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KULIKOV'S PROBLEM ON UNIVERSAL TORSION-FREE ABELIAN GROUPS

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Abstract

Let T be an abelian group and λ an uncountable regular cardinal. The question of whether there is a λ-universal group U among all torsion-free abelian groups G of cardinality less than or equal to λ satisfying $\mathrm{Ext}(G, T) = 0$ is considered. U is said to be λ-universal for T if, whenever a torsion-free abelian group G of cardinality at most λ satisfies $\mathrm{Ext}(G, T) = 0$, there is an embedding of G into U. For large classes of abelian groups T and cardinals λ, it is shown that the answer is consistently no, that is to say, there is a model of ZFC in which, for pairs T and λ, there is no universal group. In particular, for T torsion, this solves a problem by Kulikov.

1. *Introduction*

Given a class $\mathscr{C}$ of objects it is natural to ask for universal objects in $\mathscr{C}$. A universal object is an element $C \in \mathscr{C}$ such that every other object of the class $\mathscr{C}$ can be embedded into C. The existence of universal objects clearly simplifies the structure theory for $\mathscr{C}$.

On the other hand, if there are no universal objects in $\mathscr{C}$, this indicates that the class $\mathscr{C}$ has a complicated structure. Since the definition of universal objects is formulated categorically, the search for universal objects appears, as is well known, in any field of mathematics.

In this paper we focus on abelian groups, and begin with the class $\mathscr{T}\mathscr{F}_\lambda$ of all torsion-free abelian groups G of rank less than or equal to λ, where λ is a fixed cardinal. We consider the subclass $\mathscr{C} = \mathscr{T}\mathscr{F}_\lambda(T)$ of all $G \in \mathscr{T}\mathscr{F}_\lambda$ with $\mathrm{Ext}(G, T) = 0$ for some fixed abelian group T. Here $\mathrm{Ext}(\_, T)$ denotes the first derived functor of the functor $\mathrm{Hom}(\_, T)$. In 1969, Kulikov raised the problem of whether or not there are universal groups in $\mathscr{T}\mathscr{F}_\lambda(T)$ for all (uncountable) cardinals λ and torsion abelian groups T. If the group T is cotorsion, that is, $\mathrm{Ext}(\mathbb{Q}, T) = 0$, then, for any λ, there is a universal group in $\mathscr{T}\mathscr{F}_\lambda(T)$, namely the torsion-free divisible group of rank λ. Surely the restriction to classes of groups bounded by some fixed cardinal λ is necessary to find universal objects. We want to consider Kulikov's problem and its solution in the context of recently investigated cotorsion theories.

Cotorsion theories for abelian groups were introduced by Salce [**11**] in 1979. Following his notation, we call a pair $(\mathscr{F}, \mathscr{C})$ a cotorsion theory if $\mathscr{F}$ and $\mathscr{C}$ are classes of abelian groups which are maximal with respect to the property $\mathrm{Ext}(F, C) = 0$ for all $F \in \mathscr{F}$, $C \in \mathscr{C}$.

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Salce [**11**] showed that every cotorsion theory is cogenerated by a class of torsion and torsion–free groups, where $(\mathscr{F},\mathscr{C})$ is said to be cogenerated by the class $\mathscr{A}$ if

$$\mathscr{C} = \mathscr{A}^{\perp} = \{X \in \text{Mod–}\mathbb{Z} \mid \text{Ext}(A,X) = 0 \text{ for all } A \in \mathscr{A}\}$$

and

$$\mathscr{F} = {}^{\perp}(\mathscr{A}^{\perp}) = \left\{Y \in \text{Mod–}\mathbb{Z} \;\middle|\; \text{Ext}(Y,X) = 0 \text{ for all } X \in \mathscr{A}^{\perp}\right\}.$$

Examples for cotorsion theories are $(\mathscr{L}, \text{Mod–}\mathbb{Z}) = ({}^{\perp}(\mathbb{Z}^{\perp}), \mathbb{Z}^{\perp})$, where $\mathscr{L}$ is the class of all free groups, and the classical cotorsion theory

$$(\mathscr{T}\mathscr{F}, \mathscr{C}\mathscr{O}) = ({}^{\perp}(\mathbb{Q}^{\perp}), \mathbb{Q}^{\perp}),$$

where $\mathscr{T}\mathscr{F}$ is the class of all torsion-free groups and $\mathscr{C}\mathscr{O}$ is the class of all cotorsion groups. If $(\mathscr{F},\mathscr{C})$ is a cotorsion theory, then $\mathscr{C}$ is called cotorsion class and $\mathscr{F}$ is called torsion-free class.

We put $(\mathscr{F},\mathscr{C}) \leqslant (\mathscr{F}',\mathscr{C}')$ for cotorsion theories $(\mathscr{F},\mathscr{C})$ and $(\mathscr{F}',\mathscr{C}')$ if $\mathscr{C} \subseteq \mathscr{C}'$. We say that $(\mathscr{F},\mathscr{C})$ is singly cogenerated if $\mathscr{C} = G^{\perp}$ for some group G. Clearly $\mathbb{Z}^{\perp}$ and $\mathbb{Q}^{\perp}$ give rise to the maximal and minimal cotorsion theories. Moreover, Göbel, Shelah and Wallutis showed in [**5**] that any partially ordered set can be embedded into the lattice of all cotorsion classes. Hence there is no hope of characterizing cotorsion theories. However, if we restrict ourselves to torsion-free groups in $\mathscr{T}\mathscr{F}_{\lambda}(T)$, then the existence of a universal group provides a step towards classification. The existence of universal groups also contributes information about the size of singly cogenerated cotorsion classes $G^{\perp}$. If G is torsion-free of rank at most λ, and $T \in G^{\perp}$ provides a universal group C_T for $\mathscr{T}\mathscr{F}_{\lambda}(T)$, then $C_T^{\perp} \subseteq G^{\perp}$.

In a first contribution to Kulikov's problem, Strüngmann [**17**] showed that in Gödel's universe ($V = L$) for every cardinal λ and torsion abelian group T there exists a λ-universal group $G \in \mathscr{T}\mathscr{F}_{\lambda}(T)$ if T has only finitely many non-trivial bounded primary components. Moreover, if λ is finite, then this characterizes those T that give rise to universal groups even in ZFC.

In this paper we prove that there is a model of ZFC in which the generalized continuum hypothesis (GCH) holds with the property that for every abelian group T which is not cotorsion and every uncountable regular cardinal κ there is a cardinal $\lambda \geqslant \kappa$ such that the class $\mathscr{T}\mathscr{F}_{\lambda}(T)$ has no universal object. Moreover, for torsion abelian groups T (not cotorsion) of cardinality less than or equal to $\aleph_1$, there is no uncountable regular cardinal λ such that $\mathscr{T}\mathscr{F}_{\lambda}(T)$ has universal groups. This shows that the result in [**17**] is not provable under ZFC and it answers Kulikov's question consistently in the negative.

The notations are standard, and for unexplained notions in abelian group theory and set theory we refer to [**1**, **3**, **4**, **8**]. For uniformization, see [**9**] or [**13**].

2. *λ-universal groups*

In this section we introduce the notions of λ-universal groups for a given group T, and obtain some basic properties. We are mainly interested in the case when our group T is torsion, but leave T arbitrary whenever this is possible. Let $\mathscr{T}\mathscr{F}$ be the class of all torsion-free groups. For a cardinal λ we denote by $\mathscr{T}\mathscr{F}_{\lambda}(T)$ the class of all torsion-free groups G of rank at most λ such that $\text{Ext}(G,T) = 0$. Moreover, we let $\mathscr{T}\mathscr{F}(T) = \bigcup_{\lambda} \mathscr{T}\mathscr{F}_{\lambda}(T)$ be the class of all torsion-free groups G satisfying $\text{Ext}(G,T) = 0$.

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We recall from the introduction that $G \in \mathscr{C}$ is universal in the class $\mathscr{C}$ if any group in $\mathscr{C}$ embeds into G. In particular we shall use the following definition.

Definition 2.1. If T is a group and λ a cardinal, then a group G is called *λ-universal* for T if G is universal in the class $\mathscr{TF}_{\lambda}(T)$.

Kulikov [**10**, Question 1.66] posed the following question.

Problem 2.2. Do λ-universal groups exist for all uncountable cardinals and for all torsion groups?

The starting point of this paper are the following results obtained recently by Strüngmann [**17**]. For this we recall that a completely decomposable group is just a direct sum of subgroups of the rational numbers $\mathbb{Q}$.

Proposition 2.3 [**17**]. *Let T be a torsion group and λ a natural number. Then there exists a λ-universal group G for T if and only if T has only finitely many non-trivial bounded primary components. In this case, G is completely decomposable.*

Theorem 2.4 ($V = L$, see [**17**]). *If T is a torsion group with only finitely many non-trivial bounded primary components and λ is a cardinal, then there is a λ-universal group for T which is completely decomposable.*

We want to omit the case $\lambda = \aleph_0$, and therefore restrict ourselves to uncountable (regular) cardinals λ.

It is well known that any torsion group T has a basic subgroup $B \subseteq T$ which is pure, a direct sum of cyclic groups, and divisible quotient T/B. We have the following immediate lemma.

Lemma 2.5. *Let T be a torsion group, let B be a basic subgroup of T and let λ be a cardinal. A torsion-free group G is λ-universal for T if and only if G is λ-universal for B.*

Proof. Since $\operatorname{Ext}(H,T) = 0$ if and only if $\operatorname{Ext}(H,B) = 0$ for any torsion-free group H, the lemma follows immediately (see, for example, [**17**, Lemma 1.2]). □

Proposition 2.6. *If λ is a cardinal, then a torsion-free group G is λ-universal for T if and only if G is λ-universal for the reduced part of T.*

Proof. Decompose $T = D \oplus R$ into the maximal divisible subgroup D and the reduced part R. Then the lemma follows from $\operatorname{Ext}(H,D) = 0$ and $\operatorname{Ext}(H,T) = \operatorname{Ext}(H,R)$ for any group H. □

Thus it is enough to consider reduced groups. We may reduce the question about the existence of λ-universal groups for T further and by Lemma 2.7 we also assume that T is not cotorsion. Recall that a group T is called *cotorsion* if $\operatorname{Ext}(\mathbb{Q},T) = 0$.

Lemma 2.7. *There is a λ-universal group for every cardinal λ and any cotorsion group T.*

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Proof. We have $D = \bigoplus_\lambda \mathbb{Q} \in \mathcal{TF}_\lambda(T)$ for any cotorsion group T and cardinal λ, and D is λ-universal because any torsion-free group of rank at most λ embeds into D. □

From [**17**] we also note the following.

LEMMA 2.8. *Let G be any group and let T be a torsion group. Then* $\mathrm{Ext}(G, T) = 0$ *if and only if* $\mathrm{Ext}(G, T') = 0$ *for all pure subgroups T' of T such that* $|T'| \leqslant |G|$.

Thus it is no restriction to assume that $|T| \leqslant \lambda$, and we will even assume that $|T| < \lambda$ in the sequel of this paper.

3. (T, λ, γ)-*suitable groups*

Let $\gamma < \lambda$ be fixed regular infinite cardinals.

DEFINITION 3.1. Let T be a group with $|T| < \lambda$. Then a group G is called (T, λ, γ)-*suitable* if the following conditions are satisfied:

(i) $\mathrm{Ext}(G, T) \neq 0$.

(ii) There is an increasing chain $(F_i : i \leqslant \gamma)$ of free groups such that the following hold:

(a) $|F_i| \leqslant |i| + \aleph_0$ for all $i < \gamma$.

(b) F_γ / F_i is free for all $i < \gamma$.

(c) $F_\gamma / \bigcup_{i<\gamma} F_i \cong G$.

Our next lemma shows that for any group T not cotorsion there is a (T, λ, ω)-suitable group.

LEMMA 3.2. *Let T be a group with $|T| < \lambda$ and let G be a countable group such that* $\mathrm{Ext}(G, T) \neq 0$. *Then G is (T, λ, ω)-suitable. If T is not cotorsion then $\mathbb{Q}$ is (T, λ, ω)-suitable.*

Proof. Choose a free resolution

$$0 \longrightarrow K \xrightarrow{\mathrm{id}} F \longrightarrow G \longrightarrow 0$$

of G. We may assume that K and F have countable rank and write $K = \bigoplus_{i<\omega} \mathbb{Z} e_i$. Then each $F_n = \bigoplus_{i \leqslant n} e_i \mathbb{Z}$ is a direct summand of F and hence G is (T, λ, ω)-suitable. If T is not cotorsion, then $\mathrm{Ext}(\mathbb{Q}, T) \neq 0$, and hence the same arguments show that $\mathbb{Q}$ is (T, λ, ω)-suitable. □

Next we show the existence of (T, λ, γ)-suitable groups for uncountable cardinals γ. Recall that a group G is *almost-free* if all subgroups of smaller cardinality are free.

LEMMA 3.3. *If T is a group with $|T| < \lambda$ and G is almost-free with $|G| = \gamma$ such that* $\mathrm{Ext}(G, T) \neq 0$, *then G is (T, λ, γ)-suitable.*

Proof. Choose a γ-filtration $G = \bigcup_{\alpha<\gamma} G_\alpha$ of G such that each G_α is free. By [**1**, Lemma 1.4, p. 346], there is a free resolution associated with this filtration. This is to say that there are free groups $F = \bigoplus_{\alpha<\gamma} F_\alpha$ and $K = \bigoplus_{\alpha<\gamma} K_\alpha$ such that the short sequences

$$0 \longrightarrow K \longrightarrow F \longrightarrow G \longrightarrow 0$$

and

$$0 \longrightarrow \bigoplus_{\alpha<\beta} K_\alpha \longrightarrow \bigoplus_{\alpha<\beta} F_\alpha \longrightarrow G_\beta \longrightarrow 0$$

are exact for all $\beta < \gamma$. Since each G_β is free, it follows that G is (T, λ, γ)-suitable. □

LEMMA 3.4. *Let H be an epimorphic image of the group T. If G is (H, λ, γ)-suitable then G is (T, λ, γ)-suitable.*

Proof. The claim follows immediately when we note that $\mathrm{Ext}(G, T) = 0$ implies that $\mathrm{Ext}(G, H) = 0$. □

PROPOSITION 3.5. *Let $S \subseteq \gamma$ be stationary non-reflecting such that $\mathrm{cf}(\alpha) = \omega$ for all $\alpha \in S$, and assume that $\diamondsuit_S$ holds. Let T be a group which has an epimorphic image of size at most γ that is not cotorsion. Then there is a strongly γ-free group of size γ which is (T, λ, γ)-suitable.*

The proposition has an immediate corollary.

COROLLARY 3.6. *Suppose that γ and S satisfy the conditions in Proposition 3.5. If T is not cotorsion and torsion or $|T| \leqslant \gamma$, then there is a (T, λ, γ)-suitable torsion-free group.*

Proof. If $|T| \leqslant \gamma$, then the corollary follows from Proposition 3.5. If T is torsion, then we choose a basic subgroup B' of T and a countable unbounded direct summand B of B' which is therefore not cotorsion. Note that B exists since T is not cotorsion. It is well known that B is an epimorphic image of T (see [**3**, **4**, Theorem 36.1]); hence Lemma 3.4 applies and it is enough to construct a (B, λ, γ)-suitable group which follows from Proposition 3.5. □

Proof of Proposition 3.5. Let B be the epimorphic image of T as in the proposition. Then by Lemma 3.4 it is enough to construct a (B, λ, γ)-suitable group. Therefore, we may assume that $|T| \leqslant \gamma$. By Lemma 3.2, there is a countable torsion-free group R which is (T, λ, ω)-suitable. If $\lambda_n \geqslant \aleph_0$ $(n \in \omega)$ are cardinals, then, as in [**1**, Corollary 1.2, p. 182], there exist free abelian groups $K \subseteq F$ and an ascending chain K_n $(n \in \omega)$ of subgroups with $K = \bigcup_{n\in\omega} K_n$ and F/K_n is free for all $n \in \omega$, K_0 is free of rank λ_0 and K_{n+1}/K_n is free of rank λ_{n+1}. From $F/K \cong R$, it follows that $\mathrm{Ext}(F/K, T) \neq 0$. As in the proof of [**1**, Theorem 1.4, p. 185], we can construct a torsion-free group G of cardinality γ which has a γ-filtration $G = \bigcup_{\alpha<\gamma} G_\alpha$ satisfying the following conditions for all $\alpha < \beta < \gamma$:

(i) G_α is free of rank $|\alpha| + \aleph_0$.

(ii) If α is a limit ordinal, then $G_\alpha = \bigcup_{\delta<\alpha} G_\delta$.

(iii) If $\alpha \notin S$, then G_β/G_α is free of rank $|\beta| + \aleph_0$.
(iv) If $\alpha \in S$, then $\mathrm{Ext}(G_\beta/G_\alpha, T) \neq 0$.

From $\diamondsuit_S$, it follows that $\mathrm{Ext}(G, T) \neq 0$ (see [**1**, Theorem 1.15, p. 353]); hence G is (T, λ, γ)-suitable by Lemma 3.3. □

4. *The uniformization*

Besides regular cardinals $\aleph_0 \leqslant \gamma < \lambda$, we also fix a stationary subset S of λ consisting of limit ordinals of cofinality γ. To prove Theorem 4.6 and Theorem 4.9, we shall use a construction developed in [**2**]. Thus we shall concentrate on the basic steps.

DEFINITION 4.1. A *ladder system $\bar{\eta}$ on S* is a family of functions $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ such that $\eta_\delta : \gamma \longrightarrow \delta$ is strictly increasing with $\sup(\mathrm{rg}(\eta_\delta)) = \delta$, where $\mathrm{rg}(\eta_\delta)$ denotes the range of η_δ. We call the ladder system *tree-like* if, for all $\delta, v \in S$ and every $\alpha, \beta \in \gamma$, $\eta_\delta(\alpha) = \eta_v(\beta)$ implies that $\alpha = \beta$ and $\eta_\delta(\rho) = \eta_v(\rho)$ for all $\rho \leqslant \alpha$.

For a ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ on S, we can form a *tree* $B_{\bar{\eta}} \subseteq {}^{\leqslant \gamma}\lambda$ of height γ. Let $B_{\bar{\eta}} = \{\eta_\delta \restriction_\alpha : \delta \in S, \alpha \leqslant \mathrm{l}(\eta_\delta)\}$, where $\mathrm{l}(\eta_\delta)$ denotes the length of η_δ. Note that $B_{\bar{\eta}}$ is partially ordered by defining $\eta \leqslant v$ if and only if $\eta = v \restriction_{\mathrm{l}(\eta)}$.

From the ladder system $\bar{\eta}$ and a group G which is (T, λ, γ)-suitable for some group T, we want to find a new group $H_{\bar{\eta}}$. Fix a chain $\langle F_\alpha : \alpha \leqslant \gamma \rangle$ for G as in Definition 3.1. Hence $F_\gamma / \bigcup_{\alpha<\gamma} F_\alpha \cong G$. For each $\eta \in B_{\bar{\eta}}$, we let $H_\eta = F_{\mathrm{l}(\eta)}$, and, if $\eta \leqslant v \in B_{\bar{\eta}}$, then let $i_{\eta,v}$ be the inclusion map of H_η into H_v. Finally, let $H_{\bar{\eta}}$ be the direct limit of $(H_\eta, i_{\eta,v} : \eta \leqslant v \in B_{\bar{\eta}})$. More precisely, $H_{\bar{\eta}}$ equals $\bigoplus \{H_\eta : \eta \in B_{\bar{\eta}}\}/K$, where K is the subgroup generated by all elements of the form $x_\eta - y_v$, where $y_v \in H_v$, $x_\eta \in H_\eta$, $\eta \leqslant v$ and $i_{\eta,v}(x_\eta) = y_v$. Canonically, we can embed H_η into $H_{\bar{\eta}}$, and we shall therefore regard H_η as a subgroup of $H_{\bar{\eta}}$ in the sequel.

DEFINITION 4.2. Let κ be an uncountable regular cardinal. The tree $B_{\bar{\eta}}$ is called *κ-free* if for every $X \subseteq S$ with $|X| < \kappa$ there is a function $\Psi : X \longrightarrow \gamma$ such that

$$\{\{\eta_\delta \restriction_\alpha : \Psi(\delta) < \alpha \leqslant \gamma\} : \delta \in X\}$$

is a family of pairwise disjoint sets. The ladder system $\bar{\eta}$ is called *κ-free* if $B_{\bar{\eta}}$ is κ-free.

We now state some properties of the constructed group $H_{\bar{\eta}}$.

LEMMA 4.3. *If κ is an uncountable regular cardinal and $B_{\bar{\eta}}$ is a κ-free tree, then $H_{\bar{\eta}}$ is a κ-free group.*

Proof. See [**2**, Lemma 1.4]. □

LEMMA 4.4. *If S is non-reflecting, then $H_{\bar{\eta}}$ is λ-free but not free.*

Proof. Since S is non-reflecting, $\bar{\eta}$ is λ-free, and by Lemma 4.3 the group $H_{\bar{\eta}}$ is λ-free. However, if $\delta \in S$, then there exists a $v \geqslant \delta$ such that for all $\mu < \gamma$, $\eta_v \restriction_\mu \in \{\eta_\alpha \restriction_\mu : \alpha < \delta\}$. Thus [**2**, Lemma 1.5] applies, and $H_{\bar{\eta}}$ is not free. □

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We recall μ-uniformization for a ladder system $\bar{\eta}$ and a cardinal μ.

DEFINITION 4.5. If μ is a cardinal and $\bar{\eta}$ is a ladder system on S, then we say that $\bar{\eta}$ has μ-uniformization if, for every family $\{c_\delta : \delta \in S\}$, where $c_\delta : \mathrm{rg}(\eta_\delta) \longrightarrow \mu$, there exist $\Psi : \lambda \longrightarrow \mu$ and $\Psi^* : S \longrightarrow \mu$ such that $\Psi(\eta_\delta(\alpha)) = c_\delta(\eta_\delta(\alpha))$ with $\Psi^*(\delta) \leqslant \alpha < \gamma$ for all $\delta \in S$.

THEOREM 4.6. *Let T be a group with $|T| < \lambda$ and let G be (T, λ, γ)-suitable. If S is non-reflecting and $\bar{\eta}$ is a tree-like ladder system on S with $2^{(|T|^{\gamma})}$-uniformization, then there exists a torsion-free group H of size λ such that the following hold:*

(i) *H has a λ-filtration $\langle \bar{H}_\alpha : \alpha < \lambda \rangle$.*
(ii) *If $\alpha \in S$, then $\bar{H}_{\alpha+1}/\bar{H}_\alpha \cong G$.*
(iii) *If $\alpha \notin S$, then $\bar{H}_\beta/\bar{H}_\alpha$ is free for all $\alpha \leqslant \beta$.*
(iv) $\mathrm{Ext}(H, W) = 0$ *for all groups W with $|W| \leqslant |T|$.*

Proof. Let T, S, $\bar{\eta}$ and G be as stated, and choose the group

$$H_{\bar{\eta}} = \bigoplus \{H_\eta : \eta \in B_{\bar{\eta}}\}/K$$

as constructed above. Then $H_{\bar{\eta}}$ is almost-free but not free by Lemma 4.4. From [**2**, Theorem 1.7], it follows that $H_{\bar{\eta}}$ satisfies $\mathrm{Ext}(H_{\bar{\eta}}, W) = 0$ for every group W of size at most $|T|$, and it is easy to see that $H_{\bar{\eta}}$ has a λ-filtration as stated if we let $\bar{H}_\alpha = \bigcup\{(H_\eta + K)/K : \eta \in B_{\bar{\eta}}, \sup(\mathrm{rg}(\eta)) < \alpha\}$. □

We want to apply Theorem 4.6 in models of ZFC to ladder systems which have μ-uniformization for all $\mu < \lambda$. This is possible for many regular cardinals as long as $|T|$ is small enough for $2^{(|T|^{\gamma})} < \lambda$ to be obtained, which is the case when λ is strongly inaccessible or the successor of a singular cardinal. The case $\lambda = \kappa^+$ with $\gamma = \kappa$ regular is not covered by Theorem 4.6, which explains our intention to prove Theorem 4.9 next.

DEFINITION 4.7. Let κ be regular and let $\lambda = \kappa^+$. A ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ has *strong κ-uniformization* if, for every system $\bar{P} = \langle P_\alpha : \alpha < \lambda \rangle$ such that the following hold:

(i) $\emptyset \neq P_\delta \subseteq \{f \mid f : \mathrm{rg}(\eta_\delta) \longrightarrow \delta \cap \kappa\}$ if $\delta \in S$;
(ii) if $\delta \in S$ and $i < \gamma$, then

$$P_{\eta_\delta(i)} = \left\{ f \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(i+1)})} \,\middle|\, f \in P_\delta \right\};$$

(iii) if $\delta \in S$ and $i < \gamma$ is a limit ordinal, then, for every increasing sequence $\langle f_j : j < i \rangle$, $f_j \in P_{\eta_\delta(j)}$, there exists an $f_i \in P_{\eta_\delta(i)}$ which extends the union $\bigcup_{j<i} f_j$;

there exists a function $f : \lambda \longrightarrow \kappa$ such that $f \restriction_{\mathrm{rg}(\eta_\delta)} \in P_\delta$ for all $\delta \in S$.

PROPOSITION 4.8. *Let $\lambda = \gamma^+$ for some regular cardinal γ and let*

$$\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$$

be a tree-like ladder system on S such that $\bar{\eta}$ has γ-uniformization and $\diamondsuit_\gamma$ holds. Then $\bar{\eta}$ has strong γ-uniformization.

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Proof. Let $\bar{P}$ be given as in Definition 4.7 and let J be a stationary subset of γ such that $\diamondsuit_\gamma(J)$ holds. We may assume that $J = \gamma$. Thus there exists a system of diamond functions $\bar{h} = \langle h_\delta : \delta \longrightarrow \gamma \mid \delta < \gamma \rangle$ such that for every function $h : \gamma \longrightarrow \gamma$ the set $\{\delta < \gamma : h \restriction_\delta = h_\delta\}$ is stationary in γ. For each $\delta \in S$ and $i < \gamma$, we define

$$h_i^\delta : \mathrm{rg}(\eta_\delta \restriction_i) \longrightarrow \gamma, \qquad \eta_\delta(j) \longmapsto h_i(j).$$

If $\delta \in S$, then let $E_\delta = \{i < \gamma : h_i^\delta \subseteq f \text{ for some } f \in P_\delta\}$ and choose $g_i^\delta \in P_\delta$ such that $g_i^\delta \restriction_{\mathrm{rg}(\eta_\delta \restriction_i)} = h_i^\delta$ for $i \in E_\delta$. Define $f_\delta : \mathrm{rg}(\eta_\delta) \longrightarrow H(\gamma)$ for $\delta \in S$ as follows:

$$f_\delta(\eta_\delta(i)) = \left\langle g_j^\delta \restriction_{\mathrm{rg}\left(\eta_\delta \restriction_{(i+1)}\right)} : j \leqslant i,\, j \in E_\delta \right\rangle.$$

Here $H(\gamma)$ denotes the class of sets hereditarily of cardinality less than γ. Note that $H(\gamma)$ has size at most γ. By the γ-uniformization of $\bar{\eta}$ we can find $F : \lambda \longrightarrow H(\gamma)$ such that for all $\delta \in S$ there exists an $\alpha_\delta < \gamma$ with $f_\delta(\eta_\delta(i)) = F(\eta_\delta(i))$ for all $\alpha_\delta \leqslant i < \gamma$. For $i < \gamma$ let

$$F(\eta_\delta(i)) = \left\langle G_j^{\eta_\delta(i)} : j \leqslant i,\, j \in E_{\eta_\delta(i)} \right\rangle$$

for some $E_{\eta_\delta(i)} \subseteq \gamma$. Note that $F(\eta_\delta(i))$ depends only on the value $\eta_\delta(i)$ and not on δ. Moreover, F is well defined since $\bar{\eta}$ is tree-like.

We now define $f : \lambda \longrightarrow \gamma$ on $\bigcup_{\delta \in S} \mathrm{rg}(\eta_\delta)$ and arbitrarily on the complement. We use induction on $i < \gamma$. For $i = 0$ choose any member $u \in P_{\eta_\delta(0)}$ and put $f(\eta_\delta(0)) = u(\eta_\delta(0))$. Now assume that $f(\eta_\delta(j))$ has been defined for $j < i$ and $\delta \in S$ such that for $j < i$,

$$f \restriction_{\mathrm{rg}\left(\eta_\delta \restriction_{(j+1)}\right)} \in P_{\eta_\delta(j)}.$$

Put $\bar{f}_\delta = \{f(\eta_\delta(j)) : j < i\}$ and let

$$J_i^\delta = \left\{ j \in E_{\eta_\delta(i)} : G_j^{\eta_\delta(i)} \restriction_{(\mathrm{rg}(\eta_\delta \restriction_i))} \subseteq \bar{f}_\delta \right\}.$$

If $j_i^\delta = \min(J_i^\delta)$ exists, then let

$$f(\eta_\delta(i)) = G_{j_i^\delta}^{\eta_\delta(i)}(\eta_\delta(i)).$$

If $\min(J_i^\delta)$ does not exist, then we distinguish between two cases: if i is a limit ordinal, then Definition 4.7(iii) implies that there is an $f_i \in P_{\eta_\delta(i)}$ which extends

$$\bigcup_{j<i} f \restriction_{\mathrm{rg}\left(\eta_\delta \restriction_{(j+1)}\right)}.$$

If i is a successor ordinal, then Definition 4.7(ii) ensures that there is an $f_i \in P_{\eta_\delta(i)}$ extending $f \restriction_{\mathrm{rg}(\eta_\delta \restriction_i)}$. In both cases put $f(\eta_\delta(i)) = f_i(\eta_\delta(i))$. Note that f is well defined since $\bar{\eta}$ is tree-like; hence $\min(J_i^\delta)$ exists if and only if $\min(J_j^\nu)$ exists for $\eta_\delta(i) = \eta_\nu(j)$ $(\delta, \nu \in S,\, i, j < \gamma)$. It remains for us to check that $f \restriction_{\mathrm{rg}(\eta_\delta)} \in P_\delta$. By the uniformization we have

$$G_j^{\eta_\delta(i)} = g_j^\delta \restriction_{\mathrm{rg}\left(\eta_\delta \restriction_{(i+1)}\right)}$$

for all $j \leqslant i$, $j \in E_{\eta_\delta(i)}$, $i \geqslant \alpha_\delta$ and $\delta \in S$. We define

$$h : \gamma \longrightarrow \gamma, \qquad j \longmapsto f(\eta_\delta(j)).$$

By $\diamondsuit_\gamma$, there exists a $\beta_\delta \geqslant \alpha_\delta$ such that $h \restriction_{\beta_\delta} = h_{\beta_\delta}$, and hence

$$f \restriction_{\mathrm{rg}(\eta_\delta \restriction_{\beta_\delta})} = h_{\beta_\delta}^\delta \restriction_{\mathrm{rg}(\eta_\delta \restriction_{\beta_\delta})} = h_{\beta_\delta}^\delta.$$

Thus $\beta_\delta \in J^\delta_{\beta_\delta}$ and $f(\eta_\delta(i)) = G_j^{\eta_\delta(i)}$ as above. Therefore

$$f \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(\beta_\delta+1)})} = g^\delta_{\beta_\delta} \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(\beta_\delta+1)})}$$

by definition of f. By induction on $i \geqslant \beta_\delta$ it follows that $f \restriction_{\mathrm{rg}(\eta_\delta)} = g^\delta_{\beta_\delta} \in P_\delta$ and this finishes the proof. □

The extension of Theorem 4.6 is now immediate.

THEOREM 4.9. *Let $\lambda = \gamma^+$ and assume that $\diamondsuit_\gamma$ holds. Moreover, let $\bar{\eta}$ be a tree-like ladder system on a non-reflecting stationary subset S of λ whose elements have cofinality γ. If T is a group with $|T| < \lambda$ and G is (T, λ, γ)-suitable, then there is a torsion-free group H of size λ such that the following hold:*

(i) *H has a λ-filtration $\langle \bar{H}_\alpha : \alpha < \lambda \rangle$.*
(ii) *If $\alpha \in S$, then $\bar{H}_{\alpha+1}/\bar{H}_\alpha \cong G$.*
(iii) *If $\alpha \notin S$, then $\bar{H}_\beta/\bar{H}_\alpha$ is free for all $\alpha \leqslant \beta$.*
(iv) $\mathrm{Ext}(W, T) = 0$ *for all groups W with $|W| \leqslant |T|$.*

Proof. (This is similar to [**2**, Proposition 1.8].) Let S, $\bar{\eta}$ and G be as in Theorem 4.9 and choose the direct limit $H_{\bar{\eta}} = \bigoplus \{H_\eta : \eta \in B_{\bar{\eta}}\}/K$ as constructed above. Inspecting the proof of Theorem 4.6, we see that $H_{\bar{\eta}}$ is an almost-free non-free group of size λ which has the desired λ-filtration. It remains to show that $\mathrm{Ext}(H_{\bar{\eta}}, W) = 0$ for all groups W with $|W| \leqslant |T|$. We must show that any short exact sequence

$$0 \longrightarrow W \xrightarrow{\mathrm{id}} N \xrightarrow{\pi} H_{\bar{\eta}} \longrightarrow 0 \qquad (*)$$

has a splitting map $\varphi : H_{\bar{\eta}} \longrightarrow N$ such that $\varphi\pi = \mathrm{id} \restriction_{H_{\bar{\eta}}}$. Choose any set function $u : H_{\bar{\eta}} \longrightarrow N$ such that $u\pi = \mathrm{id} \restriction_{H_{\bar{\eta}}}$. As in [**2**, Proposition 1.8], the splitting maps φ of π are in one-to-one correspondence with set mappings $h : H_{\bar{\eta}} \longrightarrow W$ with $h(0) = 0$ such that for all $x, y \in H_{\bar{\eta}}$ and $z \in \mathbb{Z}$, the following hold:

(i) $zh(x) - h(zx) = zu(x) - u(zx)$.
(ii) $h(x) + h(y) - h(x+y) = u(x) + u(y) - u(x+y)$.

It is customary to denote $\mathrm{Trans}(H, W)$ to be the set of all these maps h for a fixed subgroup $H \subseteq H_{\bar{\eta}}$. Thus $(*)$ splits if and only if $\mathrm{Trans}(H_{\bar{\eta}}, W)$ is non-empty.

Fix a chain $\langle F_\alpha : \alpha \leqslant \gamma \rangle$ for G as in Definition 3.1. For $\delta \in S$, $i < \gamma$ and $h \in \mathrm{Trans}(H_{\eta_\delta \restriction_i}, W)$, let

$$\mathrm{seq}(h) : \mathrm{rg}(\eta_\delta \restriction_i) \longrightarrow W\big(\eta_\delta(j) \longmapsto h \restriction_{F_{1(\eta_\delta \restriction_j)}}, j < i\big).$$

For $\delta \in S$ let $P_\delta = \{\mathrm{seq}(h) : h \in \mathrm{Trans}(H_{\eta_\delta}, W)\}$ and for $i < \gamma$ put $P_{\eta_\delta(i)} = \{\mathrm{seq}(h) : h \in \mathrm{Trans}(H_{\eta_\delta \restriction_{(i+1)}}, W)\}$. Let $P_\alpha = \emptyset$ if it has not been defined yet ($\alpha < \lambda$). By Proposition 4.8, the ladder system $\bar{\eta}$ has strong γ-uniformization and it is easy to check that the system $\bar{P} = \langle P_\alpha : \alpha \in \lambda \rangle$ satisfies the conditions of Definition 4.7 since F_n and F_n/F_m are free for $m < n \leqslant \gamma$. Since $|W| \leqslant \gamma$, there exists a function $f : \lambda \longrightarrow W$ such that $f \restriction_{\mathrm{rg}(\eta_\delta)} \in P_\delta$ for all $\delta \in S$. We now define $h : H_{\bar{\eta}} \longrightarrow W$ by putting $h \restriction_{H_{\eta_\delta}} = f \restriction_{\mathrm{rg}(\eta_\delta)}$, and clearly h is well defined and belongs to $\mathrm{Trans}(H_{\bar{\eta}}, W)$ and therefore $(*)$ splits. □

5. *The forcing theorem*

Before we state the main theorem of this section we describe our strategy. Using class forcing we shall construct a model of ZFC satisfying GCH in which for every regular cardinal λ there exists a sequence of stationary non-reflecting subsets S_α of λ of length λ^+ on which we have 'enough' uniformization for some ladder system. Using this and the existence of (T, λ, γ)-suitable groups (for some particular γ) we can then construct, for a given group T, a sequence of torsion-free groups G_α $(\alpha < \lambda^+)$ of cardinality λ satisfying $\mathrm{Ext}(G_\alpha, T) = 0$. These G_α have λ-filtrations $\langle G_{\alpha,\delta} : \delta < \lambda \rangle$ whose quotients satisfy $\mathrm{Ext}(G_{\alpha,\delta+1}/G_{\alpha,\delta}, T) \neq 0$ for $\delta \in S_\alpha$. We show that all these groups are not subgroups of a single group $G \in \mathcal{TF}_\lambda(T)$ since this would force $\mathrm{Ext}\,(G, T) \neq 0$. Thus there cannot be any λ-universal group for the group T.

THEOREM 5.1. *Let V be a model of ZFC in which GCH holds. Then for some class forcing $\mathscr{P}$ not collapsing cardinals and preserving GCH, the following is true in $V^{\mathscr{P}}$.*

Case A*: If $\lambda > \gamma$ are infinite regular cardinals such that $\gamma = \mathrm{cf}\,(\mu)$ for $\lambda = \mu^+$, then the following hold:*

(i) *There is a normal ideal $J = J^\lambda_\gamma$ on λ.*
(ii) *There is a stationary subset $S = S^\lambda_\gamma$ of λ such that $S \notin J$.*
(iii) *If $\delta \in S$, then $\mathrm{cf}\,(\delta) = \gamma$.*
(iv) *S is non-reflecting, that is, $S \cap \alpha$ is not stationary in α for every $\alpha < \lambda$.*
(v) *If $S' \subseteq S$ is stationary in λ, then there is a stationary $S^* \in J$ such that $S^* \subseteq S'$.*
(vi) *If $S' \subseteq S$ and $S' \notin J$, then $\diamondsuit_{S'}$ holds.*
(vii) *If $S' \subseteq S$ is stationary and $S' \in J$, then there exists a tree-like ladder system on S' which has μ-uniformization for all $\mu < \kappa$ if $\lambda = \kappa^+$ and κ is singular, and for all $\mu < \lambda$ otherwise.*
(viii) *There are $S_\epsilon = S^\lambda_{\gamma,\epsilon} \in J$ for $\epsilon < \lambda^+$ such that the following hold:*
  (a) *If $\eta < \epsilon < \lambda^+$, then $S_\eta \backslash S_\epsilon$ is bounded.*
  (b) *If $\epsilon < \lambda^+$, then $S_{\epsilon+1} \backslash S_\epsilon$ is stationary.*
  (c) $J = \{S' \subseteq S : \exists\, \epsilon < \lambda^+\, \forall\, \epsilon < v < \lambda^+,\ S' \backslash S_v \text{ is not stationary}\}$.

Case B*: If $\lambda = \mathrm{cf}\,(\lambda) > \gamma$, then there is a stationary $S^* = S^{\lambda,*}_\gamma$ such that the following hold:*

(1) *If $\alpha \in S^*$, then $\mathrm{cf}\,(\alpha) = \gamma$.*
(2) *S^* is non-reflecting.*
(3) *$\diamondsuit_{S^*}$ holds.*

The proof of Theorem 5.1 will be divided into several steps. First we deal with a fixed regular cardinal λ and then we use Easton support iteration to put the forcings together. We assume knowledge about forcing, and our notation will follow [**9**], with the exception that $p \leqslant q$ means that condition q is stronger than condition p. Recall that a poset P is called *λ-complete* if for every cardinal $\kappa < \lambda$, every ascending chain

$$p_0 \leqslant p_1 \leqslant \ldots \leqslant p_\alpha, \qquad \alpha < \kappa,$$

has an upper bound. Moreover, P is said to be *λ-strategically complete* if player I has a winning strategy in the following game of length κ for every $\kappa < \lambda$. Players I

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and II alternately choose an ascending sequence

$$p_0 \leqslant p_1 \leqslant \ldots \leqslant p_\alpha, \qquad \alpha < \kappa,$$

of elements of P, where player I chooses at the even ordinals; player I wins if and only if at each stage there is a legal move and the whole sequence $\langle p_\alpha : \alpha < \kappa \rangle$ has an upper bound (see also [**14**, Definition A1.1]). Note that, if P is λ-strategically complete and G is generic over P, then $V[G]$ has no new functions from κ into V for all $\kappa < \lambda$; hence cardinals at most λ and their cofinalities are preserved.

PROPOSITION 5.2. *Let λ be a regular cardinal with $\lambda^{<\lambda} = \lambda$. For any regular $\kappa < \lambda$, there exists a poset Q of cardinality at most λ which is λ-strategically complete* (*and hence preserves all cardinals and preserves cofinalities at most λ*) *with the following property. For G generic over P, in $V[G]$ there exists a non-reflecting stationary and co-stationary subset S of λ such that every member of S has cofinality κ.*

Proof. The proof is similar to the proof of [**2**, Lemma 2.3]. We let Q be the set of all functions $q : \alpha \longrightarrow 2 = \{0, 1\}$ $(\alpha < \lambda)$ such that $q(\mu) = 1$ implies that $\mathrm{cf}(\mu) = \kappa$ and such that for all limits $\delta \leqslant \alpha$, the intersection of $q^{-1}[1]$ with δ is not stationary in δ. Then, for G generic over P,

$$S = \bigcup \{q^{-1}[1] : q \in G\}$$

will be the desired set. We have to prove that S is stationary and co-stationary in λ. Hence assume that q forces f is the name of a continuous increasing function $\bar{f} : \lambda \longrightarrow \lambda$; choose an ascending chain

$$q_o \leqslant q_1 \leqslant \ldots \leqslant q_\alpha, \qquad \alpha < \kappa,$$

such that for each α there exist $\beta_\alpha, \gamma_\alpha$ such that $q_\alpha \Vdash \bar{f}(\beta_\alpha) = \gamma_\alpha$ and

$$\mathrm{dom}(q_\alpha) \geqslant \gamma_\alpha > \mathrm{dom}(q_\mu)$$

for all $\mu < \alpha$. Let $\delta = \sup\{\gamma_\alpha : \alpha < \kappa\} = \sup\{\mathrm{dom}(q_\alpha) : \alpha < \kappa\}$ and let

$$q_i = \bigcup \{q_\alpha : \alpha < \kappa\} \cup \{(\delta, i)\}$$

for $i = 0, 1$. Then $q_i \in Q$ $(i = 0, 1)$ since $q_i^{-1}[1]$ is not stationary in δ, because δ has cofinality κ. Moreover, $q_1 \Vdash \delta \in \mathrm{rg}(f) \cap S$ and $q_0 \Vdash \delta \in \mathrm{rg}(f) \cap (\lambda \backslash S)$.

Since Q has cardinality $\leqslant \lambda$, it preserves cardinals $> \lambda$. To show that all cardinals at most λ are preserved (and their cofinalities), it suffices to prove that Q is λ-strategically complete. Let $\tau < \lambda$ be a limit ordinal. Let player I choose q_α for even α such that $\mathrm{dom}(q_\alpha)$ is a successor ordinal, say $\delta_\alpha + 1$, and $q_\alpha(\delta_\alpha) = 0$. Moreover, at limit ordinals α, player I chooses q_α to have the domain $= \sup\{\delta_\beta : \beta < \alpha\} + 1$. Then $q = \bigcup \{q_\alpha : \alpha < \mu\}$ is a member of Q because $\{\delta_\alpha : \alpha < \mu, \alpha \text{ even}\}$ is a cub in $\mathrm{dom}(q)$ which misses $q^{-1}[1]$. This is a winning strategy for player I and thus Q is λ-strategically complete. □

The next proposition is a collection of results from [**13–15**] (see also [**16**]).

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Proposition 5.3. *Let $\lambda > \gamma$ be regular infinite cardinals. Moreover, assume that $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$, and let S be a non-reflecting, stationary and co-stationary subset of λ such that each member of S has cofinality γ. Furthermore, let $\gamma = \mathrm{cf}(\kappa)$ if $\lambda = \kappa^+$. Then there exists a poset P of cardinality $\leqslant \lambda^+$ which is λ-strategically complete, satisfies the λ^+-chain condition, adds no new sequences of length less than λ, and has the following properties:*

(i) *S is non-reflecting, stationary and co-stationary in λ as an element in V^P.*

(ii) *If λ is inaccessible, then every ladder system on S has μ-uniformization for all $\mu < \lambda$; in particular, there exists a tree-like ladder system on S.*

(iii) *If $\aleph_2 \leqslant \lambda = \kappa^+$ and κ is regular, then every ladder system on S has μ-uniformization for all $\mu < \lambda$; in particular, there exists a tree-like ladder system on S.*

(iv) *If $\lambda = \aleph_1$, then there is a tree-like ladder system on S which has μ-uniformization for all $\mu < \lambda$.*

(v) *If $\lambda = \kappa^+$ and κ is singular, then there is a tree-like ladder system on S which has μ-uniformization for all $\mu < \kappa$.*

Proof. For λ inaccessible, the proof is contained in [**14**, Case A] and also for the case of $\lambda = \kappa^+$, κ regular, see [**14**, Case B]. For $\lambda = \aleph_1$, see [**13**, Theorem 1.7] and for $\lambda = \kappa^+$, κ singular, see [**15**, Theorem 2.10, Theorem 2.12]. Moreover, simpler versions with less complicated and comprehensive proofs can be found in [**12**] for all cases if we drop the requirements 'for every ladder system...', which is in fact not needed for our purposes. Finally, let us remark that co-stationarity is only needed when λ is inaccessible or a successor of a regular cardinal. $\square$

Theorem 5.4. *Let λ be a regular cardinal such that $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$. Then there is a poset P of cardinality $\leqslant \lambda^+$ satisfying the λ^+-chain condition which is λ-strategically complete and adds no new sequences of length less than λ such that in V^P for every regular $\gamma < \lambda$ with $\gamma = \mathrm{cf}(\kappa)$ if $\lambda = \kappa^+$ the statements of case* A *and case* B *of Theorem* 5.1 *hold.*

First we deduce Theorem 5.1 from Theorem 5.4.

Proof of Theorem 5.1. Let V be a model of ZFC with GCH. For any ordinal α let

$$P_\alpha = \langle P_j, \dot{Q}_i : j \leqslant \alpha,\, i < \alpha \rangle$$

be an iteration with Easton support; this is to say that we take direct limits when $\aleph_\alpha$ is regular and inverse limits elsewhere, or equivalently we have bounded support below inaccessibles and full support below non-inaccessibles. For any ordinal i, let $\dot{Q}_i$ be the forcing notion in V^{P_i} described in Theorem 5.4 for $\lambda = \aleph_i$ if $\aleph_i$ is regular, and let Q be 0 elsewhere. Let P be the direct limit of the P_α (α an ordinal). We claim that P has the desired properties. The proof is very similar to that of [**2**, Theorem 2.1], and hence we shall only state the main ingredients needed:

(i) For every κ and Easton support iteration

$$\langle P_j, \dot{Q}_i : \kappa \leqslant j \leqslant \alpha,\, \kappa \leqslant i < \alpha \rangle,$$

if each $\dot{Q}_i$ is κ-strategically complete, then so is P_α.

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(ii) $P = P_\alpha * P_{\geqslant\alpha}$, where, in V^{P_α}, $P_{\geqslant\alpha}$ is the direct limit of P^α_β (β an ordinal), with P^α_β being the Easton support iteration

$$\langle P^\alpha_j, \dot{Q}^\alpha_i : j \leqslant \beta,\ i < \beta \rangle,$$

where $\dot{Q}^\alpha_i = \dot{Q}_{\alpha+i}$.

(iii) $|P_n| = 1$ (for $n \in \omega$); if $\aleph_\delta$ is singular, $|P_\delta| \leqslant \aleph_\delta$ and $|P_\delta| \leqslant \aleph_{\delta+1}$ if $\aleph_\delta$ is regular, and hence inaccessible.

(iv) $P_{\geqslant\alpha}$ is $\aleph_\alpha$-strategically complete, and $P_{\geqslant\alpha+1}$ is even $\aleph_{\alpha+n}$- strategically complete for all $n \in \omega$.

By construction, Theorem 5.1(i)–(viii) and (1)–(3) are now satisfied in V^P. Note that stationarity is preserved in the iteration because $P_{\geqslant\alpha}$ is $\aleph_\alpha$-strategically complete. It remains to prove that V^P is a model of ZFC satisfying GCH and preserving cofinalities (and hence cardinals). This follows as in [**2**, Theorem 2.1], and hence we will omit the proof. □

It remains to prove Theorem 5.4.

Proof of Theorem 5.4. The proof follows from results in [**14**, **15**], but for the convenience of the reader we shall give some details. If $\lambda = \kappa^+$ is a successor cardinal, then the proof is immediate. There is only one γ with $\gamma = \mathrm{cf}(\kappa)$. We choose P to be the two-step iterated forcing of the two forcings from Proposition 5.2 and from Proposition 5.3 with $\gamma = \mathrm{cf}(\kappa)$. Moreover, we may assume that P also forces the sets $S^* = S^{\lambda,*}_\gamma$ satisfying Theorem 5.1(1)–(3) by an initial forcing. Note that the assumptions on λ in Theorem 5.4 are satisfied by [**7**, Exercise 12, p. 70]. If λ is inaccessible, then the argument is more complicated, since we have to deal with all the regular cardinals $\gamma < \lambda$. However, this was established in [**14**, Case B], where a stronger version of Proposition 5.3 was shown. It was proved that there is a forcing notion P such that, for all regular $\gamma < \lambda$ and given non- reflecting stationary, co-stationary subsets S_γ of λ consisting of ordinals of cofinality γ, every ladder system on S_γ has μ-uniformization for all $\mu < \lambda$. Using this stronger result and again forcing the sets $S^{\lambda,*}_\gamma$ satisfying Theorem 5.1(1)–(3), it remains to show that we can define the ideal $J = J^\lambda_\gamma$ satisfying Theorem 5.1(vi) and (viii) (Theorem 5.1(vii) is clear).

Our forcing P (from [**14**, **15**]) is the result of a $(<\lambda)$-support iteration of length λ^+, say

$$\langle P_i, \dot{Q}_j : i \leqslant \lambda^+,\ j < \lambda^+ \rangle.$$

Let us assume that P_γ forces the set $S = S^\lambda_\gamma$ and the tree-like ladder system $\bar{\eta}$ on S. In V^{P_γ}, there exists a sequence $\langle S_\epsilon = S^\lambda_{\gamma,\epsilon} : \epsilon < \lambda^+ \rangle$ such that the following hold:

(i) $S_\epsilon \subseteq S$.

(ii) $\eta < \epsilon < \lambda^+$ implies that $S_\eta \backslash S_\epsilon$ is bounded.

(iii) $\epsilon < \lambda^+$ implies that $S_{\epsilon+1} \backslash S_\epsilon$ is stationary.

Now we define $J = J^\lambda_\gamma$ as

$$J = \{ S' \subseteq S : \exists \epsilon < \lambda^+ \, \forall \epsilon < \nu < \lambda^+,\ S' \backslash S_\nu \text{ is not stationary} \}.$$

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For each $i < \lambda^+$, $\dot{Q}_i$ forces μ_i-uniformization for the ladder system

$$(\dot{A}_i, \dot{f}_i), \quad \text{where } \dot{f}_i = \langle \dot{f}^i_\delta : \delta \in \dot{S}_i \rangle.$$

Here $\dot{S}_i$ and $\dot{f}^i_\delta$ are P_i-names for a member of ${}^{A_\delta}(\mu_i)$. Thus we obtain $\Vdash$ '$\dot{S}_i \subseteq \dot{S}$ is stationary and there is $\epsilon < \lambda^+$ such that $(\forall \epsilon < \eta < \lambda^+)$

$$(\dot{S}_i \cap \dot{S}_\eta \setminus \dot{S}_\epsilon)$$

is not stationary'. A condition in $\dot{Q}_i$ is, for instance, given by $g : \alpha \longrightarrow \mu_i$ such that $\delta \in \dot{S}_i$, $\delta \leqslant \alpha$ implies that $f^i_\delta \subseteq^* g$.

It remains to show that $\diamondsuit_{S'}$ holds for $S' \notin J$. Choose $i < \lambda^+$ such that S' comes from V^{P_i}. For some $j \in (i, \lambda^+)$, Q_j is adding λ Cohen reals, and we can interpret it as adding a diamond sequence $\langle \rho_\epsilon : \epsilon \in S' \rangle$ by initial segments. Trivially, in $V^{P_{j+1}}$, $\diamondsuit_{S'}$ holds, and we may work in $V^{P_{j+1}}$ now. For χ large enough, we can find for every $x \in H(\chi(\lambda))$ an increasing continuous sequence

$$\bar{N} = \langle N_i : i < \lambda \rangle$$

of elementary submodels of $H(\chi(\lambda), \epsilon, <^*)$ of cardinality less than λ such that $x \in N_0$, $S' \in N_0$ and $\bar{N} \restriction_{(i+1)} \in N_{i+1}$ for all $i < \lambda$. Let $E = \{\delta < \lambda : N_\delta \cap \lambda = \delta\}$, which is a cub in λ. Thus, for $\delta \in E$, for every $p \in (P/P_{j+1}) \cap N_\delta$, there is a condition $p \leqslant q \in P/P_{j+1}$ which is $(N_\delta, P/P_{j+1})$ generic and forces a value to $G \cap N_\delta$. It is known that we can now replace the diamond sequence on S' which we have in $V^{P/P_{j+1}}$ by one that is preserved by forcing with P/P_{j+1}, since P/P_{j+1} adds no new subsets of λ of length less than λ, and by the strategical completeness. This finishes the proof. □

6. *Application to Kulikov's question*

Finally we show that there is a model of ZFC and GCH in which Kulikov's question has a negative answer, this is to say that, for a given group T and any cardinal λ large enough, there is no λ-universal group for T. This is in contrast to the results in Gödel's universe $V = L$; see [**17**].

DEFINITION 6.1. Let T, H be groups, where H is torsion-free with $|T| < \lambda = |H|$, where H has a λ-filtration $\{H_\alpha : \alpha < \lambda\}$. Moreover, let S be as in Theorem 5.1(ii). Then

$$S^\lambda_\gamma[H, T] = \{\delta \in S : \mathrm{Ext}(H_{\delta+1}/H_\delta, T) \neq 0\}.$$

Note that $S^\lambda_\gamma[H, T]$ depends on the filtration of H, which could be traded into an invariant of H, namely

$$\Gamma^S_H(T) = \{E \subseteq S : \text{there exists a cub } C \subseteq \lambda \text{ such that } E \cap C = S^\lambda_\gamma[H, T] \cap C\}.$$

THEOREM 6.2. *Let T, H be groups, where H is torsion-free with $|T| < \lambda = |H|$. If $S = S^\lambda_\gamma[H, T]$ and $\diamondsuit_S$ holds, then* $\mathrm{Ext}(H, T) \neq 0$.

Proof. The proof is standard, and can be found, for instance, in [**1**, Theorem 1.15, p. 353]. □

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We shall now work in the model $V^{\mathscr{P}}$ obtained in Theorem 5.1. Thus all the results will be consistent with ZFC and GCH. The symbol $\left(V^{\mathscr{P}}\right)$ indicates that the statement holds in our model $V^{\mathscr{P}}$.

THEOREM 6.3 ($V^{\mathscr{P}}$). *Let T be an abelian group with $|T| < \lambda$ and let G be (T, λ, γ)-suitable for $\gamma < \lambda$ regular, $\gamma = \mathrm{cf}(\mu)$ for $\lambda = \mu^+$. If $\lambda > 2^{(|T|^{\gamma})}$ or μ is regular, then there is no λ-universal group for T.*

Proof. Assume that there is a λ-universal group U for T. If $\lambda > 2^{(|T|^{\gamma})}$, then for $\epsilon < \lambda^+$ we apply Theorem 4.6 to $S^{\lambda}_{\gamma,\epsilon}$, T, G and the tree-like ladder system $\bar{\eta}_\epsilon$ on $S^{\lambda}_{\gamma,\epsilon}$ which comes from Theorem 5.1(vii). Note that no $S^{\lambda}_{\gamma,\epsilon}$ reflects in any $\alpha < \lambda$ and that $\bar{\eta}_\epsilon$ has $2^{(|T|^{\gamma})}$-uniformization since $\lambda > 2^{(|T|^{\gamma})}$ (and $\mu > 2^{(|T|^{\gamma})}$ if $\lambda = \mu^+$, μ singular). If μ is regular, then we apply Theorem 4.9 instead of Theorem 4.6. Note that $\diamondsuit_\mu$ holds in $V^{\mathscr{P}}$ by Theorem 5.1(1)–(3). Hence, for each $\epsilon < \lambda^+$, we obtain a torsion-free group $H_\epsilon = \bigcup_{\alpha\in\lambda} H_{\epsilon,\alpha}$ satisfying $\mathrm{Ext}(H_\epsilon, T) = 0$ and $\mathrm{Ext}(H_{\epsilon,\alpha+1}/H_{\epsilon,\alpha}, T) \neq 0$ for all $\alpha \in S^{\lambda}_{\gamma}[H_\epsilon, T] = S^{\lambda}_{\gamma,\epsilon}$. The universal group U allows an embedding $i_\epsilon : H_\epsilon \longrightarrow U$ for all $\epsilon \leqslant \lambda^+$. We claim that $S^{\lambda}_{\gamma,\epsilon}[H_\epsilon, T] \subseteq S^{\lambda}_{\gamma}[U, T]$ modulo a non-stationary set for each $\epsilon < \lambda^+$. To see this, choose a λ-filtration $U = \bigcup_{\alpha\in\lambda} U_\alpha$ of U such that $\mathrm{Ext}(U_{\alpha+1}/U_\alpha, T) = 0$ if and only if for some $\beta > \alpha$ we have $\mathrm{Ext}\left(U_\beta/U_\alpha, T\right) = 0$. Fix $\epsilon < \lambda^+$; then there is a cub $C_\epsilon \subseteq \lambda$ such that for all $\alpha \in C_\epsilon$, we have $H_{\epsilon,\alpha} = U_\alpha \cap H_\epsilon$. Thus for $\alpha < \beta \in C_\epsilon$, it follows that $H_{\epsilon,\beta}/H_{\epsilon,\alpha} = (U_\beta \cap H_\epsilon)/(U_\alpha \cap H_\epsilon) \subseteq U_\beta/U_\alpha$, and hence $\mathrm{Ext}(H_{\epsilon,\beta}/H_{\epsilon,\alpha}, T) \neq 0$ implies that $\mathrm{Ext}(U_\beta/U_\alpha, T) \neq 0$. Therefore, also, $\mathrm{Ext}(U_{\alpha+1}/U_\alpha, T) \neq 0$ and $C_\epsilon \subseteq S^{\lambda}_{\gamma}[U, T]$. Thus, by the definition of the normal ideal J (see Theorem 5.1(viii)), we have $\bar{S} = S^{\lambda}_{\gamma}[U, T] \notin J$, and therefore $\diamondsuit_{\bar{S}}$ holds by Theorem 5.1(vi). Hence $\mathrm{Ext}(U, T) \neq 0$ by Theorem 6.2, a contradiction. □

From ($V^P \Vdash$ GCH), it follows that

$$2^{(|T|^{\gamma})} \leqslant \max\{\gamma^{++}, |T|^{++}\}$$

for any group T.

COROLLARY 6.4 ($V^{\mathscr{P}}$). *Let T be a group not cotorsion with $|T| < \lambda$. If λ is strongly inaccessible, then there is no λ-universal group for T.*

Proof. Since λ is strongly inaccessible it is a limit ordinal, and we may choose $\gamma = \omega$. Moreover, for every $\alpha < \lambda$, we have $2^\alpha < \lambda$; hence $\lambda > 2^{(|T|^{\omega})}$. Lemma 3.2 implies that there is a (T, λ, ω)-suitable group for T, and hence Theorem 6.3 shows that there is no λ-universal group for T. □

COROLLARY 6.5 ($V^{\mathscr{P}}$). *Let T be a group not cotorsion such that $|T|^+ < \lambda$. If $\lambda = \mu^+$ and $\mathrm{cf}(\mu) = \omega$, then there is no λ-universal group for T.*

Proof. There is a (T, λ, ω)-suitable group by Lemma 3.2. Since $\mathrm{cf}(\mu) = \omega$, we have $\lambda > 2^{(|T|^{\omega})}$, and hence we may choose $\gamma = \omega$ and apply Theorem 6.3 to see that there is no λ-universal group for T. □

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COROLLARY 6.6 ($V^{\mathscr{P}}$). *Let T be a group not cotorsion and let H be an epimorphic image of T such that $|T| < \lambda$ and $|H| \leqslant \mathrm{cf}(\mu)$ if $\lambda = \mu^+$. If μ exists and is singular, then let*

$$2^{\left(|T|^{\mathrm{cf}(\mu)}\right)} < \lambda,$$

and $2^{(|T|^{\omega})} < \lambda$ otherwise. Then T has no λ-universal group.

Proof. We choose $\gamma = \omega$ if λ is a limit ordinal and $\gamma = \mathrm{cf}(\mu)$ if $\lambda = \mu^+$. By Theorem 5.1(1)–(3), there exists a stationary non-reflecting set $S \subseteq \gamma$ consisting of limit ordinals of cofinality ω such that $\diamondsuit_S$ holds. By assumption, we may apply Proposition 3.5 to S, H and λ, γ to obtain a (T, λ, γ)-suitable group G. Since $\lambda > 2^{(|T|^{\gamma})}$ or μ is regular, we apply Theorem 6.3 to see that there is no λ-universal group for T. □

COROLLARY 6.7 ($V^{\mathscr{P}}$). *For any group T not cotorsion and any cardinal λ, there exists a regular uncountable cardinal $\delta \geqslant \lambda$ such that there is no δ-universal group for T.*

COROLLARY 6.8 ($V^{\mathscr{P}}$). *If T is a torsion group not cotorsion with $|T|$ regular, then there is no λ-universal group for T for any regular cardinal $\lambda > |T|$.*

Proof. Let T and λ be as stated. If λ is strongly inaccessible, then Corollary 6.4 applies. Hence assume that $\lambda = \kappa^+$. Thus $|T| \leqslant \kappa$, and by Proposition 3.5 there exists a $(T, \lambda, \mathrm{cf}(\kappa))$-suitable group G for T. If κ is regular, then Theorem 6.3 shows that there is no λ-universal group for T. If κ is singular, then $\kappa > |T|^{++}$, since $|T|$ is regular. Hence

$$\lambda > 2^{\left(|T|^{\mathrm{cf}(\kappa)}\right)},$$

and again Corollary 6.8 follows from Theorem 6.3. □

COROLLARY 6.9 ($V^{\mathscr{P}}$). *There is no λ-universal Whitehead group for all $\lambda > \aleph_2$.*

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