Can. J. Math. Vol. 47 (1), 1995 pp. 132–145

DENSITIES OF ULTRAPRODUCTS OF BOOLEAN ALGEBRAS

SABINE KOPPELBERG AND SAHARON SHELAH

ABSTRACT. We answer three problems by J. D. Monk on cardinal invariants of Boolean algebras. Two of these are whether taking the algebraic density πA resp. the topological density $\mathrm{d}A$ of a Boolean algebra A commutes with formation of ultraproducts; the third one compares the number of endomorphisms and of ideals of a Boolean algebra.

In set theoretic topology, considerable effort has been put into the study of cardinal invariants of topological spaces, see *e.g.* [Ju1] and [Ho], [Ju2]. In Monk's book [Mo], similarly a systematic study of cardinal invariants of Boolean algebras is undertaken; in particular, the behaviour of these invariants with respect to algebraic constructions like taking subalgebras, quotients *etc.* is investigated. One of these is the ultraproduct construction, well known from model theory; *cf.* [ChK]. Many questions on ultraproducts are highly dependent on set theory; among the more recent results are those in Shelah' s pcf theory dealing with the possible cofinalities $\mathrm{cf}(\prod_{\alpha<\kappa}\lambda_\alpha/D)$ where the λ_α are regular cardinals, hence well-ordered in a natural way, and the ultraproduct has the resulting linear order.

Monk's book contains a list of 66 problems, three of which are answered (consistently) in this paper.

PROBLEM 9. Does there exist a system $(A_i)_{i\in I}$ of infinite Boolean algebras and an ultrafilter F on I such that $\mathrm{d}(\prod_{i\in I}A_i/F) < |\prod_{i\in I}\mathrm{d}(A_i)/F|$?

PROBLEM 12. Is it true that always $\pi(\prod_{i\in I}A_i/F) = |\prod_{i\in I}\pi(A_i)/F|$?

PROBLEM 60. Is there a Boolean algebra A such that $|\operatorname{End}A| < |\operatorname{Id}A|$?

Here πA and $\mathrm{d}A$ are the "algebraic" and the "topological" density of A, defined by

$$\mathrm{d}A = \min\{|Y| : Y \text{ a dense subset of the Stone space of } A\}$$
$$\pi A = \min\{|X| : X \text{ a dense subset of } A\}$$

(for more definitions and matters on cardinal functions, see [Mo]). Note that we are dealing only with infinite algebras and that, trivially, $\omega \le \mathrm{d}A \le \pi A$, $\mathrm{d}(\prod_{i\in I}A_i/F) \le |\prod_{i\in I}\mathrm{d}(A_i)/F|$ and $\pi(\prod_{i\in I}A_i/F) \le |\prod_{i\in I}\pi(A_i)/F|$.

Partially supported by DFG grant Ko 490/7-1 and by the Edmund Landau Center (Jerusalem) for research in Mathematical Analysis, supported by the Minerva Foundation (Germany). In addition, the first author gratefully acknowledges the hospitality of the Department of Mathematics of the Hebrew University of Jerusalem.

Publication no. 415 of the second author.

Received by the editors September 19, 1993.

AMS subject classification: 03C20, 03E10, 06E05.

Key words and phrases: Boolean algebra, ultraproduct, density, π-weight.

In Problem 60, End A is the set of all endomorphisms, Id A the set of all ideals of A.

In Section 1, we give a positive answer to Problem 12 under SCH. Here SCH is the Singular Cardinal Hypothesis: if $2^{\mathrm{cf}\,\lambda} < \lambda$ (so λ is singular), then $\lambda^{\mathrm{cf}\,\lambda} = \lambda^+$. However, $\neg$ SCH gives a negative answer to both Problems 9 and 12:

THEOREM A. *Assume we have cardinals κ, μ, and $(\lambda_\alpha)_{\alpha<\kappa}$ and an ultrafilter D on κ such that: $\kappa < \mu = \mathrm{cf}\,\mu$, $\mu^{<\mu} < \lambda_\alpha = \mathrm{cf}\,\lambda_\alpha$, and the cofinality of the ultraproduct $\prod_{\alpha<\kappa} \lambda_\alpha/D$ is less than its cardinality. Then there is a forcing notion $\mathbb{R}$ such that*

(a) $\mathbb{R}$ is μ-complete and satisfies the $(\mu^{<\mu})^+$-chain condition; hence forcing with $\mathbb{R}$ preserves all cardinalities and cofinalities outside the interval $[\mu^+, \mu^{<\mu})$

(b) for $K \subseteq \mathbb{R}$ $\mathbb{R}$-generic over V, the following holds in $V[K]$: there are Boolean algebras $(A_\alpha)_{\alpha<\kappa}$ such that $\lambda_\alpha = |A_\alpha| = \pi A_\alpha = \mathrm{d}A_\alpha$, but for the ultraproduct $A = \prod_{\alpha<\kappa} A_\alpha/D$,

$$d(A) \leq \pi(A) = \mathrm{cf}\Big(\prod_{\alpha<\kappa} \lambda_\alpha/D\Big) < \Big|\prod_{\alpha<\kappa} \lambda_\alpha/D\Big| = \Big|\prod_{\alpha<\kappa} \pi(A_\alpha)/D\Big| = \Big|\prod_{\alpha<\kappa} d(A_\alpha)/D\Big|.$$

Note that SCH is known to be independent from ZFC, modulo some large cardinal assumption (see [Ma]). And the assumption of Theorem A is a consequence of $\neg$ SCH, as follows from pcf theory. A particularly easy case is the classical one for $\neg$ SCH: assume λ is strong limit and singular, $\kappa = \mathrm{cf}\,\lambda$ satisfies $2^\kappa < \lambda$, but $\lambda^\kappa > \lambda^+$; let μ be regular such that $\kappa < \mu < \lambda$. Then there are (see [Sh, Chapter II, 1.5]) regular λ_α such that $\lambda = sup_{\alpha<\kappa}\lambda_\alpha$, $\prod_{\alpha<\kappa} \lambda_\alpha/J^{bd}_\kappa$ has true cofinality λ^+ (J^{bd}_κ the ideal of bounded subsets of κ), hence any uniform ultrafilter D on κ gives $\mathrm{cf}(\prod_{\alpha<\kappa} \lambda_\alpha/D) = \lambda^+ < |\prod_{\alpha<\kappa} \lambda_\alpha/D|$. More generally if λ violates SCH, *i.e.* for some κ, we have $2^\kappa < \lambda$ and $\lambda^\kappa > \lambda^+$, let λ' be minimal such that $\lambda'^\kappa = \lambda^\kappa$ (*i.e.* $\lambda'^\kappa \geq \lambda$); so for every cardinal $\rho < \lambda'$, we have $\rho^\kappa < \lambda'$. Now take $\mu = \kappa^+$ and find, by [Sh, Chapter II, 1.5], an appropriate family $(\lambda'_\alpha)_{\alpha<\kappa}$ with limit λ' and $\mathrm{cf}(\prod_{\alpha<\kappa} \lambda'_\alpha/J^{bd}_\kappa) = \lambda'^+$. Moreover we can replace λ'^+ by any regular cardinal in the interval $[\lambda'^+, \lambda'^\kappa]$; similarly for the strong limit case; see [Sh, Chapter VIII, §1].

Theorem 1.1 below and Theorem A show that the answer to Problem 12 is independent from ZFC. However, it has recently been shown in [RoSh 534, 2.6, 2.7] that Problem 9 has a positive answer even in ZFC.

