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# On the cofinality of the splitting number

Alan Dow[a,*], Saharon Shelah[b,1]

[a] Department of Mathematics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA
[b] Department of Mathematics, Rutgers University, Hill Center, Piscataway, NJ, 08854-8019, USA

**Abstract**

The splitting number $\mathfrak{s}$ can be singular. The key method is to construct a forcing poset with finite support matrix iterations of ccc posets introduced by Blass and Shelah (1989).
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## 1. Introduction

The cardinal invariants of the continuum discussed in this article are very well known (see [4, van Douwen, p 111]) so we just give a brief reminder. They deal with the mod finite ordering of the infinite subsets of the integers. A set $S \subset \omega$ is *unsplit* by a family $\mathcal{Y} \subset [\omega]^{\aleph_0}$ if $S$ is mod finite contained in one member of $\{Y, \omega \setminus Y\}$ for each $Y \in \mathcal{Y}$. The splitting number $\mathfrak{s}$ is the minimum cardinal of a family $\mathcal{Y}$ for which there is no infinite set unsplit by $\mathcal{Y}$ (equivalently every $S \in [\omega]^{\aleph_0}$ is *split* by some member of $\mathcal{Y}$). It is mentioned in [2] that it is currently unknown if $\mathfrak{s}$ can be a singular cardinal.

**Proposition 1.1.** *The cofinality of the splitting number is not countable.*

**Proof.** Assume that $\theta$ is the supremum of $\{\kappa_n : n \in \omega\}$ and that there is no splitting family of cardinality less than $\theta$. Let $\mathcal{Y} = \{Y_\alpha : \alpha < \theta\}$ be a family of subsets of $\omega$. Let $S_0 = \omega$ and by induction on $n$, choose an infinite subset $S_{n+1}$ of $S_n$ so that $S_{n+1}$ is not split by the family

* Corresponding author.
*E-mail addresses:* adow@uncc.edu (A. Dow), shelah@math.rutgers.edu (S. Shelah).
[1] Current address: Institute of Mathematics, Hebrew University, Givat Ram, Jerusalem 91904, Israel.

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$\{Y_\alpha : \alpha < \kappa_n\}$. If $S$ is any pseudointersection of $\{S_n : n \in \omega\}$, then $S$ is not split by any member of $\mathcal{Y}$. $\square$

One can easily generalize the previous result and proof to show that the cofinality of the splitting number is at least $\mathfrak{t}$. In this paper we prove the following.

**Theorem 1.2.** *If $\kappa$ is any uncountable regular cardinal, then there is a $\lambda > \kappa$ with* $\mathrm{cf}(\lambda) = \kappa$ *and a ccc forcing $\mathbb{P}$ satisfying that $\mathfrak{s} = \lambda$ in the forcing extension.*

To prove the theorem, we construct $\mathbb{P}$ using matrix iterations.

## 2. A special splitting family

**Definition 2.1.** Let us say that a family $\{x_i : i \in I\} \subset [\omega]^\omega$ is $\theta$-Luzin (for an uncountable cardinal $\theta$) if for each $J \in [I]^\theta$, $\bigcap\{x_i : i \in J\}$ is finite and $\bigcup\{x_i : i \in J\}$ is cofinite.

Clearly a family is $\theta$-Luzin if every $\theta$-sized subfamily is $\theta$-Luzin. We leave to the reader the easy verification that for a regular uncountable cardinal $\theta$, each $\theta$-Luzin family is a splitting family. A poset being $\theta$-Luzin preserving will have the obvious meaning. For example, any poset of cardinality less than a regular cardinal $\theta$ is $\theta$-Luzin preserving.

**Lemma 2.2.** *If $\theta$ is a regular uncountable cardinal then any ccc finite support iteration of $\theta$-Luzin preserving posets is again $\theta$-Luzin preserving.*

**Proof.** We prove this by induction on the length of the iteration. Fix any $\theta$-Luzin family $\{x_i : i \in I\}$ and let $\langle\langle\mathbb{P}_\alpha : \alpha \leq \gamma\rangle, \langle\dot{\mathbb{Q}}_\alpha : \alpha < \gamma\rangle\rangle$ be a finite support iteration of ccc posets satisfying that $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is ccc and $\theta$-Luzin preserving, for all $\alpha < \gamma$. If $\gamma$ is a successor ordinal $\beta + 1$, then for any $\mathbb{P}_\beta$-generic filter $G_\beta$, the family $\{x_i : i \in I\}$ is a $\theta$-Luzin family in $V[G_\beta]$. By the hypothesis on $\dot{\mathbb{Q}}_\beta$, this family remains $\theta$-Luzin after further forcing by $\dot{\mathbb{Q}}_\beta$.

Now we assume that $\alpha$ is a limit. Let $\dot{J}_0$ be any $\mathbb{P}_\gamma$-name of a subset of $I$ and assume that $p \in \mathbb{P}_\gamma$ forces that $|\dot{J}_0| = \theta$. We must produce a $q < p$ that forces that $\dot{J}_0$ is as in the definition of $\theta$-Luzin. There is a set $J_1 \subset I$ of cardinality $\theta$ satisfying that, for each $i \in J_1$, there is a $p_i < p$ with $p_i \Vdash i \in \dot{J}_0$. The case when the cofinality of $\alpha$ not equal to $\theta$ is almost immediate. There is a $\beta < \alpha$ such that $J_2 = \{i \in J_1 : p_i \in \mathbb{P}_\beta\}$ has cardinality $\theta$. There is a $\mathbb{P}_\beta$-generic filter $G_\beta$ such that $J_3 = \{i \in J_2 : p_i \in G_\beta\}$ has cardinality $\theta$. By the induction hypothesis, the family $\{x_i : i \in I\}$ is $\theta$-Luzin in $V[G_\beta]$ and so we have that $\bigcap\{x_i : i \in J_3\}$ is finite and $\bigcup\{x_i : i \in J_3\}$ is co-finite. Choose any $q < p$ in $G_\beta$ and a name $\dot{J}_3$ for $J_3$ so that $q$ forces this property for $\dot{J}_3$. Since $q$ forces that $\dot{J}_3 \subset \dot{J}_0$, we have that $q$ forces the same property for $\dot{J}_0$.

Finally we assume that $\alpha$ has cofinality $\theta$. Naturally we may assume that the collection $\{\mathrm{dom}(p_i) : i \in J_1\}$ forms a $\Delta$-system with root contained in some $\beta < \alpha$. Again, we may choose a $\mathbb{P}_\beta$-generic filter $G_\beta$ satisfying that $J_2 = \{i \in J_1 : p_i \restriction \beta \in G_\beta\}$ has cardinality $\theta$. In $V[G_\beta]$, let $\{J_{2,\xi} : \xi \in \omega_1\}$ be a partition of $J_2$ into pieces of size $\theta$. For each $\xi \in \omega_1$, apply the induction hypothesis in the model $V[G_\beta]$, and so we have that $\bigcap\{x_i : i \in J_{2,\xi}\}$ is finite and $\bigcup\{x_i : i \in J_{2,\xi}\}$ is co-finite. For each $\xi \in \omega_1$ let $m_\xi$ be an integer large enough so that $\bigcap\{x_i : i \in J_{2,\xi}\} \subset m_\xi$ and $\bigcup\{x_i : i \in J_{2,\xi}\} \supset \omega \setminus m_\xi$. Let $m$ be any integer such that $m_\xi = m$ for uncountably many $\xi$. Choose any condition $\bar{p} \in \mathbb{P}_\alpha$ so that $\bar{p} \restriction \beta \in G_\beta$. We prove that for each $n > m$ there is a $\bar{p}_n < \bar{p}$ so that $\bar{p}_n \Vdash n \notin \bigcap\{x_i : i \in \dot{I}\}$ and $\bar{p}_n \Vdash n \in \bigcup\{x_i : i \in \dot{I}\}$. Choose any $\xi \in \omega_1$ so that $m_\xi = m$ and $\mathrm{dom}(p_i) \cap \mathrm{dom}(\bar{p}) \subset \beta$ for all $i \in J_{2,\xi}$. Now choose any $i_0 \in J_{2,\xi}$ so that $n \notin x_{i_0}$. Next choose a distinct $\xi'$ with $m_{\xi'} = m$ so that $\mathrm{dom}(p_i) \cap (\mathrm{dom}(\bar{p}) \cup \mathrm{dom}(p_{i_0})) \subset \beta$ for all $i \in J_{2,\xi'}$. Now choose $i_1 \in J_{2,\xi'}$ so that $n \in x_{i_1}$. We now have that $\bar{p} \cup p_{i_0} \cup p_{i_1}$ is a condition that forces $\{i_0, i_1\} \subset \dot{I}$. $\square$

Next we introduce a $\sigma$-centered poset that will render a given family non-splitting.

**Definition 2.3.** For a filter $\mathfrak{D}$ on $\omega$, we define the Laver style poset $\mathbb{L}(\mathfrak{D})$ to be the set of trees $T \subset \omega^{<\omega}$ with the property that $T$ has a minimal branching node stem$(T)$ and for all stem$(T) \subseteq t \in T$, the branching set $\{k : t^\frown k \in T\}$ is an element of $\mathfrak{D}$. If $\mathfrak{D}$ is a filter base for a filter $\mathfrak{D}^*$, then $\mathbb{L}(\mathfrak{D})$ will also denote $\mathbb{L}(\mathfrak{D}^*)$.

The name $\dot{L} = \{(k, T) : (\exists t)\ t^\frown k \subset \text{stem}(T)\}$ will be referred to as the canonical name for the real added by $\mathbb{L}(\mathfrak{D})$.

If $\mathfrak{D}$ is a principal (fixed) ultrafilter on $\omega$, then $\mathbb{L}(\mathfrak{D})$ has a minimum element and so is forcing isomorphic to the trivial poset. If $\mathfrak{D}$ is principal but not an ultrafilter, then $\mathbb{L}(\mathfrak{D})$ is isomorphic to Cohen forcing. If $\mathfrak{D}$ is a free filter, then $\mathbb{L}(\mathfrak{D})$ adds a dominating real and has similarities to Hechler forcing. As usual, for a filter (or filter base) $\mathfrak{D}$ of subsets of $\omega$, we use $\mathfrak{D}^+$ to denote the set of all subsets of $\omega$ that meet every member of $\mathfrak{D}$.

**Definition 2.4.** If $E$ is a dense subset of $\mathbb{L}(\mathfrak{D})$, then a function $\rho_E$ from $\omega^{<\omega}$ into $\omega_1$ is a rank function for $E$ if $\rho_E(t) = 0$ if and only if $t = \text{stem}(T)$ for some $T \in E$, and for all $t \in \omega^{<\omega}$ and $0 < \alpha \in \omega_1$, $\rho_E(t) \leq \alpha$ providing the set $\{k \in \omega : \rho_E(t^\frown k) < \alpha\}$ is in $\mathfrak{D}^+$.

When $\mathfrak{D}$ is a free filter, then $\mathbb{L}(\mathfrak{D})$ has cardinality $\mathfrak{c}$, but nevertheless, if $\mathfrak{D}$ has a base of cardinality less than a regular cardinal $\theta$, $\mathbb{L}(\mathfrak{D})$ is $\theta$-Luzin preserving.

