Here again we extract the f-subterms from t and then replace f by $f^{w}_{\tilde{t}*}$. This gives as above a derivation of

$$\forall x\varphi(x,f^{w}_{\tilde{t}*}x) \quad Fct(\tilde{t}*;\tilde{w}) \ ,\ \Gamma \to \chi(t*) \ .$$

By induction hypothesis we then obtain a derivation of

$$\forall x_1\exists y_1 \ldots \forall x_m\exists y_m \ [\ \underset{i}{} \ \varphi(x_i,y_i) \wedge Fct(\tilde{t}*,\tilde{x};\tilde{w},\tilde{y}) \ ,\Gamma \to \chi(t*) \ .$$

Now apply ($\to\exists$) and then proceed as in case 1 above.

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ON SUCCESSORS OF SINGULAR CARDINALS

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Introduction :

We will clarify the situation for the successor of a strong limit singular cardinal λ. We find a special subset $S^*(\lambda^+)$, from which we can find which stationary subsets of λ^+ can be stopped from being stationary by μ-complete forcing (Baumgartner has done this for successor λ^+ of regular $\lambda = \lambda^{<\lambda}$).
For $\lambda = \aleph_{\omega+1}$ we succeed in continuing an induction construction done for a λ^+-free not λ^{++} (abelian) group, and similar things for transversals; on those problems see history and references in [Sh 2].
A solution of a related problem - which stationary subsets of λ^+ can be "killed" by a forcing not adding bounded subsets of λ^+-will appear in a paper by U. Avraham, J. Stavi and the author.
We also prove a result related to the title but not to the rest of the paper, improving a result of Gregory [Gr] : assuming G.C.H., for $\lambda \neq \aleph_0$, $\diamond^*_S$ holds, where $S = \{\delta < \lambda^+;\ cf\delta \neq cf\lambda\}$; hence $\diamond_{S_1}$ holds for any stationary $S_1 \subseteq S$.

For a reader interested only with the GCH, he can simplify for himself the part up to section 13. A reader interested in more general cases than those discussed in the main part has to go to the end. There we also show that the special set $S^*(\aleph_{\omega+1})$ can be stationary (even with the GCH).

The main results were announced in the AMS Notices [Sh 3].

Notation: We shall denote infinite cardinals by λ,μ,κ,χ, ordinals by $i,j,\alpha,\beta,\gamma,\xi,\zeta$ limit ordinals by δ, natural numbers by m,n,r,p,q.

Let $\bar{N}$ denote a sequence $\langle N_i : i < \lambda \rangle$ where for some $\mu, \chi \leqslant \mu$, $N_i \prec (H(\mu),\in)$; $i \subseteq N_i$, $\|N_i\| < \lambda$, $i < j \Rightarrow N_i \prec N_j$, and for limit δ, $N_\delta = \bigcup_{i<\delta} N_i$. We call this a λ-approximating sequence (for μ).

We denote by d a two-place function from one cardinal to another; $cf\delta$ is the cofinality of δ; $cf^*\delta$ is $cf\delta$ if $cf\delta < \delta$ and is ∞ otherwise. D_δ is the filter over δ generated by the closed unbounded subsets of δ (so we assume $cf\delta > \aleph_0$). If D is a filter over I, $A \subseteq B$ mod D means $I - (A - B) \in D$; similarly $A \equiv B$ mod D means $I - [(A - B) \cup (B - A)] \in D$. If $A \not\equiv \phi$ mod D, $D + A$ is the filter $\{B : B \cup (I - A) \in D\}$.

Let $CF(\delta,\kappa) = \{i < \delta : cfi = \kappa\}$, similarly $CF(\delta,<\kappa) = \bigcup_{\mu<\kappa} CF(\delta,\mu)$, $CF(\delta, \leqslant \kappa) = \bigcup_{\mu\leqslant\kappa} CF(\delta,\mu)$ $D_{\delta,\kappa} = D_\delta + CF(\delta,\kappa)$ etc.

1. Definition : 1) We say κ is good for λ if $\lambda = \lambda^{<\lambda}, \kappa = \infty$ or there is a family $\underline{P}^o_{\lambda,\kappa}$ such that

a) $|\underline{P}^o_{\lambda,\kappa}| = \lambda$

b) every member of $\underline{P}^o_{\lambda,\kappa}$ is a subset of λ of cardinality κ

c) every subset of λ of cardinality κ contains a member of $\underline{P}^o_{\lambda,\kappa}$

2) We call κ a good cofinality for λ if $\lambda = \lambda^{<\lambda}, \kappa$ is ∞ or if λ and κ are regular and there is a family $\underline{P}_{\lambda,\kappa}$ such that

a) $|\underline{P}_{\lambda,\kappa}| = \lambda$

b) every member of $\underline{P}_{\lambda,\kappa}$ is a subset of λ of cardinality $< \kappa$

c) every subset of λ of cardinality κ has a subset $\{\alpha_i : i < \kappa\}$ such that α_i is increasing and for every $j < \kappa$, $\{\alpha_i : i < j\} \in \underline{P}_{\lambda,\kappa}$

d) $\lambda = \lambda^{<\kappa}$ or $2^\mu < \lambda$ for every $\mu < \kappa$

2. Definition : 1) $Gcf(\lambda) = \{\kappa : \kappa$ is a good cofinality for $\lambda\}$

$G(\lambda) = \{\kappa : \kappa$ is good for $\lambda\}$

2) $gcf(\lambda) = \{i < \lambda : cf^*i \in Gcf(\lambda)\}$ (note that we use cf^* not cf)

3) $D^g_\lambda = D_\lambda + gcf(\lambda)$

3. Claim : 1) If $\lambda^\kappa = \lambda$ then κ is good for λ

2) If $\kappa < \infty$ is good for λ then κ is good for λ^+

3) If $\lambda = \sum_{i<\mu} \lambda_i$, $cf\mu \neq cf\kappa$, $\lambda_i (i < \mu)$ increasing and $\kappa < \infty$ is good for each λ_i then κ is good for λ

4) If $(\forall \mu < \aleph_\alpha) \mu^\kappa < \aleph_\alpha, \beta < cf\kappa$, $cf\aleph_\alpha \neq cf\kappa$ then κ is good for $\aleph_{\alpha+\beta}$ [in fact $(\forall \mu < \aleph_\alpha)\mu^\kappa \leqslant \aleph_{\alpha+\beta}$ suffice]

5) if λ,κ are regular, κ good for λ then κ is a good cofinality for λ, provided that $2^{<\kappa} \leqslant \lambda$

6) If λ,κ are regular $\lambda^{<\kappa} = \lambda$ then κ is a good cofinality for λ

7) If $\kappa < \infty$ is a good cofinality for λ then κ is a good cofinality for λ^+

8) If $\lambda = \sum_{i<\mu} \lambda_i$, $cf\mu \neq cf\kappa$, $\kappa \in Gcf(\lambda_i)$ for every $i < \mu$, λ_i increasing, and $\kappa < \infty$ then $\kappa \in Gcf(\lambda)$

9) If $(\forall \mu < \aleph_\alpha)\mu^{<\kappa} < \aleph_\alpha$, $cf\aleph_\alpha \neq \kappa$, κ regular, $\beta < \kappa$ then $\kappa \in Gcf(\aleph_{\alpha+\beta+1})$ [in fact, $(\forall \mu < \aleph_\alpha)\mu^{<\kappa} \leqslant \aleph_{\alpha+\beta+1}$ suffice].

4. Definition : For d a two-place function from δ into $\kappa (cf\delta > \aleph_0)$ we let $S_1(d) = \{\xi : \xi < \delta, \xi$ a limit ordinal such that there is an unbounded $A \subseteq \xi$ on which d is constant$\}$

$S_o(d) = \{\xi : \xi < \delta, \xi$ a limit ordinal such that there are unbounded subsets A,B of ξ, such that $(\forall b \in B)(\exists \alpha < \kappa)(\forall a \in A)[a < b \rightarrow d(a,b) \leqslant \alpha]\}$

Remark : Note that d determines δ (as Dom d) but not κ (as d is into κ, not necessarily onto κ), so if the value of κ is not clear we shall write $S_o(d,\kappa)$. In the definition of $S_1(d)$, κ has no role.

<u>5. Claim</u> : For d a two-place function from δ to κ:

1) $S_1(d) \subseteq S_o(d)$,

2) in the definition of $S_\ell(d)$ ($\ell = 0,1$) we can assume A,B have order type $cf\xi$ (and generally replace them by unbounded subsets),

3) $CF(\delta, \leq \kappa) \subseteq S_o(d)$,

4) If $\ell = 0,1$, $\xi \in S_\ell(d)$, $cf\xi > \aleph_o$, then there is $C \in D_\xi$ such that $C \subseteq S_\ell(d)$.