Problem 60 is solved in Section 8 by

THEOREM B. *Assume μ is a strong limit cardinal satisfying $\mathrm{cf}\,\mu = \omega$ and $2^\mu = \mu^+$. Then there is a Boolean algebra B such that $|B| = |\mathrm{End}\,B| = \mu^+$ and $|\mathrm{Id}\,B| = 2^{\mu^+}$.*

The organization of Sections 2 to 7 is as follows. In Section 2, we introduce a first order theory T for Boolean algebras with some extra structure which allows (*e.g.* in ultraproducts $A = \prod_{\alpha<\kappa} A_\alpha/D$ of models of T) to easily compute πA. In Section 3, we construct canonical models $A(p)$ of T from what we call valuation functions p. In sections 4 to 6, we consider the forcing notion $\mathbb{P}$ of valuation functions, determine its completeness and chain conditions, and compute $\mathrm{d}A$ and πA for the canonical algebra $A = A(P)$ constructed from a generic valuation function P. In Section 7, we prove Theorem A.

For definitions and results on set theory, see [Je]; for Boolean algebras, [Ko].

1. **Problem 12 under** SCH. We give here a positive answer to Monk's Problem 12 under SCH. Given an ultraproduct $A = \prod_{i\in\kappa} A_i/D$ of infinite Boolean algebras, we let $\lambda_i = \pi A_i$, so $\omega \leq \lambda_i$. For simplicity of notation, we will denote, in this section, by $\prod_{i\in\kappa}\lambda_i/D$ both the ultraproduct of the λ_i and its cardinality.

Note first that the answer is easy if $\lambda_i \leq 2^\kappa$ for D-almost all $i \in \kappa$ (*i.e.* if $\{i \in \kappa : \lambda_i \leq 2^\kappa\}$ is in D) and D is regular. For in this case, each A_i has an infinite set of pairwise disjoint elements, so A has cellularity at least 2^κ and, on the other hand, $\prod_{i\in\kappa}\lambda_i/D \leq 2^\kappa$, hence $2^\kappa \leq cA \leq \pi A \leq \prod_{i\in\kappa}\lambda_i/D \leq 2^\kappa$. Thus Theorem 1.1 covers the interesting case: $2^\kappa < \lambda_i$ for D-almost all i.

THEOREM 1.1 (SCH). *Assume $2^\kappa < \lambda_i = \pi A_i$ for all $i \in \kappa$ and D is an ultrafilter on κ; let $A = \prod_{i\in\kappa} A_i/D$. Then $\pi A = \prod_{i\in\kappa}\lambda_i/D$.*

PROOF. We know that $\pi A \leq \prod_{i\in\kappa}\lambda_i/D$. Let

$$\lambda = D - \lim(\lambda_i : i \in \kappa),$$

i.e. λ is the least cardinal ρ such that $\lambda_i \leq \rho$ holds for all D-almost all i. Without loss of generality, $\lambda_i \leq \lambda$ holds for all $i \in \kappa$.

CLAIM 1. If $\theta < \lambda$, then $\theta^\kappa \leq \lambda$.

To see this, pick i such that $\theta < \lambda_i$. Now if $\theta \leq 2^\kappa$, then $\theta^\kappa = 2^\kappa < \lambda_i \leq \lambda$. Otherwise, $\kappa < 2^\kappa < \theta < \theta^+ \leq \lambda_i$, $(\theta^+)^\kappa = \theta^+$ by SCH, so $\theta^\kappa \leq \theta^+ \leq \lambda_i \leq \lambda$.

CLAIM 2. $\pi A \geq \lambda$.

Otherwise pick a dense subset Y of A of size ρ, where $\rho < \lambda$, say $Y = \{y_\alpha/D : \alpha < \rho\}$ with $y_\alpha = \big(y_\alpha(i)\big)_{i\in\kappa}$ in $\prod_{i\in\kappa} A_i$ and $y_\alpha(i) \neq 0$. Since $\rho < \lambda$, we may assume without loss of generality that $\rho < \lambda_i$ for all i. So we can pick, for $i \in \kappa$, $a_i \in A_i \setminus \{0\}$ satisfying $y_\alpha(i) \not\leq a_i$, for all $\alpha < \rho$. The sequence $a = (a_i)_{i\in\kappa}$ is such that $y_\alpha/D \not\leq a/D$ for $\alpha < \rho$, a contradiction.

The theorem now follows immediately from the next three claims.

CLAIM 3. If $\pi A \geq \lambda^+$, then the assertion of the theorem holds.

For in this case, $\lambda^+ \leq \pi A \leq \prod_{i\in\kappa}\lambda_i/D \leq \lambda^\kappa/D \leq \lambda^\kappa \leq \lambda^+$, where the last inequality follows from SCH and $2^\kappa < \lambda$.

CLAIM 4. If $\pi A = \lambda$, then every function $f \in \prod_{i\in\kappa}\lambda_i/D$ is bounded below λ, modulo D.

For the proof, work as in Claim 2: fix a dense subset Y of A, $Y = \{y_\alpha/D : \alpha < \lambda\}$, $y_\alpha = \big(y_\alpha(i)\big)_{i\in\kappa}$, $y_\alpha(i) \neq 0$. Given $f \in \prod_{i\in\kappa}\lambda_i$, we know that $Y_i = \{y_\alpha(i) : \alpha < f(i)\}$ cannot be dense in A_i, since $|Y_i| \leq |f(i)| < \lambda_i = \pi A_i$. So pick $a = (a_i)_{i\in\kappa}$ where $a_i \in A_i \setminus \{0\}$ is such that $y_\alpha(i) \not\leq a_i$, for all $\alpha < f(i)$. Since Y is dense in A, pick $\alpha < \lambda$ such that $y_\alpha/D \leq a/D$. It follows that: $y_\alpha(i) \leq a_i$, for D-almost all i; $\alpha \not< f(i)$ for these i, so $f(i) \leq \alpha$; *i.e.* $f(i) \leq \alpha$ for D-almost all i. Thus f is bounded by $\alpha < \lambda$.

CLAIM 5. If $\pi A = \lambda$, then the assertion of the theorem holds.

For Claim 4 says that for every $f \in \prod_{i\in\kappa} \lambda_i$, $f/D = f'/D$ for some $f' : \kappa \longrightarrow \nu$ and some $\nu < \lambda$. By Claim 1, $\prod_{i\in\kappa} \lambda_i/D \leq \sum_{\nu<\kappa} |\nu|^\kappa \leq \lambda$. It now follows from Claim 2 that $\lambda \leq \pi A \leq \prod_{i\in\kappa} \lambda_i/D \leq \lambda$. ∎

2. **The theory T.** We sketch here a first order theory T. Its relevance for solving Problem 12 of [Mo] lies in the fact that the models $\mathfrak{A}$ of T are enlargements $(A, \dots)$ of a Boolean algebra A; the extra structure of $\mathfrak{A}$ allows to easily compute $\pi(A)$—see Remark 2.1. below. Since every ultraproduct $\mathfrak{U} = (U, \dots)$ of models of T is again a model of T, we can then similarly compute $\pi(U)$.

Let T be the first order theory (in an appropriate language) saying that, for every model $\mathfrak{A} = (A, +, \cdot, -, 0, 1, L, \leq_L, \sim, v, x)$ of T, the following hold true.

(a) $(A, +, \cdot, -, 0, 1)$ is a Boolean algebra.

(b) $L \subseteq A$ is totally ordered by $\leq_L$ and has no greatest element. (We do not require any connection between $\leq_L$ and the Boolean partial order of A, except the one stipulated by (e) below.)

(c) v is a map from A to L; for $l \in L$, $A_l = \{a \in A : v(a) <_L l\}$ is a subalgebra of A. (Hence $(A_l)_{l\in L}$ is an increasing sequence of subalgebras of A whose union is A.)

(d) $\sim$ is an equivalence relation on L and its equivalence classes are convex, with respect to $\leq_L$.