**Lemma 2.5.** *If $\mathfrak{D}$ is a free filter on $\omega$ and if $\mathfrak{D}$ has a base of cardinality less than a regular uncountable cardinal $\theta$, then $\mathbb{L}(\mathfrak{D})$ is $\theta$-Luzin preserving.*

**Proof.** Let $\{x_i : i \in \theta\}$ be a $\theta$-Luzin family with $\theta$ as in the Lemma. Let $\dot{J}$ be a $\mathbb{L}(\mathfrak{D})$-name of a subset of $\theta$. We prove that if $\bigcap\{x_i : i \in \dot{J}\}$ is not finite, then $\dot{J}$ is bounded in $\theta$. By symmetry, it will also prove that if $\bigcup\{x_i : i \in \dot{J}\}$ is not cofinite, then $\dot{J}$ is bounded in $\theta$. Let $\dot{y}$ be the $\mathbb{L}(\mathfrak{D})$-name of the intersection, and let $T_0$ be any member of $\mathbb{L}(\mathfrak{D})$ that forces that $\dot{y}$ is infinite. Let $M$ be any $< \theta$-sized elementary submodel of $H((2^{\mathfrak{c}})^+)$ such that $T_0$, $\mathfrak{D}$, $\dot{J}$, and $\{x_i : i \in \theta\}$ are all members of $M$ and such that $M \cap \mathfrak{D}$ contains a base for $\mathfrak{D}$. Let $i_M = \sup(M \cap \theta)$. If $x \in M \cap [\omega]^\omega$, then $I_x = \{i \in \theta : x \subset x_i\}$ is an element of $M$ and has cardinality less than $\theta$. Therefore, if $i \in \theta \setminus i_M$, then $x_i$ does not contain any infinite subset of $\omega$ that is an element of $M$. We prove that $x_i$ is forced by $T_0$ to also not contain $\dot{y}$. This will prove that $\dot{J}$ is bounded by $i_M$. Let $T_1 < T_0$ be any condition in $\mathbb{L}(\mathfrak{D})$ and let $t_1 = \text{stem}(T_1)$. We show that $T_1$ does not force that $x_i \supset \dot{y}$. We define the relation $\Vdash_w$ on $T_0 \times \omega$ to be the set

$$\{(t, n) \in T_0 \times \omega : \text{ there is no } T \leq T_0, \text{stem}(T) = t, \text{s.t. } T \Vdash n \notin \dot{y}\}.$$

For convenience we may write, for $T \leq T_0$, $T \Vdash_w n \in \dot{y}$ providing $(\text{stem}(T), n)$ is in $\Vdash_w$, and this is equivalent to the relation that $T$ has no stem preserving extension forcing that $n$ is not in $\dot{y}$. Let $T_2 \in M$ be any extension of $T_0$ with stem $t_1$. Let $L$ denote the set of $\ell \in \omega$ such that $T_2 \Vdash_w \ell \in \dot{y}$. If $L$ is infinite, then, since $L \in M$, there is an $\ell \in L \setminus x_i$. This implies that $T_1$ does not force $x_i \supset \dot{y}$, since $T_2 \Vdash_w j \in \dot{y}$ implies that $T_1$ fails to force that $\ell \notin \dot{y}$.

Therefore we may assume that $L$ is finite and let $\ell$ be the maximum of $L$. Define the set $E \subset \mathbb{L}(\mathfrak{D})$ according to $T \in E$ providing that either $t_1 \notin T$ or there is a $j > \ell$ such that $T \Vdash_w j \in \dot{y}$. Again this set $E$ is in $M$ and is easily seen to be a dense subset of $\mathbb{L}(\mathfrak{D})$. By the choice of $\ell$, we note that $\rho_E(t_1) > 0$. If $\rho_E(t_1) > 1$, then the set $\{k \in \omega : 0 < \rho_E(t_1{}^\frown k) < \rho_E(t_1)\}$ is in $\mathfrak{D}^+$ and so there is a $k_1$ in this set such that $t_1{}^\frown k_1 \in T_1 \cap T_2$. By a finite induction, we can choose an extension $t_2 \supseteq t_1$ so that $t_2 \in T_1 \cap T_2$ and $\rho_E(t_2) = 1$. Now, there is a set $D \in \mathfrak{D} \cap M$ contained in $\{k : t_2{}^\frown k \in T_1 \cap T_2\}$ since $M$ contains a base for $\mathfrak{D}$. Also, $D_E = \{k \in D : \rho_E(t_2{}^\frown k) = 0\}$ is in $\mathfrak{D}^+$. For each $k \in D_E$, choose the minimal $j_k$ so that $T_2{}^\frown k \Vdash j_k \in \dot{y}$. The set $\{j_k : k \in D_E\}$ is an element of $M$. This set is not finite because if it were then there would be a single $j$ such that $\{k \in D_E : j_k = j\} \in \mathcal{D}^+$, which would contradict that $\rho_E(t_2) > 0$. This means that there is a $k \in D_E^+$ with $j_k \notin x_i$, and again we have shown that $T_1$ fails to force that $x_i$ contains $\dot{y}$. □

## 3. Matrix iterations

The terminology "matrix iterations" is used in [3], see also forthcoming preprint (F1222) from the second author. The paper [3] nicely expands on the method of matrix iterated forcing first introduced in [1].

Let us recall that a poset $(P, <_P)$ is a complete suborder of a poset $(Q, <_Q)$ providing $P \subset Q$, $<_P \subset <_Q$, and each maximal antichain of $(P, <_P)$ is also a maximal antichain of $(Q, <_Q)$. Note that it follows that incomparable members of $(P, <_P)$ are still incomparable in $(Q, <_Q)$, i.e. $p_1 \perp_P p_2$ implies $p_1 \perp_Q p_2$. We use the notation $(P, <_P) <\circ (Q, <_Q)$ to abbreviate the complete suborder relation, and similarly use $P <\circ Q$ if $<_P$ and $<_Q$ are clear from the context. An element $p$ of $P$ is a reduction of $q \in Q$ if $r \not\perp_Q q$ for each $r <_P p$. If $P \subset Q$, $<_P \subset <_Q$, $\perp_P \subset \perp_Q$, and each element of $Q$ has a reduction in $P$, then $P <\circ Q$. The reason is that if $A \subset P$ is a maximal antichain and $p \in P$ is a reduction of $q \in Q$, then there is an $a \in A$ and an $r$ less than both $p$ and $a$ in $P$, such that $r \not\perp_Q q$.

**Definition 3.1.** We will say that an object $\underline{\mathbf{P}}$ is a matrix iteration if there is an infinite cardinal $\kappa$ and an ordinal $\gamma$ (thence a $(\kappa, \gamma)$-matrix iteration) such that $\underline{\mathbf{P}} = \langle\langle \mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha} : i \le \kappa, \alpha \le \gamma\rangle, \langle \dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{i,\alpha} : i \le \kappa, \alpha < \gamma\rangle\rangle$ where, for each $(i, \alpha) \in \kappa + 1 \times \gamma$ and each $j < i$,

1. $\mathbb{P}^{\underline{\mathbf{P}}}_{j,\alpha}$ is a complete suborder of the poset $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha}$ (i.e. $\mathbb{P}^{\underline{\mathbf{P}}}_{j,\alpha} <\circ \mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha}$),
2. $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{i,\alpha}$ is a $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha}$-name of a ccc poset, $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha+1}$ is equal to $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha} * \dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{i,\alpha}$,
3. for limit $\delta \le \gamma$, $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\delta}$ is equal to the union of the family $\{\mathbb{P}^{\underline{\mathbf{P}}}_{i,\beta} : \beta < \delta\}$
4. $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa,\alpha}$ is the union of the chain $\{\mathbb{P}^{\underline{\mathbf{P}}}_{j,\alpha} : j < \kappa\}$.

When the context makes it clear, we omit the superscript $\underline{\mathbf{P}}$ when discussing a matrix iteration. Throughout the paper, $\kappa$ will be a fixed uncountable regular cardinal.

**Definition 3.2.** A sequence $\vec{\lambda}$ is $\kappa$-tall if $\vec{\lambda} = \langle \mu_\xi, \lambda_\xi : \xi < \kappa\rangle$ is a sequence of pairs of regular cardinals satisfying that $\mu_0 = \omega < \kappa < \lambda_0$ and, for $0 < \eta < \kappa$, $\mu_\eta < \lambda_\eta$ where $\mu_\eta = (2^{\sup\{\lambda_\xi : \xi < \eta\}})^+$.

Also for the remainder of the paper, we fix a $\kappa$-tall sequence $\vec{\lambda}$ and $\lambda$ will denote the supremum of the set $\{\lambda_\xi : \xi \in \kappa\}$. For simpler notation, whenever we discuss a matrix iteration $\underline{\mathbf{P}}$ we shall henceforth assume that it is a $(\kappa, \gamma)$-matrix iteration for some ordinal $\gamma$. We may refer to a forcing extension by $\underline{\mathbf{P}}$ as an abbreviation for the forcing extension by $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa,\gamma}$.

For any poset $P$, any $P$-name $\dot{D}$, and $P$-generic filter $G$, $\dot{D}[G]$ will denote the valuation of $\dot{D}$ by $G$. For any ground model $x$, $\check{x}$ denotes the canonical name so that $\check{x}[G] = x$. When $x$ is an ordinal (or an integer) we will suppress the accent in $\check{x}$. A $P$-name $\dot{D}$ of a subset of $\omega$ will be said to be *nice* or *canonical* if for each integer $j \in \omega$, there is an antichain $A_j$ such that $\dot{D} = \bigcup\{\{j\} \times A_j : j \in \omega\}$. We will say that $\dot{\mathfrak{D}}$ is a nice $P$-name of a family of subsets of $\omega$ just to mean that $\dot{\mathfrak{D}}$ is a collection of nice $P$-names of subsets of $\omega$. We will use $(\dot{\mathfrak{D}})_P$ if we need to emphasize that we mean the $P$-name. Similarly if we say that $\dot{\mathfrak{D}}$ is a nice $P$-name of a filter (base) we mean that $\dot{\mathfrak{D}}$ is a nice $P$-name such that, for each $P$-generic filter, the collection $\{\dot{D}[G] : \dot{D} \in \dot{\mathfrak{D}}\}$ is a filter (base) of infinite subsets of $\omega$.

Following these conventions, the following notation will be helpful.

**Definition 3.3.** For a $(\kappa, \gamma)$-matrix $\underline{\mathbf{P}}$ and $i < \kappa$, we let $\mathbb{B}^{\underline{\mathbf{P}}}_{i,\gamma}$ denote the set of all nice $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\gamma}$-names of subsets of $\omega$. We note that this then is the nice $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\gamma}$-name for the power set of $\omega$. As usual, when possible we suppress the $\underline{\mathbf{P}}$ superscript.

For a nice $\underline{\mathbf{P}}$-name $\dot{\mathfrak{D}}$ of a filter (or filter base) of subsets of $\omega$, we let $(\dot{\mathfrak{D}})^+$ denote the set of all nice $\underline{\mathbf{P}}$-names that are forced to meet every member of $\dot{\mathfrak{D}}$. It follows that $(\dot{\mathfrak{D}})^+$ is the nice $\underline{\mathbf{P}}$-name for the usual defined notion $(\dot{\mathfrak{D}})^+$ in the forcing extension by $\underline{\mathbf{P}}$. We let $\langle\dot{\mathfrak{D}}\rangle$ denote the nice $\underline{\mathbf{P}}$-name of the filter generated by $\dot{\mathfrak{D}}$. We use the same notational conventions if, for some poset $\mathbb{P}$, $\dot{\mathfrak{D}}$ is a nice $\mathbb{P}$-name of a filter (or filter base) of subsets of $\omega$.

