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Pcf and abelian groups

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Abstract. We deal with some pcf (possible cofinality theory) investigations mostly motivated by questions in abelian group theory. We concentrate on applications to test problems but we expect the combinatorics will have reasonably wide applications. The main test problem is the "trivial dual conjecture" which says that there is a quite free abelian group with trivial dual. The "quite free" stands for "μ-free" for a suitable cardinal μ, the first open case is $\mu = \aleph_\omega$. We almost always answer it positively, that is, prove the existence of $\aleph_\omega$-free abelian groups with trivial dual, i.e., with no non-trivial homomorphisms to the integers. Combinatorially, we prove that "almost always" there are $\mathscr{F} \subseteq {}^\kappa\lambda$ which are quite free and have a relevant black box. The qualification "almost always" means except when we have strong restrictions on cardinal arithmetic, in fact restrictions which hold "everywhere". The nicest combinatorial result is probably the so-called "Black Box Trichotomy Theorem" proved in ZFC. Also we may replace abelian groups by R-modules. Part of our motivation (in dealing with modules) is that in some sense the improvement over earlier results becomes clearer in this context.

Keywords. Cardinal arithmetic, pcf, black box, negative partition relations, trivial dual conjecture, trivial endomorphism conjecture.

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Annotated content

0 Introduction, page 968

We formulate the trivial dual conjecture for μ, TDU_μ, and relate it to pcf statements and black box principles. Similarly we state the trivial endomorphism conjecture for μ, TED_μ, but postpone its treatment.

1 Preliminaries, page 977

We quote some definitions and results we shall use and state a major conclusion of this work: the Black Box Trichotomy Theorem.

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Assume $\mu \in \boldsymbol{C}_\kappa$, $\mu < \lambda < 2^\mu < 2^\lambda$, moreover $\lambda = \min\{\chi : 2^\chi > 2^\mu\}$. *Then* for any $\theta < \mu$, a black box called $\mathrm{BB}(\lambda, \mu^+, \theta, \kappa)$ holds, which for our purpose is very satisfactory.

The point is to give sufficient conditions for BB: see Observation 0.9 (2). Let $\mu \in \boldsymbol{C}_\kappa$ and $\lambda = 2^\mu$. We give sufficient conditions for the existence of μ^+-free $\mathscr{F} \subseteq {}^\kappa\mu$ of cardinality λ, which is quite helpful for our purposes, as it implies the existence of suitable black boxes. One such condition is (see Claim 3.6): the existence of $\theta < \kappa$ and $\chi < \lambda$ such that $\chi^\theta = \lambda$. Recall that by Section 2 assuming $\lambda < \lambda^{<\lambda}$ suffices (for the black box). Now assuming there is no θ as above so $\lambda = \lambda^{<\lambda}$, by older results if $\chi < \lambda \wedge \theta = \mathrm{cf}(\theta) < \kappa \Rightarrow (\chi)^{\langle\theta\rangle_{\mathrm{tr}}} < \lambda$, then $(D\ell)^*_{S^\lambda_\theta}$, hence $(D\ell)_S$ for every stationary $S \subseteq S^\lambda_\theta$.

In Claim 3.1 we consider $\theta \in (\kappa, \mu) \cap$ Reg and $\chi \in (\mu, \lambda)$ such that $\chi^{\langle\theta\rangle_{\mathrm{tr}}} = \lambda$. Here the results are less sharp. Also if $\lambda = \chi^+$, where χ is regular, then this holds; see Claim 3.12. We finish by indicating some obvious connections.

We deduce what we can on the conjecture TDU_μ.

0 Introduction

0.1 Background

We prove some black boxes, most notably the Black Box Trichotomy Theorem. Our original question is whether provably in ZFC the conjecture $\mathrm{TDU}_{\aleph_\omega}$ holds and even whether $\mathrm{TED}_{\aleph_\omega}$ holds where:

Definition 0.1. (1) Let TDU_μ, the trivial dual conjecture for $\mu > \aleph_0$, mean: there is a μ-free abelian group G, necessarily of cardinality $\geq \mu$, such that G has a trivial dual (i.e., $\mathrm{Hom}(G, \mathbb{Z}) = \{0\}$).

(2) Let TED_μ, the trivial endomorphism conjecture for μ mean: there is a μ-free abelian group with no non-trivial endomorphism, i.e., $\mathrm{End}(G)$ is trivial (that is, $\mathrm{End}(G) \cong \mathbb{Z}$).

Much is known for $\mu = \aleph_1$ (see, e.g., [8]). Note that each of the cases of Definition 0.1 implies that G is $\aleph_1$-free, not free, and much is known on the existence of μ-free, non-free abelian groups of cardinality μ (see, e.g., [4]). Also, positive answers are known for arbitrary μ under, e.g., $\boldsymbol{V} = \boldsymbol{L}$, see [8, p. 461].

Note that by singular compactness, for singular μ there are no counterexamples of cardinality μ.

By [29], if $\mu = \aleph_n$, then the answer to TDU_μ is yes, for the cardinality $\lambda = \beth_n$. It was hoped that the method would apply to many other related problems and to some extent this has been vindicated by Göbel–Shelah [6], Göbel–Shelah–Strüngman [7] and (on TED_μ, $\mu = \aleph_n$) by Göbel–Herden–Shelah [5]. But we do not know the answer for $\mu = \aleph_\omega$. Note that even if we succeed, this will not cover the results of [5–7, 29]; e.g. because there the cardinality of G is $< \beth_\omega$ when $\mu < \aleph_\omega$ and probably even more so when we deal with larger cardinals.

A natural approach is to prove in ZFC appropriate set-theoretic principles, and this is the method we try here. This raises combinatorial questions which seem interesting in their own right; our main result in this direction is the Black Box Trichotomy Theorem 1.22. But the original algebraic question has bothered me and the results are irritating: it is "very hard" not to answer yes in the following sense (later we say more on the set theory involved):

(a) Failure implies strong demands on cardinal arithmetic in many $\beth_\delta$, (e.g. if $\mathrm{cf}(\delta) = \aleph_1$, then
$$\beth_{\delta+1} = \mathrm{cf}(\beth_{\delta+1}) = (\beth_{\delta+1})^{<\beth_{\delta+1}}$$
and $\chi < \beth_{\delta+1} \Rightarrow \chi^{\aleph_0} < \beth_{\delta+1}$ – see details below).

(b) If we weaken "$\aleph_\omega$-freeness" to (so-called "stability" or "softness" and even) "$\aleph_1$-free and constructible from a ladder system $\langle C_\delta : \delta \in S \subseteq S^\lambda_{\aleph_0} \rangle$", then we can prove existence.

(c) Replacing abelian groups by R-modules, the parallel question depends on a set of regular cardinals related to the ring, $\mathrm{sp}(R)$, see Definition 4.2 (so the case of abelian groups is $R = \mathbb{Z}$). If $\mathrm{sp}(R)$ is empty, there is nothing to be done. By [27], if $\mathrm{sp}(R)$ is unbounded below some strong limit singular cardinal μ of cofinality $\aleph_0$, then TDU_{μ^+}, see Conclusion 4.16. Moreover, by [28], if $\mathrm{sp}(R)$ is infinite, say $\kappa_n < \kappa_{n+1} \in \mathrm{sp}(R)$, then by Conclusion 4.16 again TDU_μ for every μ (by the quotation Theorem 1.18). Furthermore (see Claim 3.17), we prove that: if $\aleph_0, \aleph_1, \aleph_2 \in \mathrm{sp}(R)$, then the answer for R-modules is positive.

(d) Even if the negation of $\mathrm{TDU}_{\aleph_\omega}$ is consistent with ZFC, its consistency strength is large; to some extent this follows by clause (a) above but by Section 2 we have more.

Obviously, e.g. clause (c) clearly seems informative for abelian groups; at first sight it seems helpful that for every n there is an $\aleph_n$-free non-free abelian group of cardinality $\aleph_n$, but this is not enough. More specifically this method does not at present resolve the problem because for $R = \mathbb{Z}$ we only know that $\mathrm{sp}(R)$ includes $\{\aleph_0, \aleph_1\}$ (and under MA it has no other member $< 2^{\aleph_0}$).

Still we get some information: a reasonably striking set-theoretic result is the Black Box Trichotomy Theorem 1.22 below; some abelian group theory consequences are given in Section 4.

A sufficient condition (see Conclusion 4.12) for a positive answer to TDU_μ is:

$\circledast_0$ TDU_μ if $\mathrm{BB}(\lambda, \mu, \theta, J)$ when J is $J^{\mathrm{bd}}_{\aleph_0}$ or $J^{\mathrm{bd}}_{\aleph_1 \times \aleph_0}$, see Notation 0.3 below, $\mathrm{cf}(\lambda) > \aleph_0$ and $\theta = \beth_4$.

This work will be continued in [35] and also in [34] which originally was part of the present paper, also expand on some proofs here.

Before we state the results we give some basic definitions.

0.2 Basic definitions

Recall that

Definition 0.2. We call $\chi^{\langle \partial \rangle_{\mathrm{tr}}}$ the ∂-tree power of χ, i.e., the supremum of the number of ∂-branches of a tree with $\leq \chi$ nodes and ∂ levels.

Notation 0.3. (1) For a set S of ordinals with no greatest member (e.g. a limit ordinal δ) let J^{bd}_S be the ideal $\{u : u \text{ is a bounded subset of } S\}$.

(2) For limit ordinals δ_1, δ_2 let

$$J^{\mathrm{bd}}_{\delta_1 \times \delta_2} = \{u \subseteq \delta_1 \times \delta_2 : \{\alpha < \delta_1 : \{\beta < \delta_2 : (\alpha, \beta) \in u\} \notin J^{\mathrm{bd}}_{\delta_2}\} \in J^{\mathrm{bd}}_{\delta_1}\}.$$

(3) For limit ordinals δ_1, δ_2 let $\delta_3 = \delta_2 \cdot \delta_1$ and $J^{\mathrm{bd}}_{\delta_1 * \delta_2}$ be the following ideal on δ_3:

$$\{u \subseteq \delta_3 : \{(\alpha, \beta) \in \delta_1 \times \delta_2 : \delta_2 \cdot \alpha + \beta \in u\} \in J^{\mathrm{bd}}_{\delta_1 \times \delta_2}\}.$$

Definition 0.4. (1) A sequence of sets $\bar{C} = \langle C_\alpha : \alpha \in S \rangle$ is μ-free if for every $u \in [S]^{<\mu}$ there exists a sequence $\bar{A} = \langle A_\alpha \subseteq C_\alpha : \alpha \in u \rangle$ so that the sets $\langle C_\alpha \backslash A_\alpha : \alpha \in u \rangle$ are pairwise disjoint and each A_α is bounded in C_α with respect to a given order on C_α; in the default case "every C_α is a set of ordinals with the natural order".

(2) We may replace the cardinal μ by a pair $(\mu, \bar{J})$, where $\bar{J} = \langle J_\alpha : \alpha \in S \rangle$ and J_α is an ideal on $\mathrm{otp}(C_\alpha)$ so now the condition "A_α bounded" is replaced by "$\{\mathrm{otp}(\varepsilon \cap C_\alpha) : \varepsilon \in A_\alpha\} \in J_\alpha$". If C_α is a set of ordinals of a fixed order type $\gamma(*)$ and $J_\alpha = J$ for every $\alpha \in S$ where J is an ideal on $\gamma(*)$, *then* we may replace the pair $(\mu, \bar{J})$ by the pair (μ, J). In other words, instead of the demand "A_α is bounded in C_α" we require $A'_\alpha := \{\mathrm{otp}(C_\alpha \cap \gamma) : \gamma \in A_\alpha\} \in J$.

The definition of the assertion $\mathrm{BB}(\lambda, \mu, \theta, J)$ is as follows. (BB stands for black box.) The following is a relative of [27] (and see on the history there).

Definition 0.5. Assume we are given a quadruple $(\lambda, \mu, \theta, \kappa)$ of cardinals (but we may replace λ by an ideal I on $S \subseteq \lambda = \sup(S)$ so writing λ means $S = \lambda$; also we may replace κ by an ideal J on κ and writing κ means $J = J^{\mathrm{bd}}_\kappa$). Let $\mathrm{BB}^-(\lambda, \mu, \theta, \kappa)$ mean that some pair $(\bar{C}, \bar{\boldsymbol{c}})$ satisfies the clauses (A) and (B) below; we call the pair $(\bar{C}, \bar{\boldsymbol{c}})$ a witness for $\mathrm{BB}^-(\lambda, \mu, \theta, \kappa)$. Let $\mathrm{BB}(\lambda, \mu, \theta, \kappa)$ mean that some witness $(\bar{C}, \bar{\boldsymbol{c}})$ satisfies clause (A) below and for some sequence $\langle S_i : i < \lambda\rangle$ of pairwise disjoint subsets of λ (*or* of S), each $(\bar{C} \restriction S_i, \bar{\boldsymbol{c}} \restriction S_i)$ satisfies clause (B) below, (thus replacing $S, \bar{\boldsymbol{c}}$ by $S_i, \bar{\boldsymbol{c}} \restriction S_i$) where:

(A) (a) $\bar{C} = \langle C_\alpha : \alpha \in S\rangle$ and $S = S(\bar{C}) \subseteq \lambda = \sup(S)$,
   (b) $C_\alpha \subseteq \alpha$ has order type κ,
   (c) $\bar{C}$ is μ-free (see Definition 0.4),
   (*but* when we replace κ by J, then we say "$\bar{C}$ is (μ, J)-free"),

(B) (d) $\bar{\boldsymbol{c}} = \langle \boldsymbol{c}_\alpha : \alpha \in S\rangle$,
   (e) $\boldsymbol{c}_\alpha$ is a function from C_α to θ,
   (f) if $\boldsymbol{c} : \bigcup_{\alpha\in S} C_\alpha \to \theta$, then $\boldsymbol{c}_\alpha = \boldsymbol{c} \restriction C_\alpha$ for some $\alpha \in S$,
   (*but* when we replace λ by I an ideal on S, then we demand that the set $\{\alpha \in S : \boldsymbol{c}_\alpha = \boldsymbol{c} \restriction C_\alpha\}$ is not in I).

Remark 0.6. The reader may recall that if the considered set S is a stationary subset of $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ for a regular cardinal λ and S is non-reflecting and $\bar{C} = \langle C_\alpha : \alpha \in S\rangle$ satisfies $C_\delta \subseteq \delta = \sup(C_\delta)$, $\mathrm{otp}(C_\delta) = \kappa$, *then* $\diamondsuit_S$ implies $\mathrm{BB}(\lambda, \lambda, \lambda, \kappa)$. So if $\boldsymbol{V} = \boldsymbol{L}$, then for every regular $\kappa < \lambda$, λ non-weakly compact we have $\mathrm{BB}(\lambda, \lambda, \lambda, \kappa)$.

So the consistency of (more than) having many cases of BB is known, *but* we prefer to get results in ZFC, when possible.

Variants are:

Definition 0.7. In Definition 0.5:

(1) We may replace θ by (χ, θ) which means there are $S, \bar{C}$ satisfying clause (A) of Definition 0.5 and
   (B)′ if $\bar{\boldsymbol{F}} = \langle \boldsymbol{F}_\alpha : \alpha \in S\rangle$ and $\boldsymbol{F}_\alpha$ is a function from ${}^{(C_\alpha)}\chi$ to θ, *then* for some $\bar{\boldsymbol{c}}$ we have:
      (d) $\bar{\boldsymbol{c}} = \langle \boldsymbol{c}_\alpha : \alpha \in S\rangle$,
      (e) $\boldsymbol{c}_\alpha < \theta$,

(f) if $\boldsymbol{c} : \lambda \to \chi$, *then* $\boldsymbol{c}_\alpha = \boldsymbol{F}_\alpha(\boldsymbol{c} \restriction C_\alpha)$ for some $\alpha \in S$ (*or* if we replace λ by I, then the set $\{\alpha \in S : \boldsymbol{c}_\alpha = \boldsymbol{F}_\alpha(\boldsymbol{c} \restriction C_\alpha)\}$ does not belong to the ideal I).

(2) Replacing (χ, θ) by $(\chi, 1/\theta)$, abusing notation or $\langle \chi, \theta \rangle$, means that in clause (f) we replace "$\boldsymbol{c}_\alpha = \boldsymbol{F}_\alpha(\boldsymbol{c} \restriction C_\alpha)$" by "$\boldsymbol{c}_\alpha \neq \boldsymbol{F}_\alpha(\boldsymbol{c} \restriction C_\alpha)$".

(3) We may replace μ by $\bar{C}$ and thus waive the freeness demand, i.e., $\bar{C}$ is not necessarily μ-free. Alternatively, we may replace μ by a set $\mathscr{F}$ of one-to-one functions from κ to λ when $\bar{C}$ lists $\{\mathrm{Rang}(f) : f \in \mathscr{F}\}$.

(4) Replacing κ by "$< \kappa_1$" means that in Definition 0.5 (A)(b) we require just $C_\alpha \subseteq \alpha \wedge |C_\alpha| < \kappa_1$ (and not necessarily $\mathrm{otp}(C_\alpha) = \kappa$). Replacing κ by $*$ means "$< \lambda$".

(5) We may replace θ by "$< \theta_1$" meaning "for every $\theta < \theta_1$".

Remark 0.8. (1) Note that $\mathrm{BB}(\lambda, \mu, \theta, \kappa)$ is somewhat related to $\mathrm{NPT}(\lambda, \kappa)$ from [22, Chapter II], i.e., $\mathrm{BB}(\lambda, \lambda, \theta, \kappa) \Rightarrow \mathrm{NPT}(\lambda, \kappa)$, but NPT has no "predictive" part.

(2) We shall use freely the obvious implications concerning the black boxes, e.g.

($*$) $\mathrm{BB}^-(\lambda_1, \mu_1, \theta_1, \kappa_1) \Rightarrow \mathrm{BB}^-(\lambda_2, \mu_2, \theta_2, \kappa_2)$ when $\lambda_2 = \lambda_1$, $\mu_2 \leq \mu_1$, $\theta_2 \leq \theta_1$, $\kappa_2 = \kappa_1$.

Of course, we get:

Observation 0.9. (1) *If* $\bar{C} = \langle C_\alpha : \alpha \in [\mu, \lambda) \rangle$, $C_\alpha \subseteq \mu$ *non-empty and* $2^\mu = \lambda$ *(e.g.* $\lambda = \mu^\kappa \wedge \mu \in \boldsymbol{C}_\kappa$*),* then $\mathrm{BB}(\lambda, \bar{C}, \lambda, *)$*, see Definition* 0.7 (4).

(2) *If in addition* $\mathrm{otp}(C_\alpha) = \kappa$ *and* $\bar{C}$ *is* μ_1*-free, then* $\mathrm{BB}(\lambda, \mu_1, \lambda, \kappa)$.

Proof. The proof is easy, but we shall give details.

(1) Let $S = [\mu, \lambda)$ and let $\langle S_\varepsilon : \varepsilon < \lambda \rangle$ be a partition of S into sets each of cardinality λ. Recalling Definitions 0.5, 0.7 it suffices to prove $\mathrm{BB}(\lambda, \bar{C} \restriction S_\varepsilon, \lambda, *)$ for each $\varepsilon < \lambda$; fix ε now. Clause (A) in Definition 0.5 is obvious, so we shall prove clause (B)$'$, so let $\langle \boldsymbol{F}_\alpha : \alpha \in S_\varepsilon \rangle$ and $\boldsymbol{F}_\alpha : {}^{(C_\alpha)}\lambda \to \lambda$ be given and we should choose $\bar{c} \in {}^{(S_\varepsilon)}\theta$.

Let $\bar{f} = \langle f_\alpha : \alpha \in S_\varepsilon \rangle$ list ${}^\mu\lambda$, each appearing unboundedly often (and even stationarily often if λ is regular), and choose $c_\alpha := \boldsymbol{F}_\alpha(f_\alpha \restriction C_\alpha)$. Now check.

(2) Look at the definitions. □

Discussion 0.10. We use Observation 0.9 for example in Theorem 1.31.

Recall the following definitions:

Definition 0.11. (1) If $\leq_*$ is a partial order on a set I, let $\lambda = \mathrm{tcf}(I, <_*)$ mean that λ is a regular cardinal and there is an $<_*$-increasing sequence $\langle t_\alpha : \alpha < \lambda \rangle$ which is cofinal, that is $(\forall s \in I)(\exists i < \lambda)[s \leq_* t]$.

(2) For $I, <_*$ as above let

$$\mathrm{cf}(I, <_*) = \min\{|\mathcal{P}| : \mathcal{P} \subseteq I \text{ is cofinal}\}.$$

Definition 0.12. Assume $\mu > \theta \geq \sigma = \mathrm{cf}(\sigma) \geq \mathrm{cf}(\mu)$. For J an ideal on θ (or just on a set A_* of cardinality θ) such that there is a $\subseteq$-increasing sequence of members of J of length $\mathrm{cf}(\mu)$ with union θ (or A_*):

(1) we define $\mathrm{pp}_J(\mu) = \sup\{\mathrm{tcf}(\prod_{i<\theta} \lambda_i, <_J) : \lambda_i = \mathrm{cf}(\lambda_i) \in (\theta, \mu)$ for $i < \theta$ and $\mu = \lim_J \langle \lambda_i : i < \theta \rangle\}$, where $\mu = \lim_J \langle \lambda_i : i < \theta \rangle$ means that $\mu_i < \mu$ implies $\{i < \theta : \lambda_i \notin [\mu_i, \mu]\} \in J$,

(2) we define $\mathrm{pp}_{\theta,\sigma}(\mu) = \sup\{\mathrm{tcf}(\prod_{i<\theta} \lambda_i, <_J) : J$ a σ-complete ideal on θ with $\lambda_i = \mathrm{cf}(\lambda_i) \in (\theta, \mu)$ such that $\mu = \lim_J \langle \lambda_i : i < \theta \rangle\}$,

(3) let $\mathrm{pp}_J(\mu) =^+ \chi$ mean that $\mathrm{pp}_J(\mu) = \chi$ and χ is regular and in the supremum in part (1) is attained; similarly in parts (2)–(3),

(4) let $\mathrm{pp}_J^+(\mu)$ be $(\mathrm{pp}_J(\mu))^+$ if $\mathrm{pp}_J(\mu)$ is regular and the supremum in part (1) is obtained and be $\mathrm{pp}_J(\mu)$ otherwise.

Definition 0.13. For cardinals $\lambda, \mu, \theta, \sigma$ with $\lambda \geq \mu \geq \theta \geq \sigma$ let $\mathrm{cov}(\lambda, \mu, \theta, \sigma) = \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{<\mu}$ and every $u \in [\lambda]^{<\theta}$ is included in the union of $< \sigma$ members of $\mathcal{P}\}$.

0.3 What is done

In this work we shall show that it is "hard" for $\boldsymbol{V}$ not to give a positive answer (i.e., existence) for Definition 0.1 via a case of Definition 0.5 or variants; we review below the "evidence" for this assertion. By Conclusion 4.12 (1) we know that (actually $2^{(2^{\aleph_1})}$ can be weakened):

$\odot_0$ a sufficient condition for TDU_μ is $\mathrm{BB}(\lambda, \mu, 2^{(2^{\aleph_1})^+}, J)$, where $\mathrm{cf}(\lambda) > \aleph_0$ and J is $J^{\mathrm{bd}}_{\aleph_0}$ or $J^{\mathrm{bd}}_{\aleph_1 \times \aleph_0}$ (hence also $J = J^{\mathrm{bd}}_{\aleph_1}$ suffices; noting that here κ is $\aleph_0$ or $\aleph_1$ together $\mathrm{BB}(\lambda, \mu, 2^{(2^{\aleph_1})^+}, \kappa)$ suffice).

Recall that $\boldsymbol{C}_\kappa$ is the class of strong limit singular cardinals of cofinality κ when $\kappa > \aleph_0$, and "most" of them when $\kappa = \aleph_0$ (see Definition 1.1 and Claim 1.3).

Now the first piece of the evidence given here that a failure of G.C.H. near $\mu \in \boldsymbol{C}_\kappa$ helps is the following fact:

$\circledast_1$ $\mathrm{BB}(\lambda, \mu^+, \theta, \kappa)$ if $\theta < \mu \in \boldsymbol{C}_\kappa$ and $\mu < \lambda < 2^\mu < 2^\lambda$.

(Why? By Conclusion 2.7 (1); it is a consequence of the Black Box Trichotomy Theorem 1.22.)

Note: another formulation is

$\boxdot_1$ if $\theta < \mu \in \boldsymbol{C}_\kappa$ but $\mathrm{BB}(\lambda, \mu^+, \theta, \kappa)$ fails, *then* $(2^\mu)^{<2^\mu} = 2^\mu$.

(Why? Let $\lambda_1 = \min\{\chi : 2^\chi > 2^\mu\}$, so necessarily $\mu < \lambda_1$; if $\lambda_1 < 2^\mu$, then $\mathrm{BB}(\lambda_1, \mu^+, \theta, \kappa)$ holds by $\circledast_1$, so by our assumption $\lambda_1 = 2^\mu$, so $\mu \le \chi < 2^\mu \Rightarrow 2^\chi = 2^\mu \Rightarrow (2^\mu)^\chi = 2^{\mu \cdot \chi} = 2^\chi = 2^\mu$, but this means $(2^\mu)^{<2^\mu} = 2^\mu$, as stated.)

So by $\odot_0 + \boxdot_1$, we have:

$\odot_1$ if $\mathrm{TDU}_{\aleph_\omega}$ fails, *then*

(a) a large class of cardinals satisfies a weak form of G.C.H.,

(b) more specifically, $(\mu \in \boldsymbol{C}_{\aleph_0} \cup \boldsymbol{C}_{\aleph_1}) \wedge \lambda = 2^\mu \Rightarrow \lambda = \lambda^{<\lambda}$.

For $\mathcal{T} \subseteq {}^{\sigma>}\chi$ a tree with $\le \chi$ nodes and $\le \sigma$ levels we let

$$\lim_\sigma(\mathcal{T}) = \{\eta \in {}^\sigma\chi : (\forall \varepsilon < \sigma)(\eta \restriction \varepsilon \in \mathcal{T})\},$$

and recall that the tree power $\chi^{\langle \sigma \rangle_{\mathrm{tr}}}$ is

$$\chi^{\langle \sigma \rangle_{\mathrm{tr}}} = \sup\{|\lim_\sigma(\mathcal{T})| : \mathcal{T} \subseteq {}^{\sigma>}\chi \text{ is a tree with } \le \chi \text{ nodes and } \le \sigma \text{ levels}\}.$$

We have:

$\circledast_2$ $\mathrm{BB}(2^\mu, \kappa^{+\omega+1}, \theta, J^{\mathrm{bd}}_{\kappa^+ \times \kappa})$ if $\theta < \mu \in \boldsymbol{C}_\kappa$ and $(\forall \chi)(\chi < 2^\mu \Rightarrow \chi^{\langle \kappa^+ \rangle_{\mathrm{tr}}} < 2^\mu)$.

(Why? See Claim 1.35.)

So we obtain:

$\odot_2$ if $\mathrm{TDU}_{\aleph_\omega}$ fails, *then* for every $\mu \in \boldsymbol{C}_{\aleph_0}$ there is χ such that

$$\mu < \chi < \chi^{\langle \aleph_1 \rangle_{\mathrm{tr}}} = 2^\mu$$

(see Definition 0.2), hence we have $\mu < \chi < 2^\mu$ and without loss of generality $\mathrm{cf}(\chi) = \aleph_1$, hence $\mu^{+\omega_1} \le \chi < 2^\mu$, and so G.C.H. fails quite strongly (putting us in some sense in the opposite direction to $\odot_1$)

and also

$\circledast_3$ if $\mu \in \boldsymbol{C}_\kappa$, $\theta < \mu, \lambda = 2^\mu$ and some set $\mathcal{F} \subseteq {}^\kappa\mu$ is μ_1-free of cardinality 2^μ $(= \mu^\kappa)$, *then* $\mathrm{BB}(\lambda, \mu_1, \theta, \kappa)$.

(Why? See Observation 0.9 (2).)

In Section 3 we shall give various sufficient conditions for the satisfaction of the hypotheses of $\circledast_3$. Another piece of evidence is

$\circledast_4$ $\mathrm{BB}(\lambda, \mu_1, \theta, J)$ *when*:

(a) $\theta < \mu \in \boldsymbol{C}_\kappa, \lambda = 2^\mu = \lambda^{<\lambda}$ and $\partial < \mu$,

(b) J is an ideal on $\partial = \mathrm{cf}(\partial)$ extending J^{bd}_∂, and $S \subseteq S^\lambda_\partial$ (see part (3) of Notation 0.16), $\bar{C} = \langle C_\delta : \delta \in S \rangle$ are such that

$$\delta \in S \Rightarrow C_\delta \subseteq \delta = \sup(C_\delta) \wedge \partial = \mathrm{otp}(C_\delta),$$

(c) $\bar{C}$ is μ_1-free, $\mu_1 < \lambda$, see Definition 1.2 (1A), (2), it is closed to Definition 0.4,

(d) we have

$$(\forall \alpha < \lambda)(\lambda > |\{C_\delta \cap \alpha : \delta \in S \wedge \alpha \in C_\delta\}|) \wedge (\forall \chi < \lambda)(\chi^{\langle \partial \rangle_{\mathrm{tr}}} < \lambda)$$

or $(\mathrm{D}\ell)_S$ (see Definition 1.13).

(Why? This follows from [27].)

A consequence for the present work is:

$\circledast_5$ $\mathrm{BB}(\lambda, \kappa^{+\omega}, \theta, J^{\mathrm{bd}}_{\kappa^+ \times \kappa})$ *when*:

(a) $\theta < \mu \in \boldsymbol{C}_\kappa, \lambda = 2^\mu = \lambda^{<\lambda}$,

(b) $S \subseteq S^\lambda_{\kappa^+}, \delta \in S \Rightarrow C_\delta \subseteq \delta = \sup(C_\delta) \wedge \mathrm{otp}(C_\delta) = \kappa^+$,

(c) $\langle C_\delta : \delta \in S \rangle$ is $\kappa^{+\omega}$-free and $\kappa^{+\omega} < \lambda$ which actually follows,

(d) $(\mathrm{D}\ell)_S$ or the first possibility of $\circledast_4$ (d) for $\partial = \kappa^+$.

(Why? By $\circledast_4$.)

The point of $\circledast_5$ is that we can find $\bar{C}$ as in clause (b) of $\circledast_5$ with $S \subseteq S^\lambda_{\kappa^+}$ "quite large" so we ignore the difference (in the introduction) – see Claim 1.26. In particular

$\boxdot_2$ if $\lambda = \mu^+ = 2^\mu$ and $\mu > \aleph_0$ is a strong limit cardinal of cofinality $\kappa = \aleph_0$, *then* for some $\bar{C}, S$ clauses (a)–(d) of $\circledast_5$ hold.

(Why? As in $\circledast_2$.)

Moreover:

$\boxdot_3$ if $\kappa < \chi, \kappa$ is a regular cardinal, $\lambda = \chi^+ = 2^\chi$ and $\kappa \neq \mathrm{cf}(\chi)$, *then* $\diamondsuit_S$ for every stationary $S \subseteq S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

(Why? By [32] – see Fact 1.17.)

We can conclude:

$\odot_3$ if $\mathrm{TDU}_{\aleph_\omega}$ fails and $\mu \in \boldsymbol{C}_{\aleph_0}$, *then* 2^μ is not μ^+, moreover, 2^μ is not of the form $\chi^+, \mathrm{cf}(\chi) \neq \aleph_1$,

$\circledast_6$ $\mathrm{BB}(2^\mu, \mu^+, \theta, \kappa)$ if $\theta < \mu \in \boldsymbol{C}_\kappa$ and $\chi^\sigma = 2^\mu$ for some $\sigma = \mathrm{cf}(\sigma) < \kappa$ and $\chi < 2^\mu$.

(Why? The assumptions (a)–(f) of Claim 3.6 hold for $J = J_\kappa^{\mathrm{bd}}$ and σ here standing for θ there. For example clause (d) there, "$\alpha < \mu \Rightarrow |\alpha|^\theta < \mu$" holds as μ is strong limit. So the first assumption of Conclusion 3.8 holds, and the second ($\mu^\kappa = 2^\mu$) holds as $\mu \in \boldsymbol{C}_\kappa$. So Conclusion 3.8 holds by Observation 0.9 which implies that $\circledast_6$ holds.)

$\circledast_7$ $\mathrm{BB}(2^\mu, \partial, \theta, \kappa)$ if $\theta < \mu \in \boldsymbol{C}_\kappa$ and $\partial = \sup\{\mathrm{cf}(\chi) : \mathrm{cf}(\chi) < \mu < \chi < 2^\mu$ and $\mathrm{pp}_{\mathrm{cf}(\chi)\text{-comp}}(\chi) =^+ 2^\mu\}$.

(Why? By Claim 3.1 and Observation 0.9.)

So (by $\odot_0, \circledast_6, \circledast_7$) we get:

$\odot_4$ if $\mathrm{TDU}_{\aleph_\omega}$ fails, *then* for every $\mu \in \boldsymbol{C}_{\aleph_1}$ we have:

(a) $\alpha < 2^\mu \Rightarrow |\alpha|^{\aleph_0} < 2^\mu$,

(b) for some $n, \aleph_n \leq \mathrm{cf}(\chi) < \mu \wedge \chi < 2^\mu \Rightarrow \mathrm{pp}_{\mathrm{cf}(\chi)\mathrm{comp}}(\chi) \neq^+ 2^\mu$.

By the end of Section 4:

$\odot_5$ if $\mathrm{TDU}_{\aleph_\omega}$ fails and $n \geq 3$, *then*:

(A) no $\aleph_n$-free (abelian) group G of cardinality $\aleph_n$ is Whitehead,

(B) if $\mu \in \boldsymbol{C}_{\aleph_0} \cup \boldsymbol{C}_{\aleph_1}$ and $\lambda = 2^\mu$, then $(D\ell)_{S^\lambda_{\aleph_n}}$.

Generally in [22] we suggest cardinal arithmetic assumptions as good "semi-axioms".

We have used cases of WGCH (the Weak Generalized Continuum Hypothesis, i.e., $2^\lambda < 2^{\lambda^+}$ for every λ); in [13,14,17], also in [19] and see [30,31]. Influenced also by this, Baldwin suggested adopting WGCH giving arguments parallel to the ones for large cardinals (but with no problem of consistency). So it seems reasonable to see what we can say in our context.

Note that above we get:

Claim 0.14. *Assume $\mu \in \boldsymbol{C}_\kappa$ so μ is a strong limit singular cardinal of cofinality κ.*

(1) *If $\mu^+ < 2^\mu < 2^{\mu^+}$ and $\kappa \in \{\aleph_0, \aleph_1\}$*, then *there is a μ^+-free abelian group of cardinality μ^+ with* $\mathrm{Hom}(G, \mathbb{Z}) = 0$*; note that this is iterable, i.e., if we have $\mu_{\ell+1} \in \boldsymbol{C}_{\mu_\ell^+}$ for $\ell < n, 2^{\mu_\ell} > \mu_\ell^+$ for $\ell < n$ and μ_0 is like μ above, then the conclusion applies for μ_n.*

(2) *If $\mu^+ = 2^\mu$ and $\kappa \in \{\aleph_0, \aleph_1\}$*, then *there is an $\aleph_{\omega+1}$-free abelian group of cardinality μ^+ such that* $\mathrm{Hom}(G, \mathbb{Z}) = 0$.

Proof. (1) By Theorem 1.22 there is a μ^+-free $\mathscr{F} \subseteq {}^\kappa\mu$ of cardinality μ^+ (yes! not 2^μ) hence $\mathrm{BB}(\lambda, \mu, \lambda, \kappa)$ by Conclusion 2.7 (1). By Claims 4.7 and 4.10 there is G as required; similarly for iterations.