(e) x is a map from L into A (we write x_i for $x(i)$) such that $i < l$ implies $x_i \not\leq x_l$. Moreover for $l \in L$, the set $\{x_i : i \sim l\}$ is dense for A_l in the sense that for every $a \in A_l \setminus \{0\}$ there is some $i \sim l$ satisfying $0 < x_i \leq a$. (Hence $\{x_i : i \in L\}$ is a dense subset of A.)

REMARK 2.1. Let $\mathfrak{A} = (A, \dots)$ be a model of T, ρ the cofinality of the linear order $(L, \leq_L)$ and assume that all equivalence classes in L have cardinality at most ρ. Then $\pi(A) = \rho$.

PROOF. To see that $\pi(A) \leq \rho$, fix a cofinal subset M of L of size ρ. The set

$$\{x_i : i \sim m, \text{ for some } m \in M\}$$

has size ρ and is dense in A, by (e). Assume for contradiction that A has a dense subset X of size less than ρ. Without loss of generality, $X \subseteq \{x_i : i \in L\}$; pick $l \in L$ such that $x_i \in X$ implies $i < l$. X being dense in A, there is $x_i \in X$ such that $0 < x_i \leq x_l$. So $i < l$ which is impossible by (e). ∎

In Sections 3 and 4, we will construct "standard" models $\mathfrak{A} = (A, \dots)$ of T which will roughly look like this, for some regular cardinal λ: $|A| = \lambda$, so without loss of generality, $\lambda \subseteq A$; we let $L = \lambda$ and $\leq_L$ its natural well-ordering. A will be generated by a sequence $(x_i)_{i\in\lambda}$; we then let A_l be the subalgebra of A generated by $\{x_i : i < l\}$ and define $v(a)$ to be the least i such that $a \in A_{i+1}$. Finally we will have an infinite cardinal $\mu < \lambda$ and define $i \sim l$ iff $i \leq l < i+\mu$ and $l \leq i < l+\mu$ (ordinal addition); thus the equivalence classes will have size μ. Satisfaction of condition (e) above will be guaranteed by a careful choice of the generators x_i—see Proposition 5.1. In particular, πA will be $\lambda = |A|$.

3. **Valuation functions.** We construct Boolean algebras $A(p)$ from certain functions p, the so-called valuation functions. Later the Boolean algebras $A(P)$, where P will be a generic valuation function, provide the counterexample for Problems 9 and 12 in [Mo] looked for.

We denote the three-element set consisting of the symbols $\geq, \perp, u$ = "undefined" by 3. For any set w with some linear order on it (later w will be a subset of some cardinal λ, hence well-ordered), recall that $[w]^2 = \{(i,j) : i < j \text{ in } w\}$.

Given a Boolean algebra A and a family $(x_i)_{i\in w}$ indexed by w in $A \setminus \{0\}$, we can assign to $(x_i)_{i\in w}$ the function $p\colon [w]^2 \longrightarrow 3$ defined by

$$p(i,j) = \begin{cases} \geq & \text{if } x_i \geq x_j \\ \perp & \text{if } x_i \perp x_j, \textit{ i.e. } x_i \cdot x_j = 0 \\ u & \text{otherwise.} \end{cases}$$

Clearly p has then the following properties:

(1) if $p(i,j) = \,\geq$ and $p(j,k) = \,\geq$ then $p(i,k) = \,\geq$ (where $i < j < k$)

(2) if $i < j < k$ and $\{p(i,j), p(i,k)\} = \{\perp, \geq\}$, then $p(j,k) = \perp$; similarly if $i < j < k$ and $p(i,j) = \perp$, $p(j,k) = \,\geq$, then $p(i,k) = \perp$.

Let us call a function p satisfying (1) and (2) above a *valuation function* and w its domain.

Conversely, given a valuation function $p\colon [w]^2 \longrightarrow 3$, we construct a Boolean algebra $A = A(p)$ from p as follows. Let $\mathrm{Fr}\, w$ be the free Boolean algebra on the set $\{u_i : i \in w\}$ of independent generators and let $N(p)$ be the ideal in $\mathrm{Fr}\, w$ generated by the elementary products $u_j \cdot u_i$ where $p(i,j) = \perp$ resp. $u_j \cdot -u_i$ where $p(i,j) = \,\geq$. Let then $A(p)$ (or A, for short) be the quotient algebra $\mathrm{Fr}\, w/N(p)$ and let $c\colon \mathrm{Fr}\, w \longrightarrow A(p)$ be the canonical homomorphism. Setting $x_i = c(u_i)$, for $i \in w$, we find that the x_i generate A. By the very choice of the ideal $N(p)$, $p(i,j) = \,\geq$ implies that $x_i \geq x_j$ and $p(i,j) = \perp$ implies that $x_i \perp x_j$. To see that no other relations than those imposed by p hold for the x_i, note the following general principle on construction of Boolean algebras via generators with prescribed relations.

REMARK 3.1. Let E be a set of finite partial functions from w to $\{0,1\}$ and let, for $e \in E$, q_e be the elementary product $\prod_{e(i)=1} u_i \cdot \prod_{e(i)=0} -u_i$ in $\mathrm{Fr}\, w$. Assume N is the ideal of $\mathrm{Fr}\, w$ generated by the q_e, $e \in E$. Then for any function $g\colon w \longrightarrow \{0,1\}$, there is an ultrafilter of $\mathrm{Fr}\, w/N$ including $\{x_i : g(i) = 1\} \cup \{-x_i : g(i) = 0\}$ (*i.e.* $\{x_i : g(i) = 1\} \cup \{-x_i : g(i) = 0\}$ has the finite intersection property) iff no $e \in E$ is extended by g.

This gives the following properties of the x_i in $A = A(p)$, where p is a valuation function.

REMARK 3.2. x_i is not in the ideal generated by $\{x_j : j > i\}$. In particular, $x_i \neq 0$, the x_i are pairwise distinct, and $i < j$ implies that $x_i \not\leq x_j$.

To see this, consider the function $g\colon w \longrightarrow \{0,1\}$ such that $g(k) = 1$ iff $k = i$ or ($k < i$ and $p(k,i) = \,\geq$). By Remark 3.1, let s be the ultrafilter of A induced by g. Thus $x_i \in s$ but, for $j > i$, $x_j \notin s$, which shows the claim.

REMARK 3.3. x_i is not in the subalgebra of A generated by $\{x_j : j < i\}$.

For consider the functions g and h from w to $\{0, 1\}$ where g is defined as in the proof of Remark 3.2, $h(k) = g(k)$ for $k \neq i$, but $h(i) = 0$. Let s and t be the corresponding ultrafilters of A, ϕ and ψ the homomorphisms from A to the two-element algebra corresponding to s and t. Now ϕ and ψ coincide on x_j for all $j < i$, but not on x_i.

4. **The partial order of valuation functions.** For the next sections, fix infinite cardinals λ and μ such that $\mu^{<\mu} = \mu$, $\mu < \lambda$, and λ is regular. We shall choose λ and μ somewhat more carefully in Section 7. Let $\mathbb{P}(\lambda, \mu)$ (or $\mathbb{P}$, for short) be the notion of forcing

$$\mathbb{P} = \{p : p \text{ is a valuation function and } \operatorname{dom} p \subseteq \lambda \text{ has size less than } \mu\}$$

ordered by reverse inclusion.

REMARK 4.1. $\mathbb{P}$ is μ-closed.