The main idea for controlling the splitting number in the extension by $\underline{\mathbf{P}}$ will involve having many of the subposets being $\theta$-Luzin preserving for $\theta \in \{\lambda_\xi : \xi \in \kappa\}$. Motivated by the fact that posets of the form $\mathbb{L}(\mathfrak{D})$ (our proposed iterands) are $\theta$-Luzin preserving when $\mathfrak{D}$ is sufficiently small we adopt the name $\vec{\lambda}$-thin for this next notion.

**Definition 3.4.** For a $\kappa$-tall sequence $\vec{\lambda}$, we will say that a $(\kappa, \gamma)$-matrix-iteration $\underline{\mathbf{P}}$ is $\vec{\lambda}$-thin providing that for each $\xi < \kappa$ and $\alpha \le \gamma$, $\mathbb{P}^{\underline{\mathbf{P}}}_{\xi,\alpha}$ is $\lambda_\xi$-Luzin preserving.

Now we combine the notion of $\vec{\lambda}$-thin matrix-iteration with Lemma 2.2. We adopt Kunen's notation that for a set $I$, $\mathrm{Fn}(I, 2)$ denotes the usual poset for adding Cohen reals (finite partial functions from $I$ into 2 ordered by superset).

**Lemma 3.5.** *Suppose that $\underline{\mathbf{P}}$ is a $\vec{\lambda}$-thin $(\kappa, \gamma)$-matrix iteration for some $\kappa$-tall sequence $\vec{\lambda}$. Further suppose that $\dot{\mathbb{Q}}_{i,0}$ is the $\mathbb{P}_{i,0}$-name of the poset* $\mathrm{Fn}(\lambda_\xi, 2)$ *for each $\xi \in \kappa$, and therefore $\mathbb{P}_{\kappa,1}$ is isomorphic to* $\mathrm{Fn}(\lambda, 2)$. *Let $\dot{g}$ denote the generic function from $\lambda$ onto 2 added by $\mathbb{P}_{\kappa,1}$ and, for $i < \lambda$, let $\dot{x}_i$ be the canonical name of the set $\{n \in \omega : \dot{g}(i + n) = 1\}$. Then the family $\{\dot{x}_i : i < \lambda\}$ is forced by $\underline{\mathbf{P}}$ to be a splitting family.*

**Proof.** Let $G_{\kappa,\gamma}$ be a $\mathbb{P}_{\kappa,\gamma}$-generic filter. For each $\xi \in \kappa$ and $\alpha \le \gamma$, let $G_{\xi,\alpha} = G_{\kappa,\gamma} \cap \mathbb{P}_{\xi,\alpha}$. Let $\dot{y}$ be any nice $\mathbb{P}_{\kappa,\gamma}$-name for a subset of $\omega$. Since $\dot{y}$ is a countable name, we may choose a $\xi < \kappa$ so that $\dot{y}$ is a $\mathbb{P}_{\xi,\gamma}$-name. It is easily shown, and very well-known, that the family $\{\dot{x}_i : i < \lambda_\xi\}$ is forced by $\mathbb{P}_{\xi,1}$ (i.e. $\mathrm{Fn}(\lambda_\xi, 2)$) to be a $\lambda_\xi$-Luzin family. By the hypothesis that $\underline{\mathbf{P}}$ is $\vec{\lambda}$-thin, we have, by Lemma 2.2, that $\{\dot{x}_i : i < \lambda_\xi\}$ is still $\lambda_\xi$-Luzin in $V[G \cap \mathbb{P}_{\xi,\gamma}]$. Since $\dot{y}$ is a $\mathbb{P}_{\xi,\gamma}$-name, there is an $i < \lambda_\xi$ such that $\dot{y}[G_{\xi,\gamma}] \cap \dot{x}_i[G_{\xi,\gamma}]$ and $\dot{y}[G_{\xi,\gamma}] \setminus \dot{x}_i[G_{\xi,\gamma}]$ are infinite. □

## 4. The construction of $\underline{\mathbf{P}}$

When constructing a matrix-iteration by recursion, we will need notation and language for extension. We will use, for an ordinal $\gamma$, $\underline{\mathbf{P}}^\gamma$ to indicate that $\underline{\mathbf{P}}^\gamma$ is a $(\kappa, \gamma)$-matrix iteration.

**Definition 4.1.**

(1) A matrix iteration $\underline{\mathbf{P}}^\gamma$ is an extension of $\underline{\mathbf{P}}^\delta$ providing $\delta \leq \gamma$, and, for each $\alpha \leq \delta$ and $i \leq \kappa$, $\mathbb{P}^{\underline{\mathbf{P}}^\delta}_{i,\alpha} = \mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\alpha}$. We can use $\underline{\mathbf{P}}^\gamma \restriction \delta$ to denote the unique $(\kappa, \delta)$-matrix iteration extended by $\underline{\mathbf{P}}^\gamma$.

(2) If, for each $i < \kappa$, $\dot{\mathbb{Q}}_{i,\gamma}$ is a $\mathbb{P}^{\underline{\mathbf{P}}}_{i,\gamma}$-name of a ccc poset satisfying that, for each $i < j < \kappa$, $\mathbb{P}_{i,\gamma} * \dot{\mathbb{Q}}_{i,\gamma}$ is a complete subposet of $\mathbb{P}_{j,\gamma} * \dot{\mathbb{Q}}_{j,\gamma}$, then we let $\underline{\mathbf{P}} * \langle \dot{\mathbb{Q}}_{i,\gamma} : i < \kappa \rangle$ denote the $(\kappa, \gamma+1)$-matrix $\langle \langle \mathbb{P}_{i,\alpha} : i \leq \kappa, \alpha \leq \gamma + 1 \rangle, \langle \dot{\mathbb{Q}}_{i,\alpha} : i \leq \kappa, \alpha < \gamma+1 \rangle \rangle$, where $\dot{\mathbb{Q}}_{\kappa,\gamma}$ is the $\underline{\mathbf{P}}$-name of the union of $\{\dot{\mathbb{Q}}_{i,\gamma} : i < \kappa\}$ and, for $i \leq \kappa$, $\mathbb{P}_{i,\gamma} = \mathbb{P}^{\underline{\mathbf{P}}}_{i,\gamma}$, $\mathbb{P}_{i,\gamma+1} = \mathbb{P}^{\underline{\mathbf{P}}}_{i,\gamma} * \dot{\mathbb{Q}}_{i,\gamma}$, and for $\alpha < \gamma$, $(\mathbb{P}_{i,\alpha}, \dot{\mathbb{Q}}_{i,\alpha}) = (\mathbb{P}^{\underline{\mathbf{P}}}_{i,\alpha}, \dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{i,\alpha})$.

The following, from [3, Lemma 3.10], shows that extension at limit steps is canonical.

**Lemma 4.2.** *If $\gamma$ is a limit and if $\{\underline{\mathbf{P}}^\delta : \delta < \gamma\}$ is a sequence of matrix iterations satisfying that for $\beta < \delta < \gamma$, $\underline{\mathbf{P}}^\delta \restriction \beta = \underline{\mathbf{P}}^\beta$, then there is a unique matrix iteration $\underline{\mathbf{P}}^\gamma$ such that $\underline{\mathbf{P}}^\gamma \restriction \delta = \underline{\mathbf{P}}^\delta$ for all $\delta < \gamma$.*

**Proof.** For each $\delta < \gamma$ and $i < \kappa$, we define $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\delta}$ to be $\mathbb{P}^{\underline{\mathbf{P}}^\delta}_{i,\delta}$ and $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}^\gamma}_{i,\delta}$ to be $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}^{\delta+1}}_{i,\delta}$. It follows that $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}^\gamma}_{i,\delta}$ is a $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\delta}$-name. Since $\gamma$ is a limit, the definition of $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\gamma}$ is required to be $\bigcup\{\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\delta} : \delta < \gamma\}$ for $i < \kappa$. Similarly, the definition of $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\kappa,\gamma}$ is required to be $\bigcup\{\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\gamma} : i < \kappa\}$. Let us note that $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\kappa,\gamma}$ is also required to be the union of the chain $\bigcup\{\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\kappa,\delta} : \delta < \gamma\}$, and this holds by assumption on the sequence $\{\underline{\mathbf{P}}^\delta : \delta < \gamma\}$.

To prove that $\underline{\mathbf{P}}^\gamma$ is a $(\kappa, \gamma)$-matrix it remains to prove that for $j < i \leq \kappa$, and each $q \in \mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{i,\gamma}$, there is a reduction $p$ in $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{j,\gamma}$. Since $\gamma$ is a limit, there is an $\alpha < \gamma$ such that $q \in \mathbb{P}^{\underline{\mathbf{P}}^\alpha}_{i,\alpha}$ and, by assumption, there is a reduction, $p$, of $q$ in $\mathbb{P}^{\underline{\mathbf{P}}^\alpha}_{j,\alpha}$. By induction on $\beta$ $(\alpha \leq \beta \leq \gamma)$ we note that $q \in \mathbb{P}^{\underline{\mathbf{P}}^\beta}_{i,\beta}$ and that $p$ is a reduction of $q$ in $\mathbb{P}^{\underline{\mathbf{P}}^\beta}_{j,\beta}$. For limit $\beta$ it is trivial, and for successor $\beta$ it follows from condition (1) in the definition of matrix iteration. □

We also will need the next result taken from [3, Lemma 13], which they describe as well known, for stepping diagonally in the array of posets.

**Lemma 4.3.** *Let $\mathbb{P}, \mathbb{Q}$ be partial orders such that $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$. Let $\dot{\mathbb{A}}$ be a $\mathbb{P}$-name for a forcing notion and let $\dot{\mathbb{B}}$ be a $\mathbb{Q}$-name for a forcing notion such that $\Vdash_{\mathbb{Q}} \dot{\mathbb{A}} \subset \dot{\mathbb{B}}$, and every $\mathbb{P}$-name of a maximal antichain of $\dot{\mathbb{A}}$ is also forced by $\mathbb{Q}$ to be a maximal antichain of $\dot{\mathbb{B}}$. Then $\mathbb{P} * \dot{\mathbb{A}} \lessdot \mathbb{Q} * \dot{\mathbb{B}}$*

Let us also note if $\dot{\mathbb{B}}$ is equal to $\dot{\mathbb{A}}$ in Lemma 4.3, then the hypothesis and the conclusion of the Lemma are immediate. On the other hand, if $\dot{\mathbb{A}}$ is the $\mathbb{P}$-name of $\mathbb{L}(\dot{\mathfrak{D}})$ for some $\mathbb{P}$-name of a filter $\dot{\mathfrak{D}}$, then the $\mathbb{Q}$-name of $\mathbb{L}(\dot{\mathfrak{D}})$ is not necessarily equal to $\dot{\mathbb{A}}$.