<u>6. Definition</u> : For a λ-approximating sequence $\overline{N}$ (see notation) let $S_2(\overline{N}) = \{\xi : \xi < \lambda,\ \xi$ a limit such that there is an unbounded $A \subseteq \xi$ of order type $cf\xi$ such that $(\forall i < \xi)\,[A \cap i \in N_\xi]$ and $N_\xi \cap \lambda = \xi\}$

<u>7. Claim</u> : 1) If λ is regular, $\overline{N}^o$, $\overline{N}^1$ are λ-approximating sequences for μ_o, μ_1 respectively, and $\mu_\ell > \lambda$, then $S_2(\overline{N}^1) = S_2(\overline{N}^o) \bmod D^g_\lambda$.

<u>Proof</u> : Let $\overline{N}^\ell = \langle N^\ell_i : i < \lambda \rangle$, where $N^\ell_i \prec (H(\mu_\ell), \in)$, and let $C = \{\alpha < \lambda : N^o_\alpha \cap (\bigcup_{j<\lambda} N^1_j) = (\bigcup_{j<\lambda} N^o_j) \cap N^1_\alpha = N^o_\alpha \cap N^1_\alpha$ and $N^\ell_\alpha \cap \lambda = \alpha\}$ (we do not distinguish strictly between a model N and its universe).

It is easy to check that C is a closed unbounded subset of λ.

By transitivity of equality we can assume $N^o_\alpha \prec N^1_\alpha$.

Now suppose $\xi \in C$, and $cf^*\xi \in Gcf(\lambda)$. We shall prove $\xi \in S_2(\overline{N}^o)$ iff $\xi \in S_2(\overline{N}^1)$, thus completing the proof. The "only if" part is now trivial, so we concentrate on the "if" part. Also the case $cf^*\xi = \infty$ is easy, so we assume $cf^*\xi = cf\xi < \xi$.

Let $\kappa = cf\xi < \xi$. We have just assumed $\kappa \in Gcf(\lambda)$, so the appropriate $\underline{P}_{\lambda,\kappa}$ (as in Definition 1.2) exists, hence belongs to $H(\mu_1)$, hence w.l.o.g it belongs to N^o_o, and hence, by assumption, to N^1_o.

If $\xi \in S_2(\overline{N}^1)$, then (by definition) there is an unbounded $A \subseteq \xi$ of order-type $cf\xi$, such that for every $\zeta < \xi$, $A \cap \zeta \in N^1_\xi$.

If $\lambda = \lambda^{<\kappa}$, we can assume $\underline{P}_{\lambda,\kappa} = \{B \subseteq \lambda : |B| < \kappa\} = \{B_i : i < \lambda\}$ (since $|\underline{P}_{\lambda,\kappa}| = \lambda$), and so $\underline{P}_{\lambda,\kappa} \cap N^o_\xi = \underline{P}_{\lambda,\kappa} \cap N^1_\xi = \{B_i : i < \xi\}$, hence $\zeta < \xi \Rightarrow A \cap \zeta \in N^o_\xi$, hence A witnesses that $\xi \in S_2(\overline{N}^o)$. Thus finishing.

So we are left with the case $\lambda < \lambda^{<\kappa}$. Then, by d) of Definition 1.2, $(\forall \mu < \kappa) 2^\mu < \lambda$. So, as $N^\ell_\xi \cap \lambda = \xi$, and A has order-type κ, every subset of A of power $< \kappa$ is included in a set from N^1_ξ of cardinality $<\kappa$, hence it belongs to N^1_ξ. So we can replace A by any subset of it which is unbounded in ξ. In particular, by the choice of $\underline{P}_{\lambda,\kappa}$ (see Definition 2), we can assume $A = \{\alpha_i : i < \kappa\}$, and for $j < \kappa$, $\{\alpha_i : i < j\} \in \underline{P}_{\lambda,\kappa}$ and, as mentioned above, $\{\alpha_i : i < j\} \in N^1_\xi$. But as $|\underline{P}_{\lambda,\kappa}| = \lambda$, $\underline{P}_{\lambda,\kappa} \in N^o_o$, clearly $\underline{P}_{\lambda,\kappa} \subseteq \bigcup_{i<\lambda} N^o_i$, hence (as $\xi \in C$) $\underline{P}_{\lambda,\kappa} \cap N^o_\xi = \underline{P}_{\lambda,\kappa} \cap N^1_\xi$, hence for every $j < i$, $\{\alpha_i : i < j\} \in N^o_i$. So $\{\alpha_i : i < \kappa\}$ witnesses that $\xi \in S_2(\overline{N}^o)$, and this finishes the proof of the theorem.

<u>8. Definition</u> : $S^*(\lambda) \subseteq \lambda$ is defined as $(\lambda - S_2(\overline{N})) \cap gcf(\lambda)$ for $\overline{N}$ any λ-approximating sequence for λ^+, where λ is regular. (so S^* is uniquely defined mod D_λ only).

<u>9. Definition</u> : For λ singular, a two-place function d from λ^+ to $\kappa = cf\lambda$ is called <u>normal</u> if for every $i < \kappa, \alpha < \lambda^+$, the set $\{\beta < \alpha : d(\beta,\alpha) \leq i\}$ has cardinality $<\lambda$. It is called subadditive if for $\gamma < \beta < \alpha < \lambda^+$, $d(\gamma,\alpha) \leq \max\{d(\gamma,\beta), d(\beta,\alpha)\}$.

<u>10. Claim</u> : For every singular λ, there is a normal subadditive two-place function d from λ^+ to $cf\lambda$; moreover, if $\lambda = \sum_{i<cf\lambda} \lambda_i$ (λ_i increasing), then $|\{\beta < \alpha : d(\beta,\alpha) \leq i\}| \leq \lambda_i$.

<u>Proof</u> : Easy.

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11. Claim : 1) Suppose λ is singular, $\kappa = cf\lambda$, $(\forall\mu < \lambda)(\mu^{<\chi} < \lambda)$, and d is a normal two-place function from λ^+ to κ. Then for some λ^+-approximating sequence $\bar{N}$ for λ^{++},

$$CF(\lambda^+, \leq \chi) \cap S_o(d) \subseteq S_2(\bar{N}) \bmod D_\lambda .$$

2) Suppose λ is singular, $\kappa = cf\lambda$, χ is regular and is a good cofinality for λ^+, and d is a normal two-place function from λ^+ to κ. Then for some λ^+-approximating sequence $\bar{N}$ for λ^{++}, $CF(\lambda^+,\chi) \cap S_o(d) \subseteq S_2(\bar{N})$.

Proof : 1) Choose a λ^+-approximate sequence $\bar{N}$ for λ^{++} such that $d \in N_o$, $N_i \in N_{i+1}$. Clearly $C = \{\delta < \lambda^+ : N_\delta \cap \lambda = \delta\}$ is closed and unbounded. So for every $\alpha < \lambda^+$, $i < \kappa$, the set $A^* = \{\beta < \alpha : d(\beta,\alpha) \leq i\}$ belongs to N_{i+1} and has cardinality $< \lambda$. Hence $P^\alpha_i = \{B : B \subseteq A^*, |B| < \chi\}$ belongs to N_{i+1} and has cardinality $< \lambda$, hence $P^\alpha_i \subseteq N_{i+1}$. So suppose $\delta \in S_o(d)$, and $A,B \subseteq \delta$ are witness to it (i.e. they are unbounded in δ and have order-type $cf\delta$, and for every $b \in B$, for some $i(b) < \kappa$, $(\forall a \in A)(a < b \to d(a,b) \leq i(b))$). Suppose further $\delta \in C$, $cf\delta \leq \chi$. Then $A,B \subseteq N_\delta$ (as $\delta \subseteq N_\delta$) and for every $b \in B$, $\{a : a \in A, a < b\}$ belongs to $P^b_{i(b)}$, hence to N_{i+1}, hence to N_δ. So A witnesses that $\delta \in S_2(\bar{N})$. We have just proved $\delta \in CF(\lambda^+, \leq \chi) \cap S_o(d) \Rightarrow \delta \in S_2(\bar{N})$, thus finishing the proof of the claim.

2) A similar proof.

12. Claim : Suppose λ is regular, $\kappa < \chi$, $\kappa < \lambda$, χ is a good cofinality for λ and $(\forall\mu < \chi)2^\mu < \lambda$ or $\chi = \infty$. Then for every two-place function d from λ to κ and for some λ-approximate sequence $\bar{N}$ for λ^+,

$$S_2(\bar{N}) \cap CF(\lambda,\chi) \subseteq S_1(d).$$

Proof : Choose $\bar{N}$ as λ-approximate sequence for λ^+ such that $d \in N_o$. Suppose $\delta \in S_2(\bar{N}) \cap CF(\lambda,\chi)$. We shall prove $\delta \in S_1(d)$. The case

$\chi = \infty$ is easy, so assume $\chi < \infty$.