We now build up some machinery for constructing elements of $\mathbb{P}$ with prescribed properties. Given a set r of relations of the form $x_i \geq x_j$, $x_i \perp x_j$ (where $i, j \in \lambda$; the relations may be thought of as being formulas in some formal language in the variables x_i, $i \in \lambda$), we define when a relation ρ can be derived from r and we write $r \vdash \rho$: if ρ has the form $x_k \geq x_l$, $r \vdash \rho$ iff there are $i_1, \ldots, i_m \in \lambda$ such that the relations $x_k \geq x_{i_1}, x_{i_1} \geq x_{i_2}, \ldots, x_{i_m} \geq x_l$ are all in r (in particular, $r \vdash x_i \geq x_i$); if ρ has the form $x_k \perp x_l$, $r \vdash \rho$ iff there are $\alpha, \beta \in \lambda$ such that $x_\alpha \perp x_\beta$ is in r and $r \vdash x_\alpha \geq x_k$, $r \vdash x_\beta \geq x_l$.

Call r *consistent* if no relation of the form $x_j \geq x_i$ where $i < j$ and no relation of the form $x_k \perp x_k$ is derivable from r. Given $p \in \mathbb{P}$, define $\operatorname{rel} p$, the relevant part of p, by

$$\operatorname{rel} p = \{x_i \geq x_j : p(i,j) = \geq\} \cup \{x_i \perp x_j : p(i,j) = \perp\}.$$

PROPOSITION 4.2. *If $|r| < \mu$, then r is consistent iff $r \subseteq \operatorname{rel} p$ for some $p \in \mathbb{P}$.*

PROOF. Assume first that $p \in \mathbb{P}$ and $r \subseteq \operatorname{rel} p$ where $\operatorname{dom} p = w \subseteq \lambda$. Then in the Boolean algebra $A(p)$ constructed in Section 3, all relations in r and hence all relations derivable from r are satisfied by the canonical generators $\{x_i : i \in w\}$; moreover, these generators are non-zero. Hence no relation $x_k \perp x_k$ and no relation of the form $x_j \geq x_i$, $i < j$, can be derived from r.

Conversely, if r is consistent, let w be any subset of λ such that $|w| < \mu$ and $\{i \in \lambda : x_i$ occurs in $r\} \subseteq w$. Define p: $[w]^2 \longrightarrow 3$ by

$$p(i,j) = \begin{cases} \geq & \text{iff } r \vdash x_i \geq x_j \\ \perp & \text{iff } r \vdash x_i \perp x_j \\ u & \text{otherwise.} \end{cases}$$

p is a well-defined function (*i.e.* r does not derive both $x_i \geq x_j$ and $x_i \perp x_j$, for $i < j \in w$) since otherwise, $r \vdash x_j \perp x_j$, contradicting the consistency of r. By the above definition of derivability from r, p is a valuation function. ∎

For further reference, call $p \in \mathbb{P}$ defined from a consistent set r and $w \subseteq \lambda$ as in the proof above the *canonical extension* of r over w.

We give one trivial and one not-so-trivial application of this machinery. If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over our universe V of set theory, then clearly $P_G = \bigcup G$ is a valuation function with $\operatorname{dom} P_G = \bigcup_{p \in G} \operatorname{dom} p$.

REMARK 4.3. If G is generic, then $\operatorname{dom} P_G = \lambda$.

To see this, we have to make sure that, for $i \in \lambda$, the set $D_i = \{p \in \mathbb{P} : i \in \operatorname{dom} p\}$ is dense in $\mathbb{P}$. But given $q \in \mathbb{P}$, let $w \subseteq \lambda$ be such that $|w| < \mu$ and $\operatorname{dom} q \cup \{i\} \subseteq w$. Now by Proposition 4.2, $\operatorname{rel} q$ is consistent; let p be the canonical extension of $\operatorname{rel} q$ over w. Then $p \in D_i$ and $q \subseteq p$.

PROPOSITION 4.4. *If $p, q \in \mathbb{P}$ coincide on $\operatorname{dom} p \cap \operatorname{dom} q$, then they are compatible in $\mathbb{P}$.*

PROOF. This follows from a number of claims. We write $p \vdash \cdots$ instead of $\operatorname{rel} p \vdash \cdots$ and we say that a relation, *e.g.* $x_i \geq x_j$, is in p if $p(i,j) = \geq$ *etc.*

CLAIM 1. If $p \vdash x_i \geq x_j$ where $i < j$, then $i, j \in \operatorname{dom} p$ and the relation $x_i \geq x_j$ is in p. Similarly for q and for relations of the form $x_i \perp x_j$.—The claim holds because $\operatorname{rel} p$, for $p \in \mathbb{P}$, is closed under derivations.

By Proposition 4.2 we are left with showing that the set

$$r = \operatorname{rel} p \cup \operatorname{rel} q$$

is consistent.

CLAIM 2. If $r \vdash x_i \geq x_j$, then $p \vdash x_i \geq x_j$ or $q \vdash x_i \geq x_j$ or, for some α, ($p \vdash x_i \geq x_\alpha$ and $q \vdash x_\alpha \geq x_j$) or, for some α, ($q \vdash x_i \geq x_\alpha$ and $p \vdash x_\alpha \geq x_j$).

CLAIM 3. If $r \vdash x_i \perp x_j$, then $p \vdash x_i \perp x_j$ or $q \vdash x_i \perp x_j$ or, for some α, ($p \vdash x_i \perp x_\alpha$ and $q \vdash x_\alpha \geq x_j$) or, for some α, ($q \vdash x_i \perp x_\alpha$ and $p \vdash x_\alpha \geq x_j$) (or similarly with i interchanged with j).

CLAIM 4. If $r \vdash x_i \geq x_j$ and $i, j \in \operatorname{dom} p$, then $p \vdash x_i \geq x_j$. Similarly for q and for relations of the form $x_i \perp x_j$.

The proofs are easy but require consideration of a number of cases. We give two typical examples. In Claim 3, assume *e.g.* that $p \vdash x_\gamma \perp x_\delta$, $q \vdash x_\gamma \geq x_i$ and $q \vdash x_\delta \geq x_j$. Then γ and δ are in $\operatorname{dom} p \cap \operatorname{dom} q$, $x_\gamma \perp x_\delta$ is (by Claim 1) in p, hence in q, because p and q coincide on $\operatorname{dom} p \cap \operatorname{dom} q$, and $q \vdash x_i \perp x_j$.

Similarly in Claim 4, assume *e.g.* that $p \vdash x_i \geq x_\alpha$ and $q \vdash x_\alpha \geq x_j$ where $i, j \in \operatorname{dom} p$. Since α is in $\operatorname{dom} p \cap \operatorname{dom} q$, it follows that $x_\alpha \geq x_j$ is in p, hence $p \vdash x_i \geq x_j$.

CLAIM 5. r is consistent.—For otherwise by Claim 3, we may assume that, *e.g.*, for some α, $p \vdash x_k \perp x_\alpha$ and $q \vdash x_\alpha \geq x_k$. Then k and α are in $\operatorname{dom} p \cap \operatorname{dom} q$, $x_\alpha \geq x_k$ is in q and $x_k \perp x_k$ is in p, a contradiction. ∎

PROPOSITION 4.5. *$\mathbb{P}$ satisfies the μ^+-chain condition.*

PROOF. If X is a subset of $\mathbb{P}$ of size μ^+, then by $\mu^{<\mu} = \mu$ and the Δ-lemma there are p and q in X coinciding on $\operatorname{dom} p \cap \operatorname{dom} q$. So we are finished by Proposition 4.4. ∎

5. **Computing $\pi\big(A(P)\big)$.** In this and the following section, let G be a $\mathbb{P}$-generic filter over V and P the resulting generic valuation function (see Remark 4.3). Write A for $A(P)$. We prove condition (e) of Section 2 for A, thus being able to compute $\pi(A)$ in $V[G]$.