**Lemma 4.4** (*[5, 1.9]*)**.** *Suppose that $\mathbb{P}, \mathbb{Q}$ are posets with $\mathbb{P} \lessdot \mathbb{Q}$. Suppose also that $\dot{\mathfrak{D}}_0$ is a $\mathbb{P}$-name of a filter on $\omega$ and $\dot{\mathfrak{D}}_1$ is a $\mathbb{Q}$-name of a filter on $\omega$. If $\Vdash_{\mathbb{Q}} \dot{\mathfrak{D}}_0 \subseteq \dot{\mathfrak{D}}_1$ then $\mathbb{P} * \mathbb{L}(\dot{\mathfrak{D}}_0)$ is a complete subposet of $\mathbb{Q} * \mathbb{L}(\dot{\mathfrak{D}}_1)$ if either of the two equivalent conditions hold:*

(1) $\Vdash_{\mathbb{Q}} ((\dot{\mathfrak{D}}_0)^+)_{\mathbb{P}} \subseteq \dot{\mathfrak{D}}_1^+$,
(2) $\Vdash_{\mathbb{Q}} \dot{\mathfrak{D}}_1 \cap V^{\mathbb{P}} \subseteq \langle \dot{\mathfrak{D}}_0 \rangle$ *(where $V^{\mathbb{P}}$ is the class of $\mathbb{P}$-names).*

**Proof.** Let $\dot{E}$ be any $\mathbb{P}$-name of a maximal antichain of $\mathbb{L}(\dot{\mathfrak{D}}_0)$. By Lemma 4.3, it suffices to show that $\mathbb{Q}$ forces that every member of $\mathbb{L}(\dot{\mathfrak{D}}_1)$ is compatible with some member of $\dot{E}$. Let $G$

be any $\mathbb{Q}$-generic filter and let $E$ denote the valuation of $\dot{E}$ by $G \cap \mathbb{P}$. Working in the model $V[G \cap \mathbb{P}]$, we have the function $\rho_E$ as in Definition 2.4. Choose $\delta \in \omega_1$ satisfying that $\rho_E(t) < \delta$ for all $t \in \omega^{<\omega}$. Now, working in $V[G]$, we consider any $T \in \mathbb{L}(\dot{\mathfrak{D}}_1)$ and we find an element of $E$ that is compatible with $T$. In fact, by induction on $\alpha < \delta$, one easily proves that for each $T \in \mathbb{L}(\dot{\mathfrak{D}}_1)$ with $\rho_E(\text{stem}(T)) \leq \alpha$, $T$ is compatible with some member of $E$. $\square$

**Definition 4.5.** For a $(\kappa, \gamma)$-matrix-iteration $\underline{\mathbf{P}}$, and ordinal $i_\gamma < \kappa$, we say that an increasing sequence $\langle \dot{\mathfrak{D}}_i : i < \kappa \rangle$ is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases, if for each $i < j < \kappa$

(1) $\dot{\mathfrak{D}}_i$ is a subset of $\mathbb{B}_{i,\gamma}$ (hence a nice $\mathbb{P}^{\mathbf{P}}_{i,\gamma}$-name)
(2) $\Vdash_{\mathbb{P}_{i,\gamma}} \dot{\mathfrak{D}}_i$ is a filter with a base of cardinality at most $\mu_{i_\gamma}$,
(3) $\Vdash_{\mathbb{P}_{j,\gamma}} \langle \dot{\mathfrak{D}}_j \rangle \cap \mathbb{B}_{i,\gamma} \subseteq \langle \dot{\mathfrak{D}}_i \rangle$.

Notice that a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases can be (essentially) eventually constant. Thus we will say that a sequence $\langle \dot{\mathfrak{D}}_i : i \leq j \rangle$ (for some $j < \kappa$) is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases if the sequence $\langle \dot{\mathfrak{D}}_i : i < \kappa \rangle$ is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases where $\dot{\mathfrak{D}}_i$ is the $\mathbb{P}_{i,\gamma}$-name for $\mathbb{B}_{i,\gamma} \cap \langle \dot{\mathfrak{D}}_j \rangle$ for $j < i \leq \kappa$. When $\underline{\mathbf{P}}$ is clear from the context, we will use $\vec{\lambda}(i_\gamma)$-thin as an abbreviation for $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin.

**Corollary 4.6.** *For a $(\kappa, \gamma)$-matrix-iteration $\underline{\mathbf{P}}$, ordinal $i_\gamma < \kappa$, and a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases $\langle \dot{\mathfrak{D}}_\xi : i < \kappa \rangle$, $\underline{\mathbf{P}} * \langle \dot{\mathbb{Q}}_{i,\gamma} : i \leq \kappa \rangle$ is a $\gamma + 1$-extension of $\underline{\mathbf{P}}$, where, for each $i \leq i_\gamma$, $\dot{\mathbb{Q}}_{i,\gamma}$ is the trivial poset, and for $i_\gamma \leq i < \kappa$, $\dot{\mathbb{Q}}_{i,\gamma}$ is $\mathbb{L}(\dot{\mathfrak{D}}_i)$.*

**Definition 4.7.** Whenever $\langle \dot{\mathfrak{D}}_i : i < \kappa \rangle$ is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases, let $\underline{\mathbf{P}} * \mathbb{L}(\langle \dot{\mathfrak{D}}_i : i_\gamma \leq i < \kappa \rangle)$ denote the $\gamma + 1$-extension described in Corollary 4.6.

This next corollary is immediate.

**Corollary 4.8.** *If $\underline{\mathbf{P}}$ is a $\vec{\lambda}$-thin $(\kappa, \gamma)$-matrix and if $\langle \dot{\mathfrak{D}}_i : i < \kappa \rangle$ is a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases, then $\underline{\mathbf{P}} * \mathbb{L}(\langle \dot{\mathfrak{D}}_i : i_\gamma \leq i < \kappa \rangle)$ is a $\vec{\lambda}$-thin $(\kappa, \gamma + 1)$-matrix.*

We now describe a first approximation of the scheme, $\mathcal{K}(\vec{\lambda})$, of posets that we will be using to produce the model.

**Definition 4.9.** For an ordinal $\gamma > 0$ and a $(\kappa, \gamma)$-matrix iteration $\underline{\mathbf{P}}$, we will say that $\underline{\mathbf{P}} \in \mathcal{K}(\vec{\lambda})$ providing for each $0 < \alpha < \gamma$,

(1) for each $i \leq \kappa$, $\mathbb{P}^{\mathbf{P}}_{i,1}$ is $\text{Fn}(\lambda_i, 2)$, and
(2) there is an $i_\alpha = i^{\mathbf{P}}_\alpha < \kappa$ and a $(\underline{\mathbf{P}} \restriction \alpha, \vec{\lambda}(i_\alpha))$-thin sequence $\langle \dot{\mathfrak{D}}^\alpha_i : i < \kappa \rangle$ of filter bases, such that $\underline{\mathbf{P}} \restriction \alpha + 1$ is equal to $\underline{\mathbf{P}} \restriction \alpha * \mathbb{L}(\langle \dot{\mathfrak{D}}^\alpha_i : i_\alpha \leq i < \kappa \rangle)$.

For each $0 < \alpha < \gamma$, we let $\dot{\mathfrak{D}}^\alpha_\kappa$ denote the $\underline{\mathbf{P}} \restriction \alpha$-name of the union $\bigcup \{\dot{\mathfrak{D}}^\alpha_i : i_\alpha \leq i < \kappa\}$, and we let $\dot{L}_\alpha$ denote the canonical $\underline{\mathbf{P}} \restriction \alpha + 1$-name of the subset of $\omega$ added by $\mathbb{L}(\mathfrak{D}^\alpha_\kappa)$.

Let us note that each $\underline{\mathbf{P}} \in \mathcal{K}(\vec{\lambda})$ is $\vec{\lambda}$-thin. Furthermore, by Lemma 3.5, this means that each $\underline{\mathbf{P}} \in \mathcal{K}(\vec{\lambda})$ forces that $\mathfrak{s} \leq \lambda$. We begin a new section for the task of proving that there is a $\underline{\mathbf{P}} \in \mathcal{K}(\vec{\lambda})$ that forces that $s \geq \lambda$.

It will be important to be able to construct $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequences of filter bases, and it seems we will need some help.

Sh:1127



**Definition 4.10.** For an ordinal $\gamma > 0$ and a $(\kappa, \gamma)$-matrix iteration $\underline{\mathbf{P}}$ we will say that $\underline{\mathbf{P}} \in \mathcal{H}(\vec{\lambda})$ if $\underline{\mathbf{P}}$ is in $\mathcal{K}(\vec{\lambda})$ and for each $0 < \alpha < \gamma$, if $i_\alpha = i_\alpha^{\mathbf{P}} > 0$ then $\omega_1 \leq \mathrm{cf}(\alpha) \leq \mu_{i_\alpha}$ and there is a $\beta_\alpha < \alpha$ such that

(1) for $\beta_\alpha \leq \xi < \alpha$, $i_\xi \in \{0, i_\alpha\}$,
(2) if $\beta_\alpha \leq \eta < \alpha$, $i_\eta > 0$ and $\xi = \eta + \omega_1 \leq \alpha$, then $\dot{L}_\eta \in \dot{\mathcal{D}}^\xi_{i_\xi}$, and $\mathbb{P}_{i_\xi,\xi} \Vdash \dot{\mathcal{D}}^\alpha_{i_\xi}$ has a descending mod finite base of cardinality $\omega_1$,
(3) if $\beta_\alpha < \xi \leq \alpha$, $i_\xi > 0$, and $\eta + \omega_1 < \xi$ for $\eta < \xi$, then $\{\dot{L}_\eta : \beta_\alpha \leq \eta < \alpha, \mathrm{cf}(\eta) \geq \omega_1\}$ is a base for $\dot{\mathcal{D}}^\xi_{i_\xi}$.

## 5. Producing $\vec{\lambda}$-thin filter sequences

In this section we prove this main lemma.

**Lemma 5.1.** *Suppose that $\underline{\mathbf{P}}^\gamma \in \mathcal{H}(\vec{\lambda})$ and that $\mathcal{Y}$ is a set of fewer than $\lambda$ nice $\underline{\mathbf{P}}^\gamma$-names of subsets of $\omega$, then there is a $\delta < \gamma + \lambda$ and an extension $\underline{\mathbf{P}}^\delta$ of $\underline{\mathbf{P}}^\gamma$ in $\mathcal{H}(\vec{\lambda})$ that forces that the family $\mathcal{Y}$ is not a splitting family.*

The main theorem follows easily.

**Proof of Theorem 1.2.** Let $\theta$ be any regular cardinal so that $\theta^{<\lambda} = \theta$ (for example, $\theta = (2^\lambda)^+$). Construct $\underline{\mathbf{P}}^\theta \in \mathcal{H}(\vec{\lambda})$ so that for all $\mathcal{Y} \subset \mathbb{B}_{\kappa,\theta}$ with $|\mathcal{Y}| < \lambda$, there is a $\gamma < \delta < \theta$ so that $\mathcal{Y} \subset \mathbb{B}_{\kappa,\gamma}$ and, by applying Lemma 5.1, such that $\underline{\mathbf{P}}^\theta \restriction \delta$ forces that $\mathcal{Y}$ is not a splitting family. □

We begin by reducing our job to simply finding a $(\underline{\mathbf{P}}, \vec{\lambda}(i_\gamma))$-thin sequence.

**Definition 5.2.** For a $(\kappa, \gamma)$-matrix-iteration $\underline{\mathbf{P}}^\gamma$, we say that a subset $\mathcal{E}$ of $\mathbb{B}_{\kappa,\gamma}$ is $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter subbase if, $i_\gamma < \kappa$, $|\mathcal{E}| \leq \mu_{i_\gamma}$, and the sequence $\langle\langle \mathcal{E} \cap \mathbb{B}_{i,\gamma}\rangle : i < \kappa\rangle$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin sequence of filter bases.