As $\delta \in S_2(\bar{N})$, there is a set $\{\alpha_i : i < \chi\} \subseteq \delta$, unbounded in δ, such that for every $j < \chi$, $\{\alpha_i : i < j\} \in N_\delta$. Let h be the function with domain χ, $h(i) = \alpha_i$. Clearly for $j < \chi$, $h|j \in N_\delta$.

Now we define by induction on $i < \chi$ an element x_i and function f_i as follows :

$f_i(j) = d(x_j,\delta)$ for $j < i$ (so Dom $f_i = i$)

x_i is the first ordinal which is bigger than α_i and $x_j (j < i)$ and is such that $(\forall j < i)[d(x_j,x_i) = f_i(j)]$.

This can be carried out in $H(\lambda^+)$. But now as $\mu < \chi \Rightarrow 2^\mu < \chi$, and $\mu < \chi = cf\delta \leq \delta$, clearly each f_i is in N_δ.

Note also that x_i depends only on f_i and $\{\alpha_j : j \leq i\}$ (as for $j < i$, $f_j = f_i|j$). So $x_i \in N_\delta$ for each $i < \chi$.

Now there is an unbounded $S \subseteq \chi$ and $i_o < \kappa$ such that $j \in S \Rightarrow d(x_j,\delta) = i_o$. It is easy to check that $\{x_j : j \in S\}$ witnesses that $\delta \in S_1(d)$.

From now on we concentrate on successors of strong limit singular cardinals. We can conclude e.g.

13. Conclusion : Suppose λ is a singular strong limit. Then for every normal two place function d from λ^+ to $\kappa = cf\lambda$, the following holds :

$$S_o(d) \equiv S_1(d) \cup CF(\lambda^+, \leq \kappa) \equiv \lambda^+ - S^*(\lambda^+) \bmod_{D_{\lambda^+}} .$$

(So in particular $S_o(d)$ does not depend on d (when d is normal) up to equivalence $\bmod_{D_{\lambda^+}}$).

Proof : Trivial by 5.1, 5.3, 11 and 12.

14. Claim : If λ is regular, $\kappa < \lambda$ and $(\forall\mu < \lambda)\ \mu^{<\kappa} < \lambda)$, then $CF(\lambda, \leq \kappa) \subseteq \lambda - S^*(\lambda) \bmod D_{\lambda^+}$.

Proof : We can find a λ-approximating sequence $\langle N_i : i < \lambda \rangle$ to λ^+ such that every subset of N_i of cardinality $< \kappa$ belongs to N_{i+1}. So $CF(\lambda, \leq\kappa) \subseteq S_2(\bar{N})$.

15. Claim : If $\delta \in \lambda - S_1(d)$, d a two-place function from λ to $\kappa < \mathrm{cf}\delta$, then $\mathrm{cf}\delta$ is not weakly compact.

Proof : If $\mathrm{cf}\delta$ is weakly compact then $\mathrm{cf}\delta \to (\mathrm{cf}\delta)^2_\kappa$.

16. Definition : 1) For a set $S \subseteq \lambda$ let

$$F(S) = \{\delta < \lambda : S \cap \delta \text{ is a stationary subset of } \delta\}$$

2) Define $F^n(S)$ by induction on n:

$$F^0(S) = S,\ F^{n+1}(S) = F(F^n(S)).$$

17. Claim : 1) $FF(S) \subseteq F(S)$.

2) $F(S^*(\lambda)) \subseteq S^*(\lambda)$, hence $F^n(S^*(\lambda)) \subseteq F^m(S^*(\lambda))$ if $n > m \geq 0$.

3) $\delta \in F^n(S)$ implies $\mathrm{cf}\delta \geq \aleph_n$; moreover, if $\aleph_\alpha = \min\{\mathrm{cf}\delta : \delta \in S\}$, then $\delta \in F^n(S)$ implies $\mathrm{cf}\delta \geq \aleph_{\alpha+n}$.

4) If $\alpha \leq \min\{\mathrm{cf}\delta : \delta \in \bigcup_{i<\alpha} S_i\}$, $S_i \subseteq \lambda$ then

$$F(\bigcup_{i<\alpha} S_i) = \bigcup_{i<\alpha} F(S_i) \bmod D_\lambda.$$

Proof : 1) Easy

2) By 5.4 (and second part-by induction)

3), 4) Easy.

18. Lemma : Suppose λ is a singular strong limit of cofinality κ. Then for some $C \in D_{\lambda^+}$, for every $\delta \in C$, letting $\langle \alpha_i : i < \mathrm{cf}\delta \rangle$ be increasing, continuous and converging to δ, the following holds :

$$\{i : \alpha_i \in S^*(\lambda)\} \supseteq S^*(\mathrm{cf}\delta) \bmod D_{\mathrm{cf}\delta}$$

Proof : Let d be as in 10. Then by 13, for some $C \in D_{\lambda^+}$, $S^*(\lambda^+) \cap C = S_0(d) \cap C$, so we need only deal with $S_0(d)$.

Now define a two-place function d^* from $\mathrm{cf}\delta$ to κ by : $d^*(i,j) = d(\alpha_i,\alpha_j)$. It is easy to check that

$$\{\alpha_i : i \in S_0(d^*)\} \subseteq S_0(d).$$

But by 10, $S_0(d^*) \subseteq \mathrm{cf}\delta - S^*(\mathrm{cf}\delta)$ (remember $\kappa < \mathrm{cf}\delta$), so we are finished.

19. Conclusion : 1) Suppose λ is a singular strong limit, χ,μ regular, $\chi\mu < \lambda$ and $(\forall \mu_1 < \mu)\mu_1^\chi < \mu$. Then $F[S^*(\lambda^+) \cap CF(\lambda^+,\chi)] \cap CF(\lambda^+,\mu)$ is not stationary.

2) If $n < \omega$ and $2^{\aleph_k} \leq \aleph_{k+n}$ for every $k < \omega$, then $F^n(S^*(\aleph_{\omega+1})) \equiv \phi \bmod D_{\aleph_{\omega+1}}$.

3) If $\aleph_\omega$ is a strong limit and $S^*(\aleph_{\omega+1})$ is stationary, then for some stationary $S \subseteq \aleph_{\omega+1}$, $F(S) = \phi$

Proof : 1) By 14 and 18.

2) Suppose $F^n(S^*(\aleph_{\omega+1}))$ is stationary. Then by 17.4 for some $k < \omega$, $F^n[S^*(\aleph_{\omega+1}) \cap CF(\aleph_{\omega+1},\aleph_k)]$ is stationary. Hence for some $\ell < \omega$, $F^n[S^*(\aleph_{\omega+1}) \cap CF(\aleph_{\omega+1},\aleph_k)] \cap CF(\aleph_{\omega+1},\aleph_\ell)$ is stationary. If $\ell \leq k+n$, this contradicts 19.3. But if $\ell > k+n$, then $(\forall \mu < \aleph_\ell)\mu^{\aleph_k} < \aleph_\ell$ (since $2^{\aleph_k} \leq \aleph_{k+n}$), hence we get a contradiction by 19.1. So in all cases we get a contradiction; hence $F^n(S^*(\aleph_{\omega+1}))$ is not stationary.

3) Since $S^*(\aleph_{\omega+1})$ is stationary, for some $k < \omega$, $S^*(\aleph_{\omega+1}) \cap CF(\aleph_{\omega+1},\aleph_k)$ is stationary. Let $2^{\aleph_k} = \aleph_{k+n}$ ($n < \omega$ since $\aleph_\omega$ is a strong limit). So $k+n < \ell < \omega$ implies $(\forall \mu < \aleph_\ell)\mu^{\aleph_k} < \aleph_\ell$; hence, by 19.1, $F(S) \subseteq CF(\aleph_{\omega+1}, \leq \aleph_{k+n})$, where $S = S^*(\aleph_{\omega+1}) \cap CF(\aleph_{\omega+1},\aleph_k)$. But by 17.1, $F^{n+1}(S) \subseteq F(S)$, hence $\delta \in F^{n+1}(S)$ implies $\mathrm{cf}\delta \leq \aleph_{k+n}$, and by 17.2 $\delta \in F^{n+1}(S)$ implies $\mathrm{cf}\delta \geq \aleph_{k+n+1}$ (since $\delta \in S \Rightarrow \mathrm{cf}\delta = \aleph_k$), so we get that there is no $\delta \in F^{n+1}(S)$, i.e. $F^{n+1}(S) = \phi$. Since $F^0(S) = S$ is stationary, for some ℓ, $F^\ell(S)$ is stationary but $F(F^\ell(S)) = F^{\ell+1}(S)$ is not; $F^\ell(S)$ is as required.