(2) The proof is similar. □

Note that we can prove $\mathrm{TDU}_{\aleph_{\omega+1}}$ if the answer to the following is positive:

Conjecture 0.15. If $\lambda = \lambda^{<\lambda} > \kappa^+$, $\kappa = \mathrm{cf}(\kappa)$ and $\lambda \neq \aleph_1$ (or at least $\lambda \geq \beth_\omega$ replacing the assumption $\lambda \neq \aleph_1$), then $(D\ell)_{S^\lambda_\kappa}$.

Related works are [32] and Göbel–Herden–Shelah ([5]).

Notation 0.16. (0) For sets let $u_1 \backslash u_2 \backslash u_2$ mean $(u_1 \backslash u_2) \backslash u_3$.

(1) Usually $\bar{C} = \langle C_\delta : \delta \in S \rangle$ with $S = S(\bar{C})$.

(2) A club of a limit ordinal δ (e.g. usually a regular cardinal) is a closed unbounded subset.

(3) $S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

Definition 0.17. Let $\bar{C} = \langle C_\delta : \delta \in S \rangle$ and λ be a regular cardinal.

(1) $\bar{C}$ is a weak λ-ladder system *when* S is a stationary subset of (the regular cardinal) λ and $\delta \in S \Rightarrow C_\delta \subseteq \delta$.

(2) $\bar{C}$ is a λ-ladder system *when* λ is regular, S is a stationary subset of λ and $C_\delta \subseteq \delta = \sup(C_\delta)$ for $\delta \in S$.

(3) $\bar{C}$ is a strict λ-ladder system *when* in addition $\mathrm{otp}(C_\delta) = \mathrm{cf}(\delta)$.

(4) $\bar{C}$ is a strict (λ, κ)-ladder system *when* in addition $S \subseteq S^\lambda_\kappa$.

(5) $\bar{C}$ is shallow *when* $\alpha \in \bigcup_{\delta \in S} C_\delta \Rightarrow \sup(S) > |\{C_\delta \cap \alpha : \delta \in S \text{ and } \alpha \in C_\delta\}|$.

(6) In parts (1)–(3) we may omit the "λ" when clear from the content or replace λ by S.

1 Preliminaries

Most of our results involve $\mu \in \boldsymbol{C}$ where:

Definition 1.1. (1) Let

$$\boldsymbol{C} = \{\mu : \mu \text{ is a strong limit singular cardinal and } \mathrm{pp}(\mu) =^+ 2^\mu\},$$

recalling Definition 0.12 for $=^+$.

(2) Let $\boldsymbol{C}_\kappa = \{\mu \in \boldsymbol{C} : \mathrm{cf}(\mu) = \kappa\}$.

Note that Observation 1.4 (2) below, which relies on Definition 1.2 (1) and (1A), repeats Definition 0.4.

Definition 1.2. (1) The set $\mathscr{F} \subseteq {}^\kappa\mu$ is called (θ, σ, J)-free where J is an ideal on κ *when* $[f_1 \neq f_2 \in \mathscr{F} \Rightarrow \{i : f_1(i) = f_2(i)\} \in J]$ and every $\mathscr{F}' \subseteq \mathscr{F}$ of cardinality $< \theta$ is $[J, \sigma]$-free which means that:

- there is a sequence $\langle u_f : f \in \mathscr{F}' \rangle$ of members of J such that for every pair $(\gamma, i) \in \mu \times \kappa$ the set $\{f \in \mathscr{F}' : f(i) = \gamma \wedge i \notin u_f\}$ has cardinality $< 1 + \sigma$.

(1A) We may replace "$\mathscr{F} \subseteq {}^\kappa\mu$" by a sequence $\bar{C} = \langle C_\delta : \delta \in S \rangle$, C_δ a set of order type κ, or even just a set $\{C_\delta : \delta \in S\}$, meaning that the definition applies to $\{f_\delta : \delta \in S\}$ where for $\delta \in S$, f_δ is an increasing function from κ onto C_δ; similarly for the other parts.

(2) If $\sigma = 1$, we may omit it. If $J = J^{\rm bd}_\kappa$, we may omit it, so we may say "$\mathscr{F} \subseteq {}^\kappa\mu$ is θ-free". Lastly, "$\mathscr{F}$ is free" means $\mathscr{F}$ is $|\mathscr{F}|^+$-free.

(3) If J is not an ideal on κ but is a subset of $\mathscr{P}(\kappa)$, *then* we replace "$u_f \in J$" by "$(u_f \in J) \Leftrightarrow (\emptyset \in J)$" and $u_f \subseteq \kappa$, of course.

(4) We say a *sequence* $\langle f_\alpha : \alpha < \alpha^* \rangle$ of members of ${}^\kappa\mu$ is (θ, J)-free *when* $J \subseteq \mathscr{P}(\kappa)$ and for every $w \subseteq \alpha^*$ of cardinality $< \theta$ the sequence $\bar{f} \restriction w$ is J-free which means that there is a sequence $\langle u_{f_\alpha} : \alpha \in w \rangle$ of subsets of κ such that $(u_f \in J) \Leftrightarrow (\emptyset \in J)$ and

$$\alpha \in w \wedge \beta \in w \wedge \alpha < \beta \wedge i \in \kappa \backslash u_{f_\alpha} \wedge i \in \kappa \backslash u_{f_\beta} \Rightarrow f_\alpha(i) < f_\beta(i).$$

Again if $J = J^{\rm bd}_\kappa$, then we may omit it.

(5) We say $\mathscr{F} \subseteq {}^\kappa\mu$ is normal *when* $f_1, f_2 \in \mathscr{F} \wedge f_1(i_1) = f_2(i_2) \Rightarrow i_1 = i_2$. We say $\mathscr{F} \subseteq {}^\kappa\mu$ is tree-like *when* it is normal and moreover

$$f_1 \in \mathscr{F} \wedge f_2 \in \mathscr{F}_1 \wedge i < \kappa \wedge f_1(i) = f_2(i) \Rightarrow f_1 \restriction i = f_2 \restriction i.$$

(6) For $\mathscr{F} \subseteq {}^\kappa\mu$ and an ideal J on κ let (issp stands for instability spectrum)

$$\mathrm{issp}_J(\mathscr{F}) = \{(\theta_1, \theta_2) : \kappa \leq \theta_1 < \theta_2 \text{ and for some } u \subseteq \mu \text{ of cardinality} \leq \theta_1 \text{ we have } \theta_2 \leq |\{\eta \in \mathscr{F} : \{i < \kappa : \eta(i) \in u\} \in J^+\}|\}.$$

(7) Let $\theta \in \mathrm{issp}_J(\mathscr{F})$ mean $(< \theta, \theta) \in \mathrm{issp}_J(\mathscr{F})$ where $(< \theta_1, \theta_2) \in \mathrm{issp}_J(\mathscr{F})$ means that $(\theta'_1, \theta_2) \in \mathrm{issp}_J(\mathscr{F})$ for some $\theta'_1 < \theta_1$. For $J = J^{\rm bd}_\kappa$ we may omit J.

(8) If we write $\mathrm{issp}_J(\langle \eta_s : s \in I \rangle)$, we mean $\mathrm{issp}_J(\{\eta_s : s \in I\})$ *but* demand $s_1 \neq s_2 \in I \Rightarrow \eta_{s_1} \neq \eta_{s_2}$.

Recall the following:

Claim 1.3. (a) *We have $\mu \in \boldsymbol{C}$ and, moreover,*

$$\mathrm{pp}_{J^{\mathrm{bd}}_{\mathrm{cf}(\mu)}}(\mu) =^{+} 2^{\mu}$$

when μ *is a strong limit singular cardinal of uncountable cofinality.*

(b) *If* $\mu = \beth_\delta > \mathrm{cf}(\mu)$ *and* $\delta = \omega_1$ *or just* $\mathrm{cf}(\delta) > \aleph_0$, then $\mu \in \boldsymbol{C}_{\mathrm{cf}(\mu)}$ *and for a club of* $\alpha < \delta$ *we have* $\beth_\alpha \in \boldsymbol{C}$.

(c) *If* $\mu \in \boldsymbol{C}_\kappa$ *and* $\chi \in (\mu, 2^\mu)$ *or just* $\kappa = \mathrm{cf}(\mu) < \mu$ *and* $\chi \in (\mu, \mathrm{pp}^{+}_{J^{\mathrm{bd}}_\kappa}(\mu))$, *see Definition* 0.12 (5), then *there is a* μ^+*-free* $\mathcal{F} \subseteq {}^\kappa\mu$ *of cardinality* χ, *even* $<_{J^{\mathrm{bd}}_\kappa}$*-increasing* μ^+*-free sequence of length* χ; *moreover if* $(\prod_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa})$ *is* χ^+*-directed and* $\mathcal{F}_* \subseteq \prod_{i<\kappa} \lambda_i$ *is such that* $\mathcal{F}_*$ *is cofinal* or $(\mathcal{F}_*, <_{J^{\mathrm{bd}}_\kappa})$ *is well ordered of cardinality* $> \chi$, then *we can demand* $\mathcal{F} \subseteq \mathcal{F}_*$ *(and there is such a sequence* $\langle \lambda_i : i < \kappa \rangle$*).*

Proof. Clause (a) holds by [22, Chapter II, Section 5], [22, Chapter VII, Section 1] and clause (b) by [22, Chapter IX, Section 5] and clause (c) holds by [22, Chapter II, Claim 2.3, p. 53, and Claim 1.5A, p. 51]. □

Observation 1.4. (1) *If* J *is a* σ*-complete ideal on* κ, $\mathcal{F} \subseteq {}^\kappa\mu$, $\theta_0 < \theta_1 < \theta_2$, $(\theta_1, \theta_2) \in \mathrm{issp}_J(\mathcal{F})$ *and* $\mathrm{cov}(\theta_1, \theta_0, \kappa^+, \sigma) < \mathrm{cf}(\theta_2)$ *recalling Definition* 0.13 (*e.g.* $\theta_1 < \theta_0^{+\omega}, \theta_1 < \mathrm{cf}(\theta_2)$), then $(< \theta_0, \theta_2) \in \mathrm{issp}_J(\mathcal{F})$.

(2) *If in addition* $\mathcal{F}$ *is tree-like,* $\alpha < \kappa \Rightarrow \alpha \in J$ *and* κ *is regular,* then *the condition* $\mathrm{cov}(\theta_1, \theta_0, \kappa^+, \kappa) < \mathrm{cf}(\theta_2)$ *suffices.*

(3) *Assume* J *is an ideal on* κ *and* $\mathcal{F} \subseteq {}^\kappa\mu$ *is* (θ, σ, J)*-free. If* $\sigma = \mathrm{cf}(\sigma)$ *and* $\kappa < \sigma$, then *for every* $\mathcal{F}' \subseteq \mathcal{F}$ *of cardinality* $< \theta$ *we can find* $\langle u_f : f \in \mathcal{F}' \rangle$ *as in Definition* 1.2 (1) *and a partition* $\bar{\mathcal{F}}' = \langle \mathcal{F}'_\varepsilon : \varepsilon < \varepsilon(*) \le |\mathcal{F}'| \rangle$ *of* $\mathcal{F}'$ *into sets each of cardinality* $< \sigma$ *such that*

$$\langle \{ f(i) : \text{for some } i \text{ we have } f \in \mathcal{F}'_\varepsilon, i \in \kappa \backslash u_f \} : \varepsilon < \varepsilon(*) \rangle$$

is a sequence of pairwise disjoint subsets of μ. *If we waive "*$\kappa < \sigma$*" still for each* $i < \kappa$, *there is such an* $\bar{\mathcal{F}}^i$ *which can serve for this* i.

(4) *If* J *is a* κ*-complete ideal on* κ *and* $\mathcal{F} \subseteq {}^\kappa\mu$ *is* (θ, κ^+, J)*-free hence*

$$f_1 \neq f_2 \in \mathcal{F} \Rightarrow \{ i < \kappa : f_1(i) = f_2(i) \} \in J,$$

then $\mathcal{F}$ *is* (θ, J)*-free.*

Proof. (1) This should be clear as in [22, Chapter II, Section 6], but we give details.

To this end, let $\mathcal{P}$ exemplify $\mathrm{cov}(\theta_1, \theta_0, \kappa^+, \sigma)$, i.e., $\mathcal{P} \subseteq [\theta_1]^{<\theta_0}$ has cardinality $\mathrm{cov}(\theta_1, \theta_0, \kappa^+, \sigma)$ and every $u \in [\theta_1]^{\leq \kappa}$ is included in the union of $< \sigma$ members of $\mathcal{P}$.

By the assumption "$(\theta_1, \theta_2) \in \mathrm{issp}_J(\mathcal{F})$" there is $\mathcal{U} \subseteq \mu$ which has cardinality $\leq \theta_1$ such that $\mathcal{F}' = \mathcal{F}'_{\mathcal{U}} := \{\eta \in \mathcal{F} : \{i < \kappa : \eta(i) \in \mathcal{U}\} \in J^+\}$ has cardinality $\geq \theta_2$.

Let g be a one-to-one function from $\mathcal{U}$ into θ_1 and fix for a while $\eta \in \mathcal{F}'$. Let $v_\eta := \{g(\eta(i)) : i < \kappa$ and $\eta(i) \in \mathcal{U}\}$, clearly it is $\in [\theta_1]^{\leq\kappa}$ hence there is $\mathcal{P}_\eta \subseteq \mathcal{P}$ of cardinality $< \sigma$ such that $v_\eta \subseteq \bigcup\{u : u \in \mathcal{P}_\eta\}$. So $\{\{i < \kappa : \eta(i) \in \mathcal{U}$ and $g(\eta(i)) \in u\} : u \in \mathcal{P}_\eta\}$ is a family of $< \sigma$ subsets of κ whose union belongs to J^+. But J is a σ-complete ideal on κ hence:

⊛ there is a $u_\eta \in \mathcal{P}_\eta$ such that $\{i < \kappa : \eta(i) \in \mathcal{U}$ and $g(\eta(i)) \in u_\eta\} \in J^+$.

So $\langle u_\eta : \eta \in \mathcal{F}' \rangle$ is well defined and $\eta \in \mathcal{F}' \Rightarrow u_\eta \in \mathcal{P}$ but

$$|\mathcal{P}| = \mathrm{cov}(\theta_1, \theta_0, \kappa^+, \sigma) < \mathrm{cf}(\theta_2)$$

and the family $\mathcal{F}'$ was chosen such that $|\mathcal{F}'| \geq \theta_2$, hence for some $u_2 \in \mathcal{P}$ the family $\mathcal{F}'' := \{\eta \in \mathcal{F}' : u_\eta = u_2\}$ has cardinality $\geq \theta_2$. But then letting

$$u_1 = \{\alpha \in \mathcal{U} : g(\alpha) \in u_2\}$$

we have

$$\begin{aligned} \mathcal{F}_* &:= \{\eta \in \mathcal{F} : \{i < \kappa : \eta(i) \in u_1\} \in J^+\} \\ &= \{\eta \in \mathcal{F} : \{i < \kappa : g(\eta(i)) \in u_2\} \in J^+\} \supseteq \mathcal{F}'' \end{aligned}$$

hence the subfamily $\mathcal{F}''$ of $\mathcal{F}$ has cardinality $\geq |\mathcal{F}''| \geq \theta_2$. So u_1 exemplifies that $(< \theta_0, \theta_2) \in \mathrm{issp}_J(\mathcal{F})$, the desired conclusion.

(2) As without loss of generality $J = J_\kappa^{\mathrm{bd}}$ and this ideal is κ-complete.

The proof of (3) is easy, too, and (4) follows from (3) and Claim 1.5 (1). □

Claim 1.5. *Let $\mathcal{F} \subseteq {}^\kappa\mu$ and J an ideal on κ be such that*

$$f_1 \neq f_2 \in \mathcal{F} \Rightarrow \{i < \kappa : f_1(i) = f_2(i)\} \in J.$$

(1) *$\mathcal{F}$ is (θ^+, J)-free if J is θ-complete.*

(2) *If $\kappa < \sigma < \lambda$,* then *$\mathcal{F}$ is (λ, σ, J)-free iff there are no regular $\partial \in [\sigma, \lambda)$ and pairwise distinct $f_\alpha \in \mathcal{F}$ for $\alpha < \partial$ such that $S = \{\delta < \partial :$ for some $\zeta \in [\delta, \partial)$ the set $\{i < \kappa : f_\zeta(i) \in \{f_\varepsilon(i) : \varepsilon < \delta\}$ belongs to $J^+\}$ is a stationary subset of ∂.*

(2A) *In part (2), the two equivalent statements imply that for no $\theta \in [\sigma, \lambda)$, $\theta \in \mathrm{issp}_J(\mathcal{F})$.*

(3) *Assume we are given a sequence* $\bar{f} = \langle f_\alpha : \alpha < \alpha_* \rangle$ *of members of* ${}^\kappa\mathrm{Ord}$ *with no repetitions, and* $\lambda = \mathrm{cf}(\lambda) > \kappa$ *and* J *is an ideal on* κ. Then $\bar{f}$ *is not* (λ, λ, J)*-free as a set iff there is an increasing sequence* $\langle \alpha_\varepsilon : \varepsilon < \lambda \rangle$ *of ordinals* $< \alpha_*$ *such that the set*

$$S = \{\varepsilon < \lambda : \mathrm{cf}(\varepsilon) \leq \kappa \text{ and } \{i < \kappa : (\exists \zeta < \varepsilon)(f_{\alpha_\varepsilon}(i) = f_{\alpha_\zeta}(i))\} \in J^+\}$$

is a stationary subset of λ.

(4) *In* (3) *if in addition* $\bar{f}$ *is tree-like, i.e.,* $f_\alpha(\varepsilon) = f_\beta(\varepsilon) \Rightarrow f_\alpha \restriction \varepsilon = f_\beta \restriction \varepsilon$ *and* $J_\kappa^{\mathrm{bd}} \subseteq J$, *then* $S \subseteq S^\lambda_{\mathrm{cf}(\kappa)}$.

Proof. Part (1) is easy and more is proved in the proof of Claim 1.8 below; part (2) is proved in proving ⊞ in the proof of Claim 3.4. Again, (2A) is easy, see Definition 1.2 (6). Part (3) follows from Observation 1.4, while part (4) is proved like part (2), see more in Claim 1.6. □

Claim 1.6. *Assume* $\lambda > \mu \geq \kappa_2 \geq \kappa_1 = \theta = \mathrm{cf}(\theta)$.

(1) $\mathcal{F} \subseteq {}^\theta\mathrm{Ord}$ *is* (κ_2, κ_1)*-free* iff $\mathcal{F}$ *is* (κ^+, κ)*-free for every regular* $\kappa \in [\kappa_1, \kappa_2)$.

(2) *There is a* $(\kappa^{+\omega+1}, \kappa)$*-free set* $\mathcal{F} \subseteq {}^\omega\mu$ *of cardinality* λ iff *for every* $n < \omega$ *there is a* (κ^{+n}, κ)*-free set* $\mathcal{F} \subseteq {}^\omega\mu$ *of cardinality* λ.

(3) *Assume* $\lambda > \mu \geq \kappa^{+\omega}, \mu > \sigma = \mathrm{cf}(\mu)$ *and* $(\forall \alpha < \mu)(|\alpha|^\chi < \mu)$. *If* $\mathcal{F}_\varepsilon \subseteq {}^\sigma\mu$ *has cardinality* λ *for* $\varepsilon < \chi$, then *we can find* $\mathcal{F} \subseteq {}^\sigma\mu$ *of cardinality* λ *such that:*

- *if for some* ε, $\mathcal{F}_\varepsilon$ *is* (κ_2, κ_1)*-free,* then $\mathcal{F}$ *is* (κ_2, κ_1)*-free.*

Remark 1.7. See Claims 1.5 and 3.4.

Proof. Part (1) is a consequence of Claim 1.5 (2), and part (2) follows from Observation 3.10 (1A).

(3) Let $\langle \lambda_i : i < \sigma \rangle$ be increasing with limit $\mu, \lambda_i = \lambda_i^\chi$ and let

$$\mathrm{cd}_i : \mathscr{H}_{\leq \chi}(\lambda_i) \to \lambda_i$$

be one-to-one and onto; and let $\mathcal{F}_\varepsilon = \{f^\varepsilon_\alpha : \alpha < \lambda\}$. Lastly, $f_\alpha \in {}^\sigma\mu$ is defined by $f_\alpha(i) = \mathrm{cd}_i(\langle f^\varepsilon_\alpha \cap (\lambda_i \times \lambda_i) : \varepsilon < \chi \rangle)$. □

In particular recalling Notation 0.3 (2):

Claim 1.8. (1) *Assume* $\mathcal{F} \subseteq {}^\kappa\mu$ *is* $(\theta, \kappa^{++}, J^{\mathrm{bd}}_\kappa)$*-free and* $\kappa = \mathrm{cf}(\kappa) < \mu$. Then *we can find* $\mathcal{G} \subseteq {}^{(\kappa^+ \times \kappa)}\mu$ *of cardinality* $|\mathcal{F}|$ *such that* $\mathcal{G}$ *is* $(\theta, J^{\mathrm{bd}}_{\kappa^+ \times \kappa})$*-free.*

(2) *If* $\lambda = \mathrm{cf}(\lambda) > \mu > \kappa = \mathrm{cf}(\kappa)$, *there is a* θ*-free* $\mathscr{F} \subseteq {}^{\kappa}\mu$ *of cardinality* $\geq \lambda$, $S \subseteq S^{\lambda}_{\kappa}$ *is stationary and for simplicity* $\delta \in S \Rightarrow \mu \cdot \delta = \delta$, then *there is a* θ*-free strict* S*-ladder system* $\langle C_\delta : \delta \in S \rangle$.

(2A) *In part* (2) *also for every* $\sigma = \mathrm{cf}(\sigma) \in (\kappa, \lambda)$ *and stationary* $S \subseteq S^{\lambda}_{\sigma}$ there *is a* $(\theta, J_{\sigma * \theta})$*-free strict* S*-ladder system* $\langle C_\delta : \delta \in S \rangle$.

Proof. (1) If $\kappa^+ = \mu$, then the construction below gives $\mathscr{G} \subseteq {}^{\kappa^+ \times \kappa}(\kappa^+ + \mu)$ rather than $\mathscr{G} \subseteq {}^{\kappa^+ \times \kappa}(\mu)$, but this is enough so we shall ignore this point. For $f \in \mathscr{F}$ let $g_f : \kappa^+ \times \kappa \to \mu$ be defined by:

$(*)$ for $\zeta < \kappa^+, i < \kappa$ we let $g_f(\zeta, i) = \kappa^+ \cdot f(i) + \kappa \cdot \zeta + i$.

Let $\mathscr{G} = \{g_f : f \in \mathscr{F}\}$. Now:

$(*)$ if $f_1 \neq f_2 \in \mathscr{F}$, then $g_{f_1} \neq g_{f_2}$ and moreover

$$\{(\zeta, i) \in \kappa^+ \times \kappa : g_{f_1}(\zeta, i) = g_{f_2}(\zeta, i)\} \in J^{\mathrm{bd}}_{\kappa^+ \times \kappa}.$$

(Why? By Definition 1.2 (1) we know $i(*) := \sup\{i < \kappa : f_1(i) = f_2(i)\} < \kappa$ and hence we have $\{(\zeta, i) \in \kappa^+ \times \kappa : g_{f_1}((\zeta, i)) = g_{f_2}((\zeta, i))\} \subseteq \{(\zeta, i) : \zeta < \kappa^+$ and $i < i(*)\} \in J^{\mathrm{bd}}_{\kappa^+ \times \kappa}$, so we are done.)

$(*)_2$ assume $\mathscr{G}' \subseteq \mathscr{G}$ is of cardinality $< \theta$ and we shall find $\langle u^1_g : g \in \mathscr{G}' \rangle$ as required.

(Why? We can choose $\mathscr{F}' \subseteq \mathscr{F}$ of cardinality $< \theta$ such that $\mathscr{G}' = \{g_f : f \in \mathscr{F}'\}$. We can apply the assumption "$\mathscr{F}$ is (θ, κ^{++})-free" and let $\langle u_f : f \in \mathscr{F}' \rangle$ be as in Definition 1.2 (1); moreover let $\langle \mathscr{F}_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ be as guaranteed in Observation 1.4 (3), so in particular $|\mathscr{F}_\varepsilon| \leq \kappa^+$.)

For each $\varepsilon < \varepsilon(*)$ let $\langle f_{\varepsilon,\iota} : \iota < |\mathscr{F}_\varepsilon| \rangle$ list the family $\mathscr{F}_\varepsilon$ with no repetitions and let $g_{\varepsilon,\iota} = g_{f_{\varepsilon,\iota}}$. First assume $|\mathscr{F}_\varepsilon| \leq \kappa$, then for $\iota < |\mathscr{F}_\varepsilon|$ we let

$$u^0_{\varepsilon,\iota} = \Big\{ i < \kappa : \text{the sequence } \langle f_{\varepsilon,\iota_1}(i) : \iota_1 \leq \iota \rangle \text{ has some repetitions } \textit{or}^1 \\ i \in \bigcup \{u_{f_{\varepsilon,\iota_1}} : \iota_1 \leq \iota\} \Big\}.$$

As J^{bd}_κ is κ-complete, clearly $u^0_{\varepsilon,\iota} \in J^{\mathrm{bd}}_\kappa$ and we let

$$u^1_{g_{\varepsilon,\iota}} := \kappa^+ \times u^0_{\varepsilon,\iota}.$$

Second, assume $|\mathscr{F}_\varepsilon| = \kappa^+$ and for each $\zeta \in [\kappa, \kappa^+)$ let $\langle \xi(\zeta, j) : j < \kappa \rangle$ list ζ without repetition and for $j < \kappa, \zeta < \kappa^+, \zeta \geq \kappa$, let

$$u^0_{\varepsilon,\zeta,j} = \{i < \kappa : \text{the sequence } \langle f_{\varepsilon,\xi(\zeta,j_1)}(i) : j_1 \leq j \rangle \text{ has some repetitions or} \\ i \in \{u_{f_{\varepsilon,\xi(\zeta,j_1)}} : j_1 \leq j\}\}$$

and for $\iota < |\mathcal{F}_\varepsilon|$ let

$$u^1_{g_{\varepsilon,\iota}} = \{(\zeta, i) : \zeta \in (\kappa + \iota, \kappa^+), i < \kappa \text{ and } i \in u^0_{\varepsilon,\zeta,j} \text{ where } j \text{ is the unique } j < \kappa \text{ such that } \iota = \xi(\zeta, j)\}.$$

Now check that $\langle u^1_{g_{\varepsilon,\iota}} : \varepsilon < \varepsilon(*)$ and $\iota < |\mathcal{F}_\varepsilon|\rangle$ is as required, i.e., witnessing the freeness of $\mathcal{F}'$.

(2) Let $\langle f_\delta : \delta \in S\rangle$ be a sequence of pairwise distinct members of $\mathcal{F}$ and for $\delta \in S$ let $\langle \alpha_{\delta,i} : i < \kappa\rangle$ be an increasing sequence of ordinals with limit δ. Lastly, let $C_\delta = \{\mu\alpha_{\delta,i} + f_\delta(i) : i < \kappa\}$ for $\delta \in S$ recalling $\delta \in S \Rightarrow \delta = \mu \cdot \delta$.

The proof of (2A) is similar. □

How is this connected to abelian groups?

Definition 1.9. (1) We say that G is an abelian group derived from $\mathcal{F} \subseteq {}^\omega\mu$ *when* G is generated by $\{x_\alpha : \alpha < \mu\} \cup \{y_{\eta,n} : \eta \in \mathcal{F}$ and $n < \omega\}$ freely except a set of equations $\Gamma = \bigcup\{\Gamma_\eta : \eta \in \mathcal{F}\}$ where each Γ_η has the form

$$\{y_{\eta,n} = a_{\eta,n} \cdot y_{\eta,n+1} + x_{\eta(n),n} : n < \omega\}$$

where $a_{\eta,n} \in \mathbb{Z}\backslash\{-1, 0, 1\}$.

(2) We say that G is an abelian group derived from $\mathcal{F} \subseteq {}^{\omega_1\times\omega}\mu$ *when* G is generated by $\{x_{\alpha,\varepsilon,n} : \alpha < \mu, \varepsilon < \omega_1, n < \omega\} \cup \{y_{\eta,\varepsilon,n} : \eta \in \mathcal{F}, \varepsilon < \omega_1, n < \omega\} \cup \{z_{\eta,n} : \eta \in \mathcal{F}, n < \omega\}$ freely except a set of equations $\Gamma = \bigcup\{\Gamma_\eta : \eta \in \mathcal{F}\}$ where each Γ_η has the form

$$\{y_{\eta,\varepsilon,n} = a_{\eta,\varepsilon,n} y_{\eta,\varepsilon,n+1} + b_{\eta,\varepsilon,n} z_{\eta,\rho_{\eta,\varepsilon}(n)} + c_{\eta,\varepsilon,n} x_{\eta(\varepsilon,n),\varepsilon,n} : \varepsilon < \omega_1, n < \omega\}$$

where one has $a_{\eta,\varepsilon,n} \in \mathbb{Z}\backslash\{-1, 0, 1\}$, $b_{\eta,\varepsilon,n} \in \mathbb{Z}\backslash\{0\}$, $c_{\eta,\varepsilon,n} \in \mathbb{Z}$, $\rho_{\eta,\varepsilon} \in {}^\omega\omega$ is increasing and $\varepsilon_1 < \varepsilon_2 < \omega_1 \Rightarrow \mathrm{Rang}(\rho_{\eta,\varepsilon_1}) \cap \mathrm{Rang}(\rho_{\eta,\varepsilon_2})$ is finite.

Remark 1.10. (1) Here choosing $\rho_{\eta,\varepsilon} \in {}^\omega(\omega + \varepsilon)$ is all right but not for Section 4.

(2) So in Definition 1.9 (1) if $a_{\eta,n} = n + 1$, considering G as a metric space with

$$\boldsymbol{d}_G(x, y) = \inf\{2^{-n} : x - y \in (n!)G\}$$

we have

$$y_{\eta,n} = \sum_{m\geq n} \frac{m!}{n!} x_{\eta(m)} \quad \text{for } \eta \in \mathcal{F},\ n < \omega.$$

In general for $n_1 < n_2$ we have

$$y_{\eta,n_1} = \left(\sum_{m=n_1}^{n_2-1} \prod_{l=n_1}^{m} a_{\eta,l} x_{\eta(m),m}\right) + \left(\prod_{m=n_1}^{n_2-1} a_{\eta,m}\right) y_{\eta,n_2}.$$

Easily (see [4] on the subject):

Claim 1.11. *If* $\mathcal{F} \subseteq {}^{\omega}\mu$ *is* θ*-free* or $\mathcal{F} \subseteq {}^{\omega_1 \times \omega}\mu$ *is* $(\theta, J^{\mathrm{bd}}_{\omega_1 \times \omega})$*-free,* then *any abelian group derived from it is* θ*-free.*

Similarly to Observation 1.4, we get

Claim 1.12. (1) *If* $\mathcal{F} \subseteq {}^{\mathrm{Dom}(J)}\mu$ *is* (θ, σ_2^+, J)*-free,* J *is a* (σ_2, σ_1^+)*-regular*[2] *and* σ_1*-complete ideal,* then $\mathcal{F}$ *is* (θ, J)*-free.*

(2) *Assume* I, J *are ideals on* S, T *respectively. If* $\mathcal{F} \subseteq {}^{S}\mu$ *is* (θ, σ, I)*-free,* π *is a function from* T *onto* S *and* $\pi''(J) := \{\{\pi(i) : i \in s\} : s \in J\} \supseteq I,$ then $\mathcal{F} \circ \pi = \{f \circ \pi : f \in \mathcal{F}\} \subseteq {}^{T}\mu$ *is* (θ, σ, J)*-free.*

Definition 1.13. (1) Let $(D\ell)_S$ mean that:

(a) $\lambda = \sup(S)$ is a regular uncountable cardinal,

(b) S is a stationary subset of λ,

(c) there is a witness $\bar{\mathcal{P}}$ by which we mean:

(α) $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha \in S \rangle$,

(β) $\mathcal{P}_\alpha \subseteq \mathcal{P}(\alpha)$ has cardinality $< \lambda$,

(γ) for every subset $\mathcal{U}$ of λ, the set $S_{\mathcal{U}} := \{\delta \in S : \mathcal{U} \cap \delta \in \mathcal{P}_\delta\}$ is a stationary subset of λ.

(2) Let $(D\ell)^*_S$ be defined similarly but in (c, γ) we demand $S \backslash S_{\mathcal{U}}$ is not stationary.

(3) We write $(D\ell)_{D,S}, (D\ell)^*_{D,S}$ when D is a normal filter on λ and replace "stationary" by "$\in D^+$".

Definition 1.14. (1) For a regular uncountable cardinal λ let

$$\check{I}[\lambda] = \{S \subseteq \lambda : \text{some pair } (E, \bar{a}) \text{ witnesses } S \in \check{I}(\lambda), \text{ see below}\}.$$

(2) We say that $(E, \bar{u})$ is a witness for $S \in \check{I}[\lambda]$ *if*:

(a) E is a club of the regular cardinal λ,

(b) $\bar{u} = \langle u_\alpha : \alpha < \lambda \rangle, u_\alpha \subseteq \alpha$ and $\beta \in u_\alpha \Rightarrow u_\beta = \beta \cap u_\alpha$,

(c) for every $\delta \in E \cap S, u_\delta$ is an unbounded subset of δ of order-type $< \delta$ (and δ is a limit ordinal).

[2] That is, there are $A_\alpha \in J$ for $\alpha < \sigma_2$ such that

$$u \subseteq \sigma_2 \wedge |u| \geq \sigma_1^+ \Rightarrow \bigcup \{A_\alpha : \alpha \in u\} = \mathrm{Dom}(J).$$

Claim 1.15. (1) *If* $\lambda = \lambda^{<\lambda}$, $\kappa = \mathrm{cf}(\kappa) < \lambda$, $\alpha < \lambda \Rightarrow |\alpha|^{\langle\kappa\rangle_{\mathrm{tr}}} < \lambda$ *and* $S \subseteq S^\lambda_\kappa$ *is a stationary subset of* λ, then $(D\ell)_S$.

(2) *If* μ *is a strong limit cardinal and* $\lambda = \mathrm{cf}(\lambda) > \mu$, then

$$\mu > \sup\{\kappa < \mu : \kappa = \mathrm{cf}(\kappa) \text{ and } (\exists\alpha < \lambda)(|\alpha|^{\langle\kappa\rangle_{\mathrm{tr}}} \geq \lambda)\}.$$

(3) *If* $\lambda = \lambda^{<\lambda} > \beth_\omega$, then $\{\kappa : \kappa = \mathrm{cf}(\kappa)$ *and* $\beth_\omega(\kappa) < \lambda$ *and* $\neg(D\ell)_{S^\lambda_\kappa}$ *or just* $\neg(D\ell)_S$ *for some stationary* $S \in \check{I}_\kappa[\lambda]\}$ *is finite where* $\check{I}_\kappa[\lambda]$ *is from Definition* 1.14.

(4) *If* $\lambda = \chi^+$ *and* $S \subseteq \lambda$ *is stationary,* then $(D\ell)^*_S$ *is equivalent to* $\diamondsuit_S$.

(5) *If* $\lambda > \kappa$ *are regular and* $S \in \check{I}_\kappa[\lambda]$ *is a stationary subset of* λ, then *there is a shallow strict* S*-club system.*

Proof. For parts (1)–(3) see [28]. Part (4) is a result of Kunen; for a proof of a somewhat more general result see [16]. For (5) see [11] or [18]. □

Discussion 1.16. (1) Of course, $(D\ell)_S$ is a relative of the diamond, see [15].

(2) $(D\ell)_S$ is consistently not equivalent to $\diamondsuit^*_S$ when λ is a limit (regular) cardinal.