**Lemma 5.3.** *For any $\underline{\mathbf{P}}^\gamma \in \mathcal{H}(\vec{\lambda})$, and any $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter base $\mathcal{E}$, there is an $\alpha \leq \gamma + \mu_{i_\gamma} + 1$ and extensions $\underline{\mathbf{P}}^\alpha, \underline{\mathbf{P}}^{\alpha+1}$ of $\underline{\mathbf{P}}^\gamma$ in $\mathcal{H}(\vec{\lambda})$, such that, $\underline{\mathbf{P}}^{\alpha+1} = \underline{\mathbf{P}}^\alpha * \mathbb{L}(\langle \dot{\mathfrak{D}}^\alpha_i : i_\alpha \leq i < \kappa\rangle)$ and $\underline{\mathbf{P}}^\alpha$ forces that $\mathcal{E} \cap \mathbb{B}_{i,\gamma}$ is a subset of $\dot{\mathfrak{D}}^\alpha_i$ for all $i < \kappa$.*

**Proof.** The case $i_\gamma = 0$ is trivial, so we assume $i_\gamma > 0$. There is no loss of generality to assume that $\mathcal{E} \cap \mathbb{B}_{i_\gamma,\gamma}$ has character $\mu_{i_\gamma}$. Let $\{\dot{E}_\xi : \xi < \mu_{i_\gamma}\} \subset \mathcal{E} \cap \mathbb{B}_{i_\gamma,\gamma}$ enumerate a filter base for $\langle \mathcal{E}\rangle \cap \mathbb{B}_{i_\gamma,\gamma}$. We can assume that this enumeration satisfies that $\dot{E}_\xi \setminus \dot{E}_{\xi+1}$ is forced to be infinite for all $\xi < \mu_{i_\gamma}$. Let $\mathcal{A}$ be any countably generated free filter on $\omega$ that is not principal mod finite. By induction on $\xi < \mu_{i_\gamma}$ we define $\underline{\mathbf{P}}^{\gamma+\xi}$ by simply defining $i_{\gamma+\xi}$ and the sequence $\langle \dot{\mathfrak{D}}^{\gamma+\xi}_i : i_{\gamma+\xi} \leq i \leq \kappa\rangle$. We will also recursively define, for each $\xi < \mu_{i_\gamma}$, a $\underline{\mathbf{P}}^{\gamma+\xi}$-name $\dot{D}_\xi$ such that $\underline{\mathbf{P}}^{\gamma+\xi}$ forces that $\dot{D}_\xi \subset \dot{E}_\xi$. An important induction hypothesis is that $\{\dot{D}_\eta : \eta < \xi\} \cup \{\dot{E}_\zeta : \zeta < \mu_{i_\gamma}\} \cup \mathcal{E}$ is forced to have the finite intersection property.

For each $\xi < \gamma + \omega_1$, let $i_\xi = 0$ and $\dot{\mathfrak{D}}^\xi_i$ be the $\underline{\mathbf{P}}^\xi$-name $\langle \mathcal{A}\rangle \cap \mathbb{B}_{i,\xi}$ for all $i \leq \kappa$. The definition of $\dot{D}_0$ is simply $\dot{E}_0$. By recursion, for each $\eta < \omega_1$ and $\xi = \eta + 1$, we define $\dot{D}_\xi$ to be the intersection of $\dot{D}_\eta$ and $\dot{E}_\xi$. For limit $\xi < \omega_1$, we note that $\mathbb{P}_{i_\gamma,\xi}$ forces that $\mathbb{L}(\langle \mathcal{A}\rangle)$ is isomorphic to $\mathbb{L}(\langle\{\dot{D}_\eta \cap \dot{E}_\xi : \eta < \xi\}\rangle)$. Therefore, we can let $\dot{D}_\xi$ be a $\underline{\mathbf{P}}^{\xi+1}$-name for the generic real added by $\mathbb{L}(\langle\{\dot{D}_\eta \cap \dot{E}_\xi : \eta < \xi\}\rangle)$. A routine density argument shows that this definition satisfies the induction hypothesis.

Sh:1127



The definition of $i_{\gamma+\omega_1}$ is $i_\gamma$ and the definition of $\dot{\mathfrak{D}}^{\gamma+\omega_1}_{i_\gamma}$ is the filter generated by $\{\dot{D}_\xi : \xi < \omega_1\}$. The definition of $\dot{D}_{\omega_1}$ is $\dot{L}_{\gamma+\omega_1}$.

Let $S$ denote the set of $\eta < \mu_{i_\gamma}$ with uncountable cofinality. We now add additional induction hypotheses:

(1) if $\zeta = \sup(S \cap \xi) < \xi$ and $\xi = \nu + 1$, then $\dot{D}_\xi = \dot{D}_\nu \cap \dot{E}_\xi$, and $i_\xi = 0$ and $\dot{\mathfrak{D}}^{\gamma+\xi}_i = \langle \mathcal{A} \rangle$ for all $i \le \kappa$

(2) if $\zeta = \sup(S \cap \xi) < \xi$ and $\xi$ is a limit of countable cofinality, then $i_\xi = 0$ and $\dot{\mathfrak{D}}^{\gamma+\xi}_i = \langle \mathcal{A} \rangle$ for all $i \le \kappa$, and $\dot{D}_\xi$ is forced by $\underline{\mathbf{P}}^{\gamma+\xi+1}$ to be the generic real added by $\mathbb{L}(\{\dot{D}_\eta \cap \dot{E}_\xi : \zeta \le \eta < \xi\})$,

(3) if $\zeta = \sup(S \cap \xi)$ and $\xi = \zeta + \omega_1$, then $i_\xi = i_\gamma$, $\dot{\mathfrak{D}}^{\gamma+\xi}_{i_\xi}$ is the filter generated by $\{\dot{E}_\xi \cap \dot{D}_\eta : \zeta \le \eta < \xi\}$ and $\dot{D}_\xi$ is $\dot{L}_{\gamma+\xi}$,

(4) if $S \cap \xi$ is cofinal in $\xi$ and $\mathrm{cf}(\xi) > \omega$, then $i_\xi = i_\gamma$ and $\dot{\mathfrak{D}}^{\gamma+\xi}_{i_\xi}$ is the filter generated by $\{\dot{D}_{\gamma+\eta} : \eta \in S \cap \xi\}$ and $\dot{D}_\xi = \dot{L}_{\gamma+\xi}$,

(5) if $S \cap \xi$ is cofinal in $\xi$ and $\mathrm{cf}(\xi) = \omega$, then $i_\xi = 0$ and $\dot{\mathfrak{D}}^{\gamma+\xi}_i = \langle \mathcal{A} \rangle$ for all $i \le \kappa$, and $\dot{D}_\xi$ is forced by $\underline{\mathbf{P}}^{\gamma+\xi+1}$ to be the generic real added by $\mathbb{L}(\{\dot{D}_{\eta_n} \cap \dot{E}_\xi : n \in \omega\})$, where $\{\eta_n : n \in \omega\}$ is some increasing cofinal subset of $S \cap (\gamma, \xi)$.

It should be clear that the induction continues to stage $\mu_{i_\gamma}$ and that $\underline{\mathbf{P}}^{\gamma+\xi} \in \mathcal{H}(\vec{\lambda}(i_\gamma))$ for all $\xi \le \mu_{i_\gamma}$, with $\beta_{\gamma_\xi} = \gamma$ being the witness to Definition 4.10 for all $\xi$ with $\mathrm{cf}(\xi) > \omega$.

The final definition of the sequence $\langle \dot{\mathfrak{D}}^\delta_i : i_\delta = i_\gamma \le i \le \kappa \rangle$, where $\delta = \gamma + \mu_{i_\gamma}$ is that $\dot{\mathfrak{D}}^\delta_{i_\gamma}$ is the filter generated by $\{\dot{L}_{\gamma+\xi} : \mathrm{cf}(\xi) > \omega\}$, and for $i_\gamma < i \le \kappa$, $\dot{\mathfrak{D}}^\delta_i$ is the filter generated by $\dot{\mathfrak{D}}^\delta_{i_\gamma} \cup (\mathcal{E} \cap \mathbb{B}_{i,\gamma})$. $\square$

**Lemma 5.4.** *Suppose that $\mathcal{E}$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter base. Also assume that $i < \kappa$ and $\alpha \le \gamma$ and $\mathcal{E}_1 \subset \mathbb{B}_{i,\alpha}$ is a $(\underline{\mathbf{P}}^\alpha, \vec{\lambda}(i_\gamma))$-thin filter base satisfying that $\langle \mathcal{E} \rangle \cap \mathbb{B}_{i,\alpha} \subset \langle \mathcal{E}_1 \rangle$, then $\mathcal{E} \cup \mathcal{E}_1$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter subbase.*

**Proof.** Let $\mathcal{E}_2$ be equal to $\mathcal{E} \cup \mathcal{E}_1$. The fact that each member of the sequence $\langle \dot{\mathcal{D}}_j = \langle \mathcal{E}_2 \cap \mathbb{B}_{j,\gamma} \rangle : j < \kappa \rangle$ is a name of a filter base with character at most $\mu_{i_\gamma}$ is immediate. Now we verify that if $j_1 < j_2 < \kappa$, then $\Vdash_{\mathbb{P}_{j_2,\gamma}} \dot{\mathcal{D}}_{j_2} \cap \mathbb{B}_{j_1,\gamma} \subset \dot{\mathcal{D}}_{j_1}$. Let $\dot{b} \in \mathbb{B}_{j_2,\gamma}$ and suppose there are $p \in \mathbb{P}_{j_2,\gamma}$, $\dot{E}_0 \in \mathcal{E} \cap \mathbb{B}_{j_2,\gamma}$, and $\dot{E}_1 \in \mathcal{E}_1$ such that $p \Vdash b \cap \dot{E}_0 \cap \dot{E}_1$. It suffices to produce an $\dot{E} \in \langle \mathcal{E}_2 \rangle \cap \mathbb{B}_{j_1,\gamma}$ satisfying that $p \Vdash \dot{b} \cap \dot{E} = \emptyset$. First, using that $\mathcal{E}$ is $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin, choose $\dot{E}_2 \in \langle \mathcal{E} \rangle \cap \mathbb{B}_{j_1,\gamma}$ such that $p \Vdash (\dot{b} \setminus \dot{E}_0) \cap \dot{E}_2 = \emptyset$. Equivalently, we have that $p \Vdash (\dot{b} \cap \dot{E}_2) \subset \dot{E}_0$, and therefore $p \Vdash (\dot{b} \cap \dot{E}_2) \cap \dot{E}_1 = \emptyset$. Since $\dot{E}_1$ is a $\mathbb{P}_{j_2,\alpha}$-name, there is a $\mathbb{P}_{j_1,\alpha}$-name (which we can denote as) $(\dot{b} \cap \dot{E}_2) \restriction \alpha$ satisfying that $p \Vdash \dot{E}_2 \cap (\dot{b} \cap \dot{E}_2) \restriction \alpha$ is empty and that $p \Vdash (\dot{b} \cap \dot{E}_2) \subset (\dot{b} \cap \dot{E}_2) \restriction \alpha$. Now using that $\mathcal{E}_1$ is $(\underline{\mathbf{P}}^\alpha, \vec{\lambda}(i_\gamma))$-thin, choose $\dot{E}_3 \in \langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{j_1,\alpha}$ so that $p \Vdash \dot{E}_3 \cap (\dot{b} \cap \dot{E}_2) \restriction \alpha$ is empty. Naturally we have that $p \Vdash \dot{E}_3 \cap (\dot{b} \cap \dot{E}_2)$ is also empty. This completes the proof since $\dot{E}_2 \cap \dot{E}_3$ is in $\langle \mathcal{E}_2 \rangle \cap \mathbb{B}_{j_1,\gamma}$. $\square$

Let $\underline{\mathbf{P}}^\gamma \in \mathcal{H}(\vec{\lambda})$ and let $\dot{y} \in \mathbb{B}_{\kappa,\gamma}$. For a family $\mathcal{E} \subset \mathbb{B}_{\kappa,\gamma}$ and condition $p \in \underline{\mathbf{P}}^\gamma$ say that $p$ forces that $\mathcal{E}$ measures $\dot{y}$ if $p \Vdash_{\underline{\mathbf{P}}^\gamma} \{\dot{y}, \omega \setminus \dot{y}\} \cap \langle \mathcal{E} \rangle \ne \emptyset$. Naturally we will just say that $\mathcal{E}$ measures $\dot{y}$ if 1 forces that $\mathcal{E}$ measures $\dot{y}$.