Theorem 20 : Suppose $S \subseteq \lambda$ is stationary, and $S \subseteq \mathrm{gcf}(\lambda) - S^*(\lambda)$, $S \subseteq CF(\lambda,\mu)$. If P is a μ^+-complete forcing (i.e. if $\langle p_i : i < \mu \rangle$ is an increasing sequence of elements of P then some $p \in P$ is $\geq p_i$ for every i), then S is stationary even in the universe V^P.

Remark : Remember that λ-complete forcing forces the stationariness of any $S \subseteq \lambda$.

Proof : Let $\bar{N}$ be a λ'-approximate sequence for some $\lambda' > \lambda$, such that a P-name $\underset{\sim}{C}$ of a closed unbounded subset of λ, a $p \in P$, are in N_o. So trivially there is $\delta \in S$, $A \subseteq \delta$ such that $\delta = N_\delta \cap \lambda$ and A has order type $cf\delta$, and for every $\zeta < \delta$, $A \cap \zeta \in N_\delta$. Let $f : cf\delta \to A$ enumerate A, hence $\zeta < cf\delta$ implies $f|\zeta \in N_\delta$.

We want to prove that not : $p \Vdash$ "$\underset{\sim}{C}$ is disjoint from S". For this it suffices to find $q \in P$ such that $p \leqslant q$ and $q \Vdash$ "$\delta \in \underset{\sim}{C}$" (since $\delta \in S$). We can assume that a well-ordering $<^*$ of $P \cup P \times \lambda$ belongs to N_o. Now we define by induction on $i < cf\delta$, $p_i \in N_\delta$. We let $p_o = p$, and for i a limit, p_i is the $<^*$-first p' which is $\geqslant p_j$ for every j (which exists since P is μ^+-complete). We let p_{i+1}, β_i be such that (p_{i+1}, β_i) is the $<^*$-first pair (p', β') such that $p' \geqslant p_i$, $\beta' \geqslant f(i)$ and $p' \Vdash \beta' \in \underset{\sim}{C}$. There is such (p', β') since $\underset{\sim}{C}$ was a P-name of an unbounded subset of λ. It is easy to check that $p_i, \beta_i \in P \cap N_\delta$, so $\beta_i < \delta$. Hence $\delta = \sup\{\beta_i : i < cf\delta\}$. Since P is μ^+-complete, there is $q \in P$, $p_i \leqslant q$ for every $i < cf\delta$. So q force $\underset{\sim}{C} \cap \delta$ to be unbounded below δ. But $\underset{\sim}{C}$ was a P-name of a closed subset of δ. Hence $q \Vdash$ "$\delta \in \underset{\sim}{C}$". So we are finished.

21. Theorem : Suppose $\mu < \lambda$, μ regular. Then there is a μ-complete forcing P, such that in V^P $S^*(\lambda)$ is not stationary.

Proof : First assume $\lambda = \lambda^{<\lambda}$, so $\underline{P} = \{B \subseteq \lambda : |B| < \lambda\} = \{B_i : i < \lambda\}$, each $B \in \underline{P}$ appearing in $\{B_i : i < \lambda\}$ λ times, and let $\bar{B} = \langle B_i : i < \lambda \rangle$. Clearly there is a λ-approximating sequence $\bar{N}$ of λ^+, with $\bar{B} \in N_o$; and then $\underline{P} \cap N_\delta = \{B_i : i < \delta\}$ for a closed unbounded set of δ's.

So (w.l.o.g.) $S^*(\lambda) \subseteq \{\delta < \lambda : N_\delta \cap \underline{P} = \{B_i : i < \delta\}\}$.

$P = \{\eta = \langle \alpha_i : i \leqslant \zeta \rangle$, an increasing, continuous sequence, where $B_{\alpha_{i+1}} = \{\alpha_j : j \leqslant i\}\}$. The order on P is : $\eta_1 < \eta_2$ iff η_1 is an initial segment of η_2.

It is obvious that P is μ-complete; and if $G \subseteq P$ is generic, let $C[G] = \{\alpha_\delta : \delta$ limit, and $\langle \alpha_j : i \leqslant \xi \rangle \in G$, $\zeta \geqslant \delta\}$. Clearly in $V[G]$, $C[G]$ is a closed unbounded subset of λ. Now we have to prove only : $C[G] \cap S^* = \phi$, where $S^* = S^*(\lambda)^V$. Suppose, in V, for some $p \in P$, $p \Vdash$ "$\delta \in C[\underset{\sim}{G}]$" where $\delta \in S^*$. Let $p = \langle \alpha_j : j \leqslant \zeta \rangle$, so clearly for some limit $i \leqslant \zeta$, $\delta = \alpha_i$. Since $\delta \in S^*$, $N_\delta \cap \{B_i : i < \lambda\} = \{B_i : i < \delta\}$, and there is no unbounded $A \subseteq \delta$ of order type $cf\delta$, such that $\xi < \delta \Rightarrow A \cap \xi \in N_\delta$. But there is such an A namely $\{\alpha_j : j < i\}$ ($\{\alpha_j : j < j_o < i\}$ belongs to N_δ since it is $B_{j_o+1} - \{j_o\}$), contradiction. So we are finished when $\lambda = \lambda^{<\lambda}$.

If $\lambda < \lambda^{<\lambda}$, let Q be the collapsing of 2^λ to λ, i.e. $P = \{f : \text{Dom } f = \xi < \lambda$, Range $f \subseteq 2^\lambda\}$. Note that V^P may have a different $gcf(\lambda)$, but $S^*(\lambda)^{V^Q} \cap gcf(\lambda)^V = S^*(\lambda)^V$. Now in V^Q define P as before, and $Q * P$ (the composition) is as required.

22. Conclusion : Suppose λ is regular, $\mu < \lambda$ regular, $S \subseteq gcf(\lambda)$. There is a μ-complete forcing P such that in V^P, S is not stationary iff $(S - S^*(\lambda)) \cap CF(\lambda, <\mu)$ is stationary.

23. Lemma : Suppose λ is regular, $S \subseteq \lambda$ stationary, but $F(S) = \phi$ and for every $\alpha \in S$, A_α is an unbounded subset of α of order-type $cf\alpha$.

Then for every $S' \subseteq S$ with $|S'| < \lambda$, the family $\{A_\alpha : \alpha \in S'\}$ has a transversal (=one-to-one choice function). Moreover we can find $A'_\alpha \subseteq A_\alpha$ ($\alpha \in S'$), $|A'_\alpha| < cf\alpha$, such that the sets $A_\alpha - A'_\alpha$ ($\alpha \in S'$) are pairwise disjoint.

However $\{A_\alpha : \alpha \in S\}$ does not have a transversal.

Proof : See [Sh 1].

24. Lemma : Suppose λ is singular strong limit, $\kappa = cf\lambda$, $S^*(\lambda^+) = \phi \bmod D_{\lambda^+}$, and let

$$S = \{\delta < \lambda^+ : cf\delta \neq \kappa, \aleph_0, \text{ and } \lambda\omega \text{ divides } \delta\}$$

Then we can define $A_\alpha \subseteq \alpha$ $(\alpha \in S)$, A_α unbounded in α and with order-type $\kappa(cf\alpha)$ (ordinal multiplication), such that

A) $\{A_\alpha : \alpha \in S\}$ has no transversal

B) For every $S' \subseteq S$ with $|S'| < \lambda^+$, $\{A_\alpha : \alpha \in S'\}$ has a transversal. Moreover

B') For every $S' \subseteq S$ with $|S'| < \lambda^+$, there are $A'_\alpha \subseteq A_\alpha (\alpha \in S')$ such that :

(i) they are pairwise disjoint,

(ii) A'_α is a big [and even very big] subset of A_α, which means that there is a closed (in A_α) unbounded [resp. cobounded] $C \subseteq A^*_\alpha$ so that

$$(\forall\delta \in C)\ (\exists\zeta < \kappa)\ (\forall\xi)\ (\delta + \zeta \leqslant \xi < \delta + \kappa \to \xi \in A'_\alpha).$$

Proof : Stage A :

There is a normal $d : \lambda^+ \to \kappa$, $\lambda = \sum_{i<\kappa} \lambda_i, \lambda_i < \lambda$, $|\{\beta < \alpha : d(\alpha,\beta) \leqslant i\}| \leqslant \lambda_i$, such that for every $\delta < \lambda^+$, $cf\delta \neq \kappa$, there is $A \subseteq \delta$, $\sup A = \delta$, $d|A$ bounded, and each $i \in A$ is a successor.