PROPOSITION 5.1. *The following holds in $V[G]$. Let $\alpha < \lambda$ be an ordinal, $a \subseteq \alpha$ finite, $e\colon a \longrightarrow \{0,1\}$ and*

$$y = \prod_{e(i)=1} x_i \cdot \prod_{e(i)=0} -x_i > 0 \quad (in\ A).$$

Then there is $i^* \in [\alpha, \alpha+\mu)$ (ordinal addition) such that $x_{i^*} \leq y$. - In particular, the set $\{x_{i^*} : i^* \in [\alpha,\alpha+\mu)\}$ is dense for the subalgebra of A generated by $\{x_i : i < \alpha\}$.

PROOF. We do not distinguish notationally between elements of $V[G]$ and their $\mathbb{P}$-names; in particular since a and e, being finite, are in the ground model. Pick $p \in G$ such that

$$p \Vdash y = \prod_{e(i)=1} x_i \cdot \prod_{e(i)=0} -x_i > 0;$$

it suffices to prove that

$$D = \{t \in \mathbb{P} : t \leq p, \text{ and } t \Vdash x_{i^*} \leq y \text{ for some } i^* \in [\alpha, \alpha+\mu)\}$$

is dense below p. To this end, let $q \leq p$ be arbitrary. By Remark 4.3, we can fix $r \leq q$ such that $a \subseteq \operatorname{dom} r$. Then fix $i^* \in [\alpha, \alpha+\mu) \setminus \operatorname{dom} r$; this is possible by $|\operatorname{dom} r| < \mu$. We define a function s with domain $a \cup \{i^*\}$ by putting

$$s\restriction[a]^2 = r\restriction[a]^2$$
$$s(i,i^*) = \begin{cases} \geq & \text{if } i \in a \text{ and } e(i) = 1 \\ \perp & \text{if } i \in a \text{ and } e(i) = 0. \end{cases}$$

CLAIM. $s \in \mathbb{P}$, *i.e.* s is a valuation function.

Let us check just one case. Note that, for $u \in \mathbb{P}$, $u(i,j) = \geq$ implies that $u \Vdash x_i \geq x_j$ and similarly for $\perp$ instead of $\geq$ since for any generic $H \subseteq \mathbb{P}$ containing u, $u \subseteq P_H$ and thus $x_i \geq x_j$ will hold in $A(P_H)$. Assume *e.g.* $i < j$ in a, $s(i,j) = \geq$ and $s(j,i^*) = \geq$; we have to show that $s(i,i^*) = \geq$. The assumptions say that $r(i,j) = \geq$ (since $i,j \in a$) and $e(j) = 1$; we have to show that $e(i) = 1$. But if $e(i) = 0$, then: $p \Vdash 0 \neq -x_i \cdot x_j$ (because $p \Vdash 0 < y \leq -x_i \cdot x_j$), $r \Vdash 0 \neq -x_i \cdot x_j$ (since $r \leq p$), $r \Vdash x_i \geq x_j$ (by the above assumption), $r \Vdash -x_i \cdot x_j = 0$, a contradiction. Now r and s coincide on $a = \operatorname{dom} r \cap \operatorname{dom} s$, so by Proposition 4.4, pick $t \in \mathbb{P}$ extending both r and s. Then $t \leq q$ and $s \Vdash x_{i^*} \leq y$, by the very definition of s above, so $t \in D$. ∎

COROLLARY 5.2. *$\pi(A) = \lambda$ (in $V[G]$).*

PROOF. This follows from Remark 2.1 and the sketch of the model $\mathfrak{A} = (A, \dots) \models T$ following it, plus Proposition 5.1. Let us remark that Theorem 6.1 gives another proof, since $\mathrm{d}A = \lambda$, $\mathrm{d}A \leq \pi A$ holds for all Boolean algebras and $\pi A \leq |A| = \lambda$. ■

EXAMPLE 5.3. Our construction of $A = A(P)$ and Proposition 5.1 above give a counterexample to the assertion in Theorem 4.1 of [Mo], in $V[G]$. For this, let A_α be the subalgebra of A generated by $\{x_i : i < \alpha\}$; so if $\alpha \in I = \{\alpha < \lambda : \mathrm{cf}\,\alpha = \mu\}$, then by Remark 2.1 and Proposition 5.1 above, we have $\pi A_\alpha = \mu$. Moreover $A = \bigcup_{\alpha \in I} A_\alpha$ and $\pi A = \lambda$ where λ can be larger than μ^+.—In fact, the argument given in [Mo, 4.1] depends on the assumption that the chain $(A_\alpha)_{\alpha \in I}$ is continuous which is not the case here.

6. **Computing** $\mathrm{d}\big(A(P)\big)$.

Our single theorem here is the following.

THEOREM 6.1. *In $V[G]$, $A = A(P)$ satisfies $\mathrm{d}(A) = \lambda$.*

PROOF. Otherwise, the cardinal $\sigma = \mathrm{d}(A)^{V[G]}$ is less than λ. There are a $\mathbb{P}$-name u and a condition $p \in \mathbb{P}$ (in fact, $p \in G$) such that

$$p \Vdash u \text{ is a sequence } (u_\nu)_{\nu<\sigma}, \text{ each } u_\nu \text{ is an ultrafilter of } A, \text{ and } A \setminus \{0\} = \bigcup_{\nu<\sigma} u_\nu.$$

For $\alpha < \lambda$, fix $p_\alpha \in \mathbb{P}$ and $\nu_\alpha < \sigma$ such that $p_\alpha \leq p$ and

$$p_\alpha \Vdash x_\alpha \in u_{\nu_\alpha}$$

(x_α the (name of the) α-th generator of A). In the next steps, we construct stationary subsets $S_1 \supseteq S_2 \supseteq S_3 \supseteq S_4$ of λ.

STEP 1. $S_1 = \{\alpha \in \lambda : \mathrm{cf}\,\alpha = \mu\}$ is stationary in λ because $\mu < \lambda$ and λ is regular.

STEP 2. Since $\sigma < \lambda = \mathrm{cf}\,\lambda$, there are $\nu^* < \sigma$ and a stationary $S_2 \subseteq S_1$ such that $\nu_\alpha = \nu^*$, for all $\alpha \in S_2$.

STEP 3. Write $w_\alpha = \mathrm{dom}\, p_\alpha$, for $\alpha \in \lambda$. We find $\alpha^* \in \lambda$ and a stationary $S_3 \subseteq S_2$ such that for all $\alpha \in S_3$, $\alpha^* < \alpha$ and $w_\alpha \cap \alpha \subseteq \alpha^*$ hold. To this end, note that $\mathrm{cf}\,\alpha = \mu$ for $\alpha \in S_2$ and $|w_\alpha \cap \alpha| < \mu$; so pick $j_\alpha < \alpha$ satisfying $w_\alpha \cap \alpha \subseteq j_\alpha$. Apply Fodor's theorem to obtain S_3.

STEP 4. We find a stationary set $S_4 \subseteq S_3$ such that $\alpha < \beta$ in S_4 implies $w_\alpha \subseteq \beta$. To do this, define by induction $f\colon \lambda \longrightarrow \lambda$ strictly increasing and continuous such that, for all α, $\bigcup_{\nu<\alpha} w_\nu \subseteq f(\alpha)$ and let $S_4 = S_3 \cap C$ where $C = \{\alpha : f(\alpha) = \alpha\}$ is closed unbounded. Then S_4 is stationary and, for $\alpha < \beta$ in S_4, we have $w_\alpha \subseteq f(\beta) = \beta$.