(3) Trivially $(D\ell)^*_S \Rightarrow (D\ell)_S$.

For $\boxdot_3$ of Section 0, (it was previously known only when χ is regular by using partial squares which holds by [20, Section 4]).

Fact 1.17. If $\lambda = 2^\chi = \chi^+ > \kappa = \mathrm{cf}(\kappa)$ and $\kappa \neq \mathrm{cf}(\chi)$, then $\diamondsuit_{S^\lambda_\kappa}$, moreover $\diamondsuit_S$ for every stationary $S \subseteq S^\lambda_\kappa$.

Proof. See [32]. □

Now by [27, The Main Claim 1.10], this is used in Theorems 1.22 and 1.31.

Theorem 1.18. *We have* $\mathrm{BB}(\lambda, \bar{C}, (\lambda, \theta), < \mu)$ *recalling Definition* 0.7 (1), (3), (4) when:

(a) $\mu \in \mathbf{C}_\kappa, \lambda = \mathrm{cf}(2^\mu)$ *and* $\theta < \mu, \sigma = \mathrm{cf}(\sigma) < \mu$,

(b) $S \subseteq S^\lambda_\sigma$ *is stationary,*

(c) $\bar{C} = \langle C_\delta : \delta \in S\rangle, C_\delta \subseteq \delta, |C_\delta| \leq \mu$ *recalling*[3] *Definition* 0.7 (4),

(d) $\chi < 2^\mu \Rightarrow \chi^{\langle\sigma\rangle_{\mathrm{tr}}} < 2^\mu$,

(e) $\bar{C}$ *is shallow, that is,* $|\{C_\delta \cap \alpha : \alpha \in C_\delta\}| < \lambda$ *for* $\alpha < \lambda$.

[3] Actually $2^{|C_\delta|} \leq 2^\mu$ is sufficient.

Remark 1.19. (1) Of course, if $S \in \check{I}_\kappa[\lambda]$ is stationary, *then* there is a $\bar{C}$ as in clauses (c) and (e) (and, of course, (b)).

(2) There are such stationary S as $\kappa^+ < \mu < \lambda$ by [21].

Definition 1.20. We say a filter D on a set X is weakly λ-saturated *when* there is no partition $\langle X_\alpha : \alpha < \lambda \rangle$ of X such that

$$\alpha < \lambda \Rightarrow X_\alpha \in D^+ := \{Y \subseteq X : X \backslash Y \notin D\}.$$

A notable consequence of the analysis in this work is the Black Box Trichotomy Theorem 1.22.

Remark 1.21. Using $\bar{C} = \langle C_\delta : \delta \in S \rangle$ below or using $\bar{f} = \langle f_\delta : \delta \in S \rangle$ where f_δ is an increasing function from $\mathrm{otp}(C_\delta)$ onto C_δ does not make a real difference.

Black Box Trichotomy Theorem 1.22. *If $\mu \in \mathbf{C}_\kappa$ and $\kappa > \sigma = \mathrm{cf}(\sigma)$,* then *at least one of the following holds:*

$(A)_{\mu,\kappa}$ *there is a μ^+-free $\mathscr{F} \subseteq {}^\kappa\mu$ of cardinality 2^μ,*

(B) (a) $\lambda := 2^\mu = \lambda^{<\lambda}$ *(so λ is regular) and* $\chi < \lambda \Rightarrow \chi^\sigma < \lambda$,

$(b)_{\lambda,\mu,\sigma}$ *if $S \subseteq S^\lambda_\sigma$ is stationary, $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is a weak ladder system (i.e., $C_\delta \subseteq \delta$ so, e.g., the choice $C_\delta = \delta$ for $\delta \in S$ is all right),* then:

$(c)_{\lambda,\mu,\sigma}$ *letting $J^{\mathrm{nst}}_S = \{A \subseteq \lambda : A \cap S$ is not stationary in $\lambda\}$ we have*[4]

(i) $\mathrm{BB}(J^{\mathrm{nst}}_S, \bar{C}, \theta, \leq \mu)$ *for every $\theta < \mu$ provided that we have $\delta \in S \Rightarrow |C_\delta| < \mu$, see Definition* 0.7 (4),

(ii) $\mathrm{BB}(J^{\mathrm{nst}}_S, \bar{C}, (2^\mu, \theta), < \lambda)$ *for any $\theta < \mu$,*

$(C)_{\mu,\kappa}$ (a) $\lambda_2 = 2^\mu$ *is regular,* $\chi < \lambda_2 \Rightarrow \chi^\sigma < \lambda_2$ *and* $\lambda_1 = \min\{\partial : 2^\partial > 2^\mu\}$ *is (regular and)* $< 2^\mu$,

(b) *like* $(b)_{\lambda,\mu,\sigma}$ *of clause* (B) *for* $\lambda = \lambda_2$ *but* $|C_\delta| < \lambda_1$ *for* $\delta \in S$ *(so $C_\delta = \delta$ is not all right),*

(c) $\mathrm{BB}(J^{\mathrm{nst}}_S, \mu^+, \theta, \kappa)$ *for any $\theta < \mu$ and any stationary subset S of λ_1,*

(c)′ *like* $(b)_{\lambda,\mu,\sigma}$ *of* (B) *but for $\lambda = \lambda_1$, S a club or just S not in the weak diamond ideal* ([2]).

[4] What about freeness? We may get it by the choice of $\bar{C}$, also if $\bar{C}$ is a ladder system (particularly if strictly), we get a weak form, e.g., stability.

Remark 1.23. (1) If $\kappa = \aleph_0$ above, then there is no infinite cardinal $\sigma < \kappa$ as required, but the proof still gives something (e.g. for $\sigma = \aleph_1$). In this case we cannot get "for every stationary $S \subseteq S^{\lambda}_{\sigma}$", still by [28, 3.1] one has "for all but finitely many regular $\sigma < \mu$ for almost every stationary $S \subseteq S^{\lambda}_{\sigma}$"; see Claim 1.15.

(2) Assume $\mu \in \boldsymbol{C}_{\kappa}, \lambda = 2^{\mu} = \chi^{+}$. If χ is regular then (A) of Theorem 1.22 holds because by Claim 3.12, there is a shallow family $\bar{C} = \langle C_{\delta} : \delta \in S^{\lambda}_{\kappa}$, μ divides $\delta\rangle$ with $C_{\delta} \subseteq \delta = \sup(C_{\delta})$, $\mathrm{otp}(C_{\delta}) = \kappa$, and $\bar{C}$ is μ^{+}-free. If $\kappa \neq \mathrm{cf}(\chi)$ and $\lambda = \lambda^{<\lambda}$, then for every stationary $S \subseteq S^{\lambda}_{\kappa}$ we have $\diamondsuit_S$, see [32].

(3) What happens if $\lambda := 2^{\mu}$ is weakly inaccessible? Here it seems plausible to assume, for some μ_0

(*) (a) $\mu \leq \mu_0 < \lambda$,
(b) $\alpha < \lambda \Rightarrow \lambda > \mathrm{cov}(|\alpha|, \mu_0^{+}, \mu, 2)$,
(b)$^{+}$ $\alpha < \lambda \Rightarrow \lambda > \mathrm{cov}(|\alpha|, \mu_0^{+}, \mu_0^{+}, 2)$.
Now (b)$^{+}$ implies (by [3])
(c) there is $\bar{\mathcal{P}}$ such that
(α) $\bar{\mathcal{P}} = \langle \mathcal{P}_{\alpha} : \alpha < \lambda \rangle$,
(β) $|\mathcal{P}_{\alpha}| < \lambda$,
(γ) $\mathcal{P}_{\alpha} \subseteq \{u : |u| \leq \mu_0, u \text{ is a closed subset of } \alpha\}$,
(δ) if $\alpha \in u \in \mathcal{P}_{\beta}$, then $u \cap \alpha \in \mathcal{P}_{\alpha}$,
(ε) if $\delta < \lambda, \mathrm{cf}(\delta) \leq \mu_0$, then $\sup(u) = \delta$ for some $u \in \mathcal{P}_{\delta}$.

This is enough for the argument above.

(4) Does clause (b) in (*) above suffice?

Proof of Theorem 1.22. Recall that for every $\chi \in (\mu, 2^{\mu})$ there exists a μ^{+}-free $\mathcal{F} \subseteq {}^{\kappa}\mu$ of cardinality χ (see Claim 1.3 (c)).

If for some $\chi < 2^{\mu}$ we have $\chi^{\sigma} = 2^{\mu}$, then by Claim 3.6, clause (A) holds (when θ there stands for σ here), so we can assume there is no such χ. If 2^{μ} is a singular cardinal then by Observation 3.10 (3), clause (A) holds, so assume $\lambda := 2^{\mu}$ is regular. If $\lambda = \lambda^{<\lambda}$, we shall prove clause (B). Obviously clause (B, a) holds and (B, b, ii) holds by Theorem 1.18 above and clause (B, b, i) follows. Note that any strict club system $\langle C_{\delta} : \delta \in S\rangle$ is shallow as

$$|\{C_{\delta} \cap \alpha : \delta \in S \text{ satisfies } \alpha \in C_{\delta}\}| \leq |\alpha|^{<\sigma} \leq |\alpha|^{\sigma} < \lambda.$$

So assume $\lambda < \lambda^{<\lambda}$, hence necessarily there is a $\partial < \lambda$ such that $\lambda < 2^{\partial}$.

Assume $\lambda_1 = \min\{\chi : 2^{\chi} > 2^{\mu}\} < \lambda_2 := 2^{\mu}$. Then trivially clause (C, a) holds and by Conclusion 2.7 (1) clauses (c) and (c)$'$ of (C) hold. Clause (b) of (C) holds by [27], i.e., Theorem 1.18, because we are assuming $(\forall \chi < \lambda)(\chi^\sigma < \lambda)$ so clause (C) holds. □

Remark 1.24. How can the Black Box Trichotomy Theorem 1.22 help?

If possibility (A) holds for $\kappa \in \{\aleph_0, \aleph_1\}$, we have, e.g., abelian groups as in Definition 1.9; so we have $G_0 \subseteq_{\rm pr} G_1$ (that is, G_0 is a pure subgroup of the abelian group G_1) such that G_1 is torsion-free, G_0 is free, G_1 quite free, $|G_0| = \mu$ and G_1/G_0 is divisible, and a list of $|G_1| = 2^\mu$ partial endomorphisms of G_1 such that if $G_0 \subseteq_{\rm pr} G \subseteq_{\rm pr} G_1$, any endomorphism of G is included in one of the endomorphisms in the list. So by diagonalization we can build an endo-rigid group. On the other hand, possibilities (B)–(C) help in another way: as in black boxes, see [4,9]; this is continued in [35].

Recall the following:

Definition 1.25. Assume J is an ideal of κ and $\bar{f} = \langle f_\alpha : \alpha < \alpha(*)\rangle$ is a $<_J$-increasing sequence of members of ${}^\kappa{\rm Ord}$.

Let $S^{\rm gd}_{\bar{f}}$, the good set of S, be defined as follows:

$$S^{\rm gd}_{\bar{f}} = \{\delta < \lambda : {\rm cf}(\delta) > \kappa \text{ and we can find sequence } \bar{A} = \langle A_\alpha : \alpha \in u\rangle \text{ witnessing } \delta \text{ is a good point of } \bar{f}\}$$

which means:

- $u \subseteq \delta = \sup(u)$,
- $A_\alpha \in J$ for $\alpha \in u$,
- if $\alpha < \beta$ are from u and $i \in \kappa \backslash A_\alpha \backslash A_\beta$, then $f_\alpha(i) < f_\beta(i)$.

Claim 1.26. *The sequence $\bar{C}$ is $(\aleph_\kappa, J)$-free and even $(\theta^{+\kappa}, J)$-free* when:

(a) $\mu > {\rm cf}(\mu) = \kappa, \theta \in (\kappa, \mu)$ *is regular,*

(b) $\bar{\lambda} = \langle \lambda_i : i < \kappa\rangle$ *is a sequence of regular cardinals* $< \mu$ *with* $\lim_J(\bar{\lambda}) = \mu$,

(c) $J = J_{\theta * \kappa}$, *see Notation* 0.3 (3),

(d) $\lambda = {\rm tcf}(\prod_{i<\kappa} \lambda_i, <_{J^{\rm bd}_\kappa})$ *is exemplified by* $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$,

(e) $S \subseteq S^\lambda_\theta \cap S^{\rm gd}_{\bar{f}}$ *is stationary (on* $S^{\rm gd}_{\bar{f}}$, *see Definition* 1.25 *above),* $\delta \in S \Rightarrow \mu \mid \delta$,

(f) $\bar{C} = \langle C_\delta : \delta \in S\rangle$ *is a strict* λ*-ladder system such that* $C_\delta \subseteq \delta = \sup(C_\delta)$ *and* ${\rm otp}(C_\delta) = \theta$,

(g) *if* $\delta \in S, \alpha < \kappa$ *and* $i < \kappa$, *then the* $(\kappa\alpha + i)$*-th member of* C_δ *is equal to* $f_\delta(i)$ *modulo* μ.

Proof. The proof is as in Magidor–Shelah [10] where the assumptions are quite specific. □

Hence we get:

Conclusion 1.27. *Assume that* $\kappa = \mathrm{cf}(\mu) < \mu$ *and* $\lambda = \mathrm{cf}(\lambda) =^+ \mathrm{pp}_{J^{\mathrm{bd}}_\kappa}(\mu)$. Then *there is a* $(\kappa^{+\kappa+1}, J_{\kappa^+\times\kappa})$*-free strict ladder system* $\langle \eta_\delta : \delta \in S\rangle$ *for some stationary* $S \subseteq S^\lambda_{\kappa^+}$.

Remark 1.28. The statement is used in Theorem 1.31.

Proof. We shall apply Claim 1.26. As we are assuming $\mathrm{pp}_{J^{\mathrm{bd}}_\kappa}(\mu) =^+ \lambda = \mathrm{cf}(\lambda)$, there is a sequence $\bar\lambda = \langle \lambda_i : i < \kappa\rangle$ of regular cardinals $< \mu$ such that

$$\mu = \lim_J(\bar\lambda) \quad \text{and} \quad \lambda = \mathrm{tcf}\Big(\prod_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa}\Big)$$

and let $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ exemplify it; without loss of generality $\bar\lambda$ is increasing.

Now λ is regular $> \mu > \kappa^{++}$ hence by [21] there is a stationary $S \subseteq S^\lambda_{\kappa^+}$ which is from $\check I_\kappa[\lambda]$ hence by [23] without loss of generality $S \subseteq S^{\mathrm{gd}}_{\bar f}$.

As $S \in \check I_{\kappa^+}[\lambda]$, there is a strict club system

$$\bar C = \langle C_\delta : \delta \in S\rangle.$$

Easily without loss of generality $\bar C$ satisfies clause (g) of Claim 1.26. Hence by Claim 1.26, $\bar C$ is as required. □

Recall the following (see [22, Chapter II], more in [10]). Proving Claim 1.26 we in fact use:

Claim 1.29. *If* $\circledast$ *below holds,* then *we can find a* θ*-free,* (λ,κ)*-ladder system* $\bar C' = \langle C'_\delta : \delta \in S\rangle$ *such that* $(\forall \alpha \in C'_\delta)(\exists! \beta \in C_\delta)(\alpha + \mu = \beta + \mu)$. *Moreover there is* $\langle f_\delta : \delta \in S\rangle \in {}^S\mathcal{F}$ *without repetitions such that*

$$\begin{aligned} C'_\delta \subseteq \{\beta + i : \beta \in C_\delta, i < \mu \text{ and } \\ (\exists \alpha, j)(\mu \,|\, \alpha \wedge j < \mu \wedge \beta = \alpha + j \wedge \beta + i \\ = \alpha + \mathrm{cd}(\mathrm{otp}(C_\delta \cap \alpha), i, f_\delta(\mathrm{otp}(C_\delta \cap \alpha))\}, \end{aligned}$$

when

$\circledast$ (a) $S \subseteq \lambda$ *is stationary and* $\delta \in S \Rightarrow \mu \,|\, \delta$,

(b) $\bar C = \langle C_\delta : \delta \in S\rangle$ *is a* (λ,κ)*-ladder system,*

(c) $\mu < \lambda$ *and* $\mathcal{F} \subseteq {}^\kappa\mu$ *has cardinality* $\geq \lambda$ *and is* θ*-free,*

(d) $\mathrm{cd} : \kappa \times \mu \times \mu \to \mu$ *is one-to-one.*

Proof. The proof is straightforward. □

Remark 1.30. The problem in proving the conjecture $\mathrm{TDU}_{\aleph_\omega}$ is to have $(D\ell)_S$ assuming $\lambda = \lambda^{<\lambda}$; this would have solved the problem in Section 0. As in many cases here, this is very persuasive but we do not know how to prove this in full generality.

The following theorem will be useful showing that if (R is a suitable ring) $\mathrm{SP}_{\lambda,\theta}(R)$, see Definition 4.3, contains enough ideals (say $J^{\mathrm{bd}}_\kappa, J^{\mathrm{bd}}_{\kappa^+\times\kappa}, J^{\mathrm{bd}}_{\kappa^{++}\times\kappa^+}$), then $\mathrm{TDU}_{\kappa^{+\omega}}(R)$; $\mathbb{Z}$ "just" misses this criterion; see Claim 1.35.

Theorem 1.31. *For $\mu \in \mathbf{C}_\kappa$ one of the following holds:*

(A) $\mathrm{BB}(2^\mu, \mu^+, <\mu, \kappa)$,

(B) $\mathrm{BB}(\lambda, \mu^+, <\mu, \kappa)$ *where* $\lambda = \min\{\chi : 2^\mu < 2^\chi\}$,

(C) $\lambda := 2^\mu$ *satisfies* $\lambda = \lambda^{<\lambda}$ *and* $\mathrm{BB}(\lambda, \kappa^{+\omega+1}, <\mu, J_{\kappa^+\times\kappa})$,

(D) $\lambda := 2^\mu$ *satisfies* $\lambda = \lambda^{<\lambda}$ *and* $\mathrm{BB}(\lambda, \kappa^{+\omega+1}, <\mu, J_{\kappa^{++}\times\kappa^+})$ *and also*

- $\bullet_1$ *there is a* $\chi \in (\mu, \lambda)$ *such that* $\mathrm{cf}(\chi) = \kappa^+$ *and* $\chi^{\langle\kappa^+\rangle_{\mathrm{tr}}} =^+ \lambda$,
- $\bullet_2$ $\mathcal{F} \subseteq {}^{(\kappa^+)}\chi, |\mathcal{F}| = \lambda \Rightarrow (\kappa^+, \kappa^{++}) \in \mathrm{issp}(\mathcal{F})$.

Proof. First, if Theorem 1.22 case (A) or case (C) holds, then case (A) or case (B) respectively here holds too, so we can assume case (B) of Theorem 1.22 holds and in particular $\lambda := 2^\mu$ satisfies $\lambda = \lambda^{<\lambda}$.

Second, assume there is no $\chi \in (\mu, \lambda)$ such that $\lambda =^+ \chi^{\langle\kappa^+\rangle_{\mathrm{tr}}}$. Then by part (1) of Claim 1.15 we have $(D\ell)_S$ for every stationary $S \subseteq S^\lambda_{\kappa^+}$, and then by Conclusion 1.27, we can find stationary $S \subseteq S^\lambda_{\kappa^+}$ and (see Definition 0.17 (4)) a strict (λ, κ^+)-ladder system $\langle \eta_\delta : \delta \in S\rangle$ which is $(\kappa^{+\omega+1}, J_{\kappa^+\times\kappa})$-free; hence by Theorem 1.18 we have $\mathrm{BB}(\lambda, \kappa^{+\omega+1}, <\mu, J_{\kappa^+\times\kappa})$ so part (C) of the theorem holds.

Third, assume there are a $\chi_1 < \lambda$ such that $\lambda =^+ (\chi_1)^{\langle\kappa^+\rangle_{\mathrm{tr}}}$ and an $\mathcal{F} \subseteq {}^{(\kappa^+)}\mu$ of cardinality λ which is κ^{++}-free or just such that $(\kappa^+, \kappa^{++}) \notin \mathrm{issp}(\mathcal{F})$. *Then* by Claim 3.4, clause (A) of the theorem holds.

Fourth, assume that for $\ell = 1, 2$ for some $\chi_\ell < \lambda$ we have

$$(\chi_2)^{\langle\kappa^{+\ell}\rangle_{\mathrm{tr}}} =^+ 2^\lambda$$

so without loss of generality

$$\mathrm{pp}_{J^{\mathrm{bd}}_{\kappa^{+\ell}}}(\chi_\ell) =^+ 2^\lambda;$$

so the first assumption of "third" hold and its second (by Section 3), hence part (C) of the theorem holds.

So we can assume that none of the above apply, and we shall prove clause (D), first $\bullet_1$ and $\bullet_2$. By "second" above without loss of generality we can choose $\chi_1 \in (\mu, \lambda)$ such that $(\chi_1)^{\langle \kappa^+ \rangle_{\rm tr}} =^+ \lambda$ and without loss of generality $\mathrm{cf}(\chi_1) = \kappa^+$, $\mathrm{pp}_{J^{\rm bd}_{\kappa^+}}(\chi_1) =^+ \lambda$ (by [25]), so $\bullet_1$ holds.

By "third" without loss of generality there is no $\mathscr{F} \subseteq {}^{(\kappa^+)}\mu$ of cardinality λ such that $(\kappa^+, \kappa^{++}) \notin \mathrm{issp}(\mathscr{F})$, hence $\bullet_2$ holds.

Now by "fourth" we can assume there is no $\chi_2 \in (\mu, \lambda)$ with $\lambda =^+ \chi_2^{\langle \kappa^{++} \rangle_{\rm tr}}$, hence by Claim 1.15 (1) for every stationary $S \subseteq S^\lambda_{\kappa^{++}}$ we have $(D\ell)_S$. Again we apply Conclusion 1.27 with χ_2 here for μ there and we can find a stationary set $S \subseteq S^\lambda_{\kappa^{++}}$ and a strict ladder system $\langle \eta_\delta : \delta \in S \rangle$ being $(\kappa^{+\omega+1}, J_{\kappa^{++} \times \kappa^+})$-free, hence by Theorem 1.18 we have $\mathrm{BB}(\lambda, \kappa^{+\omega+1}, < \mu, J_{\kappa^{++} \times \kappa^+})$, so clause (D) of the theorem holds. So we are done. □

Claim 1.32. *Assume* $\chi < \chi^+ \leq \lambda = \mathrm{cf}(\lambda)$ *and* $\alpha < \lambda \Rightarrow \mathrm{cf}([\alpha]^{\leq \chi}, \subseteq) < \lambda$.

(1) *If* $2^\sigma < \lambda, \sigma = \mathrm{cf}(\sigma) \leq \chi$ *and* $\lambda = \lambda^{<\lambda}$, then $(D\ell)^*_{S^\lambda_\sigma}$.

(2) *We can find* $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$ *such that:*

(a) $\mathscr{P}_\alpha \subseteq \mathscr{P}(\alpha)$,

(b) $|\mathscr{P}_\alpha| < \lambda$,

(c) *if* $u \in \mathscr{P}_\alpha$, then $|u| \leq \chi$ *and* u *is a closed subset of* α,

(d) *if* $u \in \mathscr{P}_\alpha$ *and* $\beta \in u$, then $u \cap \beta \in \mathscr{P}_\beta$,

(e) *if* $\delta < \lambda, \aleph_0 < \mathrm{cf}(\delta) \leq \chi$, then $\delta = \sup(u)$ *for some* $u \in \mathscr{P}_\delta$.

Proof. Part (1) follows from [28], and for (2) see Džamonja–Shelah [3]. □

Observation 1.33. *Assume one of the following:*

(A) $\lambda = \chi^+, \chi = \mathrm{cf}(\chi) \geq \mu$,

(B) $\lambda = \chi^+ > \mu^+$, $cf([\chi]^{\leq \mu}, \subseteq) = \chi$, *see Definition* 0.11 (2).

Then *we can find* $\langle \bar{e}_\varepsilon : \varepsilon < \chi \rangle$ *such that:*

(a) $\bar{e}_\varepsilon = \langle e_{\varepsilon,\alpha} : \alpha < \lambda \rangle$,

(b) $e_{\varepsilon,\alpha} \subseteq \alpha$ *is closed,*

(c) $\sup\{\mathrm{otp}(e_{\varepsilon,\alpha}) : \alpha < \lambda\} < \mu$ *for each* $\varepsilon < \chi$,

(d) *if* $\alpha \in e_{\varepsilon,\beta}$, *then* $e_{\varepsilon,\alpha} = e_{\varepsilon,\beta} \cap \alpha$,

(e) *if* $\alpha < \lambda \wedge \mathrm{cf}(\alpha) < \mu$, *then for some* $\varepsilon < \chi$ *the set* $e_{\varepsilon,\alpha}$ *contains a club of* α,

(f) *for every* $\alpha < \lambda$ *and* $u \in [\alpha]^{<\mu}$ *for some* $\varepsilon < \chi$ *we have* $u \subseteq e_{\varepsilon,\alpha}$.

Remark 1.34. The statement is used in Claim 3.12.

Proof. First assume clause (A) holds. By [20, Section 4] or [33, Section 3.7] there is a sequence $\langle \bar{e}_\varepsilon : \varepsilon < \chi \rangle$ satisfying clauses (a), (b), (d) and

(c)′ $e_{\varepsilon,\alpha}$ has cardinality $< \chi$,

(e) if $u \subseteq \alpha < \lambda$ has cardinality $< \chi$, then $u \subseteq e_{\varepsilon,\alpha}$ for some ε,

(f)′ $\langle e_{\varepsilon,\alpha} : \varepsilon < \chi \rangle$ is $\subseteq$-increasing.

Manipulating those $\bar{e}_\varepsilon$, we get the desired conclusion (e.g. ignoring clause (f) choose $\langle e_\delta : \delta < \mu \text{ limit} \rangle$, e_δ a club of δ of order type $\mathrm{cf}(\delta)$ and for $\varepsilon < \chi \wedge \delta < \mu$ we define $\bar{e}^\delta_\varepsilon = \langle e^\delta_{\varepsilon,\alpha} : \alpha < \lambda \rangle$ by the sets $e^\delta_{\varepsilon,\alpha} := \{\gamma \in e_{\varepsilon,\alpha} : \mathrm{otp}(\gamma \cap e_{\varepsilon,\alpha}) \in e_\delta\}$, now check).

Second, assume clause (B) holds. The proof is similar using Claim 1.32, i.e., Džamonja–Shelah [3]. □

Claim 1.35. *We have*

$\circledast_2$ $\mathrm{BB}(2^\mu, \kappa^{+\omega+1}, \theta, J^{\mathrm{bd}}_{\kappa^+ \times \kappa})$ *if* $\theta < \mu \in \boldsymbol{C}_\kappa$ *and* $(\forall \chi)(\chi < 2^\mu \Rightarrow \chi^{\langle \kappa^+ \rangle_{\mathrm{tr}}} < 2^\mu)$.

Proof. See in the proof of Theorem 1.31, "second". That is, by Conclusion 1.27 there is a $(\kappa^{+\kappa+1}, J_{\kappa^+ \times \kappa})$-free ladder system $\langle C_\delta : \delta \in S \rangle$, $S \subseteq S^\lambda_{\kappa^+}$ stationary.

We claim that $\bar{C}$ exemplifies $\mathrm{BB}(\lambda, \kappa^{+\omega+1}, < \lambda, J^{\mathrm{bd}}_{\kappa^+ \times \kappa})$. Recalling the assumption

$$\chi < 2^\mu \Rightarrow \chi^{\langle \kappa^+ \rangle_{\mathrm{tr}}} < 2^\mu$$

by Claim 1.15 we have $(D\ell)_{S_1}$ for every stationary $S_1 \subseteq S$, hence by Theorem 1.18 we have clause (B) of Definition 0.7. □

Note the following (which will be useful together with Theorem 1.31, Observation 4.4, Claim 3.17).

Observation 1.36. *If* (A)*, then* (B) *where:*

(A) (a) *J_ℓ is an ideal on κ_ℓ for $\ell = 1, 2$ and $\kappa_1 = \kappa_2 \wedge J_1 \subseteq J_2$ or $J_1 \leq_{\mathrm{RK}} J_2$ or just for some function h from κ_2 onto κ_1 we have*

$$(\forall A \in J_1)(\{\beta < \kappa_2 : h(\beta) \in A\} \in J_1),$$

(b) $\bar{C}_\ell = \langle C^\ell_\alpha : \alpha \in S_\ell \rangle$, $\mathrm{otp}(C^1_\alpha) = \kappa_1$,

(c) $S_2 = \{\kappa_2 \cdot \delta : \delta \in S_1\}$ *and for* $\delta \in S_1$ *we have*

$$C^2_{\kappa_2 \cdot \delta} = \{\kappa_2 \cdot \beta + \mathrm{otp}(C^1_\delta \cap \alpha) : \alpha \in C^1_\delta \text{ and } \beta = h(\alpha)\},$$

(B) (a) *if $\bar{C}_1$ is (μ, J_1)-free,* then *$\bar{C}_2$ is (μ, J_2)-free,*

(b) *if* $\mathrm{BB}(\lambda, \mu, \theta, J_1)$ *and* $\theta = \theta^{\kappa_2}$, then $\mathrm{BB}(\lambda, \mu, \theta, J_2)$.

Proof. The proof is straightforward. □

2 Cases of weak G.C.H.

Note that if $\mu \in C_\kappa$ and $\lambda < 2^\mu < 2^\lambda$, then we can find a μ^+-free $\mathscr{F} \subseteq {}^\kappa\mu$ of cardinality λ (by the "No Hole Conclusion", [22, Chapter II, Claim 2.3, p. 53]) so by the Section Main Claim 2.2 we can deduce $\mathrm{BB}(\lambda, \mu^+, (2^\mu, \theta), \kappa)$ for $\theta < \mu$ – see Conclusion 2.7.

Observe below that if $\theta = 2, \bar{C} = \langle C_\gamma : \gamma < \lambda\rangle, C_\gamma \subseteq \mu$ (and $2^\mu < 2^\lambda$), then easily clause (β) of the conclusion of the Section Main Claim 2.2 below holds by counting – see Remark 2.3 (5). The point is to prove it for more colors, this is a relative of [27, The Main Claim 1.10] but this section is self-contained. Also Definition 2.1 repeats [27, Definition 1.9].

This section is close to [27, Section 1] hence we try to keep similar notation.

Definition 2.1. (1) $\mathrm{Sep}(\mu', \mu, \chi, \theta, \Upsilon)$ means that for some $\bar{f}$:

(a) $\bar{f} = \langle f_\varepsilon : \varepsilon < \mu'\rangle$,

(b) f_ε is a function from ${}^\mu\chi$ to θ,

(c) for every $\varrho \in {}^{\mu'}\theta$ the set $\{\nu \in {}^\mu\chi :$ for any $\varepsilon < \mu'$ we have $f_\varepsilon(\nu) \neq \varrho(\varepsilon)\}$ has cardinality $< \Upsilon$.

(2) We may omit χ if $\chi = \theta$. We write $\mathrm{Sep}(\mu, \theta, \Upsilon)$ for $\mathrm{Sep}(\mu, \mu, \theta, \theta, \Upsilon)$, and we write $\mathrm{Sep}(\mu, \theta)$ if for some $\Upsilon = \mathrm{cf}(\Upsilon) \leq 2^\mu$ we have $\mathrm{Sep}(\mu, \mu, \theta, \theta, \Upsilon)$ and write $\mathrm{Sep}(< \mu, \theta)$ if for some $\Upsilon = \mathrm{cf}(\Upsilon) \leq 2^\mu$ and some $\sigma < \mu$ we have $\mathrm{Sep}(\sigma, \mu, \theta, \theta, \Upsilon)$. Let $\mathrm{Sep}^+(\mu, \theta)$ mean $\mathrm{Sep}(\mu, \mu, \theta, \theta, \mu)$.

Section Main Claim 2.2. *Assume:*

(a) $2^\mu < 2^\lambda$,

(b) *D is a μ^+-complete filter on λ extending the co-bounded filter,*

(c) $\bar{C} = \langle C_\gamma : \gamma < \lambda\rangle, C_\gamma \subseteq \mu$,

(d) *$2 \leq \theta \leq \mu$ and $\Upsilon \leq \mu$ (or just D is Υ^+-complete, $\Upsilon \leq 2^\mu$),*

(e) $\mathrm{Sep}(\mu, \theta, \Upsilon)$,

(f) $\lambda = \min\{\partial : 2^\partial > 2^\mu\}$ *or at least:*

(f)$^-$ *we have $h_\xi \in {}^\lambda(2^\mu)$ for $\xi < (2^\mu)^+$ such that $\zeta \neq \xi \Rightarrow h_\zeta \neq_D h_\xi$.*

Then:

(α) *if χ satisfies $\gamma < \lambda \Rightarrow \chi^{|C_\gamma|} \leq \theta$,* then *we can find $\bar{f} = \langle f_\gamma : \gamma < \lambda\rangle$ satisfying $f_\gamma \in {}^{(C_\gamma)}\chi$ such that (see Remark 2.3 (1)): for every $f : \mu \to \chi$, for some $\gamma < \lambda$, $f_\gamma \subseteq f$ (and even for D^+-many $\gamma < \lambda$),*

(β) *if* $\boldsymbol{F}_\gamma : {}^{(C_\gamma)}(2^\mu) \to \theta$ *for* $\gamma < \lambda$, then *we can find* $\bar{c} = \langle c_\gamma : \gamma < \lambda \rangle \in {}^\lambda\theta$ *such that:*

($*$) *for any map* $f : \mu \to 2^\mu$, *for some* $\gamma < \lambda$, $\boldsymbol{F}_\gamma(f \restriction C_\gamma) = c_\gamma$ *(even for* D^+*-many* $\gamma < \lambda$*),*

(γ) *if* $\bar{\chi} = \langle \chi_\varepsilon : \varepsilon < \mu \rangle$ *satisfies the condition* $\gamma < \lambda \Rightarrow \prod_{\varepsilon \in C_\gamma} \chi_\varepsilon \leq \theta$, then *we can find* $\bar{f} = \langle f_\gamma : \gamma < \lambda \rangle$ *satisfying* $f_\gamma \in \prod_{\varepsilon \in C_\gamma} \chi_\varepsilon$ *such that for every* $f \in \prod_{\varepsilon < \mu} \chi_\varepsilon$, *for some* $\gamma < \lambda$, $f_\gamma = f \restriction C_\gamma$ *(and even for* D^+*-many).*

Remark 2.3. (1) Of course "for D^+ many $t \in I$ we have xx" means that D is a filter on I and $\{t \in I : t \text{ satisfies } xx\} \in D^+$, see below.

(2) For D a filter on I let $\mathrm{Dom}(D) = I$ and let $D^+ = \{A \subseteq I : I \backslash A \notin D\}$.

(3) Similarly for J an ideal on I.

(4) Note that in Section Main Claim 2.2 clause (f) implies clause (a) and even clause (f)$^-$ does. Note that clause (f) implies λ is regular (but not (f)$^-$) and clause (b) implies $\mathrm{cf}(\lambda) > \mu$.

(5) Concerning clause (β) in Section Main Claim 2.2, when $\theta = 2$, this is easy: let D be the filter of co-bounded subsets of λ, and let $\langle f_\alpha : \alpha < 2^\mu \rangle$ list ${}^\mu(2^\mu)$, each appearing λ times. Now

$$\mathscr{F} := \{\langle 1 - \boldsymbol{F}_\gamma(f_\alpha \restriction C_\gamma) : \gamma < \lambda \rangle : \alpha < 2^\mu\}$$

is a subset of ${}^\lambda 2$ of cardinality $2^\mu < 2^\lambda = |{}^\lambda 2|$. So every element $\bar{c} \in {}^\lambda 2 \backslash \mathscr{F}$ is as required. Concerning this proof we can use any filter D on λ such that $|2^\lambda / D| > 2^\mu$,

(6) In Section Main Claim 2.2 we can replace μ by any set of cardinality μ, e.g., ${}^{\omega>}\mu$. Hence replacing $\bar{C}$ by $\bar{C}' = \langle C'_\alpha : \alpha < \lambda \rangle$, $C'_\alpha = {}^{\omega>}(C_\alpha)$ in clause (β) of Section Main Claim 2.2 we can assume $\mathrm{Dom}(\boldsymbol{F}_\gamma) = \{f : f \text{ a function from } {}^{\omega>}(C_\alpha) \text{ to } 2^\mu\}$.