Pf : Let d be from 10, then $S_1(d) \equiv \phi \bmod D_{\lambda^+}$, hence there is a closed unbounded $C \subseteq \lambda^+$, $C \cap S_0(d) = \phi$. Let $C = \{\alpha_i : i < \lambda^+\}$, α_i increasing and continuous, $\alpha_0 = 0$. For each $i < \lambda^+$, we can find $A^i_\zeta \subseteq (\alpha_i, \alpha_{i+1})(\zeta < \kappa)$ such that : $|A^i_\zeta| = \lambda_\zeta$, A^i_ζ is closed (in the interval), if $\delta \in A^i_\zeta$ is a limit then $\delta = \sup(\delta \cap A^i_\zeta)$, $\alpha_{i+1} = \sup A^i_\zeta$, for some ζ.

A^i_ζ increases with ζ and $(\alpha_i, \alpha_{i+1}) = \bigcup_{\zeta<\kappa} A^i_\zeta$. Now we define d' by : if $\alpha < \beta$ then $d'(\beta,\alpha) = d(\beta,\alpha)$ if $(\exists i)(\beta \geqslant \alpha_i > \alpha)$, and otherwise $d'(\beta,\alpha) = \min\{d(\beta,\alpha), \min\{\zeta : \alpha,\beta \in A^i_\zeta\}\}$. It is easy to check that d' is as required. For showing that every $i \in A$ is a successor, use subadditivity.

Stage B :

For any $\alpha < \lambda^+$ the family

$$\underline{P}_\alpha = \{A \subseteq \alpha : |A| < \lambda, d|A \text{ is bounded}, cf(\sup A) \neq \kappa\}$$

has cardinality $\leqslant \lambda$.

Pf : Let $\alpha = \bigcup_{i<\kappa} B_i$, $|B_i| < \lambda$, B_i increasing, and let, for $i < \kappa$, $\zeta < \kappa$, $\underline{P}^\zeta_{\alpha,i} = \{A \in \underline{P}_\alpha : A \cap B_i$ unbounded in A, $d|A$ bounded by $\zeta\}$. Since $A \in P_\alpha \Rightarrow [cf(\sup A) \neq \kappa$ and $d|A$ bounded], and by the choice of the B_i's, $\underline{P}_\alpha = \bigcup_{\zeta,i<\kappa} \underline{P}^\zeta_{\alpha,i}$, it suffices to prove $|\underline{P}^\zeta_{\alpha,i}| \leqslant \lambda$ (for given $i,\zeta < \lambda$). Let $B^\zeta_i = B_i \cup \bigcup_{\beta\in B_i} \{\gamma : \gamma < \beta, d(\beta,\gamma) \leqslant \zeta\}$. Clearly $|B^\zeta_i| \leqslant |B_i| + \lambda_\zeta < \lambda$, and $A \in \underline{P}^\zeta_{\alpha,i}$ implies $A \subseteq B^\zeta_i$. So $|\underline{P}^\zeta_{\alpha,i}| \leqslant 2^{|B_i| + \lambda_i} < \lambda$, so we have proved stage B.

Stage C :

If P is a family of subsets of A each of cardinality $< \lambda$, but $|\underline{P}| \leqslant |A| = \lambda$, then there is a set $C \subseteq A$ such that

(i) $|C| = \kappa$,

(ii) $(\forall A \in P)$ $|A \cap C| < \kappa$.

This is trivial.

Stage D :

We define the A'_αs by induction on α for $\alpha \in S$. Suppose we arrive at α. Let $\langle \gamma_i : i < cf\alpha \rangle$ be increasing with limit α, $\gamma_i + \lambda \leqslant \gamma_{i+1}$.

For a set A of ordinals, let $acc(A) = \{\delta : \delta$ a limit, $\delta = \sup(A \cap \delta)\}$ (= the set of accumulation points of A). By stage B, $|\underline{P}_\alpha| \leqslant \lambda$, so by stage C we can find $c^i_\alpha \subseteq (\gamma_i, \gamma_i + \lambda)$, of power κ such that :

(*) for every $A \in P_\alpha \cup \{\bigcup\{A_\gamma : \gamma < \alpha, \gamma \in acc(A)\} : A \in P_\alpha\}$, its intersection with c^i_α has power $<\kappa$.

In fact we have to check that $|\bigcup\{A_\gamma : \gamma < \alpha, \gamma \in acc(A)\}| < \lambda$ (for $A \in P_\alpha$), but this is easy : $\lambda \in acc(A) \Rightarrow cf\lambda \leqslant |A| \Rightarrow |A_\gamma| \leqslant \kappa + cf\gamma = \kappa + |A|$, hence the set has power $\leqslant (\kappa + A)\ |A| < \lambda$. We let $A_\alpha = \bigcup_{i<cf\alpha} c^i_\alpha$.

Stage E :

$\{A_\alpha : \alpha \in S\}$ has no transversal.

Because $A_\alpha \subseteq \alpha$, by Fodor's theorem.

Stage E :

We prove (A*) from the lemma. We prove by induction on α that there are big $A'_\beta \subseteq A_\beta$ $(\beta \leqslant \alpha, \beta \in S)$, pairwise disjoint. This will clearly suffice.

Case 1 : For α a successor ordinal, it follows from the induction hypothesis on $\alpha-1$.

Case 2 : For α such that $(\exists\beta < \alpha)\ \beta + \lambda\omega > \alpha$: proof as in the first case.

Case 3 : For α a limit, $cf\alpha = \aleph_0$. Choose ordinals $\alpha_n < \alpha$, $\alpha_n < \alpha_{n+1}$, $\alpha = \bigcup \alpha_n$, $\alpha_0 = 0$. For each n, by the induction hypothesis there are big $A^n_\beta \subseteq A_\beta$ $(\beta \leqslant \alpha_n)$, pairwise disjoint.

Define A'_β, for $\beta \leqslant \alpha$, $\beta \in S$ (hence $\beta \neq 0$), by :

$$A'_\beta = A^{n+1}_\beta - (\alpha_n + \lambda), \text{ where } \alpha_n < \beta \leqslant \alpha_{n+1}$$

It is easy to check that $A'_\beta \subseteq A_\beta$ is still big, and obviously the A'_β are pairwise disjoint. Note that $\alpha \in S$, so we do not have to define A'_α.

Case 4 : For α limit, not case 2, $cf\alpha > \aleph_0$. There is $E \subseteq \alpha$, unbounded, of order type $cf\alpha$ (hence $<\lambda$) and $E = \{\beta_{i+1} : i < cf\alpha\}$ (the β_i increasing), such that $d|E_i$ is unbounded for $i < cf\alpha$, where

$E_i = \{\beta_{j+1} : j < i\}$, and each β_{i+1} is a successor ordinal. (For $cf\alpha \leqslant \kappa$, any unbounded A of order type $cf\alpha$ is as required). (Remember d is from stage A).

We can define for limit $\delta \leqslant cf\alpha$, $\beta_\delta = \sup\{\beta_{i+1} : i < \delta\}$.

Since $\beta_i + \lambda < \alpha$, we can assume w.l.o.g. $\beta_i + \lambda < \beta_{i+1}$ (by making deletions if necessary). Let $A^i_\beta \subseteq A_\beta$ be big, pairwise disjoint, for $\beta \leqslant \beta_i$ (possible by the induction hypothesis).

We now define A'_β, if $\beta \notin \bigcup_{i<cf\alpha} [\beta_i, \beta_i + \lambda) \cup \{\alpha\}$, by :

$A'_\beta = A^i_\beta - (\beta_i + \lambda)$, where $\beta_i + \lambda < \beta \leqslant \beta_{i+1}$.

Clearly, the $A'_\beta \subseteq A_\beta$ are big, pairwise disjoint and disjoint from $D =^{df} \bigcup_{i<cf\alpha} [\beta_i, \beta_{i+1} + \lambda)$. For which β's have we still not defined A'_β ? For $\beta = \beta_i$ $(i \leqslant cf\delta)$ i.e., $\beta = \beta_j$, for which $\beta \in S$, hence $cfj \neq \aleph_0, \kappa, 1$. Checking definitions we can see that for each such β, $A_\beta \cap D \subseteq A_\beta$ is big. So it suffices to find pairwise disjoint big $A'_{\beta_j} \subseteq A_{\beta_j}$ $(j \leqslant cf\delta$, j a limit). This we do by induction on j. Suppose we have defined these for every $j' < j$. For j a successor among $\{i \leqslant cf\delta : i$ a limit$\}$ or $\beta_j \notin S$, there is no problem. (Remember for j a successor, β_j is a successor, hence $\notin S$). Otherwise, note that $cfj \neq \kappa$, hence $cf(\sup(E_j)) \neq \kappa$, hence $E_j \in P_\alpha$ (see stage B). Now look at Stage D, for β_j. We chose there an increasing continuous sequence of ordinals $\langle \gamma_i : i < cf\beta_j \rangle$ converging to β_j. Since $cf\beta_j \neq \aleph_0$, there is a closed unbounded $C \subseteq cf\beta_j$, such that $i \in C \Rightarrow \gamma_i \in \{\beta_\xi : \xi < j\}$. We then defined $A_{\beta_j} = \bigcup_{i<cf\beta_j} c^i_{\beta_j}$, where $c^i_{\beta_j} \subseteq (\gamma_i, \gamma_i + \lambda)$, has order type κ, and in particular

$$[\bigcup\{A_\zeta : \zeta \in \delta, \zeta \in acc(E_j)\}] \cap c^i_{\beta_j} \quad \text{has power} < \kappa.$$

But what is $acc(E_j)$? It is just $\{\beta_{j(o)} : j(o) < j, j(o)$ a limit$\}$. So $c^i_{\beta_j} \cap [\bigcup\{A_{j(o)} : j(o) < j, j(o)$ a limit, $A_{j(o)}$ defined$\}]$ has power $< \kappa$.