Now $\mu^+ \leq \lambda$ and $\mathbb{P}$ satisfies the μ^+-chain condition. So we can find $\alpha < \beta$ in S_4 such that p_α and p_β are compatible in $\mathbb{P}$. Let r be the following set of relations:

$$r = \mathrm{rel}(p_\alpha) \cup \mathrm{rel}(p_\beta) \cup \{x_\beta \perp x_\alpha\}$$

(see the machinery in Section 4).

CLAIM. r is consistent.

By the claim and Proposition 4.2, pick then $q \in \mathbb{P}$ such that $r \subseteq \mathrm{rel}(q)$. This q will force the following statements:

$x_\beta \perp x_\alpha$

$x_\alpha \in u_{\nu_\alpha} = u_{\nu^*}$ and $x_\beta \in u_{\nu_\beta} = u_{\nu^*}$

u_{ν^*} has the finite intersection property (being an ultrafilter),

and this contradiction finishes the proof.

PROOF OF THE CLAIM. Clearly no relation $x_i \geq x_j$ where $j < i$ can have a derivation from r, since such a derivation would not use the relation $x_\beta \perp x_\alpha$; hence $x_i \geq x_j$ would be derivable from $\mathrm{rel}(p_\alpha) \cup \mathrm{rel}(p_\beta)$, contradicting the compatibility of p_α and p_β.

Now assume $r \vdash x_k \perp x_k$, for some $k \in \lambda$. A derivation witnessing this starts, without loss of generality, with the relation $x_\beta \perp x_\alpha$. So in $p_\alpha \cup p_\beta$ there are relations

$$x_{i_0} \geq x_{i_1}, \ldots, x_{i_{r-1}} \geq x_{i_r} \quad \text{where } i_0 = \alpha,\ i_r = k$$
$$x_{j_0} \geq x_{j_1}, \ldots, x_{j_{s-1}} \geq x_{j_s} \quad \text{where } j_0 = \beta,\ j_s = k.$$

Note that $\alpha = i_0 < i_1 < \cdots < i_r = k$ (since if $x_j \geq x_i$ is in $p_\alpha \cup p_\beta$, then $j < i$); similarly, $\beta = j_0 < j_1 < \cdots < j_s = k$.

We prove by induction on $t \in \{0, \ldots, r\}$ that $i_t \notin w_\beta = \mathrm{dom}\, p_\beta$; for $t = r$ this gives a contradiction because then $k = i_r \notin w_\beta$, so $k \in w_\alpha$ and $k \geq \beta$, but $w_\alpha \subseteq \beta$. First, $i_0 \notin w_\beta$: otherwise, by Step 3, $i_0 = \alpha \in w_\beta \cap \beta \subseteq \alpha^*$, contradicting $\alpha^* < \alpha$ for $\alpha \in S_3$. If $i_t \notin w_\beta$ but $i_{t+1} \in w_\beta$, then the relation $x_{i_t} \geq x_{i_{t+1}}$ must be in p_α. But then $i_{t+1} \in w_\alpha \subseteq \beta$ and again $i_{t+1} \in w_\beta \cap \beta \subseteq \alpha^* < \alpha$, a contradiction. ∎

7. **Proof of Theorem A.**

7.1 *Proof of Theorem A.* Fix $\kappa, \mu, \lambda_\alpha$ and D as given in the theorem; $\mathbb{R}$ will be the iteration of two forcing notions. In the first step, collapse $\mu^{<\mu}$ to μ with $\mathbb{Q} = Fn(\mu, \mu^{<\mu}, < \mu)$ in Kunen's notation ([Ku]). This forcing is μ-closed and satisfies the $(\mu^{<\mu})^+$-chain condition; in the resulting generic model $V[H]$, $\mu^{<\mu} = \mu$ holds. The notions of ultrafilters on κ, the cartesian product $\prod_{\alpha<\kappa} \lambda_\alpha$ *etc.* are absolute for this forcing by μ-closedness of $\mathbb{Q}$ and $\kappa < \mu$; thus all assumptions of the theorem continue to hold in $V[H]$.

Working now in $V[H]$, let, for $\alpha \in \kappa$, $\mathbb{P}_\alpha$ be the forcing notion $\mathbb{P}(\lambda_\alpha, \mu)$ defined in Section 4; let $\mathbb{P}$ be the full cartesian product $\mathbb{P} = \prod_{\alpha<\kappa} \mathbb{P}_\alpha$ with the coordinate-wise partial order. For $G \subseteq \mathbb{P}$ generic over V, $G_\alpha = {\mathrm{pr}_\alpha}^{-1}[G]$ is $\mathbb{P}_\alpha$-generic over $V[H]$ (pr_α the α-th projection). $\mathbb{P}$ is clearly μ-closed, moreover, as in the proof of Proposition 4.5, the Δ-lemma implies that $\mathbb{P}$ satisfies the μ^+-chain condition since $\mu^{<\mu} = \mu$. Thus the assumptions of the theorem, as well as $\mu^{<\mu} = \mu$, continue to hold in $V[H][G]$.

In $V[H][G]$, $P_\alpha = \bigcup G_\alpha : [\lambda_\alpha]^2 \longrightarrow 3$ is a generic valuation function. Let $A_\alpha = A(P_\alpha)$ be its associated Boolean algebra; by Sections 5 and 6, $\pi(A_\alpha) = \mathrm{d}(A_\alpha) = \lambda_\alpha$. In the standard model $\mathfrak{A}_\alpha = (A_\alpha, \ldots)$ of T (see Section 2), the predicate L is interpreted by λ_α and the equivalence classes of $\sim_L$ have size μ. So in the ultraproduct $\mathfrak{A} = \prod_{\alpha<\kappa} \mathfrak{A}_\alpha / D$, L is interpreted by $\prod_{\alpha<\kappa}\lambda_\alpha/D$ and the equivalence classes of $\sim_L$ have size $\leq |\mu^\kappa/D| = \mu$ (by $\kappa < \mu$ and $\mu^{<\mu} = \mu$). Now Remark 2.1 says that $\pi(A) = \mathrm{cf}\prod_{\alpha<\kappa}\lambda_\alpha/D$ and hence $\mathrm{d}(A) \leq \pi(A) = \mathrm{cf}(\prod_{\alpha<\kappa}\lambda_\alpha/D) < |\prod_{\alpha<\kappa}\lambda_\alpha/D| = |\prod_{\alpha<\kappa}\pi(A_\alpha)/D| = |\prod_{\alpha<\kappa}\mathrm{d}(A_\alpha)/D|$. ∎

We can prove a little more:

REMARK 7.2. In $V[H][G]$, let $A = \prod_{\alpha<\kappa}A_\alpha/D$ be the algebra constructed in subsection 7.1 and let $\lambda = \mathrm{cf}\prod_{\alpha<\kappa}\lambda_\alpha/D$. Then $\mathrm{d}(A) = \lambda$.

PROOF. Our proof will closely follow that of Theorem 6.1.

Fix a sequence $(f_\gamma)_{\gamma\in\lambda}$ in $\prod_{\alpha<\kappa}\lambda_\alpha$ such that $(f_\gamma/D)_{\gamma\in\lambda}$ is strictly increasing and cofinal in the ultraproduct $\prod_{\alpha<\kappa}\lambda_\alpha/D$. By [Sh, Chapter II], the set

$$S = \Big\{\gamma\in\lambda : \mathrm{cf}\,\gamma = \mu^+, \text{ and there is } g\in\prod_{\alpha<\kappa}\lambda_\alpha \text{ such that } g/D \text{ is the least upper bound of } \{f_\delta/D : \delta<\gamma\} \text{ and } \mathrm{cf}\, g(\alpha) = \mu^+ \text{ for all } \alpha\in\kappa\Big\}$$

is stationary; so we may assume that, for $\gamma\in S$, f_γ satisfies the requirements for g above.