Given Lemma 5.3, it will now suffice to prove:

**Lemma 5.5.** *If $\mathcal{Y} \subset \mathbb{B}_{\kappa,\gamma}$ for some $\underline{\mathbf{P}}^\gamma \in \mathcal{H}(\vec{\lambda})$ and $|\mathcal{Y}| \le \mu_{i_\gamma}$ for some $i_\gamma < \kappa$, then there is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter $\mathcal{E} \subset \mathbb{B}_{\kappa,\gamma}$ that measures every element of $\mathcal{Y}$.*

Sh:1127



In fact, to prove Lemma 5.5, it is evidently sufficient to prove:

**Lemma 5.6.** *If $\underline{\mathbf{P}}^\gamma \in \mathcal{H}(\vec{\lambda})$, $\dot{y} \in \mathbb{B}_{\kappa,\gamma}$, and if $\mathcal{E}$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter, then there is a family $\mathcal{E}_1 \supset \mathcal{E}$ measuring $\dot{y}$ that is also a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter.*

**Proof.** Throughout the proof we suppress mention of $\underline{\mathbf{P}}^\gamma$ and refer instead to component member posets $\mathbb{P}_{i,\alpha}$, $\dot{\mathbb{Q}}_{i,\alpha}$ of $\underline{\mathbf{P}}^\gamma$. Let $i_{\dot{y}}$ be minimal such that $\dot{y}$ is in $\mathbb{B}_{i_{\dot{y}},\gamma}$. Proceeding by induction, we can assume that the lemma holds for all $\dot{x} \in \mathbb{B}_{j,\gamma}$ and all $j < i_{\dot{y}}$.

We can replace $\dot{y}$ by any $\dot{x} \in \mathbb{B}_{i_{\dot{y}},\gamma}$ that has the property that $1 \Vdash \dot{x} \in \{\dot{y}, \omega \setminus \dot{y}\}$ since if we measure $\dot{x}$ then we also measure $\dot{y}$. With this reduction then we can assume that no condition forces that $\omega \setminus \dot{y}$ is in the filter generated by $\mathcal{E}$.

**Fact 1.** If $i_{\dot{y}} \le i_\gamma$, then there is a $\dot{E} \in \mathbb{B}_{i_{\dot{y}},\gamma}$ such that $\mathcal{E} \cup \{\dot{E}\}$ is contained a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter that measures $\dot{y}$.

**Proof of Fact 1.** It is immediate that $\langle \{\dot{y}\} \cup (\mathbb{B}_{i_{\dot{y}},\gamma} \cap \mathcal{E}) \rangle$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter. Therefore, by Lemma 5.4, $\mathcal{E} \cup \{\dot{y}\}$ is a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter subbase. □

We may thus assume that $0 < i_{\dot{y}}$ and that the Lemma has been proven for all members of $\mathbb{B}_{i,\gamma}$ for all $i < i_{\dot{y}}$. Similarly, let $\alpha_{\dot{y}}$ be minimal so that $\dot{y} \in \mathbb{B}_{i_{\dot{y}},\alpha_{\dot{y}}}$, and assume that the Lemma has been proven for all members of $\mathbb{B}_{i_{\dot{y}},\beta}$ for all $\beta < \alpha_{\dot{y}}$. We skip proving the easy case when $\alpha_{\dot{y}} = 1$ and henceforth assume that $1 < \alpha_{\dot{y}}$. Notice also that $\alpha_{\dot{y}}$ has countable cofinality since $\mathbb{P}_{i_{\dot{y}},\gamma}$ is ccc.

Now choose an elementary submodel $M$ of $H((2^{\lambda\cdot\gamma})^+)$ containing $\vec{\lambda}$, $\underline{\mathbf{P}}^\gamma$, $\mathcal{E}$, $\dot{y}$ and so that $M$ has cardinality equal to $\mu_{i_\gamma}$ and, by our cardinal assumptions, $M^{\lambda_j} \subset M$ for all $j < i_\gamma$. Naturally this implies that $M^\omega \subset M$.

By the inductive assumption we may assume that there is an $\mathcal{E}_1 \supset \mathcal{E}$ that is $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin and measures every element of $M \cap \mathbb{B}_{j,\gamma}$ for $j < i_{\dot{y}}$ as well as every element of $M \cap \mathbb{B}_{i_{\dot{y}},\beta}$ for all $\beta \in M \cap \alpha_{\dot{y}}$. Moreover, it is easily checked that we can assume that $\mathcal{E}_1$ is a subset of $M$. Furthermore, we may assume that $\mathcal{E}_1$ contains a maximal family of subsets of $M \cap \mathbb{B}_{i_{\dot{y}},\alpha_{\dot{y}}}$ that forms a $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin filter subbase.

**Fact 2.** There is a maximal antichain $A \subset \mathbb{P}_{i_{\dot{y}},\gamma}$ and a subset $A_1 \subset A$ such that

(1) each $p \in A_1$ forces that $\mathcal{E}_1$ measures $\dot{y}$,
(2) for each $p \in A \setminus A_1$, $p$ forces that there is an $i_p < i_{\dot{y}}$ such that $\mathbb{B}_{i_p,\gamma} \cap \langle \mathcal{E}_1 \cup \{\dot{y}\} \rangle$ is not generated by the elements in $M$,
(3) for each $p \in A \setminus A_1$, $p$ forces that there is a $j_p < i_{\dot{y}}$ such that $i_p \le j_p$ and $\mathbb{B}_{j_p,\gamma} \cap \langle \mathcal{E}_1 \cup \{\omega \setminus \dot{y}\} \rangle$ is not generated by the elements in $M$.

**Proof of Fact 2.** Suppose that $p \in \mathbb{P}_{i_{\dot{y}},\gamma}$ forces that the conclusion (2) fails. We have already arranged that $p \Vdash_{\mathbb{P}_{i_{\dot{y}},\gamma}} \dot{y} \in \langle \mathcal{E}_1 \cap \mathbb{B}_{i_{\dot{y}},\gamma} \rangle^+$. Define $\dot{E} \in \mathbb{B}_{i_{\dot{y}},\gamma}$ so that $p$ forces $\dot{E} = \dot{y}$ and each $q \in \mathbb{P}_{i_{\dot{y}},\gamma} \cap p^\perp$ forces that $\dot{E} = \omega$. It is easily checked that $\mathbb{B}_{i_{\dot{y}},\gamma} \cap \langle \mathcal{E}_1 \cup \{\dot{E}\} \rangle$ is then $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin and that $p$ forces that it measures $\dot{y}$. This condition ensures that $p$ is compatible with an element of $A_1$.

If (2) holds but (3) fails, then by a symmetric argument as in the previous paragraph we can again define $\dot{E}$ so that $\mathbb{B}_{i_{\dot{y}},\gamma} \cap \langle \mathcal{E}_1 \cup \{\dot{E}\} \rangle$ is then $(\underline{\mathbf{P}}^\gamma, \vec{\lambda}(i_\gamma))$-thin and that $p$ forces that it measures $\omega \setminus \dot{y}$. □

If by increasing $M$ we can enlarge $A_1$ we simply do so. Since $\underline{\mathbf{P}}^\gamma$ is ccc we may assume that this is no longer possible, and therefore we may also assume that $A$ is a subset of $M$. Now we choose any $p \in A \setminus A_1$. It suffices to produce an $\dot{E}_p \in \mathbb{B}_{i_{\dot{y}},\gamma}$ that can be added to $\mathcal{E}_1$ that measures $\dot{y}$ and satisfies that $q \Vdash \dot{E}_p =_\omega \omega$ for all $q \in p^\perp$. This is because we then have that $\mathcal{E}_1 \cup \{\dot{E}_p : p \in A \setminus A_1\}$ is contained in a $\vec{\lambda}(i_\gamma)$-thin filter that measures $\dot{y}$.

**Fact 3.** There is an $\alpha$ such that $\alpha_{\dot{y}} = \alpha + 1$.

**Proof of Fact 3.** Otherwise, let $j = i_p$ and for each $r < p$ in $\mathbb{P}_{i_{\dot{y}},\alpha_{\dot{y}}}$, choose $\beta \in M \cap \alpha_{\dot{y}}$ such that $r \in \mathbb{P}_{i_{\dot{y}},\beta}$, and define a name $\dot{y}[r]$ in $M \cap \mathbb{B}_{j,\gamma}$ according to $(\ell, q) \in \dot{y}[r]$ providing there is a pair $(\ell, p_\ell) \in \dot{y}$ such that $q <_j p_\ell$ and $q \restriction \beta$ is in the set $M \cap \mathbb{P}_{j,\beta} \setminus (r \wedge p_\ell \restriction \beta)^\perp$ . This set, namely $\dot{y}[r]$, is in $M$ because $\mathbb{P}_{j,\beta}$ is ccc and $M^\omega \subset M$.

We prove that $r$ forces that $\dot{y}[r]$ contains $\dot{y}$. Suppose that $r_1 < r$ and there is a pair $(\ell, p_\ell) \in \dot{y}$ with $r_1 < p_\ell$. Choose an $r_2 \in \mathbb{P}_{j,\gamma}$ so that $r_2 <_j r_1$. It suffices to show $r_2 \Vdash \ell \in \dot{y}[r]$. Let $q <_j p_\ell$ with $q \in M$. Then $r_2 \not\perp p_\ell$ implies $r_2 \not\perp q$. Since $r_2$ was any $<_j$-projection of $r_1$ we can assume that $r_2 < q$. Since $r_2 \restriction \beta$ is in $(\mathbb{P}_{j,\beta} \cap (r \wedge p_\ell \restriction \beta)^\perp)^\perp$, it follows that $q \restriction \beta \notin (r \wedge p_\ell \restriction \beta)^\perp$. This implies that $(\ell, q) \in \dot{y}[r]$ and completes the proof that $r_2 \Vdash \ell \in \dot{y}[r]$.