(7) We may wonder if clause (e) of the assumption of Section Main Claim 2.2 is reasonable; the following Claim 2.6 gives some sufficient conditions for clause (e) of Section Main Claim 2.2 to hold.

(8) In Section Main Claim 2.2 we implicitly assert that (f) $\Rightarrow$ (f)$^-$; for completeness we recall the justification (as there $(2^\mu)^+ \leq 2^\lambda$).

Observation 2.4. *We have* (f) $\Rightarrow$ (f)$^-$ *in Section Main Claim* 2.2, *i.e., we have if* $\lambda = \min\{\partial : 2^\partial > 2^\mu\}$, then *there are* $h_\xi : \lambda \to 2^\mu$ *for* $\xi < 2^\lambda$ *such that*

$$\xi < \zeta < 2^\lambda \Rightarrow h_\xi \neq h_\zeta \mod J^{\mathrm{bd}}_\lambda.$$

Proof. As $\alpha < \lambda \Rightarrow |{}^{\alpha}2| = 2^{|\alpha|} \leq 2^{\mu}$ and $\lambda \leq 2^{\mu}$, clearly ${}^{\lambda>}2 = \bigcup\{{}^{\alpha}2 : \alpha < \lambda\}$ has cardinality 2^{μ}, so there is a one-to-one function $\boldsymbol{g}$ from ${}^{\lambda>}2$ onto 2^{μ}.

Let $\langle \eta_{\xi} : \xi < 2^{\lambda} \rangle$ list ${}^{\lambda}2$ and let $h_{\xi} : \lambda \to 2^{\mu}$ be defined by $h_{\xi}(\alpha) = \boldsymbol{g}(\eta_{\xi} \restriction \alpha)$ for $\alpha < \lambda$.

Clearly $\langle h_{\xi} : \xi < 2^{\lambda} \rangle$ is as required. □

In order to give a sufficient condition for clause (e) of Section Main Claim 2.2 we recall

Definition 2.5. (1) For J an ideal on σ and cardinal μ let

$$\boldsymbol{U}_J(\mu) = \min\{|\mathscr{P}| : \mathscr{P} \subseteq [\mu]^{\leq\sigma}$$

and for every $f \in {}^{\sigma}\mu$, for some $u \in \mathscr{P}$, we have

$$\{\varepsilon < \sigma : f(\varepsilon) \in u\} \neq \emptyset \mod J.$$

(2) If $J = J^{\mathrm{bd}}_{\sigma}$ and σ is a regular cardinal, we may write $\boldsymbol{U}_{\sigma}(\mu)$.

Claim 2.6. *Clause* (e) *of Section Main Claim* 2.2 *holds, i.e.,* $\mathrm{Sep}(\mu, \theta, \Upsilon)$ *holds,* when *at least one of the following holds:*

(a) $\mu = \mu^{\theta}$ *and* $\Upsilon = \theta$,

(b) $\boldsymbol{U}_{\theta}(\mu) = \mu$ *and* $2^{\theta} < \mu$ *and* $\Upsilon = (2^{\theta})^{+}$,

(c) $\boldsymbol{U}_J(\mu) = \mu$ *where for some* σ *we have* $J = [\sigma]^{<\theta}, \theta \leq \sigma, \sigma^{\theta} \leq \mu$, $\theta^{<\sigma} < \mu$ *and* $\Upsilon = (\theta^{<\sigma})^{+}$,

(d) μ *is strong limit of cofinality* $\neq \theta, \theta < \mu$ *and* $\Upsilon = (2^{\theta})^{+}$,

(e) $\mu \geq \beth_{\omega}(\theta)$ *and* $\Upsilon = \mu$.

The proof follows from the proof of [27, 1.11] (not the statement!); however, for completeness, below we shall give a complete proof (after the proofs of Section Main Claim 2.2, Conclusions 2.7 and 2.8). We shall use mainly Claim 2.6 (d).

Proof of the Section Main Claim 2.2. It is enough to prove clause (β), as it implies the others. Why? Clearly clause (α) is a special case of clause (γ) and for clause (γ) note that without loss of generality $(\forall\varepsilon)(\chi_{\varepsilon} \leq \theta)$ hence $(\forall\varepsilon)(\chi_{\varepsilon} \leq 2^{\mu})$ so we can choose $\boldsymbol{F}_{\gamma}$ as any function from ${}^{C_{\gamma}}(2^{\mu})$ onto θ such that:

- $\boldsymbol{F}_{\gamma} \restriction \prod_{\varepsilon \in C_{\gamma}} \chi_{\varepsilon}$ is a one-to-one function.

Now by clause (β) we can find $\langle c_{\gamma} : \gamma < \lambda \rangle$ such that ($*$) there holds and for $\gamma < \lambda$ let f_{γ} be the unique $f \in \prod_{\varepsilon \in C_{\gamma}} \chi_{\varepsilon}$ such that $\boldsymbol{F}_{\gamma}(f) = c_{\gamma}$ and f_{γ} constantly zero if there is no such f.

Now check; so indeed it is sufficient to prove clause (β).

Let $\langle \boldsymbol{F}_\gamma : \gamma < \lambda \rangle$ be as in clause (β) and we shall prove that there is $\langle c_\gamma : \gamma < \lambda \rangle$ as promised therein.

By assumption (e) we have $\mathrm{Sep}(\mu, \theta, \Upsilon)$ which means (see Definition 2.1 (2)) that we have $\mathrm{Sep}(\mu, \mu, \theta, \theta, \Upsilon)$.

Let $\bar{f} = \langle f_\varepsilon : \varepsilon < \mu \rangle$ exemplify $\mathrm{Sep}(\mu, \mu, \theta, \theta, \Upsilon)$, see Definition 2.1 (1) and

$(*)_0$ for $\varrho \in {}^\mu\theta$ let $\mathrm{Sol}_\varrho := \{\nu \in {}^\mu\theta : \text{for every } \varepsilon < \mu \text{ we have } \varrho(\varepsilon) \neq f_\varepsilon(\nu)\}$

where Sol stands for solutions, so by clause (c) of the Definition 2.1 (1) of Sep it follows that:

$(*)_1$ $\varrho \in {}^\mu\theta \Rightarrow |\mathrm{Sol}_\varrho| < \Upsilon$.

Let cd be a one-to-one function from ${}^\mu(2^\mu)$ onto 2^μ such that (this is possible as $\mathrm{cf}(2^\mu) > \mu$):

$$\alpha = \mathrm{cd}(\langle \alpha_\varepsilon : \varepsilon < \mu \rangle) \Rightarrow \alpha \geq \sup\{\alpha_\varepsilon : \varepsilon < \mu\}.$$

Let $\mathrm{cd}_\varepsilon : 2^\mu \to 2^\mu$ for $\varepsilon < \mu$ be such that $\alpha < 2^\mu \Rightarrow \alpha = \mathrm{cd}(\langle \mathrm{cd}_\varepsilon(\alpha) : \varepsilon < \mu \rangle)$.

Let $\boldsymbol{H}$ be a one-to-one function from 2^μ onto ${}^\mu\theta$, such $\boldsymbol{H}$ exists as $2 \leq \theta \leq \mu$ by clause (d) of the assumption. For $\varrho \in {}^\mu\theta$ let $\mathrm{Sol}'_\varrho := \{\alpha < 2^\mu : \boldsymbol{H}(\alpha) \in \mathrm{Sol}_\varrho\}$, so

$(*)_2$ $\varrho \in {}^\mu\theta \Rightarrow |\mathrm{Sol}'_\varrho| < \Upsilon$.

Clearly in the assumption, if clause (f) holds, then clause $(\mathrm{f})^-$ holds (see Observation 2.4), so we can assume that $\langle h_\xi : \xi < (2^\mu)^+ \rangle$ are as in clause $(f)^-$ so in particular $h_\xi \in {}^\lambda(2^\mu)$.

Fix $\xi < (2^\mu)^+$ for a while.

For $\gamma < \lambda$ let

$(*)_3$ $\varrho^*_{\xi,\gamma} := \boldsymbol{H}(h_\xi(\gamma)) \in {}^\mu\theta$.

Let $\varepsilon < \mu$. Recall that $\varrho^*_{\xi,\gamma} \in {}^\mu\theta$ for $\gamma < \lambda$ and f_ε is a function from ${}^\mu\theta$ to θ so $f_\varepsilon(\varrho^*_{\xi,\gamma}) < \theta$. Hence we can consider the sequence $\bar{c}^\xi_\varepsilon = \langle f_\varepsilon(\varrho^*_{\xi,\gamma}) : \gamma < \lambda \rangle \in {}^\lambda\theta$ as a candidate for being as required (on $\langle c_\gamma : \gamma < \lambda \rangle$) in the desired conclusion $(*)$ from clause (β) of Section Main Claim 2.2. If one of them is as required, we are done. So assume towards a contradiction that for each $\varepsilon < \mu$ (recall we are fixing $\xi < (2^\mu)^+$) there is a sequence $\eta^\xi_\varepsilon \in {}^\mu(2^\mu)$ that exemplifies the failure of $\bar{c}^\xi_\varepsilon$ to satisfy condition $(*)$, hence there is a set $E^\xi_\varepsilon \in D$, so necessarily a subset of λ, such that

$(*)_4$ $\gamma \in E^\xi_\varepsilon \Rightarrow \boldsymbol{F}_\gamma(\eta^\xi_\varepsilon \restriction C_\gamma) \neq f_\varepsilon(\varrho^*_{\xi,\gamma})$.

Define $\eta^*_\xi \in {}^\mu(2^\mu)$ by

$\boxtimes_1$ $\eta^*_\xi(\alpha) = \mathrm{cd}(\langle \eta^\xi_\varepsilon(\alpha) : \varepsilon < \mu \rangle)$ for $\alpha < \mu$; so $\eta^*_\xi \in {}^\mu(2^\mu)$ for our $\xi < (2^\mu)^+$.

By clause (b) in the assumption of our Section Main Claim 2.2, the filter D is μ^+-complete hence

$(*)_5$ $E^*_\xi := \bigcap\{E^\xi_\varepsilon : \varepsilon < \mu\}$ belongs to D.

Now we vary $\xi < (2^\mu)^+$. For each such ξ we have chosen $\eta^*_\xi \in {}^\mu(2^\mu)$, and clearly the number of such η^*_ξ is $\leq |{}^\mu(2^\mu)| = (2^\mu)^\mu = 2^\mu$ hence for some η^* and unbounded $\mathcal{U} \subseteq (2^\mu)^+$ we have $\xi \in \mathcal{U} \Rightarrow \eta^*_\xi = \eta^*$.

For $\varepsilon < \mu$ we define $\eta'_\varepsilon \in {}^\mu(2^\mu)$ by $\eta'_\varepsilon(\alpha) = \mathrm{cd}_\varepsilon(\eta^*(\alpha))$ for $\alpha < \mu$. So by the choice of η^*_ξ in $\boxtimes_1$ above:

$\boxtimes_2$ if $\xi \in \mathcal{U}$, then $\varepsilon < \mu \Rightarrow \eta^\xi_\varepsilon = \eta'_\varepsilon$.

By $(*)_4 + (*)_5$, we get:

$\boxtimes_3$ if $\gamma \in E^*_\xi$ where $\xi \in \mathcal{U}$, *then* $\varepsilon < \mu \Rightarrow \boldsymbol{F}_\gamma(\eta'_\varepsilon \restriction C_\gamma) \neq f_\varepsilon(\varrho^*_{\xi,\gamma})$.

So noting $\langle \boldsymbol{F}_\gamma(\eta'_\varepsilon \restriction C_\gamma) : \varepsilon < \mu\rangle \in {}^\mu\theta$, clearly by $(*)_0$ and $\boxtimes_3$ we have:

$\boxtimes_4$ if $\gamma \in E^*_\xi$ where $\xi \in \mathcal{U}$, then $\varrho^*_{\xi,\gamma} \in \mathrm{Sol}_{\langle \boldsymbol{F}_\gamma(\eta'_\varepsilon \restriction C_\gamma):\varepsilon<\mu\rangle}$.

As ξ was any member of $\mathcal{U}$, by the choice of $\varrho^*_{\xi,\gamma}$, i.e., $(*)_3$ which says that $\varrho^*_{\xi,\gamma} = \boldsymbol{H}(h_\xi(\gamma))$ and the definition of Sol' (just before $(*)_2$), we have:

$\boxtimes_5$ if $\xi \in \mathcal{U}$, then $\gamma \in E^*_\xi \Rightarrow h_\xi(\gamma) \in \mathrm{Sol}'_{\langle \boldsymbol{F}_\gamma(\eta'_\varepsilon \restriction C_\gamma):\varepsilon<\mu\rangle}$.

Let $\bar{\xi} = \langle \xi_i : i < \Upsilon\rangle$ be a sequence of pairwise distinct members of $\mathcal{U}$, which is possible as $\mathcal{U}$ is an unbounded subset of $(2^\mu)^+$ and $\Upsilon \leq 2^\mu$ (see clause (d) of the assumption). As D is μ^+-complete and $\Upsilon \leq \mu$ or just D is Υ^+-complete, also $E^* := \bigcap\{E^*_{\xi_i} : i < \Upsilon\}$ belongs to D. By the above,

$$\gamma \in E^* \wedge i < \Upsilon \Rightarrow h_{\xi_i}(\gamma) \in \ \mathrm{Sol}'_{\langle \boldsymbol{F}_\gamma(\eta'_\varepsilon \restriction C_\gamma):\varepsilon<\mu\rangle}.$$

But by $(*)_2$ we have

$$|\mathrm{Sol}'_{\langle \boldsymbol{F}_\gamma(\eta'_\varepsilon \restriction C_\gamma):\varepsilon<\mu\rangle}| < \Upsilon,$$

hence by $\boxtimes_5$ for each $\gamma \in E^*$ we can choose $i_\gamma < j_\gamma < \Upsilon$ such that

$$h_{\xi_{i_\gamma}}(\gamma) = h_{\xi_{j_\gamma}}(\gamma).$$

As $\Upsilon \leq \mu$ and D is μ^+-complete or just D is Υ^+-complete recalling $E^* \in D$, clearly for some $i < j < \Upsilon$ the set $\{\gamma \in E^* : i_\gamma = i \wedge j_\gamma = j\}$ is $\neq \emptyset$ mod D. As $i < j$, by the choice of $\bar{\xi}$ we have $\xi_i \neq \xi_j$ and by the previous sentences

$$\{\gamma \in E^* : h_{\xi_i}(\gamma) = h_{\xi_j}(\gamma)\} \neq \emptyset \quad \mathrm{mod}\ D.$$

But this contradicts the choice of $\langle h_\zeta : \zeta < (2^\mu)^+\rangle$, i.e., clause (f)$^-$ of the assumption as well as Observation 2.4. □

Conclusion 2.7. (1) $\mathrm{BB}(\lambda, \mu^+, \theta, \kappa)$ *and if* λ *is regular, even* $\mathrm{BB}(J^{\mathrm{nst}}_\lambda, \mu^+, \theta, \kappa)$ *– see Definition* 0.5 *– holds* when $\theta < \mu \in \boldsymbol{C}_\kappa$ *and* $\mu < \lambda < 2^\mu < 2^\lambda$.

(2) $\mathrm{BB}(\lambda, \mu^+, (2^\mu, \theta), \kappa)$ *– see Definition* 0.7 *– holds* when θ, μ, λ *are as above.*

Proof. (1) Let $\Upsilon = (2^{\theta+\kappa^+})^+$, so $\Upsilon < \mu$. By case (d) of Claim 2.6, we have $\mathrm{Sep}(\mu, \theta, \Upsilon)$. Let $\langle C_\gamma : \gamma \in [\mu, \lambda)\rangle$ be a μ^+-free family of subsets of μ each of order type κ (exist by Claim 1.3 (c)) and let $\langle S_i : i < \lambda\rangle$ be a partition of $[\mu, \lambda)$ into λ (pairwise disjoint) sets each of cardinality λ, stationary if λ is regular and let $\langle \xi_{i,\alpha} : \alpha < \lambda\rangle$ list S_i in increasing order. Clearly $\langle C_\gamma : \gamma \in [\mu, \lambda)\rangle$ is a weak (λ, κ)-ladder system and is μ^+-free so is as required in clause (A) of Definition 0.5. Hence it suffices to find for each $i < \lambda$ a function $\boldsymbol{c}_i$ with domain S_i, such that $\boldsymbol{c}_i(\gamma) \in {}^{(C_\gamma)}\theta$ as in Definition 0.5.

Clearly $\lambda \geq \lambda_0 := \min\{\partial : 2^\partial > 2^\mu\}$, so if equality holds, by Observation 2.4 there are $h_\xi \in {}^\lambda(2^\mu)$ for $\xi < 2^\lambda$ such that

$$\zeta \neq \varepsilon \Rightarrow h_\zeta \neq_{J^{\mathrm{bd}}_\lambda} h_\varepsilon.$$

So we can apply Section Main Claim 2.2 (α) with D taken to be the club filter and with $\langle C_{\xi_{i,\alpha}} : \alpha \in [\mu, \lambda)\rangle$ here standing for $\bar{C}$ there; we get $\boldsymbol{c}'_i$ with domain λ. Let $\boldsymbol{c}_i$ have domain S_i, $\boldsymbol{c}_i(\xi_{i,\alpha}) = \boldsymbol{c}'_i(\alpha)$ so $\boldsymbol{c}_i$ is as required. If otherwise, i.e., $\lambda > \lambda_0$, the result "$\mathrm{BB}(\lambda, \mu^+, \theta, \kappa)$" follows by monotonicity of BB in λ.

To get "if λ is regular, then $\mathrm{BB}(J^{\mathrm{nst}}_\lambda, \mu^+, \theta, \kappa)$", let $g : \lambda \to [\mu, \lambda_0)$ be such that $g^{-1}\{\alpha\}$ is a stationary subset of λ for $\alpha \in [\mu, \lambda_0)$, let $\langle S'_i : i < \lambda\rangle$ be a partition of $[\mu, \lambda_0)$ into stationary sets and use $S''_i = \{\beta < \lambda : g(\beta) \in S'_i\rangle$, $C''_\beta = C_{g(\beta)}$ and $D = \{A \subseteq \lambda$: for club E of $\lambda_0, (\forall \beta < \lambda)(g(\beta) \in E \Rightarrow \beta \in A)\}$.

(2) The proof is similar. □

Conclusion 2.8. *Suppose we add clause* (g) *and replace clause* (b) *by* (b)$^+$ *in Section Main Claim* 2.2 *where*

(g) $\lambda = \mathrm{cf}(\lambda)$ *and* $\mathfrak{d}_\lambda > 2^\mu$, *recalling* $\mathfrak{d}_\lambda = \mathrm{cf}({}^\lambda\lambda, <_{J^{\mathrm{bd}}_\lambda})$,

(b)$^+$ λ *is regular and* D *is the club filter on* λ.

Then *we can strengthen clause* (β) *of the conclusion to:*

(β)$^+$ *if* $\boldsymbol{F}_\gamma : {}^{(C_\gamma)}(2^\mu) \to \theta$ *for* $\gamma < \lambda$ *and* $\boldsymbol{F}' : {}^\mu(2^\mu) \to {}^\lambda\lambda$, then *we can find* $\bar{c} = \langle c_\gamma : \gamma \in S_*\rangle \in {}^\lambda\theta$ *with* $S_* \in D^+$ *such that:*

(∗) *for any* $f : \mu \to 2^\mu$ *for some* $\gamma < \lambda$ *(and even for* D^+*-many* $\gamma \in S_*$*) we have*

$$\boldsymbol{F}_\gamma(f \restriction C_\gamma) = c_\gamma \quad \text{and} \quad (\boldsymbol{F}'(f))(\gamma) < \min(S_* \backslash (\gamma + 1)).$$

Proof. Note that clause (b)$^+$ here implies clause (b) from Section Main Claim 2.2, so the conclusion of Section Main Claim 2.2 holds. We do not have to repeat the proof of the Section Main Claim 2.2; just to quote it as $\mathscr{F} = \{\boldsymbol{F}'(f) : f$ a function from μ to $2^\mu\}$ is a subset of ${}^\lambda\lambda$ of cardinality $\le 2^\mu$ and so is

$$\mathscr{F}' = \{\sup\{f_i : i < \mu\} : f_i \in \mathscr{F} \text{ for } i < \mu\}.$$

Now we apply a result from Cummings–Shelah [1] saying that

$$\mathrm{cf}({}^\lambda\lambda, <_{J^{\mathrm{bd}}_\lambda}) = \mathrm{cf}({}^\lambda\lambda, <_{J^{\mathrm{nst}}_\lambda}),$$

that is,

$$\mathfrak{d}_\lambda = \mathrm{cf}({}^\lambda\lambda, <_{J^{\mathrm{nst}}_\lambda}),$$

hence there is a map $f_* \in {}^\lambda\lambda$ such that the set $\{\alpha < \lambda : f(\alpha) < f_*(\alpha)\}$ is a stationary subset of λ for every $f \in \mathscr{F}$. For $f \in \mathscr{F}'$ let $S_f = \{\delta < \lambda : \delta$ a limit ordinal and $f_*(\alpha) \le f(\delta)\}$ and now apply Section Main Claim 2.2 for the filter $\{S \subseteq \lambda : S \cup S_f \in D$, i.e., contains a club for some $f \in \mathscr{F}'\}$. $\square$

We still owe a proof of Claim 2.6 giving sufficient conditions for the condition $\mathrm{Sep}(\mu, \mu, \theta, \theta, \Upsilon)$.

Proof of Claim 2.6. Cases 1–4 below cover all the clauses (a)–(e) of Claim 2.6 recalling

$(*)_1$ $\mathrm{Sep}(\mu, \theta, \Upsilon) = \mathrm{Sep}(\mu, \mu, \theta, \theta, \Upsilon)$

and using freely the obvious

$(*)_2$ monotonicity: if $\mathrm{Sep}(\mu'_1, \mu_1, \chi_1, \theta_1, \Upsilon_1)$ and $\mu'_1 \le \mu'_2, \mu_1 = \mu_2, \chi_1 \le \chi_2, \theta_1 = \theta_1, \Upsilon_1 \le \Upsilon_2$, *then* $\mathrm{Sep}(\mu'_2, \mu_2, \chi_2, \theta_2, \Upsilon_2)$.

Clause (a) is fully covered by Case 1 using monotonicity, clause (b) follows from clause (c) for the case $\sigma = \theta$, clause (c) by Case 2, clause (d) by Case 3 letting $\sigma = \theta$ and clause (e) by Case 4.

Case 1: $\mu = \mu^\theta, \Upsilon = \theta, \chi \in [\theta, \mu]$ *and we shall prove* $\mathrm{Sep}(\mu, \mu, \chi, \theta, \theta)$. Let

$$\mathscr{F} = \Big\{ f : f \text{ is a function from } {}^\mu\chi \text{ into } \theta \text{ and for some } u \in [\mu]^\theta \text{ and a sequence } \bar\rho = \langle \rho_i : i < \theta\rangle \text{ with no repetition, } \rho_i \in {}^u\chi, \text{ we have } (\forall \nu \in {}^\mu\chi)[\rho_i \subseteq \nu \Rightarrow f(\nu) = i] \text{ and } (\forall \nu \in {}^\mu\chi)\Big[\Big(\bigwedge_{i<\theta} (\rho_i \not\subseteq \nu)\Big) \Rightarrow f(\nu) = 0\Big]\Big\}.$$

We write $f = f^*_{u,\bar\rho}$ if $u, \bar\rho$ witness that $f \in \mathscr{F}$ as above. Notice that the size of the set of such pairs $(u, \bar\rho)$ is μ^θ, and each such pair determines a unique f.

Recalling $\mu = \mu^\theta$, we clearly have $|\mathscr{F}| = \mu$. Let $\mathscr{F} = \{f_\varepsilon : \varepsilon < \mu\}$ and we let $\bar{f} = \langle f_\varepsilon : \varepsilon < \mu \rangle$. Clearly clauses (a)–(b) of Definition 2.1 (with $\mu, \mu, \chi, \theta, \theta$ here standing for $\mu', \mu, \chi, \theta, \Upsilon$ there) hold; let us check clause (c). So suppose $\varrho \in {}^\mu\theta$ and let $R = R_\varrho := \{\nu \in {}^\mu\chi$: for every $\varepsilon < \mu$ we have $f_\varepsilon(\nu) \neq \varrho(\varepsilon)\}$. We have to prove that $|R| < \theta$ (as we have chosen $\Upsilon = \theta$).

Towards a contradiction, assume that $R \subseteq {}^\mu\chi$ has cardinality $\geq \theta$ and choose $R' \subseteq R$ of cardinality θ. Hence we can find $u \in [\mu]^\theta$ such that $\langle \nu \restriction u : \nu \in R' \rangle$ is without repetitions.

Let $\{\nu_i : i < \theta\}$ list R' without repetitions and let $\rho_i := \nu_i \restriction u$ for $i < \theta$. Now let $\bar{\rho} = \langle \rho_i : i < \theta \rangle$, so $f^*_{u,\bar{\rho}}$ is well defined and belongs to $\mathscr{F}$. Hence for some $\zeta < \mu$ we have $f^*_{u,\bar{\rho}} = f_\zeta$. Now for each $i < \theta, \nu_i \in R' \subseteq R$, hence by the definition of R,

$$(\forall \varepsilon < \mu)(f_\varepsilon(\nu_i) \neq \varrho(\varepsilon))$$

and, in particular, for $\varepsilon = \zeta$, we get $f_\zeta(\nu_i) \neq \varrho(\zeta)$. But by the choice of ζ,

$$f_\zeta(\nu_i) = f^*_{u,\bar{\rho}}(\nu_i)$$

and by the definition of $f^*_{u,\bar{\rho}}$, recalling $\nu_i \restriction u = \rho_i$, we have $f^*_{u,\bar{\rho}}(\nu_i) = i$, so $i = f_\zeta(\nu_i) \neq \varrho(\zeta)$. This holds for every $i < \theta$ whereas $\varrho \in {}^\mu\theta$, a contradiction.

Case 2: $\theta \leq \chi < \mu$, $\chi^{<\sigma} < \mu$, $\chi^\theta \leq \mu$, $\sigma^\theta \leq \mu$, $\theta \leq \sigma$, $J = [\sigma]^{<\theta}$ so it is an ideal on σ, $\boldsymbol{U}_J(\mu) = \mu$, $\Upsilon = (\chi^{<\sigma})^+$ recalling Definition 2.5. We shall prove $\mathrm{Sp}(\mu, \mu, \chi, \theta, \Upsilon)$ which is more than required.

Let $\{u_\gamma : \gamma < \mu\} \subseteq [\mu]^{\leq\sigma}$ exemplify $\boldsymbol{U}_J(\mu) = \mu$. Define $\mathscr{F}$ as in Case 1 replacing "$u \in [\mu]^\theta$" by "$u \in \mathscr{P} := \bigcup\{[u_\gamma]^\theta : \chi < \mu\}$". As $\sigma^\theta \leq \mu$, we easily get $|\mathscr{P}| = \mu$ and as $\chi^\theta \leq \mu$, clearly $|\mathscr{F}| = \mu$. Let $\langle f_\varepsilon : \varepsilon < \mu \rangle$ list $\mathscr{F}$, clearly clauses (a)–(b) of Definition 2.1 hold and we shall prove clause (c).

Assume that $\varrho \in {}^\mu\theta$ and $R = R_\varrho \subseteq {}^\mu\theta$ is defined as in Case 1, and towards a contradiction assume that $|R| \geq \Upsilon = (\chi^{<\sigma})^+$. We can find $\nu^*, \langle (\alpha_\zeta, \nu_\zeta) : \zeta < \sigma \rangle$ such that:

⊞ (a) $\nu^*, \nu_\zeta \in R_\varrho$,
(b) $\alpha_\zeta < \mu$,
(c) $\nu_\zeta \restriction \{\alpha_\xi : \xi < \zeta\} = \nu^* \restriction \{\alpha_\xi : \xi < \zeta\}$,
(d) $\nu_\zeta(\alpha_\zeta) \neq \nu^*(\alpha_\zeta)$.

(Why? Obvious, as in the proof of the Erdös–Rado Theorem; let $\langle \eta_i : i < \Upsilon \rangle$ be a sequence with no repetitions of members of R. For each $j < \Upsilon$, we try to choose by induction on $\zeta < \sigma$ ordinals $i(j,\zeta), \alpha_{j,\zeta}$ such that:

(a) $i(j,\zeta) < j$ is increasing with ζ,

(b) $\alpha_{j,\zeta} = \min\{\alpha : \eta_j(\alpha) \neq \eta_{i(j,\zeta)}(\alpha)\}$,

(c) $i(j,\zeta) = \min\{i : i(j,\varepsilon) < i < j$ and $\eta_i(\alpha_{j,\varepsilon}) = \eta_j(\alpha_{j,\varepsilon})$ for $\varepsilon < \zeta\}$.

If we succeed for some j, we are done. Otherwise for each $j < \Upsilon$ there is $\xi(j) < \sigma$ such that $(i(j,\zeta), \alpha_{j,\zeta})$ is well defined iff $\zeta < \xi(j)$.

Let $\mathcal{T} = \{\langle (i(j,\zeta), \alpha_{j,\zeta}) : \zeta < \xi \rangle : j < \Upsilon$ and $\xi \leq \xi(j)\}$ which is, under $\triangleleft$, a tree with $\leq \sigma$ levels, is normal, has a root and each node has at most χ immediate successors, hence

$$|\mathcal{T}| \leq \sum_{i<\sigma} |{}^{i}\chi| = \Sigma\{\chi^{|i|} : i < \sigma\} = \chi^{<\sigma}.$$

But

$$j \mapsto \langle (i(j,\zeta), \alpha_{j,\zeta}) : \zeta < \xi(j) \rangle$$

is a one-to-one function from Υ into $\mathcal{T}$, a contradiction.)

Clearly $\langle \alpha_\zeta : \zeta < \sigma \rangle$ has no repetitions. So by the choice of $\{u_\gamma : \gamma < \mu\}$ as exemplifying $\boldsymbol{U}_J(\mu) = \mu$, i.e., the definition of $\boldsymbol{U}_J(\mu)$ and the choice of J, for some $i < \mu$ the set $u_\gamma \cap \{\alpha_\zeta : \zeta < \sigma\}$ has cardinality $\geq \theta$; choose a subset u of this intersection of cardinality θ, hence $u \in \mathcal{P}$. So $\{\nu \restriction u : \nu \in R\}$ has cardinality $\geq \theta$; without loss of generality $u = \{\alpha_{\zeta_i} : i < \theta\}$ where ζ_i, increasing with i, and let $\rho_i^* = \nu_{\zeta_i} \restriction u$ for $i < \theta$ and we can continue as in Case 1.

Case 3: $\mu > \theta \neq \mathrm{cf}(\mu)$ and $\sigma = \theta$ (or $\theta \leq \sigma \in \mathrm{Reg} \cap \mu \backslash \{\mathrm{cf}(\mu)\}$) and μ is a strong limit cardinal, $\Upsilon = (2^{<\sigma})^+$ and we shall prove $\mathrm{Sep}(\mu, \theta, \Upsilon)$. Letting $\chi = \theta$, this follows by Case 2, the main point is "$\boldsymbol{U}_J(\mu) = \mu$ where $J = [\sigma]^{<\theta}$, recalling Definition 2.5.

Let $\mathcal{P} = \bigcup\{u : u$ is a bounded subset of μ of cardinality $\leq \sigma\}$. So $\mathcal{P} \subseteq [\mu]^{\leq \sigma}$ and as μ is a strong limit cardinal clearly $\mathcal{P}$ has cardinality $\leq \mu$ and if f is a function from σ to μ, as $\sigma = \mathrm{cf}(\sigma) \neq \mathrm{cf}(\mu)$ necessarily for some $\alpha < \mu$ the set $u_* := \{\varepsilon < \sigma : f(\varepsilon) < \alpha\}$ is of cardinality σ hence it belongs to $\mathcal{P}$ (and has subsets of cardinality exactly θ which necessarily belong to μ).

Case 4: $\mu \geq \beth_\omega(\theta)$ and $\Upsilon = \mu$ and we shall prove $\mathrm{Sep}(\mu, \theta, \Upsilon)$. Let $\chi = \theta$ so we should prove $\mathrm{Sep}(\mu, \mu, \chi, \theta, \Upsilon)$. By [26] or see [28] we can find a regular $\sigma < \beth_\omega(\theta)$ which is greater than θ and is such that $\boldsymbol{U}_\sigma(\mu) = \mu$ (i.e., the ideal is J_σ^{bd}); hence $J := [\sigma]^{<\theta} \subseteq J_\sigma^{\mathrm{bd}}$ hence trivially $\boldsymbol{U}_J(\mu) = \mu$; so Case 2 applies and by monotonicity we are done. □

Discussion 2.9. We may try to strengthen the results on $\mathrm{Sep}(\mu, \theta, \kappa)$ assuming $\mu^\sigma = \mu$, a case which is unnatural for [27] but may be helpful.

3 Getting large μ^+-free subsets of ${}^\kappa\mu$

Recall that $\mu = \boldsymbol{C}_\kappa \Rightarrow \mathrm{pp}(\mu) =^+ 2^\mu$ and easily (see Observation 0.9 (2))

⊞ if $\mathcal{F} \subseteq {}^\kappa\mu$ is μ_1-free and $\lambda = |\mathcal{F}| = 2^\mu$, then $\mathrm{BB}(\lambda, \mu_1, \lambda, \kappa)$ (and hence TDU_{μ_1} holds when $\kappa \in \{\aleph_0, \aleph_1\}$).

This is a motivation of the investigation here, i.e., trying to get more cases of μ^+-free subsets for ${}^\kappa\mu$ of cardinality $\mathrm{pp}(\mu)$. In Claim 3.1 the case of our interest is $\mu = \beth_\omega, \mu < \chi < \lambda = \beth_{\omega+1}(= 2^\mu)$, $\mathrm{cf}(\chi) = \theta \in (\aleph_\omega, \mu)$.

Claim 3.1. *There is a set $\mathcal{F} \subseteq {}^\kappa\mu$ of cardinality λ satisfying $\boxtimes$ if $\circledast$ holds* where:

$\boxtimes$ (α) *the set $\mathcal{F}$ is (θ, J_1)-free, see Definition* 1.2,

(β) *$\mathcal{F}$ is $(\mu^+, (2^\theta)^+, J_1)$-free, see Definition* 1.2,

$\circledast$ (a) $\mu < \chi < \lambda$,

(b) $\kappa = \mathrm{cf}(\mu) < \mu$,

(c) *θ is regular (naturally but not necessarily $\theta = \mathrm{cf}(\chi)$),*

(d) $\kappa < \theta < \mu$ or *just $\kappa \neq \theta$ are both $< \mu$,*

(e) $\alpha < \mu \Rightarrow |\alpha|^\theta + |\alpha|^{<\kappa} < \mu$,

(f) *$J = J_1$ is a κ-complete ideal on κ,*

(g) *$\chi^{\langle\theta\rangle_{\mathrm{tr}}} \geq^+ \lambda$ as witnessed by $\mathcal{T}$, i.e., the tree $\mathcal{T}$ has θ levels, $\leq \chi$ nodes and $\geq \lambda$ distinct θ-branches,*

(h) $\mathrm{pp}_{J_1}(\mu) > \chi$.