Now note that, in $V[H][G]$, $\mathrm{d}A \leq \pi A = \lambda$ as shown in the proof of subsection 7.1; so assume for contradiction that $\mathrm{d}A < \lambda$. Thus , in $V[H][G]$,there are a $\mathbb{P}$-name u, $\sigma<\lambda$ and $p\in\mathbb{P}$ such that

$$p \Vdash u = (u_\nu)_{\nu<\sigma} \text{ is a sequence of ultrafilters of } A \text{ covering } A\setminus\{0\}.$$

For $\gamma\in S$, fix $p_\gamma \geq p$ and $\nu_\gamma\in\sigma$ such that

$$p_\gamma \Vdash y_\gamma/D \in u_{\nu_\gamma}$$

where y_γ is (a $\mathbb{P}$-name for) $(x_{f_\gamma(\alpha)})_{\alpha<\kappa}/D$ and x_i is (a $\mathbb{P}$-name for) the i-th canonical generator of A_α, for $i<\lambda_\alpha$. There is a stationary subset S_1 of S such that ν_γ is some fixed ν^*, for $\gamma\in S_1$ (because $\nu_\gamma<\sigma<\lambda$ and λ is regular). As in Step 3 in the proof of Theorem 6.1, there exists , for $\gamma\in S_1$, some $\beta_\gamma<\gamma$ such that, for D-almost all α,

$$\mathrm{dom}\, p_\gamma(\alpha)\cap f_\gamma(\alpha) \subseteq f_{\beta_\gamma}(\alpha).$$

Without loss of generality (*i.e.* by passing to a stationary subset), β_γ is some fixed β^*, for all $\gamma\in S_1$. Now $K_\gamma = \{\alpha\in\kappa : \mathrm{dom}\,p_\gamma(\alpha)\cap f_\gamma(\alpha)\subseteq f_{\beta^*}(\alpha)\}\in D$, for $\gamma\in S_1$; since $2^\kappa<\lambda$, we may assume without loss of generality that K_γ is some fixed $K^*\in D$, for $\gamma\in S_1$.

As in Step 4 of the proof of Theorem 6.1, we may assume that $\gamma<\delta$ in S_1 implies that

$$K_{\gamma\delta} = \{\alpha\in\kappa : \mathrm{dom}\,p_\gamma(\alpha)\subseteq f_\delta(\alpha)\}\in D$$

because $(f_\delta/D)_{\delta\in\lambda}$ is cofinal in $\prod_{\alpha<\kappa}\lambda_\alpha/D$.

Now $\mathbb{P}$ satisfies the μ^+-chain condition and S_1 has size $\lambda\geq\mu^+$; so fix $\gamma<\delta$ in S_1 such that p_γ and p_δ are compatible in $\mathbb{P} = \prod_{\alpha\in\kappa}\mathbb{P}_\alpha$, *i.e.* $p_\gamma(\alpha)$ and $p_\delta(\alpha)$ are compatible in $\mathbb{P}_\alpha$, for all $\alpha\in\kappa$.

We conclude as in Theorem 6.1: for all $\alpha\in K^*\cap K_{\gamma\delta}$, the set

$$r_\alpha = \mathrm{rel}\,p_\gamma(\alpha)\cup\mathrm{rel}\,p_\delta(\alpha)\cup\{x_{f_\delta(\alpha)}\perp x_{f_\gamma(\alpha)}\}$$

is consistent; so pick $q_\alpha\in\mathbb{P}_\alpha$ satisfying $r_\alpha\subseteq\mathrm{rel}\,q_\alpha$. Choose $q\in\mathbb{P}$ having α-th coordinate q_α, for $\alpha\in K^*\cap K_{\gamma\delta}$; then q forces that: $y_\delta/D\perp y_\gamma/D$, $y_\gamma/D\in u_{\nu_\gamma} = u_{\nu^*}$ and $y_\delta/D\in u_{\nu_\delta} = u_{\nu^*}$, u_{ν^*} is an ultrafilter. This gives a contradiction. ∎

8. **Proof of Theorem B.** To abbreviate the main body of the proof, we state in advance two easy lemmas. The proofs are left to the reader.

LEMMA 8.1. *Assume $h: C \longrightarrow D$ is a homomorphism of Boolean algebras, $\{c_n : n \in \omega\}$ is a partition of unity in C, and also $\{h(c_n) : n \in \omega\}$ is a partition of unity in D. Then, if $x_n \in C$ are such that $\sum^C_{n\in\omega} x_n \cdot c_n$ exists, we have $h(\sum^C_{n\in\omega} x_n \cdot c_n) = \sum^D_{n\in\omega} h(x_n \cdot c_n)$.*

Given a subalgebra C of D and $x \in D$, let $I_C(x) = \{c \in C : c\cdot x = 0\}$, an ideal of C. Call $x, y \in D$ equivalent over C (and write $x \sim_C y$) if both $I_C(x) = I_C(y)$ and $I_C(-x) = I_C(-y)$ hold, *i.e.* if x and y realize the same quantifier-free type over C.

LEMMA 8.2. *If $x, y \in D$ are equivalent over C, then there is no $c \in C \setminus \{0\}$ disjoint from $x + -y$.*

We break up the proof of Theorem B into eight preparatory steps in which certain objects are constructed or notation is fixed, plus four claims. Let $C \leq D$ denote that C is a subalgebra of D; $\bar{A}$ is the completion of A.

STEP 1. Take μ as assumed in the theorem, fix a set U of cardinality μ, and let $A = \operatorname{Fr} U$, the free Boolean algebra over U. Since $|\bar{A}| = \mu^\omega \geq \mu^+ = 2^\mu$, we have $|\bar{A}| = \mu^+$. The algebra B promised in the theorem will be a subalgebra of $\bar{A}$, generated by A and pairwise distinct elements b_i of $\bar{A}$, $i < \mu^+$. So $|B| = \mu^+$ and we know in advance that $\mu^+ \leq |\operatorname{End} B|$ and $|\operatorname{Id} B| \leq 2^{\mu^+}$.

STEP 2. Fix an enumeration $\{g_j : j < \mu^+\}$ of all homomorphisms from A into $\bar{A}$. This is possible since $|A| = \mu$ and $|\bar{A}| = \mu^+ = (\mu^+)^\mu$.

STEP 3. Fix a sequence $(\mu_n)_{n\in\omega}$ of cardinals such that $\mu = sup_{n\in\omega}\mu_n$ and $2^{\mu_n} < \mu_{n+1}$.

STEP 4. For each ordinal $i < \mu^+$, fix subsets S_{in} of i such that $i = \bigcup_{n\in\omega} S_{in}$, $S_{in} \subseteq S_{i,n+1}$ and $|S_{in}| \leq \mu_n$. This is possible since $|i| \leq \mu$.

STEP 5. Fix a sequence $(A_n)_{n\in\omega}$ of subalgebras of A such that $A = \bigcup_{n\in\omega} A_n$, $A_n \subseteq A_{n+1}$ and $|A_n| \leq \mu_n$.

STEP 6. Define a tree $T = \bigcup_{n\in\omega} T_n$ with n'th level $T_n = \mu_0 \times \cdots \times \mu_{n-1}$ where $t \leq s$ in T means that s extends t; so $|T| = \mu$. The cartesian product $F = \prod_{n\in\omega} \mu_n$ has size $\mu^\omega = \mu^+$; fix a one-one enumeration $\{f_i : i < \mu^+\}$ of F.

Split $U \subseteq A = \operatorname{Fr} U$ (*cf.* Step 1) into two disjoint subsets X and Z such that $|X| = |Z| = \mu$; then split both X and Z into pairwise disjoint subsets X_t, $t \in T$, and Z_t, $t \in T$, such that $|X_t| = \mu$ and $Z_t \neq \emptyset$.