Now assume that $\beta < \alpha_{\dot{y}}$ and $r \Vdash \dot{b} \cap \dot{E} \cap \dot{y}$ is empty for some $r < p$ in $\mathbb{P}_{i_{\dot{y}},\beta}$, $\dot{b} \in \mathbb{B}_{j,\gamma}$, and $\dot{E} \in \mathcal{E}_1 \cap \mathbb{B}_{i_{\dot{y}},\gamma}$. Let $\dot{x} = (\dot{E} \cap \dot{y})[r]$ (defined as above for $\dot{y}[r]$). We complete the proof of Fact 3 by proving that $r \Vdash \dot{b} \cap \dot{x}$ is empty. Since each are in $\mathbb{B}_{j,\gamma}$, we may choose any $r_1 <_j r$, and assume that $r_1 \Vdash \ell \in \dot{b} \cap \dot{x}$. In addition we can suppose that there is a pair $(\ell, q) \in \dot{x}$ such that $r_1 < q$. The fact that $(\ell, q) \in \dot{x}$ means there is a $p_\ell$ with $(\ell, p_\ell)$ in the name $\dot{E} \cap \dot{y}$ such that $q <_j p_\ell$. Since $r_1 \in \mathbb{P}_{j,\gamma}$ and $r_1 < q$, it follows that $r_1 \not\perp p_\ell$. Now it follows that $r_1$ has an extension forcing that $\ell \in \dot{b} \cap (\dot{E} \cap \dot{y})$ which is a contradiction. □

**Fact 4.** $i_{\dot{y}} = i_\alpha$ and so also $i_p < i_\alpha$.

**Proof of Fact 4.** Since $\mathbb{P}_{i,\alpha+1} = \mathbb{P}_{i,\alpha}$ for $i < i_\alpha$, we have that $i_\alpha \leq i_{\dot{y}}$. Now assume that $i_\alpha < i_{\dot{y}}$ and we proceed much as we did in Fact 3 to prove that $i_p$ does not exist. Assume that $r < p$ in $\mathbb{P}_{i_{\dot{y}},\alpha+1}$ and $r \Vdash \dot{b} \cap (\dot{E} \cap \dot{y})$ is empty for some $\dot{E} \in M \cap \langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{i_{\dot{y}},\gamma}$ and $\dot{b} \in \mathbb{B}_{i_p,\gamma}$. It follows from Lemma 5.4 that we can simply assume that $\dot{E} \in \mathcal{E}_1 \cap \mathbb{B}_{i_{\dot{y}},\alpha+1}$, and similarly that $\dot{b} \in \mathbb{B}_{i_p,\alpha+1}$.

Let $\dot{T}_\alpha$ be the $\mathbb{P}_{i_{\dot{y}},\alpha}$-name such that $r \restriction \alpha \Vdash r(\alpha) = \dot{T}_\alpha \in \mathbb{L}(\mathfrak{D}^\alpha_{i_{\dot{y}}})$. We may assume that there is a $t_\alpha \in \omega^{<\omega}$ such that $r \restriction \alpha \Vdash t_\alpha = \mathrm{stem}(\dot{T}_\alpha)$.

Choose any $M \cap \mathbb{P}_{i_\alpha,\alpha}$-generic filter $\bar{G}$ such that $r \restriction \alpha \in \bar{G}^+$. Since $\mathbb{P}_{i_\alpha,\alpha}$ is ccc and $M^\omega \subset M$, it follows that $M[\bar{G}]$ is closed under $\omega$-sequences in the model $V[\bar{G}]$.

In this model, define an $\mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$-name $\dot{x}$. A pair $(\ell, T_\ell) \in \dot{x}$ if $t_\alpha \leq \mathrm{stem}(T_\ell) \in T_\ell \in \mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$ and for each $\mathrm{stem}(T_\ell) \leq t \in T_\ell$, there is a pair $(\ell, q_{\ell,t}) \in M$ in the name $(\dot{y} \cap \dot{E})$ such that $q_{\ell,t} \restriction \alpha \in \bar{G}^+$, $q_{\ell,t} \restriction \alpha \Vdash t = \mathrm{stem}(q_{\ell,t}(\alpha))$, and $(q_{\ell,t} \restriction \alpha \wedge r \restriction \alpha)$ does not force (over the poset $\bar{G}^+$ ) that $t \notin \dot{T}_\alpha$. We will show that $r$ forces over the poset $\bar{G}^+$ that $\dot{x}$ contains $\dot{E} \cap \dot{y}$ and that $\dot{x} \cap \dot{b}$ is empty. This proves that $p$ forces that $\langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{i_p,\alpha+1}$ generates $\langle \mathcal{E}_1 \cup \{\dot{y}\} \rangle \cap \mathbb{B}_{i_p,\alpha+1}$ since $\dot{x}$ must be forced to be in $\langle \mathcal{E}_1 \rangle$. It then follows from Lemma 5.4 that $\mathcal{E}_1 \cap \mathbb{B}_{i_p,\gamma}$ generates $\langle \mathcal{E}_1 \cup \{\dot{y}\} \rangle \cap \mathbb{B}_{i_p,\gamma}$, contradicting the assumption on $i_p$.

To prove that $r$ forces that $\dot{x}$ contains $\dot{y} \cap \dot{E}$, we consider any $r_\ell < r$ that forces over $\bar{G}^+$ that $\ell \in \dot{y} \cap \dot{E}$. We may choose $(\ell, p_\ell) \in M$ in the name $(\dot{E} \cap \dot{y})$ such that (wlog) $r_\ell < p_\ell$. We may assume that $r_\ell \restriction \alpha$ forces a value $t$ on $\mathrm{stem}(r_\ell(\alpha))$ and that this equals $\mathrm{stem}(p_\ell(\alpha))$. Now show there is a $T_\ell \in \mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$. In fact, assume $t \in T_\ell$ with $q_{\ell,t}$ as the witness. Let $L^- = \{k : t^\frown k \notin T_\ell\}$; it suffices to show that $L^- \notin (\mathfrak{D}^\alpha_{i_\alpha})^+$.

By assumption that $q_{t,\ell}$ is the witness, there is an $r_t < (q_{\ell,t} \restriction \alpha \wedge r \restriction \alpha)$ such that $r_t \Vdash t \in \dot{T}_\alpha$ and $r_t \Vdash t = \mathrm{stem}(q_{\ell,t}(\alpha))$. By strengthening $r_t$ we can assume that $r_t$ forces a value $\dot{D} \in \dot{\mathcal{D}}^\alpha_{i_{\dot y}}$ on $\{k : t^\frown k \in \dot{T}_\alpha \cap q_{\ell,t}(\alpha)\}$. But now, it follows that $r_t$ forces that $\dot{D}$ is disjoint from $L^-$ since if $r_{t,k} \Vdash k \in \dot{D}$ for some $r_{t,k} < r_t$, $r_{t,k}$ is the witness to $(\ell, q_{\ell,t^\frown k})$ is in $(\dot y \cap \dot E)$ etc., where $q_{\ell,t^\frown k} \restriction \alpha = q_{\ell,t} \restriction \alpha$ and $q_{\ell,t^\frown k}(\alpha) = (q_{\ell,t}(\alpha))_{t^\frown k}$. Since some condition forces that $L^-$ is not in $(\dot{\mathfrak{D}}^\alpha_{i_{\dot y}})^+$ it follows that $L^-$ is not in $(\dot{\mathfrak{D}}^\alpha_{i_\alpha})^+$

Finally we must show that $r$ forces over $\bar{G}^+$ that $\dot b$ is disjoint from $\dot x$. Since each are $\mathbb{P}_{i_p,\alpha+1}$-names, it suffices to assume that $\bar r \in \bar G^+$ is some $\mathbb{P}_{i_p,\alpha+1}$-reduct of $r$ that forces some $\ell$ is in $\dot b \cap \dot x$, and to then show that $r$ fails to force that $\ell \notin \dot b \cap (\dot E \cap \dot y)$. Choose $(\ell, q_{\ell,t}) \in (\dot y \cap \dot E)$ witnessing that $\bar r \Vdash \ell \in \dot x$. That is, we may assume that $\bar r \restriction \alpha \Vdash t = \mathrm{stem}(\bar r(\alpha))$, that $q_{\ell,t} \restriction \alpha \in \bar G^+$, and $(q_{\ell,t} \wedge r \restriction \alpha)$ does not force over $\bar G^+$ that $t \notin \dot T_\alpha$. Of course this means that the condition $\bar r \wedge r \wedge [[t \in \dot T_\alpha]] \wedge q_{\ell,t}$ is not 0. This condition forces that $\ell$ is in $\dot b \cap (\dot E \cap \dot y)$ as required. □

**Fact 5.** The character of $\mathfrak{D}^\alpha_{i_\alpha}$ is greater than $\mu_{i_\gamma}$.

**Proof of Fact 5.** We know that $\mathfrak{D}^\alpha_{i_\alpha}$ is forced to have an $\omega$-closed base (in fact, descending mod finite with uncountable cofinality). Even more, $\mathbb{P}_{i_\alpha,\alpha}$ forces that for all $T \in \mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$, there is a $D \in \mathfrak{D}^\alpha_{i_\alpha}$ such that the condition $([D]^{<\omega})_{\mathrm{stem}(T)}$ is below $T$. Let $\chi_\alpha$ be the cofinality of $\alpha$ and fix a list $\{\dot D_\beta : \beta < \chi_\alpha\} \in M$ (closed under mod finite changes) of $\mathbb{P}_{i_\alpha,\alpha}$-names of elements of $\dot{\mathfrak{D}}^\alpha_{i_\alpha}$ that is forced to be a base.

Now, suppose that $\dot b \in \mathbb{B}_{i_p,\alpha+1} = \mathbb{B}_{i_p,\alpha}$ and there is an $\dot E \in \mathcal{E}_1$ and an $r < p$ forcing that $\dot b \cap (\dot E \cap \dot y)$ is empty. We prove there is an $\dot x \in \mathcal{E}_1$ and an $r_2 < r \restriction \alpha$ in $\mathbb{P}_{i_\alpha,\alpha}$ such that $r_2 \Vdash \dot b \cap \dot x$ is empty. We may assume that $r_2$ forces a value $t$ on $\mathrm{stem}(r(\alpha))$ and that, for some $\beta < \chi_\alpha$, $r_2 \Vdash (\dot D_\beta^{<\omega})_t < r(\alpha)$. Let

$$\dot x = \{(\ell, q_\ell \restriction \alpha) : (\ell, q_\ell) \in (\dot E \cap \dot y) \text{ and } q_\ell \restriction \alpha \Vdash q_\ell(\alpha) \leq (\dot D_\beta^{<\omega})_t\}.$$

It is immediate that $\dot x \in M$ and that $(r_2 \wedge r) \Vdash_{\mathbb{P}_{i_\alpha,\alpha+1}} \dot x \supseteq (\dot E \cap \dot y)$. Since $\dot E \cap \dot y$ is forced to be in $\mathcal{E}_1^+$, it follows that $\dot x$ is forced by $r_2$ to be in $\langle \mathcal{E}_1 \rangle$. Now we verify that $r_2 \Vdash \dot b \cap \dot x$ is empty. Assume that $r_3 < r_2$ in $\mathbb{P}_{i_\alpha,\alpha}$ and that $r_3 \Vdash \ell \in \dot b \cap \dot x$. We may assume there is $(\ell, q_\ell \restriction \alpha) \in \dot x$ such that $r_3 < q_\ell \restriction \alpha$. But now $r_2 \Vdash q_\ell(\alpha) \leq r(\alpha)$ and so $r_2 \wedge r \Vdash \ell \in \dot b \cap (\dot E \cap \dot y)$— a contradiction.