Claim 3.2. *In Claim* 3.1 *we can replace $\circledast$ by $\circledast'$ and $\boxtimes(\beta)$ by $\boxtimes'(\beta)'$ below, i.e., if $\circledast'$ holds, then there is $\mathcal{F} \subseteq {}^\kappa\mu$ of cardinality λ such that $\boxtimes'$ holds* where:

$\boxtimes'$ (α) *the set $\mathcal{F}$ is (θ_1, J_1)-free,*

(β)$'$ *$\mathcal{F}$ is (μ^+, σ, J_1)-free,*

$\circledast'$ (a) $\mu < \chi < \lambda$,

(b) $\kappa = \mathrm{cf}(\mu) < \mu$,

(c) *J_2 is an ideal on θ,*

(d) *$J = J_1$ is an ideal on κ,*

(e) *$\alpha < \mu \Rightarrow |\alpha|^\theta < \mu$ (hence $\theta < \mu$),*

(f) *θ_1 satisfies (α) or (β) where*

(α) *$\theta_1 \leq \theta$ and J_2 is θ_1-complete,*

(α) *J_1 is $\mathrm{cf}(\theta_1)^+$-complete and $J_2 = J^{\mathrm{bd}}_{\theta_1}$ and $\theta_1 < \kappa$ of course,*

(g) *there are $\eta_\alpha \in {}^\theta\chi$ for $\alpha < \lambda$ such that*

$$\alpha < \beta < \lambda \Rightarrow \{\varepsilon < \theta : \eta_\alpha(\varepsilon) = \eta_\beta(\varepsilon)\} \in J_2,$$

(h) *there is a (μ^+, J_1)-free $\mathcal{F} \subseteq {}^\kappa\mu$ of cardinality $\geq \chi$,*

(i) (α) *$\mathcal{P}(\theta)/J_2$ satisfies the σ-c.c.* or *just*

(β) *for some κ^+-complete ideal $J_2' \supseteq J_2$ of θ,*

$$\sigma \geq \sup\{|\mathcal{A}|^+ : \mathcal{A} \subseteq \mathcal{P}(\theta)\backslash J_2' \text{ and } A \neq B \in \mathcal{A} \Rightarrow A \cap B \in J_2\}.$$

Remark 3.3. (1) Recall Definition 1.2 where we defined notions of freeness for sets and for sequences.

(2) The proof of Claim 3.1 is written so it can be adapted to become a proof of Claim 3.2.

Proof of Claim 3.1. As $\mathrm{cf}(\mu) = \kappa < \mu$ by clause (b) of $\circledast$ and $\alpha < \mu \Rightarrow |\alpha|^\theta < \mu$ by clause (e), we can let $\langle \mu_i : i < \kappa \rangle$ be increasing with limit μ such that

$$(\mu_i)^\theta = \mu_i > 2^\theta.$$

Let $\mu_i^- = \bigcup_{j<i} \mu_j$; without loss of generality we can assume that $\mu_i^- < \mu_i < \mu$. If $(\forall \alpha < \mu)(|\alpha|^\kappa < \mu)$, we can add $(\mu_i)^{\kappa+\theta} = \mu_i$.

There is $\bar{\rho} = \langle \rho_\gamma : \gamma < \chi \rangle$ such that:

$(*)_1$ (a) $\rho_\gamma \in \prod_{i<\kappa} \mu_i$ with no repetition; moreover $\rho_\gamma(i) \in [\mu_i^-, \mu_i)$,

(b) the set $\{\rho_\alpha : \alpha < \chi\}$ is (μ^+, J_1)-free (in fact we can add that even the sequence $\langle \rho_\alpha : \alpha < \chi \rangle$ is μ^+-free, recalling Definition 1.2 (1)–(2) but this is immaterial here).

(Why? For any regular $\chi_1 \in (\mu, \chi]$ by clause (h) of the assumption $\circledast$, there is an increasing sequence $\langle \lambda_i : i < \kappa \rangle$ of regular cardinals $< \mu$ with limit μ such that $\chi_1 < \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_{J_1})$.

As we can replace $\langle \mu_i : i < \kappa \rangle$ by any subsequence of length κ, for some non-decreasing sequence, without loss of generality $\mu_i^- < \lambda_i < \mu_i$.

By the no-hole-claim (really [22, Chapter II, Claim 1.5A]) there are

$$\rho_\gamma \in \prod_{i<\kappa} [\mu_i^-, \lambda_i) \subseteq \prod_{i<\kappa} [\mu_i^-, \mu_i) \quad \text{for } \gamma < \chi_1$$

such that $\langle \rho_\gamma : \gamma < \chi_1 \rangle$ is (μ^+, J_1)-free. If χ is regular, we can use $\chi_1 := \chi$. We are left with χ is singular; *then* still by [22, Chapter II, Claim 1.5A] there is a sequence $\langle \rho_\gamma : \gamma < \chi \rangle$ as above recalling $\circledast$ (e)–(f). So we are done.)

Let $J_2 = J_\theta^{\mathrm{bd}}$ (for Claim 3.2 the ideal J_2 is given in clause (c)), and let $\mathcal{T}$ be a tree as in clause (g) of the assumption $\circledast$. Without loss of generality:

$(*)_2$ (a) $\mathcal{T} \subseteq {}^{\theta>}\chi$ and $<_{\mathcal{T}}$ is $\triangleleft$, i.e., being an initial segment,

(b) we have

$$\eta_1, \eta_2 \in \mathcal{T} \wedge \varepsilon_1 < \theta \wedge \varepsilon_2 < \theta \wedge \eta_1(\varepsilon_1) = \eta_2(\varepsilon_2)$$
$$\Rightarrow \varepsilon_1 = \varepsilon_2 \wedge \eta_1 \restriction \varepsilon_1 = \eta_2 \restriction \varepsilon_2.$$

Recall $\lim_\theta(\mathscr{T}) = \{\eta \in {}^\theta\chi : (\forall \varepsilon < \theta)(\eta \restriction \varepsilon \in \mathscr{T})\}$, so it has $\geq \lambda$ members.

Let $\langle \eta_\alpha : \alpha < \lambda \rangle$ be a sequence of pairwise distinct members of $\lim_\theta(\mathscr{T})$. Let $\mathrm{cd}_* : \bigcup\{{}^\theta(\mu_i) : i < \kappa\} \to \mu$ be one-to-one onto μ such that

$$\rho \in {}^\theta(\mu_i) \Leftrightarrow \mathrm{cd}_*(\rho) < \mu_i.$$

Let $\langle \mathrm{cd}_\varepsilon : \varepsilon < \theta \rangle$ be the sequence of functions with domain μ such that

$$\zeta = \mathrm{cd}_*(\rho) \Rightarrow \rho = \langle \mathrm{cd}_\varepsilon(\zeta) : \varepsilon < \theta \rangle.$$

Let $\mathrm{cd}'_\varepsilon(\zeta) = \mathrm{cd}_\varepsilon(\mathrm{cd}_0(\zeta))$.

Lastly, for $\alpha < \lambda$ (the second and third demands are for later claims using this proof):

$\boxtimes_1$ $\nu_\alpha \in {}^\kappa\mu$ is defined as follows:

- for $i < \kappa$, let $\nu_\alpha(i) \in [\mu_i^-, \mu_i)$ be such that $\mathrm{cd}'_\varepsilon(\nu_\alpha(i)) = \rho_{\eta_\alpha(\varepsilon)}(i)$ for every $\varepsilon < \theta$,
- if $(\forall \alpha < \mu)(|\alpha|^\kappa < \mu)$, then we can make $\nu_\alpha(i)$ also code $\nu_\alpha \restriction i$, e.g. $\mathrm{cd}_1(\nu_\alpha(i))$ codes $\nu_\alpha \restriction i$,
- if $\varrho_\alpha \in \prod_{i<\kappa} \mu_i$ for $\alpha < \lambda$ are given, then we can add that $\nu_\alpha(i)$ codes $\varrho_\alpha(i)$, too, e.g. $\varrho_\alpha(i) = \mathrm{cd}_0(\mathrm{cd}_2(\nu_\alpha(i)))$.

We shall prove that the set $\mathscr{F} = \{\nu_\alpha : \alpha < \lambda\}$ is as required. Let $\bar{\nu} = \langle \nu_\alpha : \alpha < \lambda \rangle$. Now

$\boxtimes_2$ $\bar{\nu}$ is without repetition, i.e., $\alpha < \beta < \lambda \Rightarrow \nu_\alpha \neq \nu_\beta$, and so the set $\mathscr{F}$ has cardinality λ.

(Why? If $\nu_\alpha = \nu_\beta$, then for every $\varepsilon < \theta$ and $i < \kappa$, we have

$$\rho_{\eta_\alpha(\varepsilon)}(i) = \mathrm{cd}'_\varepsilon(\nu_\alpha(i)) = \mathrm{cd}'_\varepsilon(\nu_\beta(i)) = \rho_{\eta_\beta(\varepsilon)}(i).$$

Fixing $\varepsilon < \theta$, as this holds for every $i < \kappa$, we conclude that $\rho_{\eta_\alpha(\varepsilon)} = \rho_{\eta_\beta(\varepsilon)}$. But $\langle \rho_\gamma : \gamma < \chi \rangle$ is without repetitions, hence it follows that $\eta_\alpha(\varepsilon) = \eta_\beta(\varepsilon)$. As this holds for every $\varepsilon < \theta$, we conclude that $\eta_\alpha = \eta_\beta$ but $\langle \eta_\alpha : \alpha < \lambda \rangle$ is without repetitions hence $\alpha = \beta$, so we are done.)

Now the main point is proving clauses (α) and (β) of $\boxtimes$.

Step 1. To prove clause (α) of $\boxtimes$, i.e., "$\mathscr{F}$ is (θ, J_1)-free".

Assume $w \subseteq \lambda$ and $|w| < \theta$. Recalling $(*)_1$ (b) and $\theta < \mu$, clearly the set $\{\rho_{\eta_\alpha(\varepsilon)} : \alpha \in w, \varepsilon < \theta\}$ being of cardinality $\leq \theta < \mu^+$ is free, hence there is a sequence $\langle s_{\eta_\alpha(\varepsilon)} : \alpha \in w, \varepsilon < \theta \rangle$ of members of J_1 such that: if $(\alpha_\ell, \varepsilon_\ell) \in w \times \theta$, for $\ell = 1, 2$, and $\eta_{\alpha_1}(\varepsilon_1) \neq \eta_{\alpha_2}(\varepsilon_2)$ and $i \in \kappa \backslash s_{\eta_{\alpha_1}(\varepsilon_1)} \backslash s_{\eta_{\alpha_2}(\varepsilon_2)}$ (recalling Notation 0.16 (0)), then $\rho_{\eta_{\alpha_1}(\varepsilon_1)}(i) \neq \rho_{\eta_{\alpha_2}(\varepsilon_2)}(i)$.

Now as $\langle \eta_\alpha : \alpha \in w \rangle$ is a sequence of $< \theta$ distinct θ-branches of $\mathcal{T}$ and

$$\eta_{\alpha_1}(\varepsilon_1) = \eta_{\alpha_2}(\varepsilon_2) \Rightarrow \varepsilon_1 = \varepsilon_2$$

and

$$\eta_{\alpha_1}(\varepsilon) = \eta_{\alpha_2}(\varepsilon) \Rightarrow \eta_{\alpha_1} \restriction \varepsilon = \eta_{\alpha_2} \restriction \varepsilon$$

by $(*)_2$, i.e., the choice of $\mathcal{T}$, it follows from the regularity of θ that we can find an $\varepsilon_* < \theta$ such that $\langle \eta_\alpha(\varepsilon_*)) : \alpha \in w \rangle$ has no repetitions, and define $s'_\alpha = s_{\eta_\alpha(\varepsilon_*)} \subseteq \kappa$ for $\alpha \in w$; now $\langle s'_\alpha : \alpha \in w \rangle$ is as required.

(Why? First $s'_\alpha \in J_1$ by the choice of s'_α. Second, assume $\alpha \neq \beta$ are from w and $i \in \kappa \backslash s'_\alpha \backslash s'_\beta$ and we should prove $\nu_\alpha(i) \neq \nu_\beta(i)$. Now $\eta_\alpha(\varepsilon_*) \neq \eta_\beta(\varepsilon_*)$ by the choice of ε_* and $s'_\alpha = s_{\eta_\gamma(\varepsilon_*)}$, $s'_\beta = s_{\eta_\beta(\varepsilon_*)}$ hence $i \in \kappa \backslash s_{\eta_\alpha(\varepsilon_*)} \backslash s_{\beta_\beta(\varepsilon_*)}$ so by the choice of $\langle s_{\eta_\gamma(\varepsilon)} : \gamma \in w, \varepsilon < \theta \rangle$ we have $\rho_{\eta_\alpha(\varepsilon_*)}(i) \neq \rho_{\eta_\beta(\varepsilon_*)}(i)$ hence

$$\mathrm{cd}'_{\varepsilon_*}(\nu_\alpha(i)) = \rho_{\eta_\alpha(\varepsilon_*)}(i) \neq \rho_{\eta_\beta(\varepsilon_*)}(i) = \mathrm{cd}'_{\varepsilon_*}(\nu_\beta(i))$$

which implies that $\nu_\alpha(i) \neq \nu_\beta(i)$.)

Step 2. To prove clause (β) of $\boxtimes$.

Let the subset $\mathcal{F}' \subseteq \{\nu_\alpha : \alpha < \lambda\}$ have cardinality $\leq \mu$. Choose w such that $\mathcal{F}' = \{\nu_\alpha : \alpha \in w\}$, so that $w \in [\lambda]^{\leq \mu}$ and define $u := \bigcup\{\mathrm{Rang}(\eta_\alpha) : \alpha \in w\}$. Clearly we have $u \in [\chi]^{\leq \mu}$. By the choice of $\langle \rho_\gamma : \gamma < \chi \rangle$ we can find a sequence $\langle s_\gamma : \gamma \in u \rangle$ such that $s_\gamma \in J_1$ and

$$i \in \kappa \backslash (s_{\gamma_1} \cup s_{\gamma_2}) \wedge \gamma_1 \neq \gamma_2 \wedge \{\gamma_1, \gamma_2\} \subseteq u \Rightarrow \rho_{\gamma_1}(i) \neq \rho_{\gamma_2}(i).$$

For $\alpha \in w$ let

$$t_\alpha := \{i < \kappa : \text{the set of } \varepsilon < \theta \text{ such that } i \notin s_{\eta_\alpha(\varepsilon)} \text{ belongs to } J_2 = J_\theta^{\mathrm{bd}}\}.$$

We shall show that $\bar{t} := \langle t_\alpha : \alpha \in w \rangle$ is as required in Definition 1.2 (1)–(2); that is, we have to prove that $t_\alpha \in J_1$ and that for any $\xi < \mu$ and $i_* < \kappa$ the set of $\alpha \in w$ such that $i_* \notin t_\alpha \wedge \nu_\alpha(i_*) = \xi$ is small, i.e., of cardinality $\leq 2^\theta$; these demands are proved below in $(*)_4$ and $(*)_3$ respectively. So let $\xi < \mu$ and $i_* < \kappa$ and let $v = v_{\xi, i_*} = \{\alpha \in w : i_* \notin t_\alpha \text{ and } \nu_\alpha(i_*) = \xi\}$.

First we shall prove below that:

$(*)_3$ $|v| \leq 2^\theta$.

This will do one half of proving "$\bar{t}$ is as required in Definition 1.2 (1),(2)."

Why does $(*)_3$ hold? Now if $\alpha \in v$, then $i_* \in \kappa \backslash t_\alpha$, hence (by the definition of t_α) we have $\mathcal{U}_{\alpha, i_*} := \{\varepsilon < \theta : i_* \notin s_{\eta_\alpha(\varepsilon)}\} \in J_2^+$. So if $\alpha \neq \beta$ are from v and $\varepsilon \in \mathcal{U}_{\alpha, i_*} \cap \mathcal{U}_{\beta, i_*}$ and $\eta_\alpha(\varepsilon) \neq \eta_\beta(\varepsilon)$, then we have $i_* \notin s_{\eta_\alpha(\varepsilon)}$ (as $\varepsilon \in \mathcal{U}_{\alpha, i_*}$) and $i_* \notin s_{\eta_\beta(\varepsilon)}$ (as $\varepsilon \in \mathcal{U}_{\beta, i_*}$), and hence by the choice of $\langle s_\gamma : \gamma \in u \rangle$, we have $\rho_{\eta_\alpha(\varepsilon)}(i_*) \neq \rho_{\eta_\beta(\varepsilon)}(i_*)$, so:

$(*)_4$ $\mathrm{cd}'_\varepsilon(\nu_\alpha(i_*)) = \rho_{\eta_\alpha(\varepsilon)}(i_*) \neq \rho_{\eta_\beta(\varepsilon)}(i_*) = \mathrm{cd}'_\varepsilon(\nu_\beta(i_*))$.

Recall that $\nu_\alpha(i_*) = \xi = \nu_\beta(i_*)$ because $\varepsilon \in \mathcal{U}_{\alpha,i_*} \cap \mathcal{U}_{\beta,i_*}$, but this contradicts $(*)_4$. It follows that

$$\alpha \in v \wedge \beta \in v \wedge \alpha \neq \beta \wedge \varepsilon \in \mathcal{U}_{\alpha,i_*} \cap \mathcal{U}_{\beta,i_*} \Rightarrow \eta_\alpha(\varepsilon) = \eta_\beta(\varepsilon);$$

but $\alpha \neq \beta \Rightarrow \{\varepsilon < \theta : \eta_\alpha(\varepsilon) = \eta_\beta(\varepsilon)\} \in J_2$, hence this implies

$$\alpha \in v \wedge \beta \in v \wedge \alpha \neq \beta \Rightarrow \mathcal{U}_{\alpha,i_*} \cap \mathcal{U}_{\beta,i_*} \in J_2.$$

As we have noted earlier that $\alpha \in v \Rightarrow \mathcal{U}_{\alpha,i_*} \in J_2^+$, it follows that $\mathcal{P}(\theta)/J_2$ fails the $|v|$-c.c. But for the present proof, $\mathcal{P}(\theta)$ has cardinality 2^θ, hence $\mathcal{P}(\theta)/J_2$ satisfies the $(2^\theta)^+$-c.c., and so $|v| \leq 2^\theta$, as required in $(*)_3$. For proving "$\bar{t}$ is as required in Definition 1.2", we need also the second half:

$(*)_5$ $t_\alpha \in J_1$ for $\alpha \in w$.

Why does $(*)_5$ hold? Firstly, assume $\kappa < \theta$; towards a contradiction assume that $t_\alpha \in J_1^+$. By the choice of t_α, for each $i \in t_\alpha$, the set $\{\varepsilon < \theta : i \notin s_{\eta_\alpha(\varepsilon)}\}$ belongs to J_2, but J_2, being equal to J_θ^{bd} (and recalling θ is regular), is κ^+-complete and $|t_\alpha| \leq \kappa$, hence the set

$$r_{\eta_\alpha} := \bigcup_{i \in t_\alpha} \{\varepsilon < \theta : i \notin s_{\eta_\alpha(\varepsilon)}\}$$

lies in J_2 hence we can choose $\varepsilon_\alpha < \theta$ such that $\varepsilon = \varepsilon_\alpha \Rightarrow \bigwedge_{i \in t_\alpha} i \in s_{\eta_\alpha(\varepsilon)}$, so $t_\alpha \subseteq s_{\eta_\alpha(\varepsilon_\alpha)}$, but $s_{\eta_\alpha(\varepsilon_\alpha)} \in J_1$, and hence $t_\alpha \in J_1$ as required.

Secondly, assume $\kappa > \theta$; towards a contradiction, assume $t_\alpha \in J_1^+$. Again

$$i \in t_\alpha \Rightarrow \{\varepsilon < \theta : i \notin s_{\eta_\alpha(\varepsilon)}\} \in J_2,$$

but $J_2 = J_\theta^{\mathrm{bd}}$, hence we can find $\bar{\varepsilon}_\alpha = \langle \varepsilon_{\alpha,i} : i \in t_\alpha \rangle \in {}^{(t_\alpha)}\theta$ such that

$$\varepsilon_{\alpha,i} = \sup\{\varepsilon < \theta : i \notin s_{\eta_\alpha(\varepsilon)}\} < \theta.$$

However, J_1 is κ-complete (see clause (f) of $\circledast$) hence J_1 is θ^+-complete, so for some $\varepsilon_\alpha^* < \theta$, we have

$$t'_\alpha := \{i \in t_\alpha : \varepsilon_{\alpha,i} < \varepsilon_\alpha^*\} \in J_1^+.$$

Thus

$$i \in t'_\alpha \Rightarrow \varepsilon_{\alpha,i} < \varepsilon_\alpha^* \Rightarrow \sup\{\varepsilon < \theta : i \notin s_{\eta_\alpha(\varepsilon)}\} < \varepsilon_\alpha^* \Rightarrow i \in s_{\eta_\alpha(\varepsilon_\alpha^*)}$$

and so $t'_\alpha \subseteq s_{\eta_\alpha(\varepsilon_\alpha^*)}$. But $s_{\eta_\alpha(\varepsilon_\alpha^*)} \in J_1$, while $t'_\alpha \notin J_1$, a contradiction. □

Proof of Claim 3.2. We note the points of the proof of Claim 3.1 which have to be changed. The choice of $\bar{\rho} = \langle \rho_\gamma : \gamma < \chi \rangle$, i.e., $(*)_1$ is now done by using $\circledast'$(h). Before $(*)_2$, instead of defining J_2 recall that it is given (see $\circledast'$(f)) and if J'_2 is not given (see $\circledast'$(i, β)), let $J'_2 = J_2$. After $(*)_2$, instead of choosing $\langle \eta_\alpha : \alpha < \lambda \rangle$ it is given in $\circledast'$(g) and the tree $\mathcal{T}$ disappears.

Now Step 1 says that "$\mathscr{F}$ is (θ_1, J_1)-free". Thus we have to choose ε_* as there. Of course, now $|w| < \theta_1$ as we are proving "$\mathscr{F}$ is (θ_1, J_1)-free".

First, if clause (α) of $\circledast'$(f) holds, as $\mathcal{U}^1_{\alpha,\beta} := \{\varepsilon < \theta : \eta_\alpha(\varepsilon) = \eta_\beta(\varepsilon)\} \in J_2$ for $\alpha \neq \beta$ from w, but J_2 is θ_1-complete, so

$$\{\varepsilon < \theta : \eta_\alpha(\varepsilon) = \eta_\beta(\varepsilon) \text{ for some } \alpha \neq \beta \text{ from } w\} \in J_2,$$

hence there is an $\varepsilon_* < \theta$ not in $\bigcup\{\mathcal{U}^1_{\alpha,\beta} : \alpha \neq \beta \text{ are from } \omega\}$.

Second, if clause (β) of $\circledast'$(f) holds, clearly $\theta_1 < \kappa$, so as J_1 is κ-complete it suffices to prove $\alpha < \beta < \lambda \Rightarrow s_{\alpha,\beta} = \{i < \kappa : \nu_\alpha(i) = \nu_\beta(i)\} \in J_1$ but for $\alpha \neq \beta$ we have $\eta_\alpha \neq \eta_\beta$ hence for some $\varepsilon < \theta$ we have $\eta_\alpha(\varepsilon) \neq \eta_\beta(\varepsilon)$ hence

$$s_{\alpha,\beta} \subseteq \{i < \kappa : \rho_{\eta_\alpha(\varepsilon)}(i) = \rho_{\eta_\beta(\varepsilon)}(i)\} \in J_1$$

so we are done.

Turning to Step 2, now to define t_α we use "belongs to J'_2"; then $(*)_3$ should say $|v| < \sigma$ and in the proof instead of "$\mathscr{P}(\theta)/J_2$ satisfies the $(2^\theta)^+$-c.c." we use clause $\circledast'$(i, α) if it holds and $\circledast'$(i, β), as still $\alpha \neq \beta \Rightarrow \mathcal{U}_{\alpha,i_*} \cap \mathcal{U}_{\beta,i_*} \in J_2$.

Lastly, to prove $(*)_5$ we use clause $\circledast'$(f). □

Claim 3.4. *In Claim* 3.1, $J = J_1$ *is an ideal on* κ, *so assuming* $(\forall \alpha < \mu)(|\alpha|^\kappa < \mu)$ *we can add "$\mathscr{F}$ is (Υ, J)-free"* when (a) *or* (b) *or* (c) *hold where* $(J_2 = J^{\mathrm{bd}}_\theta)$:

(a) $\Upsilon = \theta^{+\omega+1}$ *and we can choose* $\eta_\alpha \in {}^\theta\chi$ *for* $\alpha < \lambda$ *with no repetitions such that* $\theta^+ \notin \mathrm{issp}_{J_2}(\{\eta_\alpha : \alpha < \lambda\})$,

(b) $\theta^{+\omega} < \Upsilon \leq \mu$ *and we can choose* $\eta_\alpha \in {}^\theta\chi$ *for* $\alpha < \lambda$ *with no repetitions such that* $\theta < \partial = \mathrm{cf}(\partial) \wedge (< \partial, \partial) \in \mathrm{issp}_{J_2}(\{\eta_\alpha : \alpha < \lambda\}) \Rightarrow \partial \geq \Upsilon$,

(c) *there are pairwise distinct* $\eta_\alpha \in {}^\theta\chi$ *for* $\alpha < \lambda$ *and* $\varrho_\gamma \in {}^\kappa\mu$ *for* $\gamma < \chi$ *such that for every regular* $\partial \in (\theta + \kappa^+, \Upsilon)$ *we have* $\partial \notin \mathrm{issp}_{J_2}(\{\eta_\alpha : \alpha < \lambda\})$ *and* $\partial \notin \mathrm{issp}_{J_1}(\{\varrho_\gamma : \gamma < \chi\})$ but *in* $\boxtimes_1$ *of the proof of Claim* 3.1, $\nu_\alpha(i)$ *also codes* $\varrho_\alpha(i)$.

Proof. By Observation 1.4, case (a) is a special instance of (b) and by Definition 1.2 (7), case (b) is a special case of (c), so we deal with case (c) only.

We shall repeat the proof of Claim 3.1 but we use $\langle \eta_\alpha : \alpha < \chi \rangle$, $\langle \varrho_\alpha : \alpha < \lambda \rangle$ from the assumption (c).

Consider the statement:

$\boxplus$ S is not a stationary subset of ∂ *when*:

$\odot_{\partial,S}$ we have $\partial = \mathrm{cf}(\partial) \in (\theta + \kappa^+, \mu), \alpha_\varepsilon < \lambda$ for $\varepsilon < \partial$ with no repetitions and

$$S = \{\zeta < \partial : \text{for some } \xi \in [\zeta, \partial), \{i < \kappa : \nu_{\alpha_\xi}(i) \in \{\nu_{\alpha_\varepsilon}(i) : \varepsilon < \zeta\}\} \text{ belongs to } J_1^+\}.$$

It suffices to prove $\boxplus$. Why? We prove that $\{\nu_\alpha : \alpha < \lambda\}$ is ∂-free by induction on $\partial < \Upsilon$ so let $w \subseteq \lambda, |w| \le \partial$. If $\partial \le \kappa$, just note that

$$\alpha \ne \beta \in w \Rightarrow \{i < \kappa : \nu_\alpha(i) = \nu_\beta(i)\} \in J_1;$$

if $\partial < \theta$, recall $\boxtimes(\alpha)$ of Claim 3.1. If $\partial \ge \kappa^+ + \theta$ is singular, use compactness for singulars. So assume $\partial = \mathrm{cf}(\partial) \ge \kappa^+ + \theta$, so by the induction hypothesis without loss of generality $|w| = \partial$, and let $\langle \alpha_\varepsilon : \varepsilon < \partial\rangle$ list w and define S as in $\odot_{\delta,S}$ above from $\langle \alpha_\varepsilon : \varepsilon < \partial\rangle$. As we are assuming $\boxplus$, necessarily S is not a stationary subset of ∂ so let E be a club of ∂ disjoint to S. Let $\langle \varepsilon(\iota) : \iota < \partial\rangle$ list $E \cup \{0\}$ in increasing order. For each $\iota < \theta$ we apply the induction hypothesis to $w_\iota := \{\alpha_\varepsilon : \varepsilon \in [\varepsilon(\iota), \varepsilon(\iota+1))\}$ and get the sequence $\langle s_{\iota,\varepsilon} \in J_1 : \varepsilon \in w_\iota\rangle$.

Lastly, for $\varepsilon < \partial$ let ι be such that $\varepsilon \in [\varepsilon(\iota), \varepsilon(\iota+1))$ and

$$s_\varepsilon = s_{\iota,\varepsilon} \cup \{i < \kappa : \nu_\varepsilon(i) \text{ belongs to } \{\nu_{\alpha_\zeta}(i) : \zeta < \varepsilon(\iota)\}\}.$$

Why does $\boxplus$ *hold?* Towards a contradiction, suppose that S is a stationary subset of $\partial = \mathrm{cf}(\gamma) \in (\theta + \kappa^+, \Upsilon)$. Then without loss of generality:

$(*)_5$ (a) for some stationary $S_0 \subseteq S$, for every limit $\zeta \in S_0, \zeta$ can itself serve as the witness ξ (in fact we can have $S \backslash S_0$ not stationary)

(b) for some club E of ∂, if $\varepsilon < \xi$ and $E \cap (\varepsilon, \xi] \ne \emptyset$, then

$$\{i < \kappa : \nu_{\alpha_\xi}(i) \in \{\nu_{\alpha_{\varepsilon(1)}}(i) : \varepsilon(1) < \varepsilon\}\} \in J_1.$$

(Why? For clause (a) by renaming. For clause (b), it suffices to show the condition $(\forall \varepsilon < \partial)(f(\varepsilon) < \partial)$ where for $\varepsilon < \partial$, $f(\varepsilon)$ is defined to be the minimal ordinal γ such that if $\{i < \kappa : \nu_{\alpha_\varepsilon}(i) \in \{\nu_{\alpha_\zeta}(i) : \zeta < \varepsilon\}\} \in J_1^+$, then $\varepsilon < \gamma$. Now, if $\partial \notin \mathrm{issp}_{J_2}(\{\eta_\alpha : \alpha < \lambda\})$, this follows. Otherwise, $\partial \in \mathrm{issp}_{J_1}(\{\varrho_\gamma : \gamma < \chi\})$, and think.)

Clearly $\delta \in S \Rightarrow \mathrm{cf}(\delta) \le \kappa$, and because $(\forall \alpha < \mu)(|\alpha|^\kappa < \mu)$, by the second • in $\boxtimes_1$ in the proof of Claim 3.1 we know that $\nu_\alpha(i)$ determine $\nu_\alpha \restriction i$, hence easily without loss of generality:

$(*)_6$ $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \kappa \wedge \delta \in E$.

Let

$$S_1 := \{\zeta \in S_0 : \{\varepsilon < \theta : \eta_{\alpha_\zeta}(\varepsilon) \in \{\eta_{\alpha_j}(\varepsilon) : j < \zeta\}\} \text{ belongs to } J_2^+\}.$$

Case A: S_1 is a stationary subset of ∂. Firstly, assume $\kappa < \theta$. As, see above, $\zeta \in S_0 \Rightarrow \mathrm{cf}(\zeta) \le \kappa$ and $\theta > \kappa \Rightarrow J_2$ is κ^+-complete, clearly for each $\zeta \in S_1$, for some $j_\zeta < \zeta$, we have

$$\{\varepsilon < \theta : \eta_{\alpha_\zeta}(\varepsilon) \in \{\eta_{\alpha_j}(\varepsilon) : j < j_\zeta\}\} \in J_2^+.$$

By Fodor's lemma, for some $j(*)$, the set $S_2 = \{\zeta \in S_1 : j_\zeta \le j(*)\}$ is a stationary subset of ∂. Now $\{\eta_{\alpha_\zeta} : \zeta \in S_2\}$ witnesses $(<\partial, \partial) \in \mathrm{ussp}_{J_2}(\lim_\theta(\mathcal{T}))$; but this contradicts a demand in case (c) of the assumption of Claim 3.4.

Secondly, if $\theta < \kappa$ but recalling $(*)_6$ (see above), without loss of generality $\zeta \in S_0 \Rightarrow \mathrm{cf}(\zeta) = \kappa$ and now the proof is similar.

Case B: $\kappa < \theta$ and S_1 is not stationary. Necessarily $S_0 \backslash S_1$ is a stationary subset of ∂. By the definition of S_1 (and $(*)_5$) we can find $\bar{s}^* = \langle s^*_\zeta : \zeta \in (S_0 \backslash S_1)\rangle$ such that:

$(*)_7$ (a) $s^*_\zeta \in J_2$,

(b) if $\zeta_1 \neq \zeta_2$ are from $(S_0 \backslash S_1)$ and $\varepsilon \in \theta \backslash s^*_{\zeta_1} \backslash s^*_{\zeta_2}$, then $\eta_{\alpha_{\zeta_1}}(\varepsilon) \neq \eta_{\alpha_{\zeta_2}}(\varepsilon)$.

Let $\varepsilon(\zeta) = \min(\theta \backslash s^*_\zeta)$ for $\zeta \in (S_0 \backslash S_1)$. So for some stationary $S_2 \subseteq (S_0 \backslash S_1)$, we have $\zeta \in S_2 \Rightarrow \varepsilon(\zeta) = \varepsilon(*)$ and so

$(*)_8$ $\langle \eta_{\alpha_\zeta}(\varepsilon(*)) : \zeta \in S_2\rangle$ is without repetitions.

Now $(*)_2$ (b) in the proof of Claim 3.1 says that $\langle \rho_\gamma : \gamma < \chi\rangle$ is (μ^+, J_1^+)-free; apply this to the subset $\{\varrho_{\eta_{\alpha_\zeta}(\varepsilon(*))} : \zeta \in S_2\}$ which has cardinality $\partial < \mu^+$ hence (recall $(*)_8$):

$(*)_9$ some $\langle s[\eta_{\alpha_\zeta}(\varepsilon(*))] : \zeta \in S_2\rangle$ witnesses that $\langle \rho_{\eta_{\alpha_\zeta}(\varepsilon(*))} : \zeta \in S_2\rangle$ is free, i.e., we have $s_{\eta_{\alpha_\zeta}(\varepsilon(*))} \in J_1$ for $\zeta \in S_1$ and

$$\zeta \neq \xi \in S_2 \wedge i \in \kappa \backslash s[\eta_{\alpha_\zeta}(\varepsilon(*))] \backslash s[\eta_{\alpha_\xi}(\varepsilon(*))]$$
$$\Rightarrow \varrho_{\eta_{\alpha_\zeta}(\varepsilon(*))}(i) \neq \varrho_{\eta_{\alpha_\xi}(\varepsilon(*))}(i).$$

As $\kappa < \partial$, for some $i(*) < \kappa$, the set $S_3 := \{\zeta \in S_2 : i(*) \notin s[\eta_{\alpha_\zeta}(\varepsilon(*))]\}$ is a stationary subset of ∂. By $(*)_6$ we know that $\nu_\alpha(i) = \nu_\beta(i) \Rightarrow \nu_\alpha \restriction i = \nu_\beta \restriction i$ for $\alpha, \beta < \lambda, i < \kappa$; $\langle \nu_{\alpha_\varepsilon}(i(*)) : \varepsilon \in S_2\rangle$ is a sequence without repetitions, but by the choice of S we have

$$\zeta \in S_3 \Rightarrow \nu_{\alpha_\varepsilon}(i(*)) \in \{\nu_{\alpha_\zeta}(i(*)) : \zeta < \varepsilon\}.$$

However, this contradicts "$S_3 \subseteq S_2 \subseteq S_0 \subseteq S$ is a stationary subset of ∂". $\square$

Claim 3.5. *In Claim* 3.1, *$\mathscr{F}$ satisfies: for $\kappa + \theta < \partial = \mathrm{cf}(\partial) < \lambda$, we have $\mathscr{F}$ is $(\partial^+, \partial, J_1)$-free iff $(< \partial, \partial) \in \mathrm{issp}_{J_1}(\mathscr{F})$ and there are f_ε for $\varepsilon < \partial$ with no repetitions such that for stationarily many $\delta \in S^\lambda_{\leq\kappa}$ we have*

$$\{i < \kappa : f_\delta(i) \in \{f_\alpha(i) : \alpha < \delta\}\} \in J_1^+.$$

Proof. By the proof of the previous claim. □

In Claim 3.6, the case we are interested in is $\mu = \beth_{\omega_1}, \kappa = \aleph_1, \theta = \aleph_0$.