STEP 7. Here we define, for $i < \mu^+$, the elements b_i of $\bar{A}$ and then let B be the subalgebra of $\bar{A}$ generated by $A \cup \{b_i : i < \mu^+\}$. b_i is constructed out of certain elements x_{in}, y_{in}, z_{in}, $n \in \omega$, of U by putting

$$s_{in} = x_{in} + -y_{in}$$
$$d_{i,n} = s_{i,n} \cdot \prod_{m<n} -s_{im}$$
$$b_i = \sum_{n\in\omega} z_{in} \cdot d_{in}.$$

To choose the x_{in}, y_{in}, z_{in}, fix $i < \mu^+$ and $n \in \omega$; thus

$$t = f_i \restriction n$$

is an element of the tree T. Pick $z_{in} \in Z_t$ (see Step 6) arbitrarily. x_{in} and y_{in} are chosen much more carefully: we want them to be distinct elements of X_t satisfying

$$(*) \qquad \text{for all } j \in S_{in},\ g_j(x_{in}) \sim_{A_n} g_j(y_{in})$$

(*cf.* Steps 4, 2, 5, and the definition of $\sim_{A_n}$ before Lemma 8.2). This is possible since:

$|A_n| \leq \mu_n$

there are at most 2^{μ_n} equivalence classes in $\bar{A}$, with respect to $\sim_{A_n}$, since there are at most 2^{μ_n} ideals in A_n

$|S_{in}| \leq \mu_n$

the set $\{(g_j(x)/\sim_{A_n})_{j \in S_{in}} : x \in X_t\}$ has size at most 2^{μ_n}

$2^{\mu_n} < \mu = |X_t|$.

STEP 8 (REMARK). For $b \in A$, let us denote by $\operatorname{supp} b$ (the support of b) the smallest subset of U generating b. Now for $i < \mu^+$, the supports $\{\operatorname{supp} s_{in} : n \in \omega\}$ are pairwise disjoint and thus $\sum^{\bar{A}} s_{in} = 1$. It follows that the pairwise disjoint set $\{d_{in} : n \in \omega\}$ is a partition of unity in $\bar{A}$ and all d_{in} are non-zero.—Similarly, for any homomorphism $g: A \longrightarrow \bar{A}$, the sets $\{g(d_{in}) : n \in \omega\}$ and $\{g(s_{in}) : n \in \omega\}$ have the same upper bounds in A resp. $\bar{A}$.

CLAIM 1. If $j < i < \mu^+$, then $\{g_j(d_{in}) : n \in \omega\}$ is a partition of unity (in $\bar{A}$).—Otherwise, assume $a \in A^+$ and $a \cdot g_j(s_{in}) = 0$ for all n (*cf.* Step 8). Pick n so large that $a \in A_n$ and $j \in S_{in}$. Then $a \cdot g_j(x_{in} + -y_{in}) = 0$, so $a \cdot \big(g_j(x_{in}) + -g_j(y_{in})\big) = 0$, contradicting $(*)$ and Lemma 8.2.

CLAIM 2. Let g be an endomorphism of B, say $g \restriction A = g_j$ (see Step 2). Then for all $i > j$, $g(b_i) = \sum^{\bar{A}} g_j(z_{in}) \cdot g_j(d_{in})$ holds. Hence g is uniquely determined by its action on $A \cup \{b_i : i \leq j\}$.—This follows from Claim 1 and Lemma 8.1.

CLAIM 3. $|\operatorname{End} B| \leq \mu^+$.—To completely describe some $g \in \operatorname{End} B$, we have only μ^+ choices for $g \restriction A$ (Step 2) and, for $j < \mu^+$, at most $(\mu^+)^{|j|} \leq 2^\mu = \mu^+$ choices for $\big(g(b_i)\big)_{i \leq j}$, so we are finished by Claim 2.

CLAIM 4. The generators $\{b_i : i < \mu^+\}$ are ideal-independent; hence $|\operatorname{Id} B| = 2^{\mu^+}$.—We prove that, for $i \in \mu^+$ and J a finite subset of $\mu^+ \setminus \{i\}$, $b_i \not\leq \sum_{j \in J} b_j$. (It follows that the ideals I_K generated by $\{b_i : i \in K\}$ for $K \subseteq \mu^+$, are all distinct, so B has 2^{μ^+} ideals.) The argument is elementary but a little tedious and we give it in some detail. Assume for contradiction that $b_i \leq \sum_{j \in J} b_j$.

For arbitrary $n \in \omega$, we have the following situation. d_{in} is non-zero and for $j \in J$, $\{d_{jm} : m \in \omega\}$ is a partition of unity; hence there are elements $m(j) \in \omega$, for $j \in J$, such that $p = d_{in} \cdot \prod_{j \in J} d_{jm(j)}$ is non-zero. Now $b_i \cdot d_{in} \leq z_{in}$ and thus $b_i \cdot p \leq z_{in}$; similarly $b_j \cdot p \leq z_{jm(j)}$ holds for $j \in J$. It follows from $b_i \leq \sum_{j \in J} b_j$ that $z_{in} \cdot p \leq b_i \cdot p \leq \sum_{j \in J} z_{jm(j)}$.

But supp $p \subseteq X$ and z_{in}, $z_{jm(j)}$ are in Z; hence $z_{in} \leq \sum_{j \in J} z_{jm(j)}$. So $z_{in} = z_{jm(j)}$, for some $j \in J$, since $Z \subseteq U$ is independent. Since z_{in} was chosen in Step 7 from Z_t, where $t = f_i \restriction n$, and $(Z_t)_{t \in T}$ was a disjoint family, it follows that $n = m(j)$ and $f_i \restriction n = f_j \restriction n$.

We have thus shown that for every $n \in \omega$, there is some $j \in J$ satisfying $f_i \restriction n = f_j \restriction n$. But then $f_i \in \{f_j : j \in J\}$ and $i \in J$ (since the enumeration $\{f_i : i < \mu^+\}$ in Step 6 was one-one), a contradiction. ∎

References

[ChK] C. C. Chang and H. J. Keisler, *Model Theory, 3rd edition*, North Holland, 1990.

[Ho] R. Hodel, *Cardinal functions I*. In: Handbook of set-theoretic topology, (eds. K. Kunen and J. E. Vaughan), North Holland, Amsterdam, 1984.

[J] T. Jech, *Set Theory*, Academic Press, 1978.

[Ju1] I. Juhasz, *Cardinal functions in topology—ten years later*, Math. Center Tracts **123**, 1980.

[Ju2] ———, *Cardinal functions II*. In: Handbook of set-theoretic topology, (eds. K. Kunen and J. E. Vaughan), North Holland, Amsterdam, 1984.

[Ko] S. Koppelberg, *General Theory of Boolean algebras. Handbook of Boolean algebras, Part I*, North Holland, 1989.

[Ku] K. Kunen, *Set Theory*, North Holland, 1980.

[Ma] M. Magidor, *On the singular cardinals problem*, Israel J. Math. **28**(1977), 517–547.

[Mo] J. D. Monk, *Cardinal functions on Boolean algebras*, Birkhäuser, 1990.

[RoSh 534] A. Rosłanowski and S. Shelah, *F-99: Notes on cardinal invariants and ultraproducts of Boolean algebras*, preprint.

[Sh] S. Shelah, *Cardinal Arithmetic*, Oxford University Press, in press.

2. Mathematisches Institut der FU Berlin
Arnimalee 3
14195 Berlin
Germany
sabina@math.fu-berlin.de

Department of Mathematics
Hebrew University
Jerusalem, 910904
Israel
shelah@math.huji.ac.il