Let $A'_{\beta_j} = \bigcup\{c^i_{\beta_j} - \bigcup\{A_\zeta : \zeta \in S, \zeta \in acc(E_j)\} : i \in C\}$.

It is easy to check that it is a big subset of A_{β_j}, and obviously it is disjoint from $A_{\beta_{j(o)}}$, where $j(o) < j$ is a limit. So we have finished the proof.

Stage E : Suppose λ singular strong limit, $cf\lambda = \kappa$, S a stationary subset of λ^+, and every member of S divisible by $\lambda\omega$. Suppose further $A_\alpha \subseteq \alpha$, $|A_\alpha| \leq \kappa cf\alpha$ for $\alpha \in S$, and for any $\alpha_o < \lambda^+$, $\{A_\alpha : \alpha < \alpha_o\}$ has a transversal. Then we can find $A^*_\alpha \subseteq \alpha$ for $\alpha \in S$, so that $A^*_\alpha = \{\gamma(\alpha,i) : i < \kappa(cf\alpha)\}$, where $\gamma(\alpha,i)$ increase with i, (hence $|A^*_\alpha| \leq cf\alpha + \kappa\ (< \lambda)$) and for every $\alpha_o < \lambda^+$ there are pairwise disjoint $A'_\alpha \subseteq A_\alpha$ (for $\alpha < \alpha_o$, $\alpha \in S$), such that for each α for some $i_o < cf\alpha$

$$(\forall i < cf\alpha)\ (\exists \zeta < \kappa)(\forall \xi)(\zeta \leq \xi < \kappa\ \&\ i_o < i \rightarrow \gamma(\alpha,\kappa i+\xi) \in A'_\alpha).$$

Proof : For every α, choose $B^\xi_\alpha \subseteq \alpha$, B^ξ_α increase with ξ, $\alpha = \bigcup_{\xi<\kappa} B^\xi_\alpha$ and $|B^\xi_\alpha| < \lambda$. We can define functions h_o, h_1, Dom $h_\ell = \lambda^+$, so that for any $\beta_o, \beta_1 \leq \beta < \lambda^+$, $\xi < \kappa$, $A \subseteq B^\xi_{\beta_o}$, there are λ β^*'s, $\beta \leq \beta^* < \beta + \lambda$, such that $h_1(\beta^*) = \beta_1$, $h_2(\beta^*) = A$. (We define $h_\ell \restriction [\lambda i, \lambda(i+1))$ for each i; the number of possible tuples $< \beta_1, A, \beta, \xi, \beta_o >$ is $\leq \lambda$, so there is no problem).

For each $\alpha \in S$ choose an increasing sequence $\beta(\alpha,i)$ $(i < cf\alpha)$ converging to it.

First note that $(\forall \alpha_o < \alpha)$ $\alpha_o + \lambda < \alpha$ (since $\alpha \in S$) hence w.l.o.g. $\beta(\alpha,i) + \lambda < \beta(\alpha,i+1)$, and $\beta(\alpha,i)$ is divisible by λ.

Now we define by induction on $j = i\kappa + \xi$ $(i < cf\alpha,\ \xi < \kappa)$ an ordinal $\gamma(\alpha,j)$, increasing with j, such that

(i) $\beta(\alpha,i) < \gamma(\alpha,j) < \beta(\alpha,i) + \lambda$,

(ii) $h_1(\gamma(\alpha,j)) = cf\alpha$,

(iii) $h_2(\gamma(\alpha,j)) = A_\alpha \cap B^\xi_{\beta(\alpha,i)}$, and

(iv) $\gamma(\alpha,j) \notin \{A^*_{\alpha(o)} : \alpha(o) \in B^\xi_\alpha\}$.

The last condition excludes $< \lambda$ γ's, and the conditions (ii), (iii) are satisfied by λ γ's, $\beta(\alpha,i) < \gamma < \beta(\alpha,i) + \lambda$.

So we can define $A^*_\alpha = \{\gamma(\alpha,i) : i < \kappa(cf\alpha)\}$, and $\gamma(\alpha,i)$ increase with i and converge to α.

Now we are given $\alpha(o) < \lambda^+$ and have to find $A'_\alpha \subseteq A^*_\alpha$ as required. By hypothesis, there is a transversal f of $\{A_\alpha : \alpha < \alpha(o)\}$. Define $A^1_\alpha = \{\gamma(\alpha,\kappa i + \xi) : i < cf\alpha,\ f(A_\alpha) \in A_\alpha \cap B^\xi_{\beta(\alpha,i)}\}$.

Clearly it is a very big subset of A_α.

On $S \cap \alpha(o)$ we define a graph : (α_1,α_2) is an edge iff $A^1_{\alpha_1} \cap A^1_{\alpha_2} \neq \phi$.

Note :

(a) If (α_1,α_2) is an edge then $cf\alpha_1 = cf\alpha_2$ (because $\gamma \in A_{\alpha_\ell}$ implies $h_1(\gamma) = cf\alpha_\ell$).

(b) The valency of any α_1 $(= |\{\alpha_2 : (\alpha_1,\alpha_2)$ is an edge $\}|)$ is $\leq |A^*_\alpha|$.

As f is one-to-one, it suffices to prove that $f(A_{\alpha_2}) \in A_{\alpha_1}$ whenever $A_{\alpha_2} \cap A_{\alpha_1} \neq \phi$. If $\gamma = \gamma(\alpha_1,\kappa i_1 + \xi_1) = \gamma(\alpha_2,\kappa i_2 + \xi_2) \in A^1_{\alpha_1} \cap A^1_{\alpha_2}$, then $\beta = \beta(\alpha_1,i_1) = \beta(\alpha_2,i_2)$ (it is the biggest ordinal $\leq \gamma$ divisible by λ), so $A_{\alpha_1} \cap B^{\xi_1}_{\beta(\alpha_1,i_1)} = h_2(\gamma) = A_{\alpha_2} \cap B^{\xi_2}_{\beta(\alpha_2,i_2)}$, but $f(A_{\alpha_2}) \in A_{\alpha_2} \cap B^{\xi_2}_{\beta(\alpha_2,i_2)}$ (since $\gamma \in A^1_{\alpha_2}$) hence $f(A_{\alpha_2}) \in A_{\alpha_1} \cap B^{\xi_1}_{\beta(\alpha_1,i_1)} \subseteq A_{\alpha_1}$, as required.

Now we deal with each component C of the graph separately. By (a), all $\alpha \in C$ have the same cofinality, say μ, and by b), $|C| \leq \kappa + \mu$. If $\mu > \kappa$ note that each A^1_α has order type μ and is unbounded below α, hence $\alpha_1 \neq \alpha_2 = C \Rightarrow |A^1_{\alpha_1} \cap A^1_{\alpha_2}| < \mu$. So let $C = \{\alpha_\zeta : \zeta < \mu\}$, and we can define $A^*_{\alpha_\zeta} = A^1_{\alpha_\zeta} - \bigcup_{\xi<\zeta} A^1_{\alpha_\xi}$, which are as required. If $\mu \leq \kappa$, we give a similar treatment to each $\{\gamma(\alpha,\kappa i + \xi) : \xi < \kappa\}$ for $i < \mu$, $\alpha \in C$.

25. Conclusion :

1) Suppose $\aleph_\omega$ is a strong limit.

a) There is a family of $\aleph_{\omega+1}$ countable subsets of $\aleph_{\omega+1}$ which does not have a transversal, but every subfamily of cardinality $< \aleph_{\omega+1}$ has a transversal.

b) There is an abelian group [group] of power $\aleph_{\omega+1}$, which is not free, but every subgroup of cardinality $< \aleph_{\omega+1}$ is.

2) Suppose $\aleph_{\omega\ell}$ is strong limit for $\ell \leqslant n$. Then a), b) hold for $\aleph_{\omega n+1}$.