Claim 3.6. *There is a set $\mathscr{F} \subseteq {}^\kappa\mu$ of cardinality λ which is (μ^+, J)-free* when:

⊛ (a) $\theta = \mathrm{cf}(\theta) < \kappa = \mathrm{cf}(\mu) < \mu$,
 (b) $\lambda = \mu^\kappa$,
 (c) $\mu < \chi < \chi^\theta = \lambda$,
 (d) $\alpha < \mu \Rightarrow |\alpha|^\theta < \mu$,
 (e) *J is a θ^+-complete ideal on κ,*
 (f) $\mathrm{pp}_J(\mu) =^+ \lambda$.

Remark 3.7. The statement is used in Theorem 1.22.

Proof. Let $\langle \mu_i : i < \kappa \rangle$ be increasing with limit μ such that $(\mu_i)^\theta = \mu_i$ and let $\mathrm{cd}_* : {}^\theta\mu \to \mu$ and cd_ε (for $\varepsilon < \theta$) be as in the proof of Claim 3.1, noting that by clause (a) of the assumption of the claim ${}^\theta\mu = \bigcup\{{}^\theta(\mu_i) : i < \kappa\} = \mu$ and let $\mu_i^- = \bigcup\{\mu_j : j < i\}$.

As $\chi < \mathrm{pp}_J(\mu)$, by Claim 1.3 (c), i.e., [22, Chapter II] there is a sequence $\bar\rho = \langle \rho_\gamma : \gamma < \chi \rangle$ of members of ${}^\kappa\mu$ which is (μ^+, J)-free. Let $\bar\eta = \langle \eta_\alpha : \alpha < \lambda \rangle$ with $\eta_\alpha \in {}^\theta\chi$ be pairwise distinct.

Without loss of generality, $\rho_\gamma \in \prod_{i<\kappa}[\mu_i^-, \mu_i)$; we define $\nu_\alpha \in \prod_{i<\kappa}\mu_i \subseteq {}^\kappa\mu$ for $\alpha < \lambda$ by

$$\nu_\alpha(i) = \mathrm{cd}_*(\langle \rho_{\eta_\alpha(\varepsilon)}(i) : \varepsilon < \theta \rangle) \quad \text{for } i < \kappa.$$

We shall prove that $\langle \nu_\alpha : \alpha < \lambda \rangle$ is as required, i.e., $\langle \nu_\alpha : \alpha < \lambda \rangle$ is (μ^+, J)-free; this suffices as it implies $\alpha < \beta < \lambda \Rightarrow \nu_\alpha \neq \nu_\beta$ hence $\{\nu_\alpha : \alpha < \lambda\} \subseteq {}^\kappa\mu$ has cardinality $\lambda = \mu^\kappa$ (and is (μ^+, J)-free).

For $w \in [\lambda]^{\leq\mu}$, we let $u = \bigcup\{\mathrm{Rang}(\eta_\alpha) : \alpha \in w\}$, so u is a subset of χ of cardinality $\leq \mu$.

As $\bar\rho = \langle \rho_\alpha : \alpha < \chi \rangle$ is (μ^+, J)-free, there is $\bar s = \langle s_\gamma : \gamma \in u \rangle$ such that:

⊛ (α) $s_\gamma \in J$ for every $\gamma \in u$,
 (β) if $\gamma_1 \neq \gamma_2 \in u$ and $i \in \kappa \backslash (s_{\gamma_1} \cup s_{\gamma_2})$, then $\rho_{\gamma_1}(i) \neq \rho_{\gamma_2}(i)$.

Now for each $\alpha \in w$, the set $t_\alpha := \bigcup\{s_{\eta_\alpha(\varepsilon)} : \varepsilon < \theta\}$ is the union of $\leq \theta$ members of J, but J is θ^+-complete by assumption (e), hence $t_\alpha \in J$.

Suppose $\alpha_1 \neq \alpha_2$ are from w and $i \in \kappa \backslash (t_{\alpha_1} \cup t_{\alpha_2})$. Can we have the equality $\nu_{\alpha_1}(i) = \nu_{\alpha_2}(i)$? If so, then for every $\varepsilon < \theta$, we have $i \in \kappa \backslash s_{\eta_{\alpha_1}(\varepsilon)} \backslash s_{\eta_{\alpha_2}(\varepsilon)}$ and $\rho_{\eta_{\alpha_1}(\varepsilon)}(i) = \rho_{\eta_{\alpha_2}(\varepsilon)}(i)$, hence necessarily $\eta_{\alpha_1}(\varepsilon) = \eta_{\alpha_2}(\varepsilon)$. As this holds for every $\varepsilon < \theta$, we get $\eta_{\alpha_1} = \eta_{\alpha_2}$. This implies $\alpha_1 = \alpha_2$.

So $i \in \kappa \backslash (t_{\alpha_1} \cup t_{\alpha_2}) \wedge \nu_{\alpha_1}(i) = \nu_{\alpha_2}(i) \Rightarrow \alpha_1 = \alpha_2$. Thus $\langle \nu_\alpha : \alpha \in w \rangle$ is free, so we are done. □

Conclusion 3.8. *If clauses* (a)–(f) *of Claim* 3.6 *hold and* $\lambda = \mu^\kappa = 2^\mu$, then $\mathrm{BB}(\lambda, \mu^+, \lambda, J)$.

Proof. By Claim 3.6 there is $\mathscr{F} \subseteq {}^\kappa\mu$ of cardinality λ which is (μ^+, J)-free. By assumption $|\mathscr{F}| = \mu^\kappa = 2^\mu$ hence by Observation 0.9 we get $\mathrm{BB}(2^\mu, \mu^+, \chi, J)$ so we are done. □

A relative of Claim 3.6 is

Claim 3.9. *There is a* (μ^+, J_1)*-free* $\mathscr{F} \subseteq {}^\kappa\mu$ *of cardinality* λ when:

⊛ (a) $\sigma < \theta < \kappa = \mathrm{cf}(\mu) < \mu < \lambda$,

(b) (α) *J_2 is a σ^+-complete ideal on θ,*

(β) *there are λ pairwise J_2-distinct members of ${}^\theta\chi$,*

(c) $2^\kappa < \mu < \chi < \lambda$ *and* $2^\kappa < \mathrm{cf}(\lambda)$,

(d) $\alpha < \mu \Rightarrow \mathrm{cov}(|\alpha|, \theta^+, \theta^+, \sigma^+) \leq \mu$,

(e) *J_1 is a θ^+-complete ideal on κ,*

(f) $\chi < \mathrm{pp}_{J_1}(\mu)$.

Proof. By clauses (f) and (c) there is an increasing sequence $\langle \lambda_j : j < \kappa \rangle$ of regular cardinals $\in (2^\kappa, \mu)$ with limit μ such that $\chi^+ = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_{J_1})$ and we let $\lambda_i^- = \Sigma\{\lambda_j : j < i\}$ for $i < \kappa$.

By (f) and Claim 1.3 (c), without loss of generality there is a (μ^+, J_1)-free sequence $\langle \rho_\gamma : \gamma < \chi \rangle$ of members of $\prod_{j<\kappa} \lambda_j$. Let $\mathscr{P}_i \subseteq [\lambda_i]^\theta$ be a set of cardinality $\leq \mu$ such that:

$(*)_{\mathscr{P}_i}$ for every $u \in [\lambda_i]^\theta$, we can find $\zeta_u \leq \sigma$ and $u_\zeta \in \mathscr{P}_i$ for $\zeta < \zeta_u$ such that $u \subseteq \bigcup\{u_\zeta : \zeta < \zeta_u\}$.

Note that $\mathscr{P}_i$ exists by clause (d) of the assumption. Let $\mathscr{P} = \bigcup\{\mathscr{P}_i : i < \kappa\}$, so that $|\mathscr{P}| \leq \mu$, $\mathscr{P} \subseteq [\mu]^\theta$.

By clause (b, β), let $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle$ with $\eta_\alpha \in {}^\theta\chi$ be such that $\alpha < \beta < \lambda$ implies $\eta_\alpha \neq_{J_2} \eta_\beta$, i.e., $\{\varepsilon < \theta : \eta_\alpha(\varepsilon) = \eta_\beta(\varepsilon)\} \in J_2$.

Lastly, for each $\alpha < \lambda$, for each $i < \kappa$, we know that

$$\{\rho_{\eta_\alpha(\varepsilon)}(i) : \varepsilon < \theta\} \in [\lambda_i]^{\leq\theta},$$

hence we can find a sequence $\langle u^i_{\alpha,\zeta} : \zeta < \sigma\rangle$ of members of $\mathscr{P}_i$ such that

$$\{\rho_{\eta_\alpha(\varepsilon)}(i) : \varepsilon < \theta\} \subseteq \bigcup\{u^i_{\alpha,\zeta} : \zeta < \sigma\}.$$

For each $\alpha < \lambda$ and $i < \kappa$, as J_2 is a σ^+-complete ideal on θ, for some $\zeta_{\alpha,i} < \sigma$, the set $\mathscr{W}_{\alpha,i} := \{\varepsilon < \theta : \rho_{\eta_\alpha(\varepsilon)}(i) \in u^i_{\alpha,\zeta_{\alpha,i}}\}$ belongs to J_2^+. Let

$$\boldsymbol{x}_\alpha := \{(i, \zeta_{\alpha,i}, s_{\eta_\alpha(\varepsilon)}(i) \cap u^i_{\alpha,s_{\alpha,i}}) : i < \kappa \text{ and } \varepsilon \in \mathscr{W}_{\alpha,i} \subseteq \theta\}.$$

The number of possible $\boldsymbol{x}_\alpha$ is at most $\leq 2^\kappa$, but $2^\kappa < \mathrm{cf}(\lambda)$ by clause (c) of the assumption. As we can replace $\langle \eta_\alpha : \alpha < \lambda\rangle$ by $\langle \eta_\alpha : \alpha \in v\rangle$ for any $v \in [\lambda]^\lambda$, without loss of generality for some $\boldsymbol{x} = \{(i, \zeta_{i,\varepsilon}, \gamma_{i,\varepsilon}) : i < \kappa \text{ and } \varepsilon \in \mathscr{W}_i\}$, we have:

$(*)_0$ $\boldsymbol{x}_\alpha = \boldsymbol{x}$ for every $\alpha < \lambda$.

For $\alpha < \lambda$ let $\nu_\alpha \in {}^\kappa\mathscr{P}$ be defined by:

$\odot_1$ $\nu_\alpha(i) = u^i_{\alpha,\zeta_{\alpha,i}}$.

Clearly it suffices to show that:

$\odot_2$ $\bar{\nu} = \langle \nu_\alpha : \alpha < \lambda\rangle$ exemplifies the conclusion.

This follows by $(*)_1, (*)_2, (*)_3$ below:

$(*)_1$ $\nu_\alpha \in {}^\kappa\mathscr{P}$ and $|\mathscr{P}| \leq \mu$.

(Why? Obviously.)

$(*)_2$ $\nu_\alpha \neq \nu_\beta$ for $\alpha < \beta < \lambda$.

(Why? By the proof of $(*)_3$ using $w = \{\alpha, \beta\}$.)

$(*)_3$ $\{\nu_\alpha : \alpha < \lambda\}$ is (μ^+, J_1)-free.

(Why? Let $w \in [\lambda]^{\leq\mu}$; we shall prove that $\{\nu_\alpha : \alpha \in w\}$ is J_1-free. Now

$$u := \bigcup\{\mathrm{Rang}(\eta_\alpha) : \alpha \in w\} \in [\chi]^{\leq\mu},$$

recalling $\varepsilon < \theta \Rightarrow \eta_\alpha(\varepsilon) < \chi$. By the assumption on $\{\rho_\gamma : \gamma < \chi\}$, we can find a sequence $\bar{s}$ such that:

(α) $\bar{s} = \langle s_\gamma : \gamma \in u\rangle \in {}^u(J_1)$,

(β) if $\gamma_1 \neq \gamma_2$ and $\gamma_1 \in u, \gamma_2 \in u$ and $i \in \kappa \backslash s_{\gamma_1} \backslash s_{\gamma_2}$, then $\rho_{\gamma_1}(i) \neq \rho_{\gamma_2}(i)$.

For each $\alpha \in w$, let $t_\alpha := \bigcup\{s_{\eta_\alpha(\varepsilon)} : \varepsilon < \theta\}$. Now t_α is the union of $\le \theta$ members of J_1 which is a θ^+-complete ideal (by (e)), so $t_\alpha \in J_1$. It suffices to prove that $\langle t_\alpha : \alpha \in w\rangle$ witnesses $\langle \nu_\alpha : \alpha \in w\rangle$ is J_1-free, so, by the previous sentence, it suffices to prove:

$(*)'_3$ if $\alpha_1 \neq \alpha_2$ are from w and $i \in \kappa \backslash t_{\alpha_1} \backslash t_{\alpha_2}$, then $\nu_{\alpha_1}(i) \neq \nu_{\alpha_2}(i)$.

Toward a contradiction assume that $\nu_{\alpha_1}(i) = \nu_{\alpha_2}(i)$. Recalling the choice of ν_α, i.e., $\odot_1$, this means that

$$u^i_{\alpha_1,\zeta_{\alpha_1,i}} = u^i_{\alpha_2,\zeta_{\alpha_2,i}}.$$

As $\boldsymbol{x}_{\alpha_1} = \boldsymbol{x} = \boldsymbol{x}_{\alpha_2}$, see condition $(*)_0$, clearly $\mathcal{W}_{\alpha_1,i} = \mathcal{W}_{\alpha_2,i}$, but we are assuming $u^i_{\alpha_1,\zeta_{\alpha_1,i}} = u^i_{\alpha_2,\zeta_{\alpha_2,i}}$ so by the definition of $\boldsymbol{x}_{\alpha_1}, \boldsymbol{x}_{\alpha_2}$ we have

$$\varepsilon \in \mathcal{W}_{\alpha_1} = \mathcal{W}_{\alpha_2} \Rightarrow \rho_{\eta_{\alpha_1}(\varepsilon)}(i) = \rho_{\eta_{\alpha_2}(\varepsilon)}(i) \Rightarrow \eta_{\alpha_1}(\varepsilon) = \eta_{\alpha_2}(\varepsilon)$$

so $\{\varepsilon < \theta : \eta_{\alpha_1}(\varepsilon) = \eta_{\alpha_2}(\varepsilon)\} \supseteq \mathcal{W}_{\alpha_1}$ but $\mathcal{W}_{\alpha_1,i} \in J_2^+$ by the choice of $\zeta_{\alpha_1,i}$. So we get $\neg(\eta_{\alpha_1} \neq_{J_2} \eta_{\alpha_2})$, contradicting the choice of $\langle \eta_\alpha : \alpha < \lambda\rangle$.)

So $(*)'_3$ holds, and hence $(*)_3$ holds. Therefore $\odot_2$ holds, so we are done. □

Observation 3.10. (1) *Assume* $\lambda > \mu > \kappa = \mathrm{cf}(\mu)$, $\alpha < \mu \Rightarrow |\alpha|^\sigma < \mu$, *and* $\theta = \sup\{\theta_i : i < \sigma\}$ *and for each* $i < \sigma$, *there is a* θ_i*-free* $\mathcal{F} \subseteq {}^\kappa\mu$ *of cardinality* λ. Then *there is a* θ*-free* $\mathcal{F} \subseteq {}^\kappa\mu$ *of cardinality* λ.

(1A) *If* $\kappa = \sigma$, then $\alpha < \mu \Rightarrow |\alpha|^{<\sigma} < \mu$ *suffices.*

(2) *If* $\mathcal{F} \subseteq {}^\kappa\mu$ *is* θ*-free,* then *there is a normal* θ*-free* $\mathcal{F}' \subseteq {}^\kappa\mu$ *of cardinality* $|\mathcal{F}|$ *– see Definition* 1.2 (5).

(3) *If* J *is an ideal on* $\kappa, \delta < \lambda$ *and* $\langle \lambda_i : i < \delta\rangle$ *is increasing with limit* λ *and there are* (θ, J)*-free* $\mathcal{F}_i \subseteq {}^\kappa\mu$ *of cardinality* λ_i *for* $i < \delta$, then *there is a* (θ, J)*-free* $\mathcal{F} \subseteq {}^\kappa\mu$ *of cardinality* λ.

(3A) *In part* (3), *if* $f \in \mathcal{F}_i \wedge \varepsilon < \kappa$, $f(\varepsilon) \in \mathcal{U}_\varepsilon \subseteq \mu$ *and* $\mathcal{U}_\varepsilon$ *is infinite for* $\varepsilon < \kappa$, then *without loss of generality* $f \in \mathcal{F} \wedge \varepsilon < \kappa \Rightarrow f(\varepsilon) \in \mathcal{U}_\varepsilon$.

(4) *We can in part* (3) *add "*$(\mathcal{F}_i, <_J)$ *has order type* λ_i*" while changing the conclusion to "*$\mathcal{F} \subseteq {}^\kappa(\mu \times \mu)$, *is* (θ, J)*-free and* $(\mathcal{F}_i, <_J)$ *has order type* λ*".*

(5) *Similarly to* (4) *but* $\mathcal{F} \subseteq {}^\kappa\mu$ *if* $2^\kappa < \mu, \mathrm{cf}(\delta)$ *recalling Definition* 1.2 (4).

Proof. (1) By coding (separating the proof according to whether $\sigma < \kappa$ or $\sigma \ge \kappa$).

In more detail, without loss of generality, $i < \sigma \Rightarrow \theta_i < \theta$; let $\mathcal{F}_i \subseteq {}^\kappa\mu$ be θ_i-free of cardinality λ, let $\langle \eta^i_\alpha : \alpha < \lambda\rangle$ list $\mathcal{F}_i$ with no repetitions, and let the map $\mathrm{cd} : \bigcup_{\alpha<\mu} {}^\sigma\alpha \to \mu$ be one-to-one.

Case 1: $\sigma < \kappa$. For $\alpha < \lambda$ and $\varepsilon < \kappa$ the sequence $\langle \eta^i_\alpha(\varepsilon) : i < \sigma\rangle$ belongs to ${}^\sigma\mu$ hence by the present case to $\bigcup\{{}^\sigma\beta : \beta < \alpha\}$.

Let $\eta_\alpha := \langle \mathrm{cd}(\langle \eta^i_\alpha(\varepsilon) : i < \sigma\rangle) : \varepsilon < \kappa\rangle$, so $\eta_\alpha \in {}^\kappa\mu$, and clearly $\langle \eta_\alpha : \alpha < \lambda\rangle$ is as required.

Case 2: $\sigma \geq \kappa$. Let $\langle \mu_\varepsilon : \varepsilon < \kappa\rangle$ be increasing with limit μ. For $\varepsilon < \kappa$ let the map $\boldsymbol{h}_\varepsilon : \sigma \times \varepsilon \to \sigma$ be one-to-one and onto.

We define $\eta_\alpha \in {}^\kappa\mu$ as follows:

- for $\varepsilon < \kappa$ we let $\eta_\alpha(\varepsilon) = \mathrm{cd}(\langle \gamma_{\alpha,i} : i < \sigma\rangle)$ where $\gamma_{\alpha,i}$ is defined as follows: if $j < \sigma$ and $\zeta < \varepsilon$ and $i = \boldsymbol{h}_\varepsilon(j,\zeta)$, then $\gamma_{\alpha,i} = \min\{\eta^j_\alpha(\zeta), \mu_\varepsilon\}$.

First for $\varepsilon < \kappa$, $\eta_\alpha(\varepsilon)$ is well defined ($< \mu$) as $\langle \gamma_{\alpha,i} : i < \sigma\rangle \in {}^\sigma(\mu_\varepsilon) \subseteq \mathrm{dom}(\mathrm{cd})$; so indeed $\eta_\alpha \in {}^\kappa\mu$. Second, $\{\eta_\alpha : \alpha < \lambda\}$ is θ-free because if $w \subseteq \lambda, |w| < \theta$, then for some $i < \sigma$ we have $|w| < \theta_i$, hence we can find a sequence $\langle \zeta_\alpha : \alpha \in w\rangle$ of ordinals $< \kappa$ such that:

- $\alpha \in w \wedge \beta \in w \wedge \varepsilon < \kappa \wedge \varepsilon \geq \zeta_\alpha \wedge \varepsilon \geq \zeta_\beta \Rightarrow \eta^i_\alpha(\varepsilon) \neq \eta^i_\beta(\varepsilon)$.

Let $\xi_\alpha = \min\{\xi : \xi \geq \zeta_\alpha$ and $\eta^i_\alpha(\zeta_\alpha) < \mu_\xi\}$. Then, easily

- $\alpha \in w \wedge \beta \in w \wedge \varepsilon < \kappa \wedge \varepsilon \geq \xi_\alpha \wedge \varepsilon \geq \xi_\beta \Rightarrow \eta_\alpha(\varepsilon) \neq \alpha_\beta(\varepsilon)$.

So we are done.

(1A) The proof is similar using $\eta_\alpha = \langle \mathrm{cd}(\eta^i_\alpha(\varepsilon) : i \leq \varepsilon) : \varepsilon < \kappa\rangle$ for an appropriate function cd. This is all right because $\alpha < \mu \Rightarrow |\alpha|^{<\sigma} < \mu$; actually $\alpha < \mu \Rightarrow |\alpha|^{<\sigma} \leq \mu$ suffices.

(2) Easy.

(3) Let $i(*) = \min\{i : \delta \leq \lambda_i\}$ and let $\lambda^-_i = \bigcup\{\lambda_j : j < i\}$ for $i < \delta$. Further, let $\langle f^i_\alpha : \alpha < \lambda_i\rangle$ list $\mathscr{F}_i$ with no repetitions, for $\varepsilon < \kappa$ let $\mathrm{cd}_\varepsilon : \mu \times \mu \to \mu$ be one-to-one and for $\alpha < \lambda$ let $f_\alpha \in {}^\kappa\mu$ be defined by: if $\alpha \in [\lambda^-_i, \lambda_i)$ and $\varepsilon < \kappa$, then $f'_\alpha(\varepsilon) = \mathrm{cd}_\varepsilon(f^i_\alpha(\varepsilon), f^{i(*)}_i(\varepsilon))$. One can now check that this works.

(3A) Similarly but add: cd_ε maps $\mathcal{U}_\varepsilon \times \mathcal{U}_\varepsilon$ into $\mathcal{U}_\varepsilon$.

(4) As we weaken the conclusion to "there is a $<_J$-increasing sequence of length λ in ${}^\kappa(\mu \times \mu)$", the proof of part (3) suffices if we add

$\oplus$ $\mathrm{cd}_\varepsilon(\alpha_1, \alpha_2) < \mathrm{cd}_\varepsilon(\alpha'_1, \alpha'_2)$ *iff* $(\alpha_2 < \alpha'_2) \vee (\alpha_2 = \alpha'_2 \wedge \alpha_1 < \alpha'_1)$

(5) Without loss of generality $\lambda < \mu$ and $\delta = \mathrm{cf}(\delta)$.

Without loss of generality each λ_i is regular and (even $> \mu$ and also $\lambda_0 > \delta$). For each $i < \delta$ let $\bar{f}^i = \langle f^i_\alpha : \alpha < \lambda_i\rangle$ be a $<_J$-increasing sequence of members of ${}^\kappa\mu$, in the role of $\mathscr{F}_i$. Let $\langle \mu_\varepsilon : \varepsilon < \kappa\rangle$ be an increasing sequence of regular cardinals $> \kappa$ with limit μ and for $i < \delta, \alpha < \lambda_i$ let $g^i_\alpha : \kappa \to \kappa$ be

$$g^i_\alpha(\varepsilon) = \min\{\zeta < \mu : f^i_\alpha(\varepsilon) < \mu_\zeta\}.$$

Hence $\{g_\alpha : \alpha < \lambda_i\} \subseteq {}^\kappa\kappa$ has cardinality $\leq 2^\kappa$ which is $< \mu < \lambda_i = \mathrm{cf}(\lambda_i)$, so for some $g_i \in {}^\kappa\kappa$ the set $\{\alpha < \lambda_i : g^i_\alpha = g_i\}$ is unbounded in λ_i. Hence without loss of generality $i < \sigma \wedge \alpha < \lambda_2 \Rightarrow g^i_\alpha = g_i$.

Also we can replace $\langle(\lambda_i, \bar{f}^i) : i < \delta\rangle$ by its restriction to any $u \subseteq \delta$ which is unbounded in δ. Hence without loss of generality $\langle g_i : i < \delta\rangle$ is constant or with no repetitions. The latter is impossible as $\mathrm{cf}(\delta) > 2^\kappa$. Now we can just use the proof of part (3) using $\oplus$ from above. $\square$

Observation 3.11. *There is a* $\sup\{\theta_i : i < i(*)\}$*-free* $\mathcal{F} \subseteq {}^\kappa\mu$ *of cardinality* 2^μ when:

(a) $\mu \in \boldsymbol{C}_\kappa$,

(b) *for each* $i < i(*)$ *at least one of the following holds:*

(α) *for some* $\chi, \theta_i < \mu < \chi < \lambda$ *and* $\chi^{\langle\theta_i\rangle_{\mathrm{tr}}} = \lambda$ *(and the supremum is attained),*

(β) $\theta_i = \mu^+$ *and for some* χ *and* $\sigma = \mathrm{cf}(\sigma) < \kappa$ *we have* $\mu < \chi < \lambda$ *and* $\chi^\sigma = \lambda$,

(γ) *for some* $\chi, \theta_i < \mu < \chi < \lambda, \kappa \neq \mathrm{cf}(\chi) < \mu$ *and* $\mathrm{pp}_{J^{\mathrm{bd}}_\kappa}(\chi) =^+ \lambda$.

Proof. Clearly $i < i(*) \Rightarrow \theta_i \leq \mu^+$. Without loss of generality, $i(*) < \mu$.

(Why? Clearly we can replace $\langle\theta_i : i < i(*)\rangle$ by $\langle\theta_i : i \in u\rangle$ *when* $u \subseteq i(*)$ and $\sup\{\theta_i : i < i(*)\} = \sup\{\theta_i : i \in u\}$, so without loss of generality the sequence $\langle\theta_i : i < i(*)\rangle$ has no repetitions, and so $i(*) \leq \mu + 1$, and if $i(*) \geq \mu$, we can find u as above of cardinality $< \mu$.)

If for every $i < i(*)$ clause (α) or clause (γ) of (b) of the assumption holds, *then* by Claims 3.1 or 1.26 there is a θ_i-free $\mathcal{F}_i \subseteq {}^\kappa\mu$ of cardinality λ for each $i < i(*)$ and by Observation 3.10 (1) the conclusion holds. It holds by Claim 3.6 if (β) of (b) applies for some $i < i(*)$. $\square$

Claim 3.12. *If* $\mu \in \boldsymbol{C}_\kappa$, $\lambda = 2^\mu = \chi^+$ *and* χ *is regular or just* $\mathrm{cf}([\chi]^{\leq\mu}, \subseteq) = \chi$ then:

(a) *there is a* μ^+*-free* $\mathcal{F} \subseteq {}^\kappa\mu$ *of cardinality* $2^\mu = \mu^\kappa$,

hence

(b) $\mathrm{BB}(\lambda, \mu^+, \theta, \kappa)$ *for every* $\theta < \mu$.

Remark 3.13. This is actually as in [22, Chapter VII, Claim 6.5 (3), p. 100] and the no-hole claim.

Proof. By Definition 1.1 there is an ideal J on κ and a sequence $\langle\lambda_i : i < \kappa\rangle$ of regular cardinals $< \mu$ such that $\lambda = \mathrm{tcf}(\prod_{i<\kappa}\lambda_i, <_J)$. So there is a $<_J$-increasing cofinal sequence $\langle f_\alpha : \alpha < \lambda\rangle$ of members of $\prod_{i<\kappa}\lambda_i$. Let $\bar{e}'_\varepsilon = \langle e_{\varepsilon,\alpha} : \alpha < \lambda\rangle$ for $\varepsilon < \chi$ be as in Observation 1.33, that is, if χ is regular, then we apply Observation 1.33 (A) and if $\mathrm{cf}([\chi]^{\leq\mu}, \subseteq) = \chi$, then we apply Observation 1.33 (B).

Now by induction on $\alpha < \lambda$ we choose $\bar{g}_\alpha = \langle g_{\varepsilon,\alpha} : \varepsilon < \chi \rangle$ and f^*_α such that:

$\boxplus_2$ (a) $g_{\varepsilon,\alpha} \in \prod_{i<\kappa} \lambda_i$,

(b) $f^*_\alpha \in \prod_{i<\kappa} \lambda_i$,

(c) $g_{\varepsilon,\alpha} <_J f^*_\alpha$,

(d) $f^*_\gamma <_J g_{\varepsilon,\alpha}$ if $\gamma < \alpha$,

(e) $g_{\varepsilon,\alpha}(i) > \sup\{f^*_\beta(i), g_{\varepsilon,\beta}(i) : \beta \in e_{\varepsilon,\alpha}\}$ when $\lambda_i > |e_{\varepsilon,\alpha}|$.

As $(\prod_{i<\kappa} \lambda_i, <_J)$ is λ-directed, we can carry out this definition. In more detail, at stage α, first we can choose $f'_\alpha \in \prod_{i<\kappa} \lambda_i$ such that

$$\beta < \alpha \Rightarrow f_\beta <_J f'_\alpha \quad \text{as } \lambda > |\{f_\beta : \beta < \alpha\}.$$

Second, for $\varepsilon < \chi$ we choose $g_{\varepsilon,\alpha} \in \prod_{i<\kappa} \lambda_i$ such that

$$\lambda_i > |e_{\varepsilon,\alpha}| \Rightarrow g_{\varepsilon,\alpha}(i) = \sup(\{f^*_\beta(i), g_{\varepsilon,\beta}(i); \beta \in e_{\varepsilon,\alpha}\} \cup \{f'_\alpha(i) + 1\}).$$

Third, choose $f_\alpha \in \prod_{i<\kappa} \lambda_i$ such that $\varepsilon < \chi \Rightarrow g_{\varepsilon,\alpha} <_J f_\alpha$, again possible as we have $< \lambda$ demands.

Now we can prove that for any subset $u \subseteq \lambda$ of cardinality $\leq \mu$ the sequence $\langle f^*_\alpha : \alpha \in u \rangle$ is J-free (see Definition 1.2 (4)) by induction on $\mathrm{otp}(u)$, as in the proof of the no-hole claim, actually [22, Chapter II, Claim 1.5A]. □

Remark 3.14. (1) Note that Fact 1.17 is quoted in $\boxdot_3$ of Section 0 in order to show $\odot_{3.1}$, but we could also use Claim 3.12.

(2) How much partial square on λ suffices in Claim 3.12? One for cofinality $\geq \kappa$ where the ideal J is J^{bd}_κ or just κ-complete (which is all right).

(3) We may consider a parallel of Claim 3.12 when χ is not as there. So assume $\mu \in \boldsymbol{C}_\kappa, \lambda = 2^\mu = \chi^+$ and χ is singular and $\mathrm{cf}([\chi]^{\leq\mu}, \subseteq) = \lambda$.

(A) Is there $\mathrm{cf}(\chi)$-free $\mathcal{F} \subseteq {}^\kappa\mu$ of cardinality λ?

(4) If for some $\mu_1, \mu < \mu_1 < \chi$ and $\mathrm{cov}(\chi, \mu_1^+, \mu_1^+, 2) = \chi$, *then* there is a $\mathrm{cf}(\chi)$-free $\mathcal{F} \subseteq {}^\kappa\mu$ of cardinality κ.

(Why? We apply Observation 1.33 (B) with λ, μ_0 here standing for λ, χ there getting $\langle e^1_{\varepsilon,\alpha} : \alpha < \lambda, \varepsilon < \chi \rangle$, so $\mathrm{otp}(e^1_{\varepsilon,\alpha}) < \mu_0^+ < \lambda$. Let $\langle e^2_i : i < \mu_0^+ \rangle$ be such that e^2_i is a closed unbounded subset of i of order type $\mathrm{cf}(i)$ for each $i < \mu_0^+$. Now let $\bar{e} = \langle e_{i,\varepsilon,\alpha} : \alpha < \lambda, \varepsilon < \chi, i < \mu_0^+ \rangle$ be defined by

$$e_{i,\varepsilon,\alpha} = \{\beta \in e_{\varepsilon,\alpha} : \mathrm{otp}(e_{\varepsilon,\alpha}) \in e^2_i\}.$$

So $\bar{e}$ is as required except that we use $(i, \varepsilon) \in \chi \times \mu_0^+$ instead of $\varepsilon < \chi$; but as $\chi \times \mu_0^+$ has cardinality χ, this is all right.)

Now a variant of Claim 3.1 is:

Claim 3.15. *If* $\circledast$ *holds,* then *there is* $\mathcal{F}$ *such that* $\boxtimes$ *holds where:*

$\boxtimes$ (α) $\mathcal{F} \subseteq {}^{\kappa}\mu$,

(β) $|\mathcal{F}| = \lambda$,

(γ) $\mathcal{F}$ *is* (θ, J_1)*-free,*

$\circledast$ (a) $\mu < \chi < \lambda$,

(b) $\kappa = \mathrm{cf}(\mu)$,

(c) θ *is regular,*

(d) $\sigma < \kappa < \theta < \mu$,

(e) J_1 *is a* σ^+*-complete ideal on* κ,

(f) *if* $\alpha < \mu$*, then* $\mathrm{cov}(|\alpha|, \theta^+, \theta^+, \sigma^+) \leq \mu$,

or *just*

(f)$^-$ *if* $\alpha < \mu$*, then* $\boldsymbol{U}_{J_2}(|\alpha|) \leq \mu$*, see Definition* 2.5,

(g) *there is a set of* λ *pairwise* J_2*-distinct members of* ${}^{\theta}\chi$,

(h) $\mathrm{pp}_{J_1}(\mu) > \chi$.

(i) J_1 *is* θ^+*-complete.*

(j) $2^\theta < \mu$.

Proof. Combine the proofs of Claim 3.1 and Claim 3.9. □

Claim 3.16. *In Claim* 3.15:

(1) *If in* $\circledast, \partial \geq \theta$ *clause* (g) *is exemplified by* $\mathcal{F}_2 \subseteq {}^{\theta}\chi$ *which is* (∂, J_2)*-free,* $\partial < \mu$, then $\mathcal{F}$ *is* $(\partial, \theta^+, J_2)$*-free.*

(2) *If* $\mathcal{F}' \subseteq \mathcal{F}$ *has cardinality* $> \theta$, then *the set* $\bigcup\{\mathrm{Rang}(\nu) : \nu \in \mathcal{F}\}$ *has cardinality* $\geq \theta$.

(3) *Clauses* (f) *and* (e) *from* $\circledast$ *imply clause* (f)$^-$*; in fact, clause* (e)*, "*J_2 *is* σ^+*-complete", is needed only for this.*

(4) *We can in* $\circledast$ *weaken (also in part* (1)*) clause* (h) *to*

(h)$'$ *there is a* ∂*-free* $\mathcal{F} \subseteq {}^{\kappa}\mu$ *of cardinality* λ.

Proof. We leave the proof to the reader. □

Claim 3.17. *Assume that* $\mu \in \mathbf{C}_\kappa$, J *is a* κ*-complete ideal on* κ *and there is no* $(\kappa^{+\omega}, J)$*-free* $\mathscr{F} \subseteq {}^\kappa\mu$ *of cardinality* $\lambda := 2^\mu$. Then *the set*

$$\Theta = \{\theta : \theta = \mathrm{cf}(\theta) < \mu, \theta \neq \kappa \text{ and for some witness } (\chi, J),\ J \text{ is a } \theta\text{-complete ideal on } \theta \text{ and } \chi \in (\mu, \lambda) \text{ is of cofinality } \theta, \text{ we have } \mathrm{pp}_J(\chi) =^+ \lambda \text{ for some } \theta\text{-complete ideal } J \text{ on } \theta\}$$

is empty, or a singleton or of the form $\{\theta, \theta^+\}$.

Remark 3.18. This is intended to help in Section 4 in dealing with R-modules when R has at least three members together with Observation 1.36, Theorem 1.31 and Observation 4.4.

Proof. Without loss of generality λ is regular. Note that:

$(*)_1$ if $\theta \in \Theta$, then $\theta > \kappa$.

(Why? Let (χ, J) witness $\theta \in \Theta$; now by Claim 3.6 we get a contradiction to the assumption "there is no $(\kappa^{+\omega}, J)$-free $\mathscr{F} \subseteq {}^\kappa\mu$ of cardinality λ".)

Let (θ_1, χ_1, J_1) be such that

$(*)_2$ $\theta_1 \in \Theta$ and (χ_1, J_1) is a witness for $\theta_1 \in \Theta$ and χ_1 is minimal under these conditions (even varying θ_1).

If $\theta \in \Theta$, by the choice of χ_1 as minimal, by [22, Chapter II, Claim 5.4] we have:

$(*)_3$ $\alpha < \chi_1 \Rightarrow \mathrm{cov}(|\alpha|, \mu^+, \mu^+, \kappa^+) < \chi_1$.

If $\Theta = \{\theta_1\}$ or $\Theta = \{\theta_1, \theta_1^+\}$ or $\theta_2^+ = \theta_1 \wedge \Theta = \{\theta_1, \theta_2\}$, we are done; otherwise let (θ_2, χ_2, J_2) be such that

$(*)_4$ $\theta_2 \in \Theta \backslash \{\theta_1, \theta_1^+\} \wedge \theta_1 \neq \theta_2^+$ and (χ_2, J_2) witness that $\theta_2 \in \Theta$, and χ_2 is minimal under these requirements.

Now we get:

$(*)_5$ there is a (θ_1^{++}, J_1)-free set $\mathscr{F} \subseteq {}^{\theta_1}(\chi_1)$ of pairwise J_1-distinct elements of cardinality λ.

(Why? *Case 1:* $\theta_2 > \theta_1$. Necessarily $\theta_2 > \theta_1^+$ by $(*)_4$, hence such an $\mathscr{F}$ exists by Claim 3.15 with $\lambda, \chi_1, \chi_2, \kappa, \theta_1, \theta_2, J_1, J_2$ here standing for $\lambda, \mu, \chi, \sigma, \kappa, \theta, J_1, J_2$ there.

For example why does clause (d) from Claim 3.15 hold? It means "$\kappa < \theta_1 < \theta_2 < \chi_1$" and these inequalities hold by $(*)_1$ of our assumption because, first $\kappa < \theta_1$ holds by $(*)_1$, second $\theta_1 < \theta_2$ holds by the present case assumption, and third "$\theta_2 < \mu < \chi_1$" holds by $(*)_2$.

Clause (e) of Claim 3.15 means "J_1 and J_2 are κ^+-complete" which hold as $\theta_1, \theta_2 > \kappa$ by $(*)_1$ and J_ℓ is θ_ℓ^+-complete by the definition of Θ.

For clause (f) of Claim 3.15 see the proof of Case 2.

Lastly, clause (g) of Claim 3.15 means "there is a set of λ pairwise J_2-distinct members of ${}^{\theta_2}(\chi_2)$" which holds as (J_2, χ_2) witnesses $\theta_2 \in \Theta$.

The conclusion of Claim 3.15 gives a family $\mathcal{F} \subseteq {}^{\theta_1}(\chi_1)$ of cardinality λ which is (θ_2, J_1)-free, but $\theta_2 \geq \theta_1$ by "First", and $\theta_2 \neq \theta_2^+$ by $(*)_4$ so we are done.

Case 2: $\theta_2 \leq \theta_1$. Again by $(*)_4, \theta_2^+ < \theta_1$. Hence by Claim 3.9 with $\lambda, \chi_1, \chi_2, \kappa, \theta_1, \theta_2, J_1, J_2$ here standing for $\lambda, \mu, \chi, \sigma, \kappa, \theta, J_1, J_2$ there, we have finished the proof of $(*)_5$.)

Now by $(*)_5$ we can apply case (c) of Claim 3.4 and so we are done. □

Claim 3.19. *If* (A)*, then* (B) *where:*

(A) (a) *J is a σ^+-complete ideal on κ,*

(b) *$\mathcal{F}_i \subseteq {}^\kappa\mu$ has cardinality λ for $i < \sigma$,*

(c) $\mu = \mu^\sigma$ or $(\forall i)[\mathcal{F}_i \subseteq \prod_{\varepsilon<\kappa} \lambda_\varepsilon]$ *and* $\varepsilon < \kappa \Rightarrow (\lambda_\varepsilon)^\sigma < \mu$,

(B) *there is $\mathcal{F} \subseteq {}^\kappa\mu$ of cardinality λ such that:*

(a) *$\mathcal{F}$ is $(\theta_2, \theta_1) - J$-free* when *at least one $\mathcal{F}_i$ is (θ_1, θ_2)-free,*

(b) *$\mathcal{F}$ is $(\theta_n, \theta_0) - J$-free* when $\theta_0 < \cdots < \theta_n$ *and for each $\ell < n$ for some $i < \sigma$ the set $\mathcal{F}_i$ is $(\theta_{\ell+1}, \theta_\ell) - J$-free.*

Proof. The proof is straightforward. □

4 On the μ-free trivial dual conjecture

We shall look at the following definition.

Definition 4.1. (1) For a ring R and a cardinal μ, let $\mathrm{sp}_\mu(R)$ be the class of regular cardinals κ such that there is a witness $(\bar{G}, h)$ where "$(\bar{G}, h)$ is a witness for $\mathrm{sp}_\mu(R)$" means:

⊛ (a) $\bar{G} = \langle G_i : i \leq \kappa + 1\rangle$,

(b) $\bar{G}$ is an increasing continuous sequence of free left R-modules,

(c) if $i < j \leq \kappa + 1$ and $(i, j) \neq (\kappa, \kappa + 1)$, then G_j/G_i is free,

(d) h is a homomorphism from G_κ to R as left R-modules,

(e) h cannot be extended to a homomorphism from $G_{\kappa+1}$ to R,

(f) $|G_{\kappa+1}| \leq \mu$.

(2) For a ring R and cardinals $\mu \geq \theta$, we define $\mathrm{sp}_{\mu,\theta}(R) = \mathrm{sp}^1_{\mu,\theta}(R)$ similarly, replacing "free" by "θ-free" in clauses (b) and (c). Writing $\mathrm{sp}_{<\mu}(R)$ or $\mathrm{sp}_{<\mu,\theta}(R)$ means that "$|G_{\kappa+1}| < \mu$" in clause (f).

Definition 4.2. (1) Let $\mathrm{sp}(R) = \bigcup\{\mathrm{sp}_\mu(R) : \mu$ a cardinal$\} = \{\kappa : \kappa$ is a regular cardinal such that for some $\bar{G}$ the conditions ⊛ (a)–(e) from Definition 4.1 (1) hold$\}$.

(2) Let $\mathrm{sp}_1(R) = \bigcap\{\mathrm{sp}^1_\theta(R) : \theta$ a cardinal$\}$ where $\mathrm{sp}^1_\theta(R) = \{\kappa : \kappa$ is regular such that for some μ, we have $\kappa \in \mathrm{sp}_{\mu,\theta}(R)\}$.

The next definition is similar to Definition 4.1 (adding the parameter "$r \in R$"), but replacing the cardinal κ by a set of ideals on κ, that is:

Definition 4.3. (1) Let $\mathrm{sp}^2_{\lambda,\theta}(R)$ be the set of cardinals κ such that

$$J^{\mathrm{bd}}_\kappa \in \mathrm{SP}_{\lambda,\theta}(R),$$

see below.

(2) $\mathrm{SP}_{\lambda,\theta}(R)$ is the set of ideals J on some κ such that for every $r \in R\backslash\{0\}$, there exists a witness $(\bar{G}, h)$ (for r), where "$(r, \bar{G}, h)$ is a witness for $\mathrm{SP}_{\lambda,\theta}(R)$" and $(\bar{G}, h)$ witness $\mathrm{SP}_{\lambda,\theta}(R)$ (for r)" means that $(r, \bar{G}, h)$ possesses the following properties:

⊛ (a) $\bar{G} = \langle G_i : i \le \kappa + 1\rangle$ is a sequence of (left) R-modules,
   (b) $G_\kappa = \bigoplus\{G_i : i < \kappa\} \subseteq G_{\kappa+1}$,
   (c) if $u \in J$, then $G_{\kappa+1}/\bigoplus\{G_i : i \in u\}$ is a θ-free (left) R-module,
   (d) G_i is a θ-free R-module and $G_i \neq 0$ for simplicity,
   (e) $|G_{\kappa+1}| + \kappa \le \lambda$,
   (f) h is a non-zero homomorphism from G_κ to ${}_RR$, i.e., R as a left module,
   (g) there is no homomorphism h^+ from $G_{\kappa+1}$ to ${}_RR$ such that

$$x \in G_\kappa \Rightarrow h^+(x) = h(x)r.$$

(3) Omitting θ means replacing "θ-free" by "free"; omitting θ and λ means for some λ; writing "$< \lambda$" has the obvious meaning.

Observation 4.4. (1) *If $J_1 \subseteq J_2$ are ideals on κ,* then $J_1 \in \mathrm{SP}_{\lambda,\theta}(R)$ *implies* $J_2 \in \mathrm{SP}_{\lambda,\theta}(R)$.

(2) *If J_ℓ is an ideal on κ_ℓ for $\ell = 1, 2$ and $J_1 \le_{\mathrm{RK}} J_2$,* then *the above holds.*

Proof. The proof is straightforward. □

Remark 4.5. (1) If R is a principal ideal domain, *then* in Definition 4.3 without loss of generality $r = 1$.

(2) In Definition 4.3 (2), if κ is regular for $J = J^{\rm bd}_\kappa$, we may replace clause (c) by "$i < \kappa \Rightarrow G_{\kappa+1}/\bigoplus\{G_j : j < i\}$ is a θ-free R-module"; in general, we may replace J by a directed subset of $\mathscr{P}(\kappa)$ generating it.

(3) Note that if $J \in {\rm SP}_{\lambda,\theta}(R)$, then $\lambda \geq |R|$ because by clause (c) of Definition 4.3 (2) we know that $G_{\kappa+1}$ is θ-free hence is of cardinality $\geq |R|$ (except when $G_{\kappa+1}$ is zero contradicting clause (g) there) and $\lambda \geq |G_{\kappa+1}|$ by clause (e) there.

As in Definition 0.1:

Definition 4.6. Let ${\rm TDU}_{\lambda,\mu}(R)$ mean that R is a ring and there is a μ-free left R-module G of cardinality λ with ${\rm Hom}_R(G, R) = \{0\}$, that is, with no non-zero homomorphism from G to R as left R-modules.

Claim 4.7. *A sufficient condition for* ${\rm TDU}_{\lambda,\mu}(R)$ *is:*

$\circledast$ (a) *R is a ring with unit* $(1 = 1_R)$,

(b) $J \in {\rm SP}_{\chi,\mu}(R)$ *so is an ideal on* κ,

(c) $\bar{C} = \langle C_\delta : \delta \in S\rangle$ *is such that* ${\rm otp}(C_\delta) = \kappa$ *and* $C_\delta \subseteq \delta$ *where S is an unbounded subset of* λ,

(d) $\lambda > |R| + \chi$ *is regular, or at least* ${\rm cf}(\lambda) > |R| + \chi + \kappa$ *and* $\mu > \kappa$,

(e) ${\rm BB}(\lambda, \bar{C}, \Upsilon, J)$ *where* $\Upsilon = 2^{(2^{|R|+\chi})^+}$, $\kappa \leq (2^\chi)^+$ *and* $\chi < \lambda$, *so we have* $I_* = J^{\rm bd}_S$ *recalling* $J^{\rm bd}_S = \{\mathcal{U} : \mathcal{U} \subseteq S$ *and* ${\rm sup}(\mathcal{U}) < {\rm sup}(S))\}$,

(f) $\bar{C}$ *is* (μ, J)*-free, recalling Definition* 1.2 (1A).

Remark 4.8. (1) In the present definition of the set ${\rm SP}_{\lambda,\theta}(R)$, we need to use ${\rm BB}(\lambda, \bar{C}, \Upsilon, J)$ before applying SP in Claim 4.7. But normally it suffices to have a version of BB with fewer colors and weaker demands on $|G_i|$, for example:

(A) Use ${\rm BB}(\lambda, \bar{C}, (\chi_*, \theta), J)$ and $\chi_* = \Pi\{|R|^{\chi_i} : i < \kappa\}$, where

$$\chi_i = |G_i| + \sup\{|{\rm Hom}(G_j, {}_RR)| : j < \kappa\}.$$

(B) We define ${\rm SP}_{\lambda,\bar{\chi},\sigma,\theta}(R)$ as in Definition 4.3 (2) where $\bar{\chi} = \langle \chi_i : i < \kappa\rangle$ and write χ if $(\forall i)(\chi_i = \chi)$ but instead of (e) and (f)–(g),

(e)′ $|G_{\kappa+1}| \leq \lambda$ and $|{\rm Hom}(G_i, R)| \leq \chi_i$,

(f)′ $\bar{h} = \langle h_i : i < \sigma\rangle$, $h_i \in {\rm Hom}(G_\kappa, {}_RR)$ and if $i < j < \sigma$, then $h_i - h_j$ cannot be extended to any $h' \in {\rm Hom}(G_{\kappa+1}, {}_RR)$.

(C) In Claim 4.7, we change

(b)′ $\kappa \in \mathrm{SP}_{\lambda,\chi,\sigma,\theta}$ or ($\bar{C}$ is tree-like, $\kappa \in \mathrm{SP}_{\lambda,\bar{\chi},\sigma,\theta}$ and $J \in \mathrm{SP}_{\lambda,\bar{\chi},\sigma,\theta}$ is an ideal on κ),

(e)′ $\mathrm{BB}(\lambda, \bar{C}, (\chi, \sigma), J)$.

(2) $\mathrm{BB}(\lambda, \bar{C}, (\chi, 1/\sigma), J)$ is sufficient for the correct version of Definition 4.3, see Definition 0.7 (2); really we need there to use $\theta = 2^\kappa$ and the guessing is of an initial segment of the possibilities, i.e., in Definition 4.3 we need: without loss of generality $|G_i| \le \kappa$ for every i, given $f_\varepsilon \in \mathrm{Hom}(G_\kappa, {}_R R)$ for $\varepsilon < \varepsilon(*) < 2^\kappa$ we can find, e.g., a permutation π of κ, inducing

$$G^\pi_\kappa \supseteq \bigoplus \{G_i : i < \kappa\}$$

such that none of them can be extended to $f \in \mathrm{Hom}(G^\pi_\kappa, {}_R R)$. This means we can use "very few colors" as in [24, Appendix, Section 1], i.e., Definition 0.7 (2A).

(3) See $\odot_0$ in Section 0.

(4) We may use only tree-like sequences of sets $\bar{C}$ (in Definition 4.7 (c)) and in $\mathrm{BB}(\lambda, \bar{C}, (\bar{\chi}, \sigma), J)$ (in (e)′ of (C) above).

(5) In the proof of Claim 4.7, if we demand that G_i $(i < \kappa)$ is free, then we can save on χ, using free R-modules G^*_α.

(6) The beginning of the proof can be stated separately.

Proof of Claim 4.7. Without loss of generality $\bar{C}$ is normal, see Definition 1.2 (5). By Definitions 0.5 and 0.7 of $\mathrm{BB}(\lambda, \bar{C}, \Upsilon, J)$, there is a sequence $\langle S_\varepsilon : \varepsilon < \lambda\rangle$ of λ pairwise disjoint subsets of $S = S(\bar{C})$ such that $\mathrm{BB}^-(\lambda, \bar{C} \restriction S_\varepsilon, \Upsilon, J)$ for each $\varepsilon < \lambda$.

Without loss of generality $\delta \in S \Rightarrow C_\delta \cap S = \emptyset$, moreover S is a set of limit ordinals and each C_δ is a set of successor ordinals and we let $C_* = \bigcup\{C_\delta : \delta \in S\}$. We say that D is $\bar{C}$-closed when $D \subseteq C_* \cup S$ and $\delta \in D \cap S \Rightarrow C_\delta \subseteq D$. So for every $B' \subseteq C_* \cup S$ there is a $\bar{C}$-closed $B'' \subseteq C_* \cup S$ such that

$$B' \subseteq B'' \wedge |B''| \le |B'| + \kappa.$$

We can put λ of the sets S_i together, i.e.,

$\boxplus_1$ we can replace $\langle S_i : i < \lambda\rangle$ by $\langle \bigcup\{\{S_i : i \in \mathcal{U}_\zeta\} : \zeta < \lambda\}\rangle$ provided that $\langle \mathcal{U}_\zeta : \zeta < \lambda\rangle$ is a partition of λ with each $\mathcal{U}_\zeta$ non-empty).

Also

$\boxplus_2$ we can replace $\langle C_\delta : \delta \in S\rangle$ by $\langle C_\delta \backslash h(\delta) : \delta \in S\rangle$ when $\delta \in S \Rightarrow h(\delta) \in C_\delta$, hence without loss of generality

$\boxplus_3$ (a) $\varepsilon < \lambda \wedge S' \subseteq S_\varepsilon \wedge |S'| < \lambda \Rightarrow \mathrm{BB}^-(\lambda, \bar{C} \restriction (S_\varepsilon \backslash S'), \Upsilon, J)$,

(b) if $\alpha < \lambda$, *then* for λ ordinals $\varepsilon < \lambda$ we have $\delta \in S_\varepsilon \Rightarrow \alpha < \min(C_\delta)$.

Without loss of generality

$\circledast_0$ $\chi \geq |R| + \kappa$ and $\lambda > 2^\chi$.

(Why? We have $\chi \geq |R|$ because $\mathrm{SP}_{\chi,\mu}(R) \neq \emptyset$ by clause (b) of the assumption, using Remark 4.5 (3). The "and" holds as $\lambda \geq \Upsilon$ by the first phrase of clause (e) of the assumption and $\Upsilon > 2^\chi$ by the second phrase of clause (e) of the assumption.)

$\circledast_1$ There is a μ-free R-module G_* of cardinality $\chi_* := (2^\chi)^+$ such that:

(a) $G_* = \bigoplus\{G_{*,\varepsilon} : \varepsilon < \chi_*\}$,

(b) if G is a μ-free R-module of cardinality $\leq \chi$, *then* G is isomorphic to $G_{*,\varepsilon}$ for χ_* ordinals $\varepsilon < \chi_*$ (actually we need just that for any element $r \in R\backslash\{0_R\}$ there is a sequence $\langle G_i : i \leq \kappa + 1\rangle$ satisfying $\circledast$ of Definition 4.3 (2) with χ, μ here standing for λ, θ there),

(c) $G_{*,\varepsilon}$ is a μ-free R-module of cardinality $\leq \chi$ for each $\varepsilon < \chi_*$.

(Why? Because the number of such G up to isomorphism is $\leq 2^{|R|+\chi} = 2^\chi$ and $\kappa \leq (2^\chi)^+ = \chi_*$.)

Let $E = \{(\varepsilon, \zeta) : \varepsilon, \zeta < \chi_*$ and $G_{*,\varepsilon} \cong G_{*,\zeta}\}$, so E is an equivalence relation on χ_* and $\varepsilon/E := \{\zeta < \chi_* : \varepsilon E \zeta\}$ is the equivalence class of $\varepsilon < \chi_*$ under E. For $\varepsilon < \chi_*$, let f^1_ε be an isomorphism from $G_{*,\min(\varepsilon/E)}$ onto $G_{*,\varepsilon}$.

$\circledast_2$ For any $r \in R\backslash\{0\}$ let $\boldsymbol{x}_r = \{(\bar{G}, h) : (\bar{G}, h)$ witness $J \in \mathrm{SP}_{\chi,\theta}(R)$ for r, see Definition 4.3 (2)$\}$.

$\circledast_3$ $H_* := \bigoplus\{G^*_\alpha : \alpha \in C_*\} \oplus \bigoplus\{K^*_\delta : \delta \in S\}$, where

$\bullet_1$ each G^*_α is isomorphic to G_* under g^1_α,

$\bullet_2$ K^*_δ isomorphic to G_* for $\delta \in S$ under g^2_δ,

$\bullet_3$ for $\varepsilon < \chi_*$ let $G_{\alpha,\varepsilon} = g^1_\alpha(G_{*,\varepsilon})$, $K_{\delta,\varepsilon} = g^2_\alpha(G_{*,\varepsilon})$.

$\circledast_4$ Let $K_{<\delta} = \bigoplus\{G^*_\alpha : \alpha \in C_\delta\}$ for $\delta \in S$, which has cardinality χ_* as $\kappa \leq \chi_*$ by clause (e) of the assumption.

$\circledast_5$ For every $B \subseteq C_* \cup S$ let

$$H_B := \bigoplus\{G^*_\alpha : \alpha \in B \cap C_*\} \oplus \bigoplus\{K^*_\delta : \delta \in S \cap B\}.$$

We easily see that

$\circledast_6$ for every $x \in H_*$ there is a $\bar{C}$-closed set $D^*_x \subseteq C_* \cup S$ of cardinality $\leq \kappa$ such that $x \in H_{D^*_x}$, in fact there is a minimal one.

Let

$\circledast_7$ (a) $\langle (x_i, r_i) : i < \lambda \rangle$ list the pairs (x, r) such that $x \in H_*, r \in R \backslash \{0_R\}$,

(b) by $\circledast_6$ and $\boxplus_3$ without loss of generality

$$\delta \in S_i \Rightarrow \sup(D^*_{x_i}) < \min(C_\delta),$$

$\circledast_8$ $H_{<\alpha} := \bigoplus \{G^*_\beta, K^*_\delta : \beta \in C_* \cap \alpha \text{ and } \delta \in S \cap \alpha\}$.

For $\delta \in S$ let $\beta(\delta, \iota)$ be the ι-th member of C_δ.

For $\delta \in S$, clearly $\text{Hom}(K_{<\delta}, {}_R R)$ is a set of cardinality $\leq 2^{\chi_*} = \Upsilon$. Also any $f \in \text{Hom}(K_{<\delta}, {}_R R)$ is determined by $\langle f \restriction G^*_\alpha : \alpha \in C_\delta \rangle$. Hence by clause (e) of the assumption, for each $i < \lambda$, we can find $\langle h^1_\delta : \delta \in S_i \rangle$ such that:

$\circledast_9$ (a) if $\delta \in S$, then $h^1_\delta \in \text{Hom}(K_{<\delta}, {}_R R)$,

(b) if $i < \lambda$ and $h \in \text{Hom}(H_*, {}_R R)$, then for some (even stationarily many) $\delta \in S_i$, we have $h^1_\delta \subseteq h$,

$\circledast_{10}$ for $\delta \in S_i$:

(a) let $x^*_\delta = x_i, r^*_\delta = r_i$,

(b) let $\bar{N}^\delta = \langle N^\delta_\iota : \iota \leq \kappa + 1 \rangle$ and h^*_δ be, for r^*_δ, as guaranteed in Definition 4.3 (2), with N^δ_i here standing for G_i there, so $h^*_\delta \in \text{Hom}(N^\delta_\kappa, {}_R R)$,

(c) for $\iota < \kappa$, let $\varepsilon(\delta, \iota) = \text{Min}\{\varepsilon < \chi_* : G_{*,\varepsilon} \cong N^\delta_\iota\}$ and let $f^0_{\delta,\iota}$ be an isomorphism from N^δ_ι onto $G_{*,\varepsilon(\delta,\iota)}$.

(Why is this possible? By clause (b) of the assumption.)

Now we have:

$\circledast_{11}$ for $\delta \in S_i$ and $\iota < \kappa$ we can choose $\varepsilon_{\delta,\iota,1} < \varepsilon_{\delta,\iota,2} < \chi_*$ from

$$Y = Y_{\delta,\iota} = \{\zeta < \chi_* : G_{*,\varepsilon(\delta,\iota)} \cong G^\delta_{*,\zeta}\}$$

such that

$$h^1_\delta \circ g^1_{\beta(\delta,\iota)} \circ f^1_{\varepsilon_{\delta,\iota,1}} \circ f^0_{\delta,\iota} = h^1_\delta \circ g^1_{\beta(\delta,\iota)} \circ f^1_{\varepsilon_{\delta,\iota,2}} \circ f^0_{\delta,i}.$$

(Why? Note that $\min(Y) = \varepsilon(\delta, \iota)$ and

- $h^1_\delta \in \text{Hom}(K_{<\delta}, {}_R R)$ hence

$$h^1_\delta \restriction G^*_{\beta(\delta,\iota)} \in \text{Hom}(G^*_{\beta(\delta,\iota)}, {}_R R),$$

- $g^1_{\beta(\delta,\iota)}$ is an isomorphism from G_* onto $G^*_{\beta(\delta,\iota)}$ hence

$$h^1_\delta \circ g^1_{\beta(\delta,\iota)} \in \text{Hom}(G_*, {}_R R),$$

- f^1_ε, see before $\circledast_2$, is an isomorphism from $G_{*,\min(Y)}$ onto $G_{*,\varepsilon} \subseteq G_*$ for $\varepsilon \in Y$,
- $\langle h^1_\delta \circ g^1_{\beta(\delta,\iota)} \circ f^1_\varepsilon : \varepsilon \in Y \rangle$ is a sequence of members of $\mathrm{Hom}(G_{*,\min(Y)}, {}_R R)$,
- $\mathrm{Hom}(G_{*,\min(Y)}, {}_R R)$ has cardinality

$$\leq |R|^{|G_{*,\min(Y)}|} \leq |G_*| \leq 2^{\chi + |R|},$$

whereas

$$|Y| = \chi_* = (2^\chi)^+.$$

Hence we can choose $\varepsilon_{\delta,\iota,1}, \varepsilon_{\delta,\iota,2}$ such that:

- $\varepsilon_{\delta,\iota,1} < \varepsilon_{\delta,\iota,2}$ are members of Y satisfying

$$h^1_\delta \circ g^1_{\beta(\delta,\iota)} \circ f^1_{\varepsilon_{\delta,\iota,2}} = h^1_\delta \circ g^1_{\beta(\delta,\iota)} \circ f^1_{\varepsilon_{\delta,\iota,1}}.$$

So the desired conclusion of $\circledast_{11}$ holds.)

Let $g^2_{\delta,\iota}$ be the following embedding of N^δ_ι into H_*, in fact, into $G^*_{\beta(\delta,\iota)}$ (recalling $f^0_{\delta,\iota}$ is an isomorphism from N^δ_ι onto $G_{*,\min(Y)}$):

$(*)_0$ $g^2_{\delta,\iota}(x) = g^1_{\beta(\delta,\iota)} \circ f^1_{\varepsilon_{\delta,\iota,2}} \circ f^0_{\delta,\iota}(x) - g^1_{\beta(\delta,\iota)} \circ f^1_{\varepsilon_{\delta,\iota,1}} \circ f^0_{\delta,\iota}(x)$ for $x \in G^\delta_\iota$.

Let g^3_δ be the embedding of N^δ_κ into H_* extending $g^2_{\delta,\iota}$ for each $\iota < \kappa$, so:

$(*)_1$ (a) g^3_δ is an embedding of N^δ_κ into $K_{<\delta} \subseteq H_*$,
(b) $h^1_\delta \restriction \mathrm{Rang}(g^3_\delta)$ is zero.

Let g^4_δ be the following homomorphism from N^δ_κ into H_*:

$(*)_2$ $g^4_\delta(x) = g^3_\delta(x) + h^*_\delta(x)x^*_\delta$ for $x \in N^\delta_\kappa$.

(Why? Recalling $x^*_\delta \in H_{<\delta}$ is from $\circledast_{10}$ (a), $h^*_\delta \in \mathrm{Hom}(N^\delta_\kappa, {}_R R)$ is from $\circledast_{10}$ (b) so $h^*_\delta(x) \in R$ hence $h^*_\delta(x)x^*_\delta \in H_*$ indeed.)

By the choice of $H_{<\delta}$ as $\delta \in S_i \Rightarrow x^*_\delta = x_i \in H_{D^*_{x_i}} \subseteq H_{<\min(C_\delta)} \subseteq H_{<\delta}$ using $\circledast_7$ (b) clearly:

$(*)_3$ g^4_δ is an embedding of N^δ_κ into $H_{<\delta}$.

So by $(*)_1$ and $(*)_2$ we have:

$(*)_4$ if h is a homomorphism from H into ${}_R R$ where $K_{<\delta} \subseteq H \subseteq H_*$ such that $h^1_\delta \subseteq h \wedge h(x^*_\delta) = r^*_\delta$, then $x \in N^\delta_\kappa \Rightarrow h(g^4_\delta(x)) = h^*_\delta(x)r^*_\delta$.

Let $\alpha_{\delta,\kappa} < \chi_*$ be such that $G_{*,\alpha_{\delta,\kappa}} \cong N^\delta_{\kappa+1}$, and let $f^0_{\delta,\kappa}$ be an isomorphism from $N^\delta_{\kappa+1}$ onto $G_{*,\alpha_{\delta,\kappa}}$, and recalling $\circledast_3, \bullet_2$ it follows that $g^2_\delta \circ f^0_{\delta,\kappa}$ embeds $N^\delta_{\kappa+1}$ into $K^*_\delta \subseteq H_*$ hence letting

$$f^4_{\delta,\kappa} = f^0_{\delta,\kappa} \restriction N^\delta_\kappa$$

we have that $g^2_\delta \circ f^4_{\delta,\kappa} - g^4_\delta$ is a homomorphism from N^δ_κ into H_* (actually an embedding).

Let

$(*)_5$ $L_\delta = \{g^2_\delta \circ f^4_{\delta,\kappa}(x) - g^4_\delta(x) : x \in N^\delta_\kappa\}$.

Clearly L_δ is an R-submodule of H_*. Now by the choice of $(\bar{N}^\delta, r^*_\delta, h^*_\delta)$ we shall show:

$(*)_6$ there is no homomorphism h from H_* into ${}_RR$ such that $h^1_\delta \subseteq h$, $h(x^*_\delta) = r^*_\delta$ and $h \restriction L_\delta = 0_{L_\delta}$ that is constantly zero.

(Why? Toward a contradiction assume h is a counterexample.

$\oplus_{6.1}$ if $x \in \text{Rang}(g^3_\delta)$, then $x \in K_{<\delta}$ and $h(x) = h^1_\delta(x) = 0$.

(Why? Note $\text{Rang}(g^3_\delta) \subseteq K_{<\delta}$ hence $x \in K_{<\delta}$ by $(*)_1$ (a), $h \supseteq h^1_\delta$ by the choice of h and $\text{Dom}(h^1_\delta) = K_{<\delta}$ by $\circledast_9$ (a) hence

$$h \restriction \text{Rang}(g^3_\delta) = h^1_\delta \restriction \text{Rang}(g^3_\delta).$$

So as $x \in \text{Rang}(g^3_\delta)$ by the assumption of $\oplus_{6.1}$, clearly we have $h(x) = h^1_\delta(x)$. But $h^1_\delta \restriction \text{Rang}(g^3_\delta)$ is constantly zero by $(*)_1$ (b) and $x \in \text{Rang}(g^3_\delta)$ so $h^1_\delta(x) = 0$, so we are done.)

$\oplus_{6.2}$ $x \in N^\delta_\kappa \Rightarrow h(g^4_\delta(x)) = h^*_\delta(x) r^*_\delta$.

(Why? The assumptions of $(*)_1$ say that $h^1_\delta \subseteq h^+ \wedge h(x^*_\delta) = r^*_\delta$ which hold by the assumption of $(*)_6$, but the conclusion of $(*)_4$ is what we claim in $\oplus_{6.2}$.)

$\oplus_{6.3}$ If $x \in N^\delta_\kappa$, then $h((g^2_\delta \circ f^4_{\delta,\kappa})(x)) = h(g^4_\delta(x))$.

(Why? As we are assuming $h \restriction L_\delta$ is constantly zero and by the choice of L_δ in $(*)_5$.)

$\oplus_{6.4}$ If $x \in N^\delta_\kappa$, then $h((g^2_\delta \circ f^0_{\delta,\kappa})(x)) = h(g^4_\delta(x))$.

(Why? As $f^4_{\delta,\kappa} \subseteq f^0_{\delta,\kappa}$ and $\oplus_{6.3}$.)

$\oplus_{6.5}$ If $x \in N^\delta_\kappa$, then $h((g^2_\delta \circ f^0_{\delta,\kappa})(x)) = h^*_\delta(x) r_\delta$.

(Why? By $\oplus_{6.2}$ and $\oplus_{6.4}$.)

Recalling g^2_δ is from $\circledast_3$ and $f^0_{\delta,\kappa}$ is from after $(*)_4$:

$\oplus_{6.6}$ Define $h' : N^\delta_{\kappa+1} \to {}_R R$ by $h'(x) = h((g^2_\delta \circ f^0_{\delta,\kappa})(x))$.

$\oplus_{6.7}$ (a) h' is indeed a function from $N^\delta_{\kappa+1}$ to ${}_R R$,

(b) moreover it is an R-module homomorphism.

(Why? As $f^0_{\delta,\kappa}$ is a homomorphism from $N^\delta_{\kappa+1}$ into $G_{*,\alpha_{\delta,\kappa}}$ and g^2_δ is a homomorphism from $G_* \supseteq G_{*,\alpha_{\delta,\kappa}}$ into H_* and h is a homomorphism from H_* to ${}_R R$.)

$\oplus_{6.8}$ h' extends the mapping $x \mapsto h^*_\delta(x) r_\delta$ for $x \in N^\delta_\kappa$.

(Why? By $\oplus_{6.5}$.)

Now $\oplus_{6.7}$ and $\oplus_{6.8}$ contradict the choice of h^*_δ, r^*_δ in $\circledast_{10}$. So condition $(*)_6$ indeed holds.)

Lastly, let:

$(*)_7$ (a) $L := \Sigma\{L_\delta : \delta \in S\}$, a sub-module of H_*,

(b) $H := H_*/L$, a module of cardinality λ.

Then we have:

$(*)_8$ $\mathrm{Hom}(H, {}_R R) = 0$.

(Why? Assume $h \in \mathrm{Hom}(H, {}_R R)$ is not constantly zero, so we can define a homomorphism $h^+ \in \mathrm{Hom}(H_*, {}_R R)$ by $h^+(x) = h(x + L)$ hence also h^+ is not constantly zero. Let $x \in H_*$ be such that $h^+(x) \neq 0$, so for some $i < \lambda$ we have

$$(x_i, r_i) = (x, h^+(x)).$$

By the choice of $\langle h^1_\delta : \delta \in S_i\rangle$ the set $\{\delta \in S_i : h \restriction K_{<\delta} = h^1_\delta\}$ is unbounded in λ, so for some $\delta \in S_i$ we have:

$\oplus_{8.1}$ $h \restriction K_{<\delta} = h^1_\delta$,

and by $(*)_6$ we are done as $h^+ \restriction L_\delta$ is zero.)

$(*)_9$ H is a μ-free R-module.

(Why? Let $H^1 \subseteq H$ be of cardinality $< \mu$. So for some $H^2 \subseteq H_*$ of cardinality $< \mu$, we have $H^1 = \{x + L : x \in H^2\}$.

So we have $H^1 \subseteq (H^2 + L)/L$, and it is enough to prove that $(H_B + L)/L$ is free for every $\bar{C}$-closed $B \subseteq C^* \cup S$ because for every H^1, H^2 as above for some $\bar{C}$-closed set $B \subseteq C_* \cup S$ of cardinality $< \mu$ (see before $\oplus_1$) we have $H^2 \subseteq H^3 := H_B$, see $\circledast_5$. By clause (f) of the claim's assumption there is $\bar{u} = \langle u_\delta : \delta \in B \cap S\rangle$ such that $u_\delta \in J$ and

$$\delta_1 \neq \delta_2 \in B \cap \delta \wedge \iota_1 \in (\kappa \backslash u_{\delta_1}) \wedge \iota_2 \in (\kappa \backslash u_{\delta_2}) \Rightarrow \beta(\delta_i, \iota_1) \neq \beta(\delta_2, \iota_2)$$

recalling $\bar{C}$ is normal. The rest is clear.)

By $(*)_7$, $(*)_8$ and $(*)_9$ we are done. □

Claim 4.9. (1) *In Claim* 4.7 *if* $\mu = \lambda$ *(i.e., for* $\bar{C}$ *the cardinality and degree of freeness coincide, naturally in clause* (b) *we have* $J \in \mathrm{SP}_\chi(R)$*), we can also deduce* $\lambda \in \mathrm{sp}_\lambda(R)$.

(2) *In Claim* 4.7 *it suffices to assume:*

$\circledast'$ *as in* $\circledast$ *of Claim* 4.7 *omitting* (d) *and strengthening clause* (b) *to*

(b)$'$ $\kappa \in \mathrm{sp}_{\leq\lambda,\mu}(R)$*, see Definition* 4.1*,*

(c)$'$ *like* (c) *but* $\bar{C}$ *is tree-like, that is,*

$$\alpha \in C_{\delta_1} \cap C_{\delta_2} \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha.$$

Proof. This is clear. □

Claim 4.10. (1) *For* $R = \mathbb{Z}$*, we have:*

(a) $J^{\mathrm{bd}}_{\aleph_0}$ *belongs to* $\mathrm{SP}_{\aleph_0}(R)$*,*

(b) $J^{\mathrm{bd}}_{\aleph_1}$ *belongs to* $\mathrm{SP}_{\aleph_1}(R)$*,*

(c) $J^{\mathrm{bd}}_{\aleph_1 * \aleph_0}$ *belongs to* $\mathrm{SP}_{\aleph_1}(R)$*,*

(d) *if* $2^{\aleph_0} = \aleph_1$ *or* $2^{\aleph_1} < 2^{\aleph_2}$, then $J^{\mathrm{bd}}_{\aleph_2}$ *belongs to* $\mathrm{SP}_{\aleph_2}(R)$*,*

(e) *if* $2^{\aleph_0} = \aleph_1$ *or* $2^{\aleph_1} < 2^{\aleph_2}$, then $J^{\mathrm{bd}}_{\aleph_2 * \aleph_1}$ *belongs to* $\mathrm{SP}_{\aleph_2}(R)$.

(2) *Similarly, if* R *is a proper subring of* $\mathbb{Q}$.

Remark 4.11. (1) If we want the proof of TDU_μ to be more direct, we have to add $\mathrm{Hom}(G_{\kappa+1}/G_\kappa) = 0$, otherwise we have to "iterate".

(2) Claim 4.10 does not seem new but we could not find a direct quote. Clauses (b)–(c) follow essentially from [12] and clauses (d)–(e) are the parallel for $\aleph_2$ instead of $\aleph_1$; we can continue for higher $\aleph_i$ inductively.

(3) This is closely related to "G is derived from $\mathcal{F}$", see Definition 1.9.

(4) Can we use this to prove $\mathrm{TDU}_{\lambda,\aleph_{\omega+1}}(\mathbb{Z})$ for some λ? Can we do it assuming CH? Can we do it assuming there $k < \omega$ such that $2^{\aleph_\ell} = \aleph_{\ell+1}$ for $\ell < k$?

Proof of Claim 4.10. For part (1) let $R = \mathbb{Z}$ and $a \in \mathbb{Z}$ be a prime, $a_n = a$ (or we can use, e.g., $a_n = n!$), for part (2) let $a \in R$ be a prime such that $\frac{1}{a} \notin R$ and $a_n = a$; but we could use any $\langle a_n : n < \omega \rangle$ such that $a_n R \subset R$. We have to check Definition 4.3. Note that here the r in Definition 4.3 is without loss of generality 1, because any ideal of $\mathbb{Z}$ is principal, see Remark 4.5 (1).

Clause (a): Let $G_{\omega+1}$ be the abelian group generated by $\{x_n, y_n : n < \omega\}$ freely except for the equations

$$a_n y_{n+1} = y_n - x_n \quad \text{for } n < \omega.$$

Let $G_n = Rx_n$ and $G_\omega = \bigoplus\{Rx_k : k < \omega\}$.

Letting $a_{<n} = \prod_{\ell<n} a_\ell$ so that $a_0 = 1$, we have

$$G_{\omega+1} \models a_{<(n+1)} y_{n+1} = y_0 + \sum_{\ell \le n} a_{<\ell} x_\ell.$$

We now define $h \in \mathrm{Hom}(G_\omega, R)$ by choosing $h(x_n)$ by induction on n so that: if $b \in \mathbb{Z}$, then for some n, computing in $\mathbb{Q}$, the sum $b + \sum_{\ell\le n} a_{<\ell} h(x_\ell)$ is not in $a_{<(n+1)}R$, i.e., not divisible by $a_{<(n+1)}$ in R.

Clause (b): Let $\eta_\alpha \in {}^\omega 2$ for $\alpha < \omega_1$ be pairwise distinct. Let G_{ω_1+1} be the abelian group freely generated by

$$\{x_i : i < \omega_1\} \cup \{y_\eta : \eta \in {}^{\omega>}2\} \cup \{z_{\alpha,n} : \alpha < \omega_1, n < \omega\}$$

freely except for the equation

$\oplus_1$ $a_n z_{\alpha,n+1} = z_{\alpha,n} - y_{\eta_\alpha \restriction n} - x_{\omega\alpha+n}$ for $\alpha < \omega_1, n < \omega$.

For $\alpha < \omega_1$ let $G_\alpha := Rx_\alpha$ and $G_{\omega_1} = \bigoplus\{Rx_\beta : \beta < \omega_1\}$.

Clause (c): As in clause (b) note that for $A \in J$ we let

$$G_A = \bigoplus\{Rx_{\omega\alpha+n} : (\alpha, n) \in A\}.$$

Clause (d): For each $\alpha < \omega_2$ let $\langle \varrho_{\alpha,\varepsilon} : \varepsilon < \omega_1\rangle$ be a sequence of pairwise distinct members of ${}^\omega 2$. Let $\langle \nu_\alpha : \alpha < \omega_2\rangle$ be a sequence of increasing functions from ω_1 to ω_1 of length ω_1 such that for all $\alpha < \beta < \omega_2$ for some $\varepsilon < \omega_1$ we have $\{\nu_\alpha(\zeta) : \zeta \in [\varepsilon, \omega_1)\} \cap \{\nu_\beta(\zeta) : \zeta \in [\varepsilon, \omega_1)\} = \emptyset$.

Let G_{ω_2+1} be the R-module generated by

$$X = \{z_{\alpha,\varepsilon,n} : \alpha < \omega_2, \varepsilon < \omega_1, n < \omega\} \cup \{y_\zeta : \zeta < \omega_1\}$$
$$\cup \{x_{\alpha,\varrho} : \alpha < \omega_2, \varrho \in {}^{\omega>}2\} \cup \{t_\alpha : \alpha < \omega_2\}$$

freely except for the equations:

$\oplus_3$ $a_n z_{\alpha,\varepsilon,n+1} = z_{\alpha,\varepsilon,n} - y_{\nu_\alpha(\omega\varepsilon+n)} - x_{\alpha,\varrho_{\alpha,\varepsilon}\restriction n} - t_{\omega_1\alpha+\omega\varepsilon+n}$ for $\alpha < \omega_2$ and $\varepsilon < \omega_1, n < \omega_0$.

For $\alpha < \omega_2$ let

$$G_\alpha = \bigoplus\{Rt_\beta : \beta \in [\omega_1\alpha, \omega_1\alpha + \omega_1)\} \quad \text{and} \quad G_{\omega_2} = \bigoplus\{G_\alpha : \alpha < \omega_2\}.$$

Then we have:

$\boxplus_2$ $G_{\omega_2+1}/G_{\omega_2}$ is $\aleph_2$-free.

(Why? Let $H_* = \bigoplus\{Ry_\varepsilon : \varepsilon < \omega_1\}$ and for $\alpha < \omega_2$ we let H_α be the subgroup of G_{ω_2+1} generated by $G_{\omega_2} \cup H_* \cup \{z_{\alpha,\varepsilon,n} : \varepsilon < \omega_1, n < \omega\} \cup \{x_{\alpha,\varrho} : \varrho \in {}^{\omega>}2\}$.

For $\alpha \leq \omega_2$ let $H_{<\alpha} = \Sigma\{H_\beta : \beta < \alpha\}$. Then clearly

$\boxplus_3$ $G_{\omega_2+1} = H_{<\omega_2}$ and $\langle H_{<\alpha} : \alpha \leq \omega_2\rangle$ is $\subseteq$-increasing continuous.

Hence it suffices to prove for $\alpha < \aleph_2$:

$\boxplus^4_\alpha$ $H_{<\alpha}/G_{\omega_2}$ is free.

(Why? Without loss of generality $\alpha \geq \omega_1$, let $\langle \beta(\xi) : \xi < \omega_1\rangle$ list $\{\beta : \beta < \alpha\}$ with no repetitions. We can easily find a sequence $\bar\zeta = \langle \zeta_\beta : \beta < \alpha\rangle$ such that the sets $\mathcal{U}_\beta := \{\nu_\beta(\varepsilon) : \varepsilon \in [\zeta_\beta, \omega_1)\}$ for $\beta < \alpha$ are pairwise disjoint. Without loss of generality ω^ω divide ζ_β and we let $\mathcal{U} = \omega_1 \setminus \bigcup\{\mathcal{U}_\beta : \beta < \alpha\}$. Moreover, without loss of generality $\xi_1 < \xi_2 \Rightarrow \mathrm{Rang}(\nu_{\beta(\xi_1)}) \cap \{\nu_{\beta(\xi_2)}(\varepsilon) : \varepsilon \in [\zeta_{\beta(\xi_2)}, \omega_1)\} = \emptyset$.

For $\xi \leq \omega_1$ let $H_{\alpha,\xi}$ be the subgroup of $H_{<\alpha}$ generated by

$$\begin{aligned} G_{\omega_2} \cup \{&z_{\gamma,\varepsilon,n} : \gamma \in \{\beta(\zeta) : \zeta < \xi\} \text{ and } \varepsilon < \omega_1, n < \omega\} \\ &\cup \{y_\gamma : \gamma \in \mathcal{U}\} \\ &\cup \{y_{\nu_\gamma(\varepsilon)} : \varepsilon \in [\zeta_\gamma, \aleph_1) \text{ for some } \gamma \in \{\beta(\zeta) : \zeta < \xi\}\} \\ &\cup \{x_{\gamma,\varrho} : \gamma \in \{\beta(\zeta) : \zeta < \xi\} \text{ and } \varrho \in {}^{\omega>}2\}. \end{aligned}$$

So we have $G_{\omega_2} \subseteq H_{\alpha,0} = \bigoplus\{Ry_\zeta : \zeta \in \mathcal{U}\} \oplus G_{\omega_2}$ hence $H_{\alpha,0}/G_{\omega_2}$ is free; also $H_{\alpha,\omega_1} = H_{<\alpha}$ and $\langle H_{\alpha,\xi} : \xi \leq \omega_1\rangle$ is $\subseteq$-increasing continuous. Hence it suffices to prove, for each $\xi < \omega_1$, that $H_{\alpha,\xi+1}/H_{\alpha,\xi}$ is free. Let $H'_{\alpha,\xi}$ be the subgroup of $H_{\alpha,\xi+1}$ generated by $H_{\alpha,\xi} \cup \{x_{\beta(\xi),\varrho} : \varrho \in {}^{\omega>}2\}$. Now

$$H_{\alpha,\xi} \subseteq H'_{\alpha,\xi} \subseteq H_{\alpha,\xi+1}.$$

It is easy to see that $H'_{\alpha,\xi}/H_{\alpha,\xi}$ is countable and free.

Also $H_{\alpha,\xi+1}/H'_{\alpha,\xi}$ is free, in fact $\{z_{\beta(\xi),\varepsilon,n} : \varepsilon \in [\zeta_{\beta(\xi)}, \omega_1),\ n < \omega\}$ is a free basis. Putting those together $\boxplus^4_\alpha$ holds hence $\boxplus_2$ is true.))

$\boxplus_5$ some $h_0 \in \mathrm{Hom}(G_{\omega_2}, {}_RR)$ has no extension $h_2 \in \mathrm{Hom}(G_{\omega_2+1}, {}_RR)$.

(Why? For $\alpha < \omega_2$ let $W_\alpha = \{t_{\omega_1\alpha+\varepsilon} : \varepsilon < \omega_1\}$ and $Y_\alpha = \{y_{\nu_\alpha(\varepsilon)} : \varepsilon < \omega_1\}$. For $\ell = 1, 2$ let K^ℓ_α be the subgroup of G_{ω_2+1} generated by:

- $\{y'_{\alpha,\varepsilon} : \varepsilon < \omega_1\}$ when $\ell = 1$ and $y'_{\alpha,\varepsilon} = y_{\nu_\alpha(\varepsilon)} + t_{\omega_1\cdot\alpha+\varepsilon}$,
- $\{x_{\alpha,\rho} : \rho \in {}^{\omega>}2\} \cup \{y'_{\alpha,\varepsilon} : \varepsilon < \omega_1\}$ for $\ell = 2$,
- $\{z_{\alpha,\varepsilon,n} : \varepsilon < \omega_1, n < \omega\} \cup \{x_{\alpha,\rho} : \rho \in {}^{\omega>}2\} \cup \{y'_{\alpha,\varepsilon} : \varepsilon < \omega_1\}$ when $\ell = 3$

so $K^1_\alpha \subseteq K^2_\alpha \subseteq K^3_\alpha \subseteq G_{\omega_2+1}$.

Let $L^\ell_\alpha = \mathrm{Hom}(K^\ell_\alpha, \mathbb{Z})$ for $\ell = 1, 2, 3$.

Let $L_\alpha = \{f \restriction K^1_\alpha : f \in L^3_\alpha\}$. Clearly L_α is a submodule of L^1_α. As in the proof of clause (b), $L_\alpha \not\subseteq L^1_\alpha$. Let $u_\alpha = u(\alpha) = \mathrm{Rang}(\nu_\alpha)$. We now define a function $\boldsymbol{F}_\alpha : {}^{u(\alpha)}R \to L^1_\alpha / L_\alpha$ as follows: for $f \in {}^{u(\alpha)}R$ let $g_f \in \mathrm{Hom}(K^1_\alpha, {}_R R)$ be defined by $g_f(y'_{\alpha,\varepsilon}) = f(\nu_\alpha(\varepsilon))$ and then $\boldsymbol{F}_\alpha(f) = g_f + L_\alpha \in L^1_\alpha / L_\alpha$. Obviously:

$(*)_{5.1}$ $\boldsymbol{F}_\alpha$ is a homomorphism from ${}^{u(\alpha)}R$ onto L^1_α / L_α.

Now consider:

$(*)_{5.2}$ it suffices to find $\bar{g}^* = \langle g^*_\alpha : \alpha < \omega_2 \rangle$ such that $g^*_\alpha \in L^1_\alpha$ and for every $f \in {}^{\omega_1}R$ for some $\alpha < \omega_2$ we have $\boldsymbol{F}_\alpha(f \restriction u_\alpha) \neq g^*_\alpha + L_\alpha$.

Why is $(*)_{5.2}$ enough? Let $f_\alpha \in {}^{u(\alpha)}R$ be such that $\boldsymbol{F}_\alpha(f_\alpha) = g^*_\alpha + L_\alpha$. We define $h_0 \in \mathrm{Hom}(G_{\omega_2}, R)$ by:

$(*)_{5.3}$ $h_0(t_{\omega_1\alpha+\varepsilon}) = -f_\alpha(\nu_\alpha(\varepsilon))$ for $\alpha < \omega_2, \varepsilon < \omega_1$.

Toward contradiction assume $h_2 \in \mathrm{Hom}(G_{\omega_2+1}, R)$ extends h_0. Define the function $f : \omega_1 \to R$ by $f(\varepsilon) = h(y_\varepsilon)$. Now for each $\alpha < \omega_2$, clearly

$$h_2 \restriction K^1_\alpha \in \mathrm{Hom}(K^1_\alpha, {}_R R) = L^1_\alpha$$

hence $h_2 \restriction K^1_\alpha \in L_\alpha$. Now let $f'_\alpha = f \restriction u_\alpha \in {}^{u(\alpha)}R$ so $f'_\alpha(\nu_\alpha(\varepsilon)) = h_2(y_{\nu_\alpha(\varepsilon)})$ for $\varepsilon < \omega_1$. Recall that

$$\varepsilon < \omega_1 \Rightarrow -f_\alpha(\nu_\alpha(\varepsilon)) = h_0(t_{\omega_1\alpha+\varepsilon}) = h_2(t_{\omega_1\alpha+\varepsilon})$$

by $(*)_{5.3}$ and by $h_2 \supseteq h_0$. So $f''_\alpha := f'_\alpha - f_\alpha \in {}^{u(\alpha)}R$ satisfies

$$f''_\alpha(\nu_\alpha(\varepsilon)) = f'_\alpha(\nu_\alpha(\varepsilon)) - f_\alpha(\nu_\alpha(\varepsilon)) = h_2(y_{\nu_\alpha(\varepsilon)}) + h_2(t_{\omega_1\alpha+\varepsilon}) = h_2(y'_{\alpha,\varepsilon}),$$

hence $g_{f''_\alpha} = h_2 \restriction K^1_\alpha$ which (as we said above) belongs to L_α. It follows that $g_{f'_\alpha} - g_{f_\alpha} \in L_\alpha$, that is, $\boldsymbol{F}_\alpha(f'_\alpha) = \boldsymbol{F}_\alpha(f_\alpha) \in L^3_\alpha / L^1_\alpha$, hence by the choice of f_α above, $\boldsymbol{F}_\alpha(f'_\alpha) = g^*_\alpha + L_\alpha$, but $f'_\alpha = f \restriction u_\alpha$.

As this holds for every $\alpha < \omega_2$, the function f contradicts the present assumption that $\langle g^*_\alpha : \alpha < \omega_2 \rangle$ are as in $(*)_{5.2}$, so there is no h_2 as above, hence indeed it suffices to find

- $\bar{g}^*$ as in $(*)_{5.2}$.

Why does such a $\bar{g}^*$ exist? The proof splits into cases.

Case 1: $2^{\aleph_1} < 2^{\aleph_2}$. By renaming without loss of generality:

$\odot$ $\bigcup\{u_\alpha : \alpha < \omega_2\} = \omega_1$.

We note that $\{\langle \boldsymbol{F}_\alpha(f \restriction u_\alpha) : \alpha < \omega_2\rangle : f \in {}^{\omega_1}R\}$ is a subset of $\prod_{\alpha<\omega_2} L^1_\alpha / L_\alpha$ but the former has cardinality $\leq |R|^{\aleph_1} \leq 2^{\aleph_1}$ and the latter has cardinality $\geq 2^{\aleph_2}$ (actually equal) but we are assuming $2^{\aleph_1} < 2^{\aleph_2}$ in the present case, so we can find $\langle g_\alpha : \alpha < \omega_2\rangle \in \prod_{\alpha<\omega_2} L_\alpha$ which is $\neq \langle \boldsymbol{F}_\alpha(f \restriction u_\alpha) : \alpha < \omega_2\rangle$ for every $f \in {}^{\omega_1}R$.

Case 2: $2^{\aleph_0} = \aleph_1$. Without loss of generality $\rho_{\alpha,\varepsilon} = \rho_\varepsilon$ for $\alpha < \omega_2, \varepsilon < \omega_1$. Now choose $\bar{\nu}$ such that:

$\odot_1$ (a) $\bar{\nu} = \langle \nu_\alpha : \alpha < \omega_2\rangle$,

(b) $\nu_\alpha : \omega_1 \to \omega_1$ is increasing,

(c) if $\beta < \alpha < \omega_2$, then for some $\varepsilon < \omega_1$ we have

$$\nu_\alpha \restriction (\omega\varepsilon + \omega) = \nu_\beta \restriction (\omega\varepsilon + \omega)$$

but $\nu_\alpha(\omega\varepsilon + \omega) \neq \nu_\beta(\omega\varepsilon + \omega)$,

(d) if $\alpha \neq \beta$, then $\mathrm{Rang}(\nu_\alpha) \cap \mathrm{Rang}(\nu_\beta)$ is countable.

(Why? For example choose ν_α by induction on $\alpha < \omega_2$ so that $\mathrm{Rang}(\nu_\alpha)$ is a non-stationary subset of ω_1 and the relevant parts of (a)–(d) hold.)

Now choose h_* such that:

$\odot_2$ (a) $h_* : {}^{\omega_1 >}2 \to {}^{\omega}R$,

(b) let $h^*_n : {}^{\omega_1 >}2 \to R$ for $n < \omega$ be such that $h_*(\nu) = \langle h^*_n(\nu) : n < \omega\rangle$,

(c) if $\varepsilon < \omega_1, \varrho \in {}^{\omega\cdot\varepsilon+\omega}2, \varrho_\ell = \nu^\frown\langle \ell\rangle$ for $\ell = 0, 1$, then the following set of equations is not solvable in R:

- $a_n z_{n+1} = z_n - (h^*_n(\varrho_1) - h^*_n(\varrho_0))$ for $n < \omega$.

This is as in the proof of case (b).

Now we choose h_0 satisfying:

$\odot_3$ h_0 is the homomorphism from G_{ω_2} to ${}_RR$ such that

- $h_0(t_{\omega_1\alpha+\omega\varepsilon+n}) = h^*_n(\nu_\alpha \restriction (\omega \cdot \varepsilon + \omega + 1))$.

Toward a contradiction assume that $h \in \mathrm{Hom}(G_{\omega_2+1}, {}_RR)$ extends h_0. We define a two-place relation E on ω_2 by:

$\odot_4$ $\alpha E \beta$ *iff*

(a) $\nu_\alpha \restriction \omega = \nu_\beta \restriction \omega$,

(b) $h_2(x_{\alpha,\varrho}) = h_2(x_{\beta,\varrho})$ for $\varrho \in {}^{\omega>}2$.

Clearly E is an equivalence relation with $\leq 2^{\aleph_0}$ equivalence classes, so in our case $\aleph_1$ equivalence classes so there are $\alpha \neq \beta$ such that $\alpha E \beta$. By $\odot_1$ (c) there is ε such that $\nu_\alpha \restriction (\omega\cdot\varepsilon+\omega) = \nu_\beta \restriction (\omega\cdot\varepsilon+\omega)$ and $\nu_\alpha(\omega\cdot\varepsilon+\omega) \neq \nu_\beta(\omega\cdot\varepsilon+\omega)$. Without loss of generality $\nu_\alpha(\omega \cdot \varepsilon + \omega) = 1$ and $\nu_\beta(\omega \cdot \varepsilon + \omega) = 0$.

For each n, consider the equations in $\oplus_3$ for $(\alpha, \varepsilon, n), (\beta, \varepsilon, n)$; apply h_2 and subtract them. The differences $h(y_{\nu_\alpha(\omega\varepsilon+n)}) - h(y_{\nu_\beta(\omega\varepsilon+n)})$ cancel by the choice of ε. Also the differences $h_2(x_{\alpha,\varrho_\varepsilon \restriction n}) - h_2(x_{\beta,\varrho_\varepsilon \restriction n})$ cancel as $\alpha E \beta$.

Lastly, by the choice of h_0 recalling $h_0 \subseteq h_2$ we have

$$h_2(t_{\omega_1 \cdot \alpha + \omega \cdot \varepsilon + n}) - h_2(t_{\omega_1 \cdot \beta + \omega \cdot \omega + n}) \\ = h_n^*(\nu_\alpha \restriction (\omega \cdot \varepsilon + \omega + 1)) - h_n^*(\nu_\beta \restriction (\omega \cdot \varepsilon + \omega + 1)).$$

Hence the substitution $z_n \mapsto h_2(z_{\alpha,\varepsilon,n}) - h_2(z_{\beta,\varepsilon,n})$ solves the equations in $\odot_2$ (c) for

- $\varrho_1 = \nu_\alpha \restriction (\omega \cdot \varepsilon + \omega + 1), \varrho_0 = \nu_\beta \restriction (\omega \cdot \varepsilon + \omega + 1)$.

So we get a contradiction to $\odot_2$ (c).

Clause (e): As in clause (d). □

Conclusion 4.12. (1) TDU_λ *holds* when $\mathrm{BB}(\lambda, \mu, 2^{(2^{\aleph_1})^+}, J)$, *where* $\mathrm{cf}(\lambda) > \aleph_1$ *and* $J \in \{J^{\mathrm{bd}}_{\aleph_0}, J^{\mathrm{bd}}_{\aleph_1 * \aleph_0}\}$.

(2) *Similarly for* $\mathrm{BB}(\lambda, \mu, (2^{\mathrm{Dom}(J)}, 2^{\mathrm{Dom}(J)}), J)$.

Proof. (1) By Claims 4.7 and 4.10.

(2) Similarly by Claim 4.15 below and Claim 4.10. □

Remark 4.13. (1) The number, $2^{(2^{\aleph_1})^+}$ of colors is an artifact of the proof. Actually 2 and even the so-called "$1/\theta$ colors" (as in [24, Appendix, Section 1], Definition 0.7 (2)) should suffice, see Remark 4.5.

(2) See Claim 1.8. But we can quote in Section 0 cases of BB with 2 instead of $\beth_4$ or just $2^{(2^{\aleph_1})^+}$ colors.

We can get more than in Claim 4.7.

Definition 4.14. For cardinals λ, θ, σ for $\iota \in \{0, 1\}$ let $\mathrm{SP}^{3+\iota}_{\lambda,\theta,\sigma}(R)$ be the set of ideals J on some κ such that for every $r \in R \setminus \{0\}$ some pair $(\bar{G}, \bar{h})$ witnesses it for r where "$(\bar{G}, \bar{h})$ witness $\mathrm{SP}^{3+\iota}_{\lambda,\theta,\sigma}(R)$ for r" means:

$\oplus$ (a) $\bar{G} = \langle G_i : i < \kappa + 1 + \sigma \rangle$ is a sequence of R-modules each of cardinality $\leq \lambda$,

(b) $G_\kappa = \bigoplus \{G_i : i < \kappa\}$ and $\zeta < \sigma \Rightarrow G_\kappa \oplus_R R \subseteq G_{\kappa+1+\zeta}$,

(c) if $u \in J$ and $\zeta < \sigma$, then $G_{\kappa+1+\zeta} / \bigoplus \{G_i : i \in u\}$ is a θ-free left R-module,

(d) G_i is a θ-free left R-module,

(e) $\bar{h} = \langle h_\zeta : \zeta < \sigma\rangle$ and h_ζ is a homomorphism from G_κ to ${}_R R$ for $\zeta < \sigma$,

(f) if $\iota = 0$, for every homomorphism h from G_κ to ${}_R R$ there is $\zeta < \sigma$ such that:

- no homomorphism h^+ from $G_{\kappa+1+\zeta}$ to ${}_R R$ satisfies

$$x \in G_\kappa \Rightarrow h^+(x) = h(x) + h_\zeta(x)r,$$

(f)$_1$ if $\iota = 1$, then for every homomorphism h from G_κ to ${}_R R$ there is $\varepsilon < \sigma$ such that for every $\zeta < \sigma, \zeta \neq \varepsilon$ we have as above:

- no homomorphism h^+ from $G_{\kappa+1+\zeta}$ to ${}_R R$ satisfies

$$x \in G_\kappa \Rightarrow h^+(x) = h(x) + h_\zeta(x)r.$$

Claim 4.15. *A sufficient condition for* $\mathrm{TDU}_{\lambda,\mu}(R)$ *(i.e., there is a μ-free left R-module G of cardinality λ with* $\mathrm{Hom}_R(G, R) = \{0\}$*) is $\circledast_0$ and also $\circledast_1$ where:*

$\circledast_0$ (a) *R is a ring with unit* $(1 = 1_R)$,

(b) *$J \in \mathrm{SP}^3_{\chi,\theta,\sigma}(R)$ is an ideal on κ,*

(c) *$\bar{C} = \langle C_\delta : \delta \in S\rangle$ is such that* $\mathrm{otp}(C_\delta) = \kappa$ *and $C_\delta \subseteq \delta$,*

(d) *$\lambda > |R| + \chi$ is regular or at least* $\mathrm{cf}(\lambda) > |R| + \chi$ *and $\mu > \kappa$,*

(e) $\mathrm{BB}(\lambda, \bar{C}, (2^{|R|+\chi}, \sigma), J)$, *see Definition* 0.7 (1),

(f) *$\bar{C}$ is (μ, J)-free (but see Claim* 1.8*),*

$\circledast_1$ *similarly replacing clauses* (b) *and* (e) *by* (b)$'$ *and* (e)$'$ *where*

(b)$'$ $J \in \mathrm{SP}^4_{\chi,\theta,\sigma}(R)$,

(e)$'$ $\mathrm{BB}(\lambda, \bar{C}, \langle 2^{|R|+\chi}, \sigma\rangle, J)$, *see Definition* 0.7 (2).

Proof. Assuming $\circledast_\iota$, the proof is similar to the proof of Claim 4.7 with some changes. First of all, instead of $\circledast_1$ we use

$\circledast'_0$ let $(\bar{G}^r, \bar{h}^r)$ witness Definition 4.14 for $r \in R\backslash\{0\}$,

$\circledast'_1$ G_* is a μ-free R-module and for some ordinal $\varepsilon(*) \leq |R| + \kappa$:

(a) $G_* = \bigoplus\{G_{*,\varepsilon} : \varepsilon < \varepsilon(*)\}$ is a μ-free R-module $G_{*,\varepsilon}$ of cardinality $\leq \chi$ for $\varepsilon < \varepsilon(*)$,

(b) if $r \in R\backslash\{0\}$, then for some sequence $\bar{G}^r = \langle G^r_j : j < \kappa + 1 + \sigma\rangle$ as in Definition 4.14 we have: if $j < \kappa$, then

$$\varepsilon(*) = \mathrm{otp}\{\varepsilon < \varepsilon(*) : G^r_j \cong_{f^*_{r,j}} G_{*,\varepsilon}\}$$

hence

(c) $|G_*| \leq \chi + \kappa + |R|$.

Secondly, after $\circledast_8$ we choose $\langle \eta_\delta : \delta \in S_i \rangle$ such that $\eta_\delta \in {}^\kappa\varepsilon(*)$ and

$$j < \kappa \Rightarrow G_{*,\eta_\delta(j)} \cong G^{r_i}_j .$$

Thirdly, we choose $\langle \zeta^1_\delta : \delta \in S_i \rangle$ such that:

$\circledast'_{9.1}$ (a) $\zeta^1_\delta < \sigma$,

(b) if $h \in \mathrm{Hom}(H_*, {}_RR)$, *then* for unboundedly many $\delta \in S_i$ we have $\zeta^1_\delta \neq \bar{c}^1_\delta(h \restriction \bigcup_{\alpha \in C_\delta} G^*_\alpha)$ – see below,

$\circledast_{9.2}$ for $\delta \in S_i$ and $h \in \mathrm{Hom}(K_{<\delta}, {}_RR)$, we define $\boldsymbol{c}^1_\delta(h)$ to be the minimal $\zeta < \sigma$ satisfying $\odot^i_{\delta,\zeta}$ below, and zero if there is no such ζ,

$\odot^i_{\delta,\zeta}$ there is $f \in \mathrm{Hom}(G^{r_i}_{\kappa+1+\zeta}, {}_RR)$ such that:

(α) $f(z) = r_i$,

(β) if $j < \kappa$, then $x \in G^{r_i}_j \Rightarrow f(x) = h(f^*_{r_i,j}(x))$.

The rest is similar. □

Conclusion 4.16. *Assume that* $J^{\mathrm{bd}}_{\kappa_n \times \omega} \in \mathrm{SP}_{\lambda_n,\theta_n}(R)$ *and* $\kappa_n < \kappa_{n+1}$ *for* $n < \omega$. Then*, for some* λ*, for every large enough* n*,* $\mathrm{TDU}_{\lambda,\theta_n^{+\omega+1}}$ *holds.*

Remark 4.17. If we use [26], then we need "$\sum_n \kappa_n$ is strong limit" but instead we use [28].

Proof of Conclusion 4.16. We shall use Claim 4.7 freely. Let $\mu \in \boldsymbol{C}_{\aleph_0}$ be greater than λ_n for each n, and let $\sigma_n < \mu$ be large enough.

Case 1: There is λ' *such that* $\lambda' < 2^\mu < 2^{\lambda'}$. Then we can apply Conclusion 2.7.

Case 2: 2^μ *is singular* or *just there is a* μ^+*-free* $\mathscr{F} \subseteq {}^\omega\mu$ *of cardinality* 2^λ. By Observation 0.9 (2).

Case 3: neither Case 1 nor Case 2. By Theorem 1.22,

$$\lambda = 2^\mu = \lambda^{<\lambda} \quad \text{and} \quad \lambda = \mathrm{tcf}\Big(\prod_{m<\omega} \lambda_m, <_{J^{\mathrm{bd}}_\omega}\Big)$$

for some regular $\lambda_m < \mu$ increasing with $m < \omega$ and let $\langle f_\alpha : \alpha < \lambda \rangle$ exemplify this. Let $S_{\mathrm{gd}} = S^{\mathrm{gd}}_{\bar{f}}$, see Definition 1.25, and $S'_{\mathrm{gd}} = \{\delta \in S_{\mathrm{gd}} : \mathrm{cf}(\delta) > \aleph_0$ and δ is divisible by $\mu\}$.

For each $n < \omega, \delta \in S_* = S'_{\mathrm{gd}} \cap S^\lambda_{\kappa_n}$, let $C_{\delta,n}$ be a club of δ of order type κ_n and let

$$C^n_\delta = \{\mu^\alpha + \eta_\delta(n) : \alpha \in C_\delta \text{ and } n < \omega\}$$

So $\langle C^n_\delta : \delta \in S^n_\delta \rangle$ is a strict (λ, κ_n)-ladder system, i.e., we have $\mathrm{otp}(C^n_\delta) = \kappa$ and $C^n_\delta \subseteq \delta = \sup(C^n_\delta)$. By Claim 1.26 we know that $\bar{C}^n$ is $(\kappa_n^{+\kappa_n}, J^\kappa_{\kappa_n \times \omega})$-free (see Notation 0.3 (3) and Definition 1.2). Now by [27, Theorem 1.10], [28, Theorem 3.1] or Claim 1.15 (3) it follows that for every n large enough, we have $\mathrm{BB}(\lambda, \bar{C}^n, (\lambda, \theta_*), \kappa_n)$, where $\theta_* < \mu$ is large enough. $\square$

Conclusion 4.18. *If the ideal* $J = J^{\mathrm{bd}}_\kappa$ *belongs to* $\mathrm{SP}_{\lambda,\mu}(R)$, then TDU_μ *holds.*

Proof. Left to the reader. $\square$

Remark 4.19. Now we can check all the promises from Section 0.

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