let $a_\nu = 0$; otherwise, let a_ν be as in (3), when $H = \cup\{H_\mu : \mu<\nu\}$, and $G = \langle S \cup \{\theta(a_\mu): \theta \in \mathcal{D}', \mu<\nu\}\rangle_*$. (We can suppose that S is closed under φ and φ^{-1}.)

Define

$A_\nu = \langle S \cup \{\theta(a_\mu): \mu \leq \nu, \theta \in \mathcal{D}'\}\rangle_*$

$H_\nu = \langle y \in K: \exists\theta \in \mathcal{D}' \exists x \in A_\nu \exists q \in \mathbf{Z} \exists \mu \leq \nu$ s.t. $q\overline{\psi}_\mu(a_\mu) = x + \theta(y)\rangle$.

By Claim (1). H_ν has cardinality $< 2^{\aleph_0}$. This completes the inductive step in the construction.

Now we must verify that A has no undesirable endomorphisms, where $A = \bigcup_{\nu < 2^{\aleph_0}} A_\nu =$

$$\langle S \cup \{\theta(a_\nu) : \nu < 2^{\aleph_0},\ \theta \in \mathcal{D}'\}\rangle_* .$$

Suppose, to the contrary, that $\psi : A \to A$ is undesirable; then $\psi \restriction \mathbf{Z}^{(\omega)} = \psi_\nu$ for some ν such that $\psi_\nu \notin \mathcal{D}$, by (2). Hence at stage ν we chose a_ν such that $\psi(a_\nu) = \overline{\psi}_\nu(a_\nu)$ does not belong to A_ν. Moreover, our definition of H_μ for $\mu \geq \nu$ is such that $\psi(a_\nu) \notin A_{\mu+1}$. Thus $\psi(a_\nu) \notin A$, which is a contradiction.

We are therefore reduced to proving the three claims.

Proof of (1): In fact, there is an $n \in \omega$ such that $\theta \restriction U_n$ is injective (where we also denote by θ its natural extension to $\mathbf{Z}^\omega$.) Suppose $\theta = \sum_{j=r}^{s} k_j \varphi^j$ where $r \leqq s$ and $k_r \neq 0$. If $r \geqq 0$, let $n = 0$; if $r < 0$, let $n = -mr$. Then one can easily verify by induction on $i \in \omega$ that if $x \in U_n$ and $\theta(x) = y \in \mathbf{Z}^\omega$, the coordinate $x(i)$ is uniquely determined by y.

Proof of (2): Note first that we can assume that $\psi_\nu \restriction \mathbf{Z}^{(\omega)} \cap U_n = \theta \restriction \mathbf{Z}^{(\omega)} \cap \mathbf{U}_n$, where $n \equiv 0 \pmod m$ and $\theta \restriction \mathbf{Z}^{(\omega)} \cap U_n$ is injective; moreover we have:

$$A/\overline{\psi}_\nu(A) \cong (A/\overline{\psi}_\nu(A \cap U_n))/(\overline{\psi}_\nu(A)/\overline{\psi}_\nu(A \cap U_n))$$

and $rk(\overline{\psi}_\nu(A)/\overline{\psi}_\nu(A \cap U_n)) = rk(A/A \cap U_n) \equiv 0 \pmod m$. So it suffices to consider the case when $\psi_\nu = \theta$ for some $\theta \in \mathcal{D}'$. Now if $\theta = k_r \varphi^r$ for some $r \in \omega$, then (for any $A \supseteq \mathbf{Z}^{(\omega)}$) it is clear that $rk(A/\theta(A \cap U_n))$ is either divisible by m or infinite (depending on whether $k_r = \pm 1$ or not). So we are left with the case of $\theta = \sum_{j=r}^{s} k_j \varphi^j$ where $k_r \neq 0$, $k_s \neq 0$ and $r < s$. We are also given $n \equiv 0 \pmod m$ such that $\theta \restriction U_n$ is one-one. We claim that the rank of $K/K \cap \theta(\overline{\mathbf{Z}^{(\omega)}} \cap U_n)$ is infinite. If this is true for all such θ, then we can choose S so that it contains, for each such θ, a countably infinite subset of $\mathbf{K}$ which is independent modulo $\theta(\overline{\mathbf{Z}^{(\omega)}} \cap U_n)$.

So it remains to prove the claim. For this purpose we first assert the following subclaim:

(#) given a prime $p > |k_s|$, $t \in \omega$ and $y \in \mathbf{Z}^t$, there exists $y(t) \in \mathbf{Z}$ such that if

$w = (y(0), \cdots, y(t-1), y(t), 0, 0, 0, \cdots)$.

then either $w \notin \theta(U_n)$ or there are infinitely many $i \in \omega$ with $\theta^{-1}(w)(i)$ not divisible by p.