Proof : 1 a), 2 a). It is easy to see this after reading Milner and Shelah [MS].

1 b), 2 b) are easy to see.

26. Claim : Suppose λ is strong limit, $cf\lambda = \aleph_0$, $\mu < \kappa$, μ regular and : P is μ-complete or among any μ members of P there are μ which are pairwise compatible.

If in V^P λ is still a strong limit cardinal, then

$$S^*(\lambda^+)^V \cap CF(\lambda,\mu)^V, \quad S^*(\lambda^+)^{V^P} \cap CF(\lambda,\mu)^{V^P}$$

are equal (i.e., for some representation they are equal).

Proof : Let $d : \lambda^+ \to \kappa$ be normal. Clearly it is still normal in V^P. By 13 it suffices to prove that the truth value of "$\alpha \in S_1(d)$" is not changed, which is quite easy.

27. Claim : If χ is supercompact, $\lambda > \chi$, $cf\lambda < \chi$, then $S^*(\lambda^+)$ is stationary.

Proof : Let $d : \lambda^+ \to cf\lambda$ be normal and subadditive, and suppose $C \subseteq \lambda^+$ is closed and unbounded.

Suppose $N \prec (H(\lambda^{++}), \in)$, $cf\lambda + 1 \subseteq N$, $C,d \in N$, $\|N\| < \chi$ and every subset of $N \cap \lambda^+$ belongs to N (this is possible as χ is supercompact).

Let $\delta^* = \sup(N \cap \lambda^+)$. Clearly $cf\delta^*$ is the successor of a singular cardinal of cofinality $cf\lambda$ so $cf\delta^* > cf\lambda$. Clearly $C \cap N$ is unbounded, hence $\delta^* \in C$; so it suffices to prove $\delta^* \notin S_0(d)$.

So suppose $A \subseteq \delta^*$ is unbounded, and $d|A$ is bounded by ζ. Let $A = \{\beta_i : i < \delta^*\}$, β_i increasing. We may assume, w.l.o.g., for each i there is γ_i, $\beta_i < \gamma_i < \beta_{i+1}$, $\gamma_i \in N$. Let $\zeta_i = \text{Max}\,\{\zeta, d(\beta_{i+1},\gamma_i), d(\gamma_i,\beta_i)\} < cf\lambda < cf\delta^*$. So (w.l.o.g.) $\zeta_i = \zeta^*$ for every i. Now if $i < j$, then by the subadditivity :

$$d(\gamma_i,\gamma_j) \leqslant \max\,\{d(\gamma_j,\beta_{j+1}), d(\beta_{j+1},\beta_{i+1}), d(\beta_{i+1},\gamma_i)\} \leqslant \zeta^* .$$

So $d|\{\gamma_i : i < cf\delta^*\}$ is bounded, but the set necessarily belongs to N, and, as $N \prec (H(\lambda^{++}), \in)$, there is an unbounded $B \subseteq \lambda^+$ on which d is bounded, giving an easy contradiction to normality.

28. Remark : We in fact prove that if d is a subadditive function, with domain α^*, $\alpha \leqslant \alpha^*$, and d is bounded on some unbounded $A \subseteq \alpha$, then every unbounded $A' \subseteq \alpha$ has an unbounded subset $A'' \subseteq A' \subseteq \alpha$ such that $d|A''$ is bounded.

29. Conclusion : If ZFC + "$\exists$ a supercompact" is consistent then the following is consistent :

ZFC + GCH + "$S^*(\aleph_{\omega+1})$ is stationary".

Proof : Suppose χ is supercompact, and also (w.l.o.g.) GCH holds. Let λ be the first singular cardinal $> \chi$. By 27 we can choose a regular $\mu < \chi$ such that $S^*(\lambda^+) \cap CF(\lambda^+,\mu)$ is stationary. We use Levy collapsing P to collapse every $\mu' < \mu$ to $\aleph_0$ (by finite conditions). So now, in V^P, μ is $\aleph_1$. By 26, in V^P, $S^*(\lambda^+)^{V^P} \supseteq S^*(\lambda^+)^V \cap CF(\lambda^+,\mu)^V$, and the latter obviously remains stationary. Now collapse χ to $\aleph_1$ by a Q which is $\aleph_1$-complete. Again $S^*(\lambda^+)^V \cap CF(\lambda^+,\mu)^V$ remains stationary and is still included in $S^*(\lambda^+)^{P*Q}$.

$\diamondsuit_\lambda$ is not a strong requirement

30. Definition : Let λ be a regular cardinal and $E \subseteq \lambda$ a stationary

set in it.

(1) $\diamondsuit^*_\lambda(E)$. There is $< W_\alpha : \alpha \in E >$ such that for every α, W_α is a family of subsets of α with $|W_\alpha| \leq |\alpha|$, and for every $X \subseteq \lambda$ there is a closed and unbounded $C \subseteq \lambda$ such that $X \cap \alpha \in W_\alpha$ for all $\alpha \in C \cap E$.

(2) $\diamondsuit_\lambda(E)$. There is $< S_\alpha : \alpha \in E >$ such that $S_\alpha \subseteq \alpha$, and for every $X \subseteq \lambda$, $\{\alpha : X \cap \alpha = S_\alpha\}$ is stationary in λ.

<u>31. Theorem</u> : (Kunen) : (1) For stationary $E \subseteq \lambda$, $\diamondsuit^*_\lambda(E)$ implies $\diamondsuit_\lambda(E)$.

(2) For $E_1 \subseteq E_2 \subseteq \lambda$, $\diamondsuit_\lambda(E_1)$ implies $\diamondsuit_\lambda(E_2)$ and $\diamondsuit^*_\lambda(E_2)$ implies $\diamondsuit^*_\lambda(E_1)$.

<u>32. Theorem</u> : Suppose $\lambda = 2^\mu = \mu^+$ and for some regular $\kappa < \mu$, either

(i) $\mu^\kappa = \mu$, or

(ii) μ is singular $\kappa \neq cf\mu$ and for every $\delta < \mu, |\delta|^\kappa < \mu$

<u>Then</u> $\diamondsuit^*_\lambda(E(\kappa))$ where $E(\kappa)$ is the stationary subset $\{\alpha < \lambda : cf\alpha = \kappa\}$.

<u>Remark</u> : Case (i) is due to Gregory [Gr].

<u>Proof</u> : Let $<A_\alpha : \alpha < \lambda>$ be a list of all bounded subsets of λ each appearing λ times (there are λ such subsets as $\lambda = 2^\mu = \mu^+$)

<u>Case (i)</u> : For $\alpha \in E(\kappa)$ let W_α be the set of all unions of no more than κ subsets of α belonging to $< A_\beta : \beta < \alpha>$.

$(W_\alpha = \{\cup Y : |Y| \leq \kappa, x \in Y \rightarrow x \subseteq \alpha, x \in \{A_\beta : \beta < \alpha\}\})$.

Given $X \subseteq \lambda$, let C be $\{\alpha_i \mid i < \lambda\}$ where α_o is any successor less than λ, $\alpha_\delta = \bigcup_{\beta<\delta} \alpha_\beta$ for limit δ, and α_{i+1} is the least $\alpha > \alpha_i$ such that for some $\gamma < \alpha$, $A_\gamma = X \cap \alpha_i$.

Now $C' = \{\delta : \delta = \cup \{\alpha_i : \alpha_i < \delta\}\}$ is closed unbounded, and for $\delta \in C \cap E(\kappa)$ there are $i(j)$ and $\gamma_j < \delta$ $(j < \kappa)$ such that

$\bigcup_{j<\kappa} \alpha_{i(j)} = \delta$, $X \cap \alpha_{i(j)} = A_{\gamma_j}$. So $X \cap \delta = \bigcup_{j<\kappa} A_{\gamma_j} \in W_\delta$.

<u>Case (ii)</u> : For δ such that $cf\delta = \kappa$, let $\delta = \bigcup_{j<\mu} V^\delta_j$, where $<V^\delta_j : j < \mu>$ is increasing and for $j < \mu$, $|V^\delta_j| < \mu$.

Let W_δ be $\{\bigcup_{\alpha \in Q} A_\alpha : (\exists j < \mu)\ Q \subseteq V^\delta_j, |Q| \leq \kappa\}$.

Given $X \subseteq \lambda$ let $f : \lambda \rightarrow \lambda$ be such that $X \cap \alpha = A_{f(\alpha)}$ $f(\alpha) > \sup f(\beta)$.

There exists a closed unbounded $C \subseteq \lambda$ such that for $\alpha \in C$, $\beta < \alpha$ implies $f(\beta) < \alpha$.

Let $\delta \in C \cap E(\kappa)$, and for increasing $<\delta_i : i < \kappa>$, $\delta = \bigcup_{i<\kappa} \delta_i$.

There exists j such that .

$\kappa = |V^\delta_j \cap \{f(\delta_i) : i < \kappa\}|$ hence $X \cap \delta = \cup\{X \cap \delta_i : i < \kappa, f(\delta_i) \in V^\delta_j\} \in W_\delta$.

<u>33. Conclusion</u> : (GCH) If $\lambda > \aleph_o$, then $\diamondsuit^*_{\lambda^+}(E(\kappa))$ holds, whenever $\kappa \neq cf\lambda$. In particular $\diamondsuit_\lambda$ holds.

<u>Final comments</u>

1) The restriction "λ strong limit" in most cases can be weakened at the expense of complicating the results : assuming $(\forall \mu < \lambda)$ $\mu^{<\chi} < \lambda$, and restricting ourselves to $CF(\lambda^+, <\chi)$ or $CF(\lambda^+, \leq \chi)$.

2) A more serious question is whether we can, in 7, replace D^g_λ by D_λ. This remains open.

Note that the natural notion is $S_2(\bar{N})$, and that for regular λ, $I^+(\lambda) = \{A \subseteq \lambda$: for some λ-approximating sequence $\bar{N}$, $A \subseteq S_2(\bar{N})\}$ is always a normal ideal. Similarly

$I^-(\lambda) = \{A \subseteq \lambda : A \cap B \equiv \phi \bmod D_\lambda$ for every $B \in I^+(\lambda)\}$

is a normal ideal. The meaning of claim 7 is that $I^+(\lambda)$ is $\{A : A \subseteq A_o \bmod D_\lambda\}$ for some A_o, when $gcf(\lambda) = \lambda$. Another formulation of our question is whether this always holds.

However, we can meanwhile just formulate the later theorems in terms of $I^+(\lambda)$ instead of $S^*(\lambda)$ (and the changes in the proofs are minor). By the way it may be more natural to use $S_3(\bar{N}) = \{\delta$: there is a function h, Dom h = cfδ, Range h an unbounded subset of δ, $(\forall i < \mathrm{cf}\delta)$ $h|i \in N_\delta$, and $N_\delta \cap \lambda = \delta\}$ (in gcf(λ) it does not matter).

3) Why were we interested mainly in $\aleph_{\omega+1}$ and not in e.g. $\aleph_{\omega+2}$? The answer is that several inductive proofs work for successors of regular cardinal, and it was not clear whether they fail at successors of singulars. (But see remarks 5 and 6 below).

4) It may be of interest to mention our original line of thought, which is not so transparent from the present paper.

We want to prove that $S_2(\bar{N})$ is quite "big", where $\bar{N}$ is an $\aleph_{\omega+1}$-approximating sequence for $\aleph_{\omega+1}$, assuming GCH. So we let $d : \aleph_{\omega+1} \to \aleph_0$ be normal, and using the Erdös-Rado theorem $(2^{\aleph_n})^+ \to (\aleph_{n+1})^2_{\aleph_0}$, prove that if $C \subseteq \aleph_{\omega+1}$ is closed of order type $(2^{\aleph_n})^+$ then it contains C_1 of order type $\aleph_{n+1}$, with d constant on C_1. C_1' (the set of accumulation points of C_1) is $\subseteq S_2(\bar{N})$ and is a closed subset of C of order type $\aleph_{n+1}$. This proves that $S_2(\bar{N})$ is in some sense big.

5) We can try to generalize 4) to other cardinals.

Let $\kappa = \mathrm{cf}\, \aleph_\alpha < \aleph_\alpha$.

<u>Definition</u> : Call an (n+1)-place function d from $\aleph_{\alpha+n}$ to κ <u>normal</u> if for every $\alpha_0 < \ldots < \alpha_n < \aleph_{\alpha+n}$ there is $k < n$ such that

$$\{\alpha < \aleph_{\alpha+n} : d(\alpha_0,\alpha_1,\ldots,\alpha_{k-1},\alpha,\alpha_{k+1},\ldots,\alpha_n) = d(\alpha_0,\ldots,\alpha_k,\ldots,\alpha_n)\}$$

has cardinality $< \aleph_\alpha$.

<u>Claim</u> : There is a normal function $d : \aleph_{\alpha+n} \to \kappa$.

<u>Proof</u> : By induction on n.

<u>Lemma</u> : Let $\bar{N}$ be an $\aleph_{\alpha+n}$-approximating sequence for $\aleph_{\alpha+n+1}$, C a closed subset of $\aleph_{\alpha+n}$ of order type $\beth_{n+1}(\kappa + \mu)^+$, where $\mu < \lambda$.

Then C has a closed subset of order type μ^+ which is included in $S_2(\bar{N})$.

<u>Proof</u> : Let $d \in N_0$, $d : \aleph_{\alpha+n} \to \kappa$, d normal. By the Erdös-Rado theorem $(\beth_{n+1}(\kappa + \mu)^+ \to (\mu^+)^{n+1}_\kappa)$ there is $C_1 \subseteq C$ of order type μ^+ on which d is constant. If $\delta \in C_1'$, then $C_1 \cap \delta$ witnesses that $\delta \in S_2(\bar{N})$.

6) Suppose $\aleph_\alpha$ is strong limit, $\kappa = \mathrm{cf}\,\aleph_\alpha$, γ a successor ordinal, $\kappa \leq \mu < \aleph_\alpha$ and $\beth_\gamma(\mu) < \aleph_\alpha$. If $\bar{N}$ is a $\aleph_{\alpha+\gamma}$-approximating sequence for $\aleph_{\alpha+\gamma+1}$, and $C \subseteq \aleph_{\alpha+\gamma}$ has order type $\beth_\gamma(\mu)^+$, then C has a closed subset C_1 of order type μ^+ which is included in $S_2(\bar{N})$.

<u>Proof</u> : We prove a somewhat stronger statement :

If $C \subseteq \aleph_{\alpha+\beta}$, $\beta \leq \gamma$ a successor ordinal, and C has order type$\geq \beth_\beta(\mu)^+$, then there is $C_1 \subseteq C \cap S_2(\bar{N})$ of order type μ^+, such that for some $\ell < n$, if $\alpha_0 < \ldots < \alpha_n \in C_1$ then

$$(H(\aleph_{\alpha+\gamma+1}),\in) \models \varphi(\alpha_0,\ldots,\alpha_n) \; \& \; |\{x : \varphi(\alpha_0,\ldots,\alpha_{\ell-1},x,\alpha_\ell,\ldots,\alpha_n)\}| < \aleph_\alpha$$

(This implies $C_1' \subseteq S_2(\bar{N})$).

We prove this by induction on β. For finite β this was done above, and the induction step is easy.

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P A U L B E R N A Y S

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Paul Bernays was born on October 17th 1888 in London; he died after a short illness on September 18th 1977 in Zürich. He was the son of Julius and Sara Bernays, née Brecher. His father was a businessman and - as he states in the curriculum vitae appended to his thesis - he was of jewish confession and a citizen of Switzerland. (1) Soon after the birth of Paul, the family moved to Paris and from there to Berlin. It is in Berlin that he attended school, from 1895 to 1907. He seems to have been quite happy at school, a gifted, well adapted child accepting the prevailing cultural values in literature as well as in music. It was indeed his musical talent that first attracted attention; he tried his hand at composing, but being never quite satisfied with what he achieved, he decided on a scientific career. He studied engineering at the Technische Hochschule Charlottenburg for one semester, then realizing (and convincing his parents) that pure mathematics was what he wanted to do, he transferred to the University of Berlin. His main teachers were: Schur, Landau, Frobenius and Schottky in mathematics; Riehl, Stumpf and Cassirer in philosophy, Planck in physics. After four semesters, he moved to Göttingen; there he attended lectures on mathematics by Hilbert, Landau, Weyl and Klein, on physics by Born, and on philosophy by Leonard Nelson. Nelson was the center of the Neu-Friessche Schule - Bernays was quite an active member of the group and stayed in contact with it all his life. His first publication - in 1910 - was "Das Moralprinzip bei Sidgwick und bei Kant", published in the Abhandlungen der Friesschen Schule. (2) There were two further publications in 1913 in the same Abhandlungen, one "Ueber den transzendentalen Idealismus", the other "Ueber die Bedenklichkeiten der neueren Relativitätstheorie". (3,4) Though we no longer share the difficulties discussed by Bernays, it is remarkable how calmly he takes part in otherwise rather heated controversies. There is no doubt that Bernays was deeply influenced by Nelson - by his liberal socialism as well as by his revised version of Kant's imperative demanding the permanent readiness to act according to duty (Nelson lived from 1882 to 1927).

In the spring of 1912 Bernays received his doctorate with a dissertation (written with Landau) on analytic number theory - the exact title being: "Ueber die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht-quadratischen Diskriminante." (1)