The conclusion now follows from Lemma 5.4. □

**Definition 5.7.** For each $t \in \omega^{<\omega}$, define that $\mathbb{P}_{i_\alpha,\alpha}$-name $\dot E_t$ according to the rule that $r \Vdash \ell \in \dot E_t$ providing $r \in \mathbb{P}_{i_\alpha,\alpha}$ forces that there is a $\dot T$ with $r \Vdash \dot T \in \mathbb{L}(\dot{\mathfrak{D}}^\alpha_{i_\alpha})$, $r \Vdash t = \mathrm{stem}(\dot T)$, and $r \cup \{(\alpha, \dot T)\} \Vdash \ell \notin \dot y$.

**Fact 6.** There is a $\dot T \in \mathbb{L}(\dot{\mathfrak{D}}^\alpha_{i_\alpha}) \cap M$ such that $p \restriction \alpha$ forces the statement: $\dot E_t \in \mathcal{E}_1$ *for all* $t$ *such that* $\mathrm{stem}(\dot T) \leq t \in \dot T$.

**Proof of Fact 6.** By elementarity, there is a maximal antichain of $\mathbb{P}_{i_\alpha,\alpha}$ each element of which decides if there is a $\dot T$ with $\dot E_t \in \mathcal{E}_1$ for all $t \in \dot T$ above $\mathrm{stem}(\dot T)$. Since $p \in A \setminus A_1$ it follows that there is an $i_p < i_\alpha$ as in condition (2) of Fact 2. Let $t_0 \in \omega^{<\omega}$ so that $p \restriction \alpha \Vdash t_0 = \mathrm{stem}(p(\alpha))$. By the maximum principle, there is a $\dot b \in \mathbb{B}_{i_p,\gamma}$ and a $\dot E_0 \in \mathcal{E}_1$ satisfying that $p \Vdash \dot b \cap \dot E_0 \cap \dot y$ is empty, while $p \Vdash \dot b \cap \dot E$ is infinite for all $\dot E \in \langle \mathcal{E}_1 \rangle$. This means that $p$ forces that $\dot b \cap \dot E_0$ is an element of $\langle \mathcal{E}_1 \rangle^+$ that is contained in $\omega \setminus \dot y$. As in the proof of Lemma 5.4, there is an $\dot E_2 \in \langle \mathcal{E}_1 \rangle \cap \mathbb{B}_{i_p,\gamma}$ such that $p$ forces that $\dot b \cap \dot E_2$ is contained in $\dot E_0$. We also have that $(\dot b \cap \dot E_2) \restriction \alpha$ is forced to be contained in $\omega \setminus \dot{y}$. It now follows that $p \restriction \alpha$ forces that for all $t_0 \le t \in p(\alpha)$, $p \restriction \alpha$ forces that $\dot{E}_t$ contains $(\dot{b} \cap \dot{E}_2) \restriction \alpha$ and so is in $\langle \mathcal{E}_1 \rangle^+$. Since $\dot{E}_t$ is also measured by $\mathcal{E}_1$, we have that $p \restriction \alpha$ forces that such $\dot{E}_t$ are in $\mathcal{E}_1$. This completes the proof. $\square$

Now we show how to extend $\mathcal{E}_1 \cap \mathbb{B}_{i_\alpha,\gamma}$ so as to measure $\dot{y}$. Let $\beta = \sup(M \cap \alpha)$. By Fact 5, $\beta < \alpha$ and by the definition of $\mathcal{H}(\vec{\lambda})$, $M \cap \dot{\mathfrak{D}}^\alpha_{i_\alpha}$ is a subset of $\langle \dot{\mathfrak{D}}^\beta_{i_\beta} \rangle$, $\dot{L}_\beta \in \dot{\mathfrak{D}}^\alpha_{i_\alpha}$, and $i_\beta = i_\alpha$. We also have that the family $\{\dot{L}_\xi : \mathrm{cf}(\xi) \ge \omega_1 \text{ and } \beta_\alpha \le \xi \in M \cap \beta\}$ is a base for $\dot{\mathfrak{D}}^\beta_{i_\beta}$. For convenience let $q <_M p$ denote the relation that $q$ is an $M \cap \mathbb{P}_{i_\alpha,\alpha+1}$-reduct of $p$. Let $\bar{p}$ be any condition in $\mathbb{P}_{i_\beta,\beta+1}$ satisfying that $\bar{p} \restriction \beta = p \restriction \alpha$ and $\bar{p} \restriction \beta \Vdash \mathrm{stem}(\bar{p}(\beta)) = t_\alpha$; recall that $p \restriction \alpha \Vdash t_\alpha = \mathrm{stem}(p(\alpha))$.

Let us note that for each $q \in M \cap \mathbb{P}_{\alpha,i_\alpha+1}$, $q \restriction \alpha = q \restriction \beta$ and $q \restriction \beta \Vdash q(\alpha)$ is also a $\mathbb{P}_{\beta,i_\beta}$-name of an element of $\mathbb{L}(\dot{\mathfrak{D}}^\beta_{i_\beta})$. Let $\dot{x}$ be the following $\mathbb{P}_{i_\beta,\beta+1}$-name

$$\dot{x} = \{(\ell, q \restriction \beta \cup \{(\beta, q(\beta))\}) : (\ell, q) \in \dot{y} \cap M \text{ and } q <_M p\}.$$

We will complete the proof by showing that there is an extension of $p$ that forces that $\mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\}$ measures $\dot{y}$ and that 1 forces that $\langle \mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\} \rangle \cap \mathbb{B}_{i_{\dot{y}},\beta+1}$ is $\vec{\lambda}(i_\gamma)$-thin. Here $\dot{x}[\dot{L}_\beta]$ abbreviates the $\mathbb{P}_{i_\beta,\beta+1}$-name

$$\{(\ell, r) : (\exists q)\ (\ell, q) \in \dot{x},\ q \restriction \beta = r \restriction \beta,\ \text{and } r \Vdash \mathrm{stem}(q(\beta)) \in \dot{L}^{<\omega}_\beta\}.$$

The way to think of $\dot{x}[\dot{L}_\beta]$ is that if $\bar{p}$ is in some $\mathbb{P}_{i_\alpha,\alpha}$-generic filter $G$, then $\dot{y}[G]$ is now an $\mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$-name, $L^{<\omega}_\beta = (\dot{L}_\beta[G])^{<\omega}$ is in $\mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$, and $(\dot{x}[\dot{L}_\beta])[G]$ is equal to $\{\ell : L^{<\omega}_\beta \not\Vdash \ell \notin \dot{y}\}$. We will use the properties of $\dot{x}$ to help show that $\mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\}$ is $\vec{\lambda}(i_\gamma)$-thin. This semantic description of $\dot{x}[\dot{L}_\beta]$ makes clear that $\bar{p} \cup \{(\alpha, (\dot{L}_\beta)^{<\omega})\} \in \mathbb{P}_{i_\alpha,\alpha+1}$ forces that $\dot{x}[\dot{L}_\beta]$ contains $\dot{y}$. This implies that $\mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\}$ measures $\dot{y}$.

Claim: It is forced by $\bar{p}$ that $\omega \setminus \dot{x}$ is not measured by $\mathcal{E}_1$.

Each element of $\mathcal{E}_1$ is in $M$ and simple elementarity will show that for any condition in $q$ in $M$ that forces $\dot{E} \cap (\omega \setminus \dot{y})$ is infinite, the corresponding $\bar{q} = q \restriction \alpha \cup \{(\beta, q(\alpha))\}$ will also force that $\dot{E} \cap (\omega \setminus \dot{x})$ is infinite.

It follows from Fact 5, with $\omega \setminus \dot{x}$ playing the role of $\dot{y}$, that $\mathcal{E}_1 \cup \{\omega \setminus \dot{x}\}$ is $\vec{\lambda}(i_\gamma)$-thin. Recall that $q \Vdash \dot{x} = \emptyset$ for all $q \perp \bar{p}$. Now to prove that $\mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\}$ is also $\vec{\lambda}(i_\gamma)$-thin, we prove that

$$\langle \mathcal{E}_1 \cup \{\omega \setminus \dot{x}\} \rangle \cap \mathbb{B}_{i,\alpha} = \langle \mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\} \rangle \cap \mathbb{B}_{i,\alpha}$$

for all $i < i_\alpha$. In fact, first we prove

$$\langle \mathcal{E}_1 \cup \{\omega \setminus \dot{x}\} \rangle \cap \mathbb{B}_{i,\beta} = \langle \mathcal{E}_1 \cup \{\omega \setminus (\dot{x}[\dot{L}_\beta])\} \rangle \cap \mathbb{B}_{i,\beta}$$

for all $i < i_\alpha$.

We begin with this main Claim.

**Claim 1.** *If $\dot{b} \in \mathbb{B}_{i,\beta}$ $(i < i_\beta)$ and there is an $\dot{E} \in \mathcal{E}_1 \cap \mathbb{B}_{i_\alpha,\beta}$ and a $\bar{p} \ge q \in \mathbb{P}_{i_\beta,\beta+1}$ such that $q \Vdash \dot{b} \cap (\dot{E} \setminus \dot{x}) = \emptyset$ then $q \restriction \beta \Vdash (\exists \dot{E} \in \mathcal{E}_1)\ \dot{b} \cap \dot{E} = \emptyset$.*

**Proof of Claim 1.** We may assume that $q \restriction \beta$ forces a value $t$ on $\mathrm{stem}(q(\beta))$. Recall that $q \restriction \beta$ forces the statement: there is a $\dot{D} \in M \cap \dot{\mathfrak{D}}^\alpha_{i_\alpha}$ such that $(\dot{D}^{<\omega})_t \le q(\beta)$. The definition of $\dot{x}$ ensures that $q \restriction \beta \cup \{(\alpha, (\dot{D}^{<\omega})_t)\} \Vdash \dot{b} \cap (\dot{E} \setminus \dot{y})$ is empty. There is a $\mathbb{P}_{i_\alpha,\alpha}$-name $\dot{E}_1 \in M$ such that $q \restriction \alpha \Vdash \dot{E}_1 = \{\ell :\ (\dot{D}^{<\omega})_t \not\Vdash \ell \notin (\dot{E} \setminus \dot{y})\}$. By assumption $q \restriction \alpha \Vdash \dot{E}_1 \in \langle \mathcal{E}_1 \rangle$. Since $\dot{b}$ is also a $\mathbb{P}_{i,\alpha}$-name, we have that $q \restriction \alpha \Vdash \dot{b} \cap \dot{E}_1 = \emptyset$. $\square$

Now assume that $\dot{b} \in \mathbb{B}_{i_\beta,\beta}$ and $q \Vdash \dot{b} \cap (\dot{E} \cap (\omega \setminus (\dot{x}[\dot{L}_\beta])))$ is empty for some $q < \bar{p}$ in $\mathbb{P}_{i_\beta,\beta+1}$. By Lemma 5.4 it suffices to assume that $\dot{E} \in \mathbb{B}_{i_\beta,\beta}$. To prove that $q$ forces that $\dot{b} \notin \langle \mathcal{E}_1 \rangle^+$, it suffices to prove that there is some $\dot{E}_1 \in \mathcal{E}_1$ such that $q \Vdash \dot{b} \cap (\dot{E}_1 \cap (\omega \setminus \dot{x}))$ is finite. We proceed by contradiction.

We may again assume that $q \restriction \beta$ forces that $q(\beta)$ is $(\dot{D}^{<\omega})_t$ for some $t \supset t_\alpha$ and some $\dot{D} \in \dot{\mathfrak{D}}^\alpha_{i_\alpha} \cap M$. