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Trivial and non-trivial automorphisms of $\mathcal{P}(\omega_1)/[\omega_1]^{<\aleph_0}$

by

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Abstract. The following statement is shown to be independent of set theory with the Continuum Hypothesis: There is an automorphism of $\mathcal{P}(\omega_1)/[\omega_1]^{<\aleph_0}$ whose restriction to $\mathcal{P}(\alpha)/[\alpha]^{<\aleph_0}$ is induced by a bijection for every $\alpha \in \omega_1$, but the automorphism itself is not induced by any bijection on ω_1.

1. Introduction. For any set X let $\mathcal{P}(X)/\mathcal{F}in$ represent the Boolean algebra of all subsets of X modulo the ideal of finite subsets of X. Let $A \equiv^* B$ denote that $A \triangle B$, the symmetric difference of A and B, is finite, and for $A \subseteq X$, let $[A]$ denote the equivalence class $\{B \subseteq X \mid A \equiv^* B\}$. A homomorphism

$$\Psi : \mathcal{P}(X)/\mathcal{F}in \to \mathcal{P}(Y)/\mathcal{F}in$$

is called *trivial* if there is a function $\psi : Y \to X$ such that $[\Psi(A)] = [\psi^{-1}A]$. Let $\mathbb{AUT}_\kappa$ denote the set of all automorphisms of $\mathcal{P}(\kappa)/\mathcal{F}in$. For $\Psi \in \mathbb{AUT}_\kappa$ let $\mathcal{T}(\Psi)$ denote, as in [8, §2], the ideal of all subsets $X \subseteq \kappa$ such that $\Psi\restriction\mathcal{P}(X)/\mathcal{F}in$ is trivial.

The study of $\mathbb{AUT}_\omega$ was initiated by W. Rudin [5, 6] who showed that the Continuum Hypothesis can be used to construct non-trivial autohomeomorphisms of $\beta\mathbb{N}\setminus\mathbb{N}$, in other words, using Stone duality, homeomorphisms $\beta\mathbb{N}\setminus\mathbb{N}$ such that the automorphism of $\mathcal{P}(\mathbb{N})/\mathcal{F}in$ they induce is not trivial. A further advance was provided by S. Shelah [7] who showed that it is consistent with set theory that $\mathcal{T}(\Psi)$ is not proper—in other words, $\omega \in \mathcal{T}(\Psi)$—for every $\Psi \in \mathbb{AUT}_\omega$; in more conventional terminology, every $\Psi \in \mathbb{AUT}_\omega$ is trivial. B. Veličković [11] later showed that the conjunction of OCA and MA implies that the same is true for every $\Psi \in \mathbb{AUT}_{\omega_1}$ and, assuming PFA,

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the same is true for every $\Psi \in \mathbb{AUT}_\kappa$. It was later shown in [9] that it is consistent that $\mathcal{T}(\Psi)$ contains an infinite set for every $\Psi \in \mathbb{AUT}_\omega$, yet there are Ψ such that $\mathcal{T}(\Psi)$ is proper.

However, finding extensions of Rudin's result of the existence on non-trivial automorphisms of $\mathcal{P}(\kappa)/\mathcal{F}in$ has proven to be much harder. In [10] it is shown that if $\kappa > 2^{\aleph_0}$ and κ is less than the first inaccessible cardinal then for every $\Psi \in \mathbb{AUT}_\kappa$ there is a set $X \in \mathcal{T}(\Psi)$ such that $|\kappa \setminus X| \leq 2^{\aleph_0}$. On the other hand, it has been shown by P. Larson and P. McKenney [4] that if $\kappa \leq 2^{\aleph_0}$ and $\Psi \in \mathbb{AUT}_\kappa$ and $[\kappa]^{\aleph_1} \subseteq \mathcal{T}(\Psi)$ then Ψ is trivial. It follows that if κ is an uncountable cardinal less than the first inaccessible and $\Psi \in \mathbb{AUT}_\kappa$ is non-trivial then there is $X \in [\kappa]^{\aleph_1}$ such that $\Psi\restriction\mathcal{P}(X)/\mathcal{F}in$ is also non-trivial.

These results leave open the question of whether or not it is consistent that there is some $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\mathcal{T}(\Psi)$ is proper. Of course, this question must be formulated properly because an easy solution is to use Rudin's result under the Continuum Hypothesis and find a $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\omega \notin \mathcal{T}(\Psi)$. Hence the proper formulation is in [10, Question 7.2]: Is it consistent that there is some $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $[\omega_1]^{\aleph_0} \subseteq \mathcal{T}(\Psi)$ and $\mathcal{T}(\Psi)$ is proper? A positive answer will be provided by Theorem 1.1. On the other hand, Theorem 4.2 will provide the following companion to Veličković's result from [11] under the conjunction of OCA and MA: It is even consistent with the Continuum Hypothesis that $\mathcal{T}(\Psi)$ is not proper for any $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\mathcal{T}(\Psi) \supseteq [\omega_1]^{\aleph_0}$. The following are the main results to be proved:

THEOREM 1.1. *Assuming $\diamondsuit^+_{\omega_1}$ (see Definition 2.1) there is $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\mathcal{T}(\Psi) \supseteq [\omega_1]^{\aleph_0}$ yet Ψ is not trivial.*

THEOREM 1.2. *The Continuum Hypothesis, and even $\diamondsuit_{\omega_1}$, does not imply that there is $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\mathcal{T}(\Psi)$ is a proper ideal containing $[\omega_1]^{\aleph_0}$.*

In §3 the methods of §2 are modified to obtain results giving more information on the possible structure of $\mathcal{T}(\Psi)$.

2. Proof of Theorem 1.1

DEFINITION 2.1. Let $H_{<\aleph_0}(X)$ be the hereditarily finite sets with the elements of X considered as atoms—in other words, $H_{<\aleph_0}(X) = \bigcup_{n\in\omega} A_n(X)$ where $A_0(X) = X$ and $A_{n+1}(X) = [A_n(X)]^{<\aleph_0}$. Following the proof of R. Jensen and K. Kunen [1] that there is a Kurepa family if $V = L$, a family $\{D_\xi\}_{\xi\in\omega_1}$ will be said to be a $\diamondsuit^+_{\omega_1}$ *sequence* if

- each D_ξ is a countable model of set theory without the power set axiom,
- $\xi + 1 \subseteq D_\xi$,

- for each $X \subseteq H_{<\aleph_0}(\omega_1)$ there is a club $C \subseteq \omega_1$ such that $X \cap H_{<\aleph_0}(\xi) \in D_\xi$ and $C \cap \xi \in D_\xi$ for every $\xi \in C$,
- $\emptyset = D_{\xi+1} = D_{\xi+\omega}$ for each $\xi \in \omega_1$.

The last clause is not part of the usual definition, but will permit us to avoid technical difficulties that would complicate the proof of Theorem 1.1. Thanks to the use of $H_{<\aleph_0}(\omega_1)$ instead of ω_1 we avoid having to make remarks about coding when trapping more complicated sets, such as functions, instead of just subsets of ω_1.

The following theorem was first proved by R. Jensen and is documented in handwritten notes [2]. A proof can also be found in [3].

THEOREM 2.2 (R. Jensen). *There is a $\diamondsuit^+_{\omega_1}$ sequence in the constructible universe.*

DEFINITION 2.3. Suppose that $\sqsubset$ is a tree ordering on $\omega_1 \times \omega$ whose αth level is $\{\alpha\} \times \omega$. If $t \in \{\alpha\} \times \omega$ then α will be denoted by $\mathbf{ht}(t)$. If $\alpha \in \mathbf{ht}(t)$ then $t[\alpha]$ will denote the unique element of $\{\alpha\} \times \omega$ such that $t[\alpha] \sqsubset t$.

Let $\mathfrak{R}$ denote the set of all functions R such that there is some $C(R)$ such that

$$C(R) \subseteq \omega_1 \text{ is closed,} \tag{2.1}$$

$$(\forall \xi)\ \{\xi+1, \xi+\omega\} \cap C(R) = \emptyset, \tag{2.2}$$

$$\mathbf{domain}(R) = C(R) \times \omega, \tag{2.3}$$

$$(\forall t \in \mathbf{domain}(R))\ R(t) \subseteq \mathbf{ht}(t), \tag{2.4}$$

$$(\forall t \sqsubset s)\ R(t) = R(s) \cap \mathbf{ht}(t). \tag{2.5}$$

If $R \in \mathfrak{R}$ and $\eta \in C(R)$ then define $R \perp \eta = R \restriction (C(R) \cap (\eta+1)) \times \omega$ and note that $R \perp \eta \in \mathfrak{R}$. Let

$$\mathfrak{R}_\xi = \{R \in \mathfrak{R} \mid \sup(C(R)) \leq \xi \text{ and } (\forall \zeta \in C(R) \cap \xi+1)\ R \perp \zeta \in D_\zeta\}$$

noting that the dependence on $\sqsubset$ has been suppressed in the notation. Note also that it may happen that $\mathfrak{R}_\xi \neq \emptyset$ even when $D_\xi = \emptyset$.

NOTATION 2.4. For any function F and A a subset of the domain of F let $F\langle A \rangle$ denote the image of A under F.

The main part of the proof will be to construct the tree order $\sqsubset$ as well as mappings π_t for $t \in \omega_1 \times \omega$ and $\psi_\xi : \mathfrak{R}_\xi \to \mathfrak{R}_\xi$ for each $\xi \in \omega_1$. This will be accomplished by constructing tree orderings $\sqsubset_\xi$ on $\xi \times \omega$, π_t for $t \in \xi \times \omega$ and $\psi_\xi : \mathfrak{R}_\xi \to \mathfrak{R}_\xi$ by induction on ξ so that

(1) if $\eta \in \xi$ then $\sqsubset_\eta = \sqsubset_\xi \cap [\eta \times \omega]^2$,
(2) π_t is an involution of $\mathbf{ht}(t)$ such that $\pi_t\langle\zeta\rangle = \zeta$ for every limit ordinal $\zeta \in \mathbf{ht}(t)$,
(3) if $\xi + \omega \in \mathbf{ht}(t)$ then $\pi_t(\xi + i) = \xi + i$ for all but finitely many $i \in \omega$,

(4) if $t \sqsubset_\xi s$ then $\pi_t \subseteq^* \pi_s$,
(5) if $\eta \in \xi$ then $\psi_\eta \subseteq \psi_\xi$,
(6) if $R \in \mathfrak{R}_\xi$ (to be precise, it must be specified that $\mathfrak{R}_\xi$ is defined using the tree ordering $\sqsubset_\xi$ in (2.5) of Definition 2.3) then
   - $C(R) = C(\psi_\xi(R))$,
   - $\pi_t\langle R(t)\rangle \equiv^* \psi_\xi(R)(t)$ for all $t \in T_\xi$ such that $\mathbf{ht}(t) \geq \sup(C(R))$,

(7) if $R \in \mathfrak{R}_\xi$ and $\eta \in C(R)$ then $\psi_\xi(R)\perp\eta = \psi_\xi(R\perp\eta)$.

It will furthermore be assumed that if ξ is a limit ordinal then the following conditions also hold:

(8) if $\mathcal{C} \in D_\xi$ is a maximal antichain in $\sqsubset_\xi$ then for all $t \in \{\xi\} \times \omega$ there is some $\zeta \in \xi$ such that $t[\zeta] \in \mathcal{C}$,
(9) if $g \in D_\xi$ is a function with domain $\xi \times \omega$ such that $g(t) : \mathbf{ht}(t) \to \xi$ and ([1]) for each $t \in \xi \times \omega$ there is s such that $\mathbf{ht}(s) = \xi$ and $t \sqsubset_{\xi+1} s$ then for every $\mu \in \xi$ there is some η such that
   - $\xi > \eta > \mu$,
   - $g(s[\eta + \omega])(\eta) \neq \pi_t(\eta)$.

(10) if $\mathcal{A} \in [\mathfrak{R}_\xi]^{<\aleph_0}$ and $t \in \xi \times \omega$ then there is t^* such that
   - $\mathbf{ht}(t^*) = \xi$,
   - $t \sqsubset_{\xi+1} t^*$,
   - $\pi_{t^*}\langle R(t^*)\rangle = \psi(R)(t^*)$ for all $R \in \mathcal{A}$.

If this induction can be completed, then let the tree order $\sqsubset$ be defined to be $\bigcup_{\xi\in\omega_1} \sqsubset_\xi$ and note that condition (8) implies that $\mathbb{S} = (\omega_1 \times \omega, \sqsubset)$ is a Suslin tree. Let $\psi : \mathfrak{R} \to \mathfrak{R}$ be defined by

$$\psi(R) = \bigcup_{\xi\in\omega_1} \psi_\xi(R\perp\xi)$$

using (5) and (7) to conclude that ψ is a well defined function from $\mathfrak{R}$ to itself.

Observe that if $\dot{A}$ is an $\mathbb{S}$-name for a subset of ω_1 then, since $\mathbb{S}$ is a Suslin tree, it is possible to find a club $C \subseteq \omega_1$ and R with domain $C \times \omega$ such that if $t \in C \times \omega$ then $R(t) \subseteq \mathbf{ht}(t)$ and for each $\xi \in C$ and each $t \in \{\xi\} \times \omega$,

$$t \Vdash_{\mathbb{S}} \text{“}\dot{A} \cap \xi = R(t)\text{”}.$$

Given $R \in \mathfrak{R}$ and letting $\dot{G}$ be a name for the generic set on $\mathbb{S}$ define

$$R(\dot{G}) = \bigcup_{\xi\in\omega_1} R(\dot{G}_\xi)$$

([1]) In applications it will always be the case that if $t \sqsubset s$ then $g(t) \subseteq g(s)$, but there is no need to assume this at this stage.

where $\dot{G}_\eta$ is a name for the element of $\{\eta\} \times \omega$ satisfying

$$1 \Vdash_{\mathbb{S}} \text{“}\{\dot{G}_\eta\} = \dot{G} \cap \{\eta\} \times \omega\text{”}.$$

Hence every subset $A \subseteq \omega_1$ in an $\mathbb{S}$ generic extension is equal to $R(\dot{G})$ for some $R \in \mathfrak{R}$. Given a generic set $G \subseteq \mathbb{S}$ let Ψ be the function from $\mathcal{P}(\omega_1)/\mathcal{F}in$ to $\mathcal{P}(\omega_1)/\mathcal{F}in$ defined by $\Psi([R(\dot{G})]) = [\psi(R)(\dot{G})]$ for $R \in \mathfrak{R}$. Furthermore, in $V[G]$ let π_ξ be defined to be $\pi_{\dot{G}_\xi}$.

Claim 2.5.

(2.6) $1 \Vdash_{\mathbb{S}}$ “*$\dot{\Psi}$ is a well defined automorphism of $\mathcal{P}(\omega_1)/\mathcal{F}in$ such that $(\forall \xi \in \omega_1)\ \dot{\Psi}\restriction\mathcal{P}(\xi)/\mathcal{F}in$ is induced by $\dot{\pi}_\xi$*”.

Moreover, $1 \Vdash_{\mathbb{S}}$ “*$\dot{\Psi}$ is non-trivial*”.

Proof. Since it has already been established that if $G \subseteq \mathbb{S}$ is generic over V then in $V[G]$ we have

$$\mathcal{P}(\omega_1) = \{R(\dot{G}) \mid R \in \mathfrak{R} \cap V\},$$

the first point to verify is that Ψ is well defined. So suppose that R and R' are in $\mathfrak{R}$ and

$$t \Vdash_{\mathbb{S}} \text{“}R(\dot{G}) \equiv^* R'(\dot{G})\text{”} \tag{2.7}$$

but

$$t \Vdash_{\mathbb{S}} \text{“}\psi(R)(\dot{G}) \not\equiv^* \psi(R'(\dot{G}))\text{”}.$$

By extending t if necessary, it may be assumed that there is some $\eta \in \omega_1$ such that $t \Vdash_{\mathbb{S}}$ “$\psi(R)(\dot{G}) \cap \eta \not\equiv^* \psi(R')(\dot{G}) \cap \eta$”, and hence there is some $\eta \in \omega_1$ such that $t \Vdash_{\mathbb{S}}$ “$(\psi(R)\perp\eta)(\dot{G}) \not\equiv^* (\psi(R')\perp\eta)(\dot{G})$”. By condition (7) it follows that $t \Vdash_{\mathbb{S}}$ “$\psi(R\perp\eta)(\dot{G}) \not\equiv^* \psi(R'\perp\eta)(\dot{G})$”. By condition (6),

$$t \Vdash_{\mathbb{S}} \text{“}\pi_t\langle(R\perp\eta)(\dot{G})\rangle \not\equiv^* \pi_t\langle(R\perp\eta)(\dot{G})\rangle\text{”},$$

and hence, $t \Vdash_{\mathbb{S}}$ “$(R\perp\eta)(\dot{G}) \not\equiv^* (R\perp\eta)(\dot{G})$”, contradicting condition (4) and 2.7. The fact that Ψ is one-to-one has a similar proof.

To see that Ψ is an automorphism suppose that $t \Vdash_{\mathbb{S}}$ “$R(\dot{G}) \subseteq^* R'(\dot{G})$” but $t \Vdash_{\mathbb{S}}$ “$\psi(R(\dot{G})) \not\subseteq^* \psi(R'(\dot{G}))$”. As in the argument for well definedness, it can be assumed that there is some $\eta \in \omega_1$ such that $t \Vdash_{\mathbb{S}}$ “$(\psi(R)\perp\eta)(\dot{G}) \not\subseteq^* (\psi(R')\perp\eta)(\dot{G})$”. But condition (7) then yields the contradiction that

$$t \Vdash_{\mathbb{S}} \text{“}\psi(R\perp\eta)(\dot{G}) \not\subseteq^* \psi(R'\perp\eta)(\dot{G})\text{”}.$$

Since each π_t is an involution, it follows easily that so is Ψ. Hence that Ψ is a surjection. To see that Ψ is not trivial, it suffices to show that there is no $g : \omega_1 \to \omega_1$ in $V[G]$ such that $\pi_\xi \subseteq g$ for all $\xi \in \omega_1$. To this end suppose that $s \Vdash_{\mathbb{S}}$ “$\dot{g} : \omega_1 \to \omega_1$” and note that since $\mathbb{S}$ is Suslin, there is a club $B \subseteq \omega_1$ such that for each $\beta \in B$ and $t \in \{\beta\} \times \omega$ there is some $\bar{g}(t) : \beta \to \beta$

such that

$$t \Vdash_{\mathbb{S}} \text{“}\dot{g}\restriction\beta = \bar{g}(t)\text{”}.$$

Let g with domain $\omega_1 \times \omega$ be defined by

$$g(t) = \begin{cases} \bar{g}(t) & \text{if } \mathbf{ht}(t) \in B, \\ \bar{g}\big(t[\sup(B \cap \mathbf{ht}(t))]\big) & \text{otherwise.} \end{cases}$$

Then use $\diamondsuit^+_{\omega_1}$ to find $\xi \in \omega_1$ and $s^* \in \{\xi\} \times \omega$ such that

- $\xi \in B \setminus \mathbf{ht}(s)$,
- $B \cap \xi$ is cofinal in ξ,
- $g\restriction(B \times \omega) \in D_\xi$,
- $s \sqsubset_\xi s^*$.

Then apply condition (9) to deduce that there are infinitely many $\gamma \in \xi$ such that

$$\pi_{s^*}(\gamma) \neq g(s^*[\gamma + \omega])(\gamma) = g(s^*)(\gamma).$$

Since $s^* \Vdash_{\mathbb{S}}$ “$\dot{g}\restriction\xi = g(s^*)$”, it follows that $s^* \Vdash_{\mathbb{S}}$ “$\dot{g} \not\supseteq^* \pi_{s^*} = \pi_\xi$” as required. ∎

To begin the induction let $\sqsubset_{\omega+1}$ be an arbitrary tree order on $(\omega+1)\times\omega$ and let $\pi_t(k) = k$ for each $k \in \mathbf{ht}(t)$. Let $\psi_{\omega+1}(R) = R$ for each $R \in \mathfrak{R}_\omega$. It is immediate that conditions (1) to (7) and (10) all hold. Since ω is not a limit of limit ordinals, (8) and (9) are not relevant at this stage.

A very similar argument works if ξ is a limit ordinal and $\sqsubset_{\xi+1}$, $\psi_{\xi+1}$ and $\{\pi_t\}_{\mathbf{ht}(t)\leq\xi}$ have been constructed. In this case let $\sqsubset_{\xi+\omega+1}$ be an arbitrary tree order extending $\sqsubset_{\xi+1}$. If $\xi < \mathbf{ht}(t) < \xi + \omega$ let π_t be defined by

$$\pi_t(\gamma) = \begin{cases} \pi_{t[\xi]}(\gamma) & \text{if } \gamma \leq \xi, \\ \gamma & \text{if } \gamma > \xi. \end{cases}$$

Let $\psi_{\xi+\omega+1} = \psi_\xi$ noting that $D_{\xi+\omega} = \emptyset$, and hence there are no further requirements on $\psi_{\xi+\omega+1}$ since $(\xi + \omega + 1) \cap C(R) \subseteq \xi + 1$ for all $R \in \mathfrak{R}$. It is again immediate that conditions (1) to (7) all hold. Note that (8) and (9) are again not relevant at this stage since $D_{\xi+\omega} = \emptyset$. In order for (10) to hold it is necessary to define π_t appropriately for $t \in \{\xi + \omega\} \times \omega$.

To do this, let $\{R_j\}_{j\in\omega}$ enumerate $\mathfrak{R}_\xi = \mathfrak{R}_{\xi+\omega}$ and let

$$f : (\xi + \omega) \times \omega \to \{\xi + \omega\} \times \omega$$

be a one-to-one function such that $t \sqsubset_{\xi+\omega+1} f(t, k)$ for each t and k. Let ξ^- be the largest ordinal that is a limit of limit ordinals and $\xi^- \leq \xi$. From Definition 2.3 it follows that

$$(\forall R \in \mathfrak{R}_\xi)\ \sup(C(R)) \leq \xi^-. \tag{2.8}$$

Now fix $t \in (\xi+\omega)\times\omega$ and $k \in \omega$. Let $\rho \in \xi^-$ be a limit ordinal larger than the maximal element of the finite set of all $\gamma \in \xi^-$ such that

$$(\exists j \le k)\ \pi^{-1}_{t[\xi]}(\gamma) \in R_j(t[\xi]) \quad \text{if and only if} \quad \gamma \notin \psi_\xi(R_j)(t[\xi]). \tag{2.9}$$

It follows that

$$R_j(t[\xi]) \cap \rho = R^*_j(t[\rho]), \tag{2.10}$$

$$\psi_\xi(R_j)(t[\xi]) \cap \rho = \psi_\xi(R^*_j)(t[\rho]), \tag{2.11}$$

where $R^*_j = R_j \perp \sup(C(R_j)\cap\rho)$. Then apply (10) and the induction hypotheses to find t^{**} such that $\mathbf{ht}(t^{**}) = \xi$ and $t[\rho] \sqsubset_\xi t^{**}$ such that

$$\pi_{t^{**}}\langle R_j(t^{**})\rangle = \psi_\xi(R_j)(t^{**}) \tag{2.12}$$

for each $j \le k$. Then define $\pi_{f(t,k)}$ by

$$\pi_{f(t,k)}(\gamma) = \begin{cases} \gamma & \text{if } \xi \le \gamma < \xi+\omega, \\ \pi_{t[\xi]}(\gamma) & \text{if } \rho \le \gamma < \xi, \\ \pi_{t^{**}}(\gamma) & \text{if } \gamma \in \rho. \end{cases}$$

It must first be established that $\pi_{f(t,k)}$ is an involution. This follows from the fact both

$$\pi_{t[\xi]}\restriction[\rho,\xi) \quad \text{and} \quad \pi_{t^{**}}\restriction\rho \tag{2.13}$$

are involutions of their domains since ρ is a limit ordinal and (2) holds.

Then, by (3) and the fact that $\xi = \xi^- + \omega\cdot m$ for some $m \in \omega$, it follows that $\pi_{f(t,k)}(\gamma) = \pi_t(\gamma)$ for all but finitely many $\gamma \in \mathbf{ht}(t)$; so (4) holds. Next,

$$\begin{aligned} &\pi_{t^{**}}\langle R_j(t[\xi])\rangle \cap \rho = \pi_{t^{**}}\langle R_j(t[\xi])\cap\rho\rangle = \pi_{t^{**}}\langle R^*_j(t[\rho])\rangle \\ &= \pi_{t^{**}}\langle R_j(t^{**})\rangle\cap\rho = \psi_\xi(R_j)(t^{**})\cap\rho = \psi_\xi(R^*_j)(t[\rho])\cap\rho = \psi_\xi(R^*_j)(t[\xi])\cap\rho. \end{aligned} \tag{2.14}$$

The first, second, fourth and last equalities follow from (2), (2.10), (2.12) and (2.11) respectively. The others follow from the definition of t^{**} and β. Hence $f(t,k)$ witnesses that (10) holds for t and $\mathcal{A} = \{R_j\}_{j\le k}$. To see this keep in mind that (2.8) holds and note that (2.14) implies that

$$\begin{aligned} \pi_{f(t,k)}\langle R_j(f(t,k))\rangle &= \big(\pi_{t[\xi]}\langle R_j(t[\xi])\rangle \cap [\rho,\xi)\big) \cup \big(\pi_{t^{**}}\langle R_j(t[\xi])\rangle \cap \rho\big) \\ &= \big(\psi_\xi(R_j)(t[\xi]) \cap [\rho,\xi)\big) \cup \big(\psi_\xi(R^*_j)(t[\xi])\cap\rho\big) = \psi_\xi(R_j)(f(t,k)) \end{aligned} \tag{2.15}$$

for each $j \le k$.

So now suppose that $\xi \in \omega_1$ is an arbitrary limit of limit ordinals such that all of the induction hypotheses hold for all $\eta \in \xi$. First, let

$$\mathfrak{R}^* = \{R \in \mathfrak{R}_\xi \mid C(R)\cap\xi \text{ is cofinal in } \xi \text{ or } \sup(C(R)) < \xi\}$$

or, in other words, $C(R) \notin \mathfrak{R}^*$ if $\xi \in C(R)$ and ξ has an immediate predecessor in $C(R)$. The first step will be to find $\sqsubset_{\xi+1}$, $\{\pi_t\}_{t\in\{\xi\}\times\omega}$ and $\psi_{\xi+1}\restriction\mathfrak{R}^*$ such that

(11) conditions (1), (2), (3), (4), (8) and (9) all hold,
(12) $\psi_\eta \subseteq \psi_{\xi+1}\restriction\mathfrak{R}^*$ for each $\eta \leq \xi$,
(13) the versions of (6), (7) and (10) in which $\mathfrak{R}_\xi$ is replaced by $\mathfrak{R}^*$ all hold.

To do this, begin by letting

- $\xi_n \in \xi$ be such that $\lim_{n\to\infty} \xi_n = \xi$,
- $\{t_n\}_{n\in\omega}$ enumerate infinitely often $\xi \times \omega$,
- $\{R_n\}_{n\in\omega}$ enumerate $\mathfrak{R}^*$,
- $\{\mathcal{C}_n\}_{n\in\omega}$ enumerate the antichains of $\sqsubset_\xi$ belonging to D_ξ,
- $\{g_n\}_{n\in\omega}$ enumerate infinitely often all the functions g belonging to D_ξ such that $g(t) : \mathbf{ht}(t) \to \xi$ for each $t \in \xi \times \omega$.

Now fix n and construct a sequence $\{b_n(j)\}_{j\in\omega} \subseteq \xi \times \omega$ and involutions $\{\theta_j\}_{j\in\omega}$ such that (denoting $b_n(i)$ by $b(i)$ to simplify notation)

(14) $t_n \sqsubset_\xi b(0)$,
(15) $b(i) \sqsubset_\xi b(i+1)$,
(16) $\mathbf{ht}(b(j))$ is a limit ordinal at least as large as ξ_j,
(17) there is some $s \in \mathcal{C}_j$ such that $s \sqsubset^* b(j+1)$,
(18) $\theta_0 = \pi_{b(0)}$ and the domain of θ_{i+1} is $[\mathbf{ht}(b(i)), \mathbf{ht}(b(i+1))$, and
  - $\theta_{i+1}(\gamma) = \pi_{b(i+1)}(\gamma)$ for all γ such that $\mathbf{ht}(b(i))+\omega \leq \gamma < \mathbf{ht}(b(i+1))$,
  - $\theta_{i+1}(\gamma) = \pi_{b(i+1)}(\gamma)$ for all but finitely many γ such that $\mathbf{ht}(b(i)) \leq \gamma < \mathbf{ht}(b(i)) + \omega$,

(19) for all $j \in \omega$ there is $k \in \omega$ such that
$$\theta_{j+1}(\mathbf{ht}(b(j)) + k) \neq g_j\big(b(j+1)[\mathbf{ht}(b(j)) + \omega]\big)(\mathbf{ht}(b(j) + k).$$

Furthermore, if we let $R_{j,i} = R_j \perp \sup(C(R_j) \cap b(i))$, then

(20) $\pi_{b(i)}\langle R_{j,i}(b(i))\rangle = \bigcup_{k\leq i} \theta_k\langle R_{j,i}(b(i))\rangle = \psi_\xi(R_{j,i})(b(i))$ for all $j \leq n$ and all integers i,
(21) $\pi_{b(i+1)}\langle R_{j,i+1}(b(i+1)) \setminus \mathbf{ht}(b(i))\rangle = \theta_{i+1}\langle R_{j,i+1}(b(i+1)) \setminus \mathbf{ht}(b(i))\rangle = \psi_\xi(R_{j,i+1})(b(i+1)) \setminus \mathbf{ht}(b(i))$ for all $j \leq i$.

If this can be done, then define $t \sqsubset_{\xi+1} (\xi, n)$ if and only if there is some j such that $t \sqsubset_\xi b(j)$. Then define $\pi_{(\xi,n)} = \bigcup_{j\in\omega} \theta_j$. Conditions (1) to (4) are immediate. Conditions (8) and (9) follow from (17) and (19) respectively and so (11) holds. Then for $R \in \mathfrak{R}^*$ define

$$\psi_{\xi+1}(R) = \begin{cases} \bigcup_{\eta\in\xi} \psi_\xi(R\perp\eta) & \text{if } \sup(C(R) \cap \xi) = \xi, \\ \psi_\xi(R) & \text{if } \sup(C(R) \cap \xi) < \xi. \end{cases}$$

It is immediate that $C(R) = \psi_{\xi+1}(C(R))$ and (12) holds. To see that (13) holds observe that (7) follows directly from the construction, (6) follows from condition (21), and (10) follows from (20). Then choose $\{b_m(i)\}$ similarly for all $m \in \omega$.

To construct $\{b(i)\}_{i\in\omega}$ use (10) to let $b(0)$ be such that $t_n \sqsubset_\xi b(0)$ and $\pi_{b(0)}\langle R_{j,0}(b(0))\rangle = \psi_\xi(R_{j,0})(b(0))$ for $j \le n$. Let $\theta_0 = \pi_{b(0)}$. It follows that conditions (14) to (16) all hold. Conditions (17), (19) and (21) do not apply in this case. Conditions (18) and (20) are immediate.

Now suppose that $b(i)$ is given. First find $s \in \mathcal{C}_i$ such that either $s \sqsubset_\xi b(i)$ or $b(i) \sqsubset_\xi s$. Let $s^* = \max_{\sqsubset_\xi}(s, b(i))$. Then find a limit ordinal $\Xi \ge \xi_i$ such that $\mathbf{ht}(s^*) + \omega < \Xi$. Using (10) of the induction hypothesis let $b(i+1)$ be such that

- $\mathbf{ht}(b(i+1)) = \Xi$,
- $s^* \sqsubset_\xi b(i+1)$,
- $\pi_{b(i+1)}\langle R_{j,i+1}(b(i+1))\rangle = \psi_\xi(R_{j,i+1})(b(i+1))$ for $j \le \max(i,n)$.

It follows that (15) and (16) both hold, and (14) is no longer relevant. The choice of s guarantees that (17) holds. Let u_m denote $\mathbf{ht}(b(i)) + m$. Using (3) let $K \in \omega$ be such that $\pi_{b(i+1)}(u_m) = u_m$ for $m > K$. Find ([2]) $\ell_1 > \ell_0 > K$ such that $u_{\ell_0} \in R_j(b(i+1))$ if and only if $u_{\ell_1} \in R_j(b(i+1))$ for all $j \le \max(i,n)$. Then let

$$\theta_{i+1} = \pi_{b(i+1)} \restriction [\mathbf{ht}(b(i)), \mathbf{ht}(b(i+1)))$$

if either $g_i(b(i+1))(u_{\ell_0}) \ne u_{\ell_0}$ or $g_i(b(i+1))(u_{\ell_1}) \ne u_{\ell_1}$. Otherwise define θ_{i+1} with domain $[\mathbf{ht}(b(i)), \mathbf{ht}(b(i+1)))$ by

$$\theta_{i+1}(\delta) = \begin{cases} \pi_{b(i+1)}(\delta) & \text{if } \delta \notin \{u_{\ell_0}, u_{\ell_1}\}, \\ u_{\ell_1} & \text{if } \delta = u_{\ell_0}, \\ u_{\ell_0} & \text{if } \delta = u_{\ell_1}. \end{cases}$$

Observe that

$$\theta_{i+1}\langle R_{j,i+1}(b(i+1))\rangle = \psi_\xi(R_{j,i+1})(b(i+1)) \cap [\mathbf{ht}(b(i)), \mathbf{ht}(b(i+1)))$$

for each $j \le \max(i,n)$. Therefore (18)–(21) all hold. This completes the induction.

It remains to define $\psi_\xi(R)$ for $R \in \mathfrak{R}_\xi \setminus \mathfrak{R}^*$. In other words, $\psi_\xi(R)$ must be defined when $R \in \mathfrak{R}_\xi$, $\xi \in C(R)$, but $\mu(R) = \sup(C(R) \cap \xi) < \xi$. In this case $\psi_\xi(R)(t)$ must be defined for each $t \in \{\xi\} \times \omega$. Note, however, that $\psi(R)(t) \cap \mu(R)$ must be equal to $\psi(R \perp \mu(R))(\mu(R))$ in order for (2.5) to hold. Hence it suffices to define

$$\psi(R)(t) = \psi(R\perp\mu(R))(\mu(R)) \cup \big([\mu(R), \xi) \cap \pi_t(R)\big).$$

Observe that

$$(2.16)\qquad (\forall t \in \{\xi\} \times \omega)\ \pi_t\langle R(t)\rangle \setminus \mu(R) = \psi_\xi(R)(t) \setminus \mu(R)$$

([2]) The reader wondering why the argument presented here does not apply to ω_2 assuming $\diamondsuit^+_{\omega_2}$, thereby contradicting the results of [10], will note that this is the key point that does not extend beyond ω_1.

and hence (6) holds. Conditions (5) and (7) are immediate. To see that (10) holds let $\mathcal{A} \in [\mathfrak{R}_\xi]^{<\aleph_0}$ and $t \in T_\xi$ be such that $\mathbf{ht}(t) < \xi$. Let

$$\mathcal{A}^* = (\mathcal{A} \cap \mathfrak{R}^*) \cup \{R \perp \mu(R) \mid R \in \mathcal{A} \setminus \mathfrak{R}^*\}$$

and note that $\mathcal{A}^* \subseteq \mathfrak{R}^*$. It is therefore possible to use the version of (10) for $\mathfrak{R}^*$ to find $t^* \sqsupset_{\xi+1} t$ such that $\mathbf{ht}(t^*) = \xi$ and $\pi_{t^*}\langle R(t^*)\rangle = \psi(R)(t^*)$ for all $R \in \mathcal{A}^*$. Then applying (2.16) yields $\pi_{t^*}\langle R(t^*)\rangle = \psi(R)(t^*)$ for all $R \in \mathcal{A}$ as required.

3. Other results on $\mathcal{T}(\Psi)$. The methods of §2 can be modified to exert more control over $\mathcal{T}(\Psi)$. This section sketches arguments exhibiting two extreme possibilities for $\mathcal{T}(\Psi)$.

THEOREM 3.1. *It is consistent that there is $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\mathcal{T}(\Psi)$ is a proper ideal, $[\omega_1]^{\leq\aleph_0} \subseteq \mathcal{T}(\Psi)$ but $\mathcal{T}(\Psi)$ is not a σ-ideal—in other words, ω_1 can be covered by countably many elements from $\mathcal{T}(\Psi)$.*

Proof. The only change needed to the proof of §2 is to choose disjoint sets B_n such that $\omega_1 = \bigcup_{n\in\omega} B_n$ and $B_n \cap [\xi, \xi+\omega)$ is infinite for every $\xi \in \omega_1$ and then to add to (2) the requirement that for every $n \in \omega$ and for all but finitely many $\beta \in B_n \cap \mathbf{ht}(t)$ the equality $\pi_t(\beta) = \beta$ holds. This will guarantee that each B_n belongs to $\mathcal{T}(\Psi)$ but requires modifying (10) of §2 to

(10) if $\mathcal{A} \in [\mathfrak{R}_\xi]^{<\aleph_0}$ and $m \in \omega$ and $t \in \xi \times \omega$ then there is $t^* \sqsupset_{\xi+1} t$ such that $\mathbf{ht}(t^*) = \xi$ and $\pi_{t^*}\langle R(t^*)\rangle = \psi(R)(t^*)$ for all $R \in \mathcal{A}$ and $\pi_{t^*}(\beta) = \beta$ for each $\beta \in \bigcup_{j\leq m} B_j \setminus \mathbf{ht}(t)$.

The u_{ℓ_i} required to satisfy (19) will have to come from $\bigcup_{j>m} B_j$ where m is now an additional parameter in the enumeration following (13). ■

THEOREM 3.2. *It is consistent that there is $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $[\omega_1]^{\leq\aleph_0} = \mathcal{T}(\Psi)$.*

Proof. To prove this, it will be necessary to use $\diamondsuit^+_{\omega_1}$ to trap uncountable partial functions from ω_1 to ω_1 and not just bijections. This will of course require weakening (2) because it cannot be expected that any interval of the form $[\xi, \xi+\omega)$ will contain more than one member of the domain of the trapped function, as is necessary in choosing the u_{ℓ_i} to satisfy (19). On the other hand, dispensing with (2) entirely might create problems in finding the limit ρ to satisfy (2.9) because satisfying (2.13) would no longer be automatic. Nevertheless, the following modification of (10) of §2 allows requirement (2) to be removed from the construction:

(10) if $\mathcal{A} \in [\mathfrak{R}_\xi]^{<\aleph_0}$ and $t \in \xi\times\omega$ then there is $t^* \sqsupset_{\xi+1} t$ such that $\mathbf{ht}(t^*) = \xi$ and $\pi_{t^*}\langle R(t^*)\rangle = \psi(R)(t^*)$ for all $R \in \mathcal{A}$, and furthermore $\zeta = \pi_{t^*}\langle\zeta\rangle$.

It is easy to check that the construction of §2 actually does yield this stronger induction hypothesis.

Next modify (9) of §2 to

(9) if $g \in D_\xi$ is a function with domain $\Gamma \times \omega$ for some cofinal subset Γ of ξ and if $g(t) : \Delta_t \to \gamma$ with Δ_t a cofinal subset of γ for each $\gamma \in \Gamma$ and $t \in \{\gamma\} \times \omega$, then for each $t \in \{\xi\} \times \omega$,

$$(\forall \beta \in \xi)(\exists \gamma \in \Gamma)(\exists \delta \in \Delta_{t[\gamma]})\ \beta < \delta \text{ and } g(t[\gamma])(\delta) \neq \pi_t(\delta).$$

In choosing the u_{ℓ_i} required to satisfy (19) it can no longer be expected that they will come from $[\mathbf{ht}(b(j)), \mathbf{ht}(b(j)+\omega))$. However, if it is only required that they belong to $\Delta_{b_n(j+1)}$, the construction can proceed as before. ■

4. Proof of Theorem 1.2

Notation 4.1. Let $\mathbb{C}(X)$ denote the set of countable partial functions from X to 2 ordered by inclusion.

Theorem 4.2. *Given bijections $\pi_\xi : \xi \to \xi$ for each $\xi \in \omega_1$ such that*

(i) *if $\xi \in \eta$ then $\pi_\xi \equiv^* \pi_\eta \restriction \xi$,*
(ii) *there is no $\pi : \omega_1 \to \omega_1$ such that $\pi_\eta \equiv^* \pi \restriction \eta$ for all $\eta \in \omega_1$,*
(iii) *$G \subseteq \mathbb{C}(\omega_1)$ is generic,*

there is no set $B \subseteq \omega_1$ such that

$$\pi_\xi^{-1}(B) \equiv^* \bigcup_{g \in G} g^{-1}\{1\} \cap \xi \quad \text{for each } \xi \in \omega_1.$$

Proof. Suppose that $\dot{B}$ is a $\mathbb{C}(\omega_1)$ name such that

$$1 \Vdash_{\mathbb{C}(\omega_1)} \text{“}(\forall \xi \in \omega_1)\ \dot{B} \cap \xi \equiv^* \bigcup_{g \in \dot{G}} \pi_\xi \langle g^{-1}\{1\} \rangle\text{”}$$

where $\dot{G}$ is a name for the generic set. Let $\mathfrak{M} = (M, \dot{B}, \{\pi_\xi\}_{\xi \in \omega_1}, \in)$ be a countable elementary submodel of $(H(\aleph_2), \dot{B}, \{\pi_\xi\}_{\xi \in \omega_1}, \in)$ and let $\mu = M \cap \omega_1$.

Claim 4.3. *For all $g \in \mathbb{C}(\omega_1) \cap M$ there is $h \in \mathbb{C}(\omega_1) \cap M$ such that $g \subseteq h$ and*

$$h \Vdash_{\mathbb{C}(\omega_1)} \text{“}\dot{B} \cap \mathbf{domain}(h \setminus g) \neq \pi_\mu \langle (h \setminus g)^{-1}\{1\} \rangle\text{”}. \tag{4.1}$$

Proof. Suppose that $g \in \mathbb{C}(\omega_1) \cap M$ is a counterexample to the claim. Without loss of generality there is $\alpha \in \mu$ such that $\mathbf{domain}(g) = \alpha$. If $\alpha \in \delta \in \mu$ and $X \subseteq [\alpha, \delta)$ then define $F_{X,\delta} \in \mathbb{C}(\omega_1)$ to be the function extending g with domain δ such that if $\alpha \in \eta \in \delta$ then $F_{X,\delta}(\eta) = 1$ if and only if $\eta \in X$. It follows from the failure of (4.1) that if $\alpha \leq \beta < \delta$ then

$$F_{\{\beta\},\delta} \Vdash_{\mathbb{C}(\omega_1)} \text{“}\dot{B} \cap [\alpha, \delta) = \{\pi_\mu(\beta)\}\text{”},$$

and hence it is possible to define in $\mathfrak{M}$ a function θ by letting $\theta(\beta)$ be the unique ordinal such that

$$F_{\{\beta\},\delta} \Vdash_{\mathbb{C}(\omega_1)} \text{“}\dot{B} \cap [\alpha, \delta) = \{\theta(\beta)\}\text{”}$$

for all $\delta > \beta$ and noting that $\theta(\beta)$ is defined for each $\beta \geq \alpha$. Then

$$\text{(4.2)} \qquad \mathfrak{M} \models \theta : [\alpha, \omega_1) \to [\alpha, \omega_1) \text{ and } (\forall \beta > \alpha)(\forall \delta > \beta) \quad F_{\{\beta\},\delta} \Vdash_{\mathbb{C}(\omega_1)} \text{“}\dot{B} \cap [\alpha, \delta) = \{\theta(\beta)\}\text{”}.$$

By hypothesis (ii) of the theorem, there must be ξ such that

$$\text{(4.3)} \qquad \mathfrak{M} \models \pi_\xi \not\equiv^* \theta \restriction \xi,$$

and since $\theta \subseteq \pi_\mu$ it follows $\pi_\xi \not\equiv^* \pi_\mu \restriction \xi$, contradicting (i). ■

By Claim 4.3 it is easy to find a sequence $\{h_n\}_{n \in \omega}$ of conditions in $\mathbb{C}(\omega_1) \cap M$ such that $h_n \subseteq h_{n+1}$ and

$$h_{n+1} \Vdash_{\mathbb{C}(\omega_1)} \text{“}\dot{B} \cap \mathbf{domain}(h_{n+1} \setminus h_n) \neq \pi_\mu \langle (h_{n+1} \setminus h_n)^{-1}\{1\} \rangle\text{”},$$

and then to let $h = \bigcup_n h_n$. It follows that $h \Vdash_{\mathbb{C}(\omega_1)}$ “$\dot{B} \cap \mu \not\equiv^* \pi_\mu \langle h^{-1}\{1\} \rangle$” as required. ■

Proof of Theorem 1.2. Let V be a model of the Continuum Hypothesis and let G be a subset of $\mathbb{C}(\omega_2)$ that is generic over V. Then $\diamondsuit_{\omega_1}$ holds in $V[G]$. Given $\Psi \in \mathbb{A}UT_{\omega_1}$ such that $\mathcal{T}(\Psi) \supseteq [\omega_1]^{\aleph_0}$ let $X \in [\omega_2]^{\aleph_1}$ be such that for each $\xi \in \omega_1$ there is $\pi_\xi \in V[G \cap \mathbb{C}(X)]$ such that $\Psi \restriction \mathcal{P}(\xi)/\mathcal{F}in$ is induced by π_ξ. If $\mathcal{T}(\Psi)$ is not a proper ideal in $V[G \cap \mathbb{C}(X)]$ then it is not a proper ideal in $V[G \cap \mathbb{C}(\omega_2)]$ either, so assume that $\mathcal{T}(\Psi)$ is a proper ideal in $V[G \cap \mathbb{C}(X)]$. Then let $\mu = \sup(X) + 1$ and apply Theorem 4.2 to conclude that if

$$B \in \Psi([\{\beta \in \omega_1 \mid (\exists g \in G)\ g(\mu + \beta) = 1\}])$$

then there is some $\xi \in \omega_1$ with $\pi_\xi^{-1}(B) \not\equiv^* g^{-1}\{1\} \cap \xi$ for all $g \in G \cap \mathbb{C}(\mu + \omega_1)$. A standard argument shows that no countably closed forcing can add a set Z such that for every $\xi \in \omega_1$ there is $g \in G \cap \mathbb{C}(\mu + \omega_1)$ such that $\pi_\xi^{-1}(Z) \equiv^* g^{-1}\{1\} \cap \xi$. Hence $[\{\beta \in \omega_1 \mid (\exists g \in G)\ g(\mu + \beta) = 1\}]$ has no image under Ψ in $V[G]$, contradicting $\Psi \in \mathbb{AUT}_{\omega_1}$. ■

5. Open questions. An examination of Veličković's proof of [11, Theorem 3.1] shows that it is consistent that there is some $\Psi \in \mathbb{AUT}_\omega$ such that $\mathcal{T}(\Psi)$ is an ultrafilter. His proof does not generalize to answer the following question though.

Question 5.1. Is it consistent that there is $\Psi \in \mathbb{AUT}_{\omega_1}$ such that $\mathcal{T}(\Psi)$ is an ultrafilter? Can the question be answered when ω_1 is replaced by some other uncountable cardinal?

It was mentioned in the introduction that it is shown in [10] that if $\kappa > 2^{\aleph_0}$ and κ is less than the first inaccessible cardinal then for every $\Psi \in \mathbb{AUT}_\kappa$ there is a set $X \in \mathcal{T}(\Psi)$ such that $|\kappa \setminus X| \leq 2^{\aleph_0}$. The following question remains open though.

QUESTION 5.2. Is it consistent that κ is at least as large as the first inaccessible cardinal and there is $\Psi \in \mathbb{AUT}_\kappa$ such that $T(\Psi)$ is a proper ideal and $[\kappa]^{<\kappa} \subseteq \mathcal{T}(\Psi)$?

However, it will be noted that [10, remark following Question 7.4] is strengthened by the following. Recall that if κ is weakly compact then every tree of height κ whose levels all have size less than κ has a branch of length κ.

PROPOSITION 5.3. *If κ is a weakly compact cardinal then every Ψ such that $[\kappa]^{<\kappa} \subseteq \mathcal{T}(\Psi)$ is trivial.*

Proof. If $\Psi \in \mathbb{AUT}_\kappa$ is a counterexample to the proposition then note first that there is an unbounded set $S \subseteq \kappa$ and a finite $F \subseteq \kappa$ such that for each $\xi \in S$ there is a one-to-one function $\pi_\xi : \xi \setminus F \to \xi$ such that π_ξ induces $\Psi \restriction \mathcal{P}(\xi)/\mathcal{F}in$. To see this simply choose a continuous sequence $\{\mathfrak{M}_\xi\}_{\xi\in\kappa}$ of elementary submodels of $(H(\kappa^+), \Psi, \in)$ such that the set of elements of κ in the universe of $\mathfrak{M}_\xi$ is an ordinal $\mu_\xi \in \kappa$, and if ξ has uncountable cofinality, then the universe of $\mathfrak{M}_\xi$ is closed under countable subsets. Note that since $[\kappa]^{<\kappa} \subseteq \mathcal{T}(\Psi)$, for each $\xi \in \kappa$ there is some $\pi : \mu_\xi \to \kappa$ that induces $\Psi \restriction \mathcal{P}(\mu_\xi)/\mathcal{F}in$. Note also that if ξ has uncountable cofinality and $\pi^{-1}(\kappa \setminus \mu_\xi)$ is infinite then there is some infinite $Z \subseteq \pi^{-1}(\kappa \setminus \mu_\xi)$ such that $Z \in \mathfrak{M}_\xi$. By elementarity there are η and θ in $\mathfrak{M}_\xi$ such that

$$\mathfrak{M}_\xi \models Z \subseteq \eta \text{ and } \theta \text{ induces } \Psi \restriction \mathcal{P}(\eta)/\mathcal{F}in.$$

But then $\theta\langle Z\rangle \subseteq \mu_\xi$, contradicting $\theta \restriction \eta \equiv^* \pi \restriction \eta$. Therefore $F_\xi = \pi^{-1}(\kappa \setminus \mu_\xi)$ is finite and π_ξ can be defined to be $\pi \restriction (\xi \setminus F_\xi)$. There is then some fixed F such that

$$S = \{\mu_\xi \mid F_\xi = F \text{ and } \xi \in \kappa \text{ and } \mathbf{cof}(\xi) \geq \omega_1\}$$

satisfies the requirement.

Let $\{\sigma_\xi\}_{\xi\in\kappa}$ be an increasing enumeration of S and let

$$L_\xi = \{\pi : \sigma_\xi \setminus F \to \sigma_\xi \mid \pi \equiv^* \pi_{\sigma_\xi}\}$$

and note (3) that $|L_\xi| \leq 2^{|\sigma_\xi|} < \kappa$. Then let $T = (\bigcup_{\xi\in\kappa} L_\xi, \subseteq)$.

Observe that $L_\eta \neq \emptyset$ since $\pi_{\sigma_\eta} \in L_{\sigma_\eta}$, and distinct elements of L_η are incomparable under $\subseteq$. Hence it suffices to check that if $\xi \in \eta \in \kappa$ then

$$(\forall \pi \in L_\eta)(\exists \theta \in L_\xi)\ \theta \subseteq \pi \tag{5.1}$$

(3) Note also that if L_ξ were to be defined as $\{\pi : \sigma_\xi \to \kappa \mid \pi \equiv^* \pi_{\sigma_\xi}\}$, as would be natural, then it would not be the case that $|L_\xi| < \kappa$.

since this will establish that L_η is precisely the ηth level of T. But (5.1) is immediate since $\theta = \pi\restriction(\sigma_\xi \setminus F) \in L_\xi$. Therefore T is a tree of height κ with levels of cardinality less than κ and no branches of length κ, contradicting the fact that κ is weakly compact. ∎

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