To prove (#), notice first that $y \in \mathbf{Z}^t$ uniquely determines $x \in \mathbf{Z}^{t-mr} \cap U_n$ (if it exists) such that $y(i) = \sum_{j=r}^{s} k_j x(i-mj)$ for $i = 0, \cdots, t-1$. Choose $y(t)$ so that $x(t-mr)$ is - if it exists - not divisible by p. Now suppose $\Theta(x) = w$ where $x \in U_n$; and suppose only finitely many coordinates of x are not divisible by p. Let q be maximal such that $x(q) \not\equiv 0 \pmod p$. Then $w(q+ms) \equiv k_s x(q) \pmod p$ so $w(q+ms) \not\equiv 0 \pmod p$ and hence $q \leq t-ms$. But by choice of $y(t)$, $q \geq t-mr$, which contradicts $r < s$.

Finally, we shall show how to define by induction $y \in K$ so that for all $d \in \omega - \{0\}$, $dy \notin \Theta(\overline{Z^{(\omega)}} \cap U_n)$; it will then be clear how to extend the argument to construct an infinite subset of K linearly independent mod $\Theta(\overline{\mathbf{Z}^{(\omega)}} \cap U_n)$. Suppose that $y(0),\cdots,y(t-1)$ have been defined so that $y(i) \in i!\mathbf{Z}$; suppose also that we have chosen primes $q_1,\cdots,q_{t-1}$ and are requiring that all future coordinates be divisible by them. At stage t we consider a certain integer $d \neq 0$. Choose a prime q_t larger than the t^{th}-prime and such that $q_t > \max\{|k_s|, d\cdot q_1\cdot q_2\cdots q_{t-1}t!\}$. Apply (#) - and its proof - to $dy \in \mathbf{Z}^t$ to find $y(t) \in q_1\cdot q_2\cdots q_{t-1}\cdot t!\mathbf{Z}$ such that $\Theta^{-1}(dy(0),\cdots,dy(t),0,0,\cdots)$ - if it exists - has infinitely many coordinates not divisible by q_t.

Let $y \in K$ be the element constructed in this way so that every d is considered at infinitely many stages t.

Suppose that $x \in Z^{\omega} \cap U_n$ such that $\Theta(x) = dy$ $(d \neq 0)$ and consider $w = (dy(0),\cdots,dy(t),0,0,\cdots)$. If $\hat{w}$ denotes the image of w in $(\mathbf{Z}/q_t\mathbf{Z})^{\omega}$, then $(dy)\hat{} = w$, so $\Theta^{-1}(w)\hat{} = \Theta^{-1}(\hat{w}) = \Theta^{-1}(dy)\hat{} = \hat{x}$. Thus there are infinitely many primes p such that infinitely many cordinates of x are not divisible by p. Hence $x \notin \overline{Z^{(\omega)}}$.

<u>**Proof of (3)**</u>: Let G, H, ψ_ν be as in (3), and suppose that the conclusion of (3) fails to hold. Then

(*) for every $x \in K-H$, there exists

$a_x \in G$, $q_x \in \mathbf{Z}$, and $\Theta_x \in D'$ such that

$q_x\bar{\psi}_\nu(x) = a_x + \Theta_x(x)$.

We shall impose on $K(= \prod_{n\in\omega} n!\mathbf{Z})$ the product topology, where each factor $(n!\mathbf{Z})$ is given the discrete topology; thus, for any $a \in K$, the sets $a + (U_n \cap K)$ constitute a basis of neighborhoods of a. For each $\Theta \in D'$, and $q \in \mathbf{Z}$, let

$$T_{\Theta,q} = \{x\in K\colon q\bar{\psi}_\nu(x) = \Theta(x)\}.$$

Notice that $T_{\Theta,q}$ is closed in the product topology on K, since homomorphisms are continuous with respect to the Z-adic topology, and since if x belongs to the closure of $T_{\Theta,q}$ in the product topology, then also x belongs to the closure of $T_{\Theta,q}$ in the Z-adic topology (because of the definition of

K) .

We claim that

(4) There exists $\Theta \in D'$, $q \in \mathbf{Z}$, such that $T_{\Theta,q}$ is somewhere dense

(i.e., - in view of the note above - $T_{\Theta,q}$ has non-empty interior). Suppose for the moment that (4) is true. Then there exists $\Theta \in D'$, $a \in K$, $q \in \mathbf{Z}$, $n \in \omega$ such that $a + (U_n \cap K) \subseteq T_{\Theta,q}$. By taking differences we may suppose that $a = 0$. So $(U_n \cap K) \subseteq T_{\Theta,q}$. Now let $x \in U_n \cap \mathbf{Z}^{(\omega)}$; then for some $r > 0$, $r!x \in K$, so $q\Psi_\nu(x) = \Theta(x)$. Since this holds for all elements of $U_n \cap \mathbf{Z}^{(\omega)}$, we must have $\Theta = q\Theta'$ for some $\Theta' \in D'$ and hence $\psi_\nu \in D$: a contradiction. So it remains only to give the

Proof of (4): Suppose, to the contrary, that for every $\Theta \in D'$ and $q \in Z$, $T_{\Theta,q}$ is nowhere dense. We shall construct a subset P of K of cardinality $2^{\aleph_0}$ such that for every $x_1 \neq x_2$ in P and every $\Theta \in D'$, $q \in \mathbf{Z}$, we have $x_1 - x_2 \notin T_{\Theta,q}$. This will yield a contradiction because, by a counting argument, (*) implies that there exist $\Theta \in D'$, $q \in \mathbf{Z}$, $a \in G$ and $x_1 \neq x_2$ in P-H such that

$$q\bar{\psi}_\nu(x_\ell) = a + \Theta(x_\ell)$$

for $\ell = 1,2$; but then (subtracting),

$q\bar{\psi}_\nu(x_1-x_2) = \Theta(x_1-x_2)$, i.e., $x_1 - x_2 \in T_{\Theta,q}$.

So it remains to construct P . To do this, we shall define by induction on n a strictly increasing sequence $\{k_n : n\epsilon\omega\}$ of natural numbers, and for each $\eta \in {}^n2$ (i.e., η is a function from $\{0,\cdots,n-1\}$ to $\{0,1\}$) , an element $a_\eta \in \prod_{j<k_n} j!\mathbf{Z}$, such that if $r < n$ then $a_\eta \restriction k_r = a_{\eta\restriction r}$. (In the end we shall let $P = \{a \epsilon K: \exists\zeta \in {}^\omega 2 \text{ s.t. for all } j \geq 1 , a(j-1) = a_{\zeta\restriction j}(j-1)\}$.)

Let $\{\Theta_j : j\epsilon\omega\}$ be an enumeration of D', and let $\{q_j : j \in \omega\}$ be an enumeration of $\mathbf{Z}$. Suppose that for some $n \geq 0$ we have chosen $k_0,\cdots,k_n$ and a_η (for $\eta \in \bigcup_{k=0}^{n} {}^k2$) as above and such that in addition

(**) If $\eta_1 \neq \eta_2 \in {}^n2$, then for all $x_1, x_2 \in K$ such that $x_\ell \restriction k_n = a_\eta$ (=1,2) , we have

$x_1 - x_2 \notin T_{\Theta_i,q_j}$ for any $i,j \leq n$.

(This will insure that P has the desired property.) Now we must define k_{n+1} and a_η for $\eta \in {}^{n+1}2$. To do this we define by induction on $p \leq 2^{2(n+1)}$ an increasing sequence

$k_n = c_0 \leq \cdots \leq c_{2^{2n+2}}$

and elements $a_{\eta,p} \in \Pi\{j!\mathbf{Z}: j < c_p\}$ for each $\eta \in {}^{n+1}2$ such that $a_{\eta,p+1} \restriction c_p = a_{\eta,p}$. Let $\{(\eta_p,\zeta_p): p \in 2^{2n+2}\}$ be an enumeration of ${}^{n+1}2 \times {}^{n+1}2$. Suppose that we have chosen

$a_{\eta,\ell}$ and c_ℓ for $\ell \leq p < 2^{2n+2}$. and that $\eta_p \neq \zeta_p$. If $S = \cup\{T_{\theta_i, q_j} : i,j \leq n+1\}$, then S is nowhere dense. So for any open subset V of K there is an open subset $W \subseteq V$ such that $W \cap S = \phi$. Letting V be

$$\{x \in K: x \restriction c_p = a_{\eta_p,p} - a_{\zeta_p,p}\}$$

we have such a W , which we may suppose is of the form $z + U_s$ for some $s > c_p$. Then let $c_{p+1} = s$, and let

$a_{\eta_p,p+1} = a_{\eta_p,p}{}^\frown(z(c_p),\cdots,z(c_{p+1}-1))$;

$a_{\zeta_p,p+1} = a_{\zeta_p,p}{}^\frown(0,\cdots,0)$; and $a_{\mu,p+1}$ = any extension of $a_{\mu,p}$ for $\mu \in {}^{n+1}2 - \{\eta_p,\zeta_p\}$. (This definition will insure that (**) holds for the pair (η_p,ζ_p) .)

This completes the inductive step in the construction of P , and thus completes the proof of (4). Hence the proof of (3) is complete, and therefore the proof of the Main Theorem is complete.

REMARK. Francis Oger [3] has solved the analogous problem for ordered abelian groups. In fact, if $G = \{a \in \mathbf{Z}^\omega : a(2k) = 0$ for all but finitely many $k)\}$, then $\mathbf{Z} \times \mathbf{Z} \times G \cong G$ but $\mathbf{Z} \times G \not\cong G$ as <u>ordered</u> groups (where $\mathbf{Z}$ has the usual order, and all products are given the lexicographical order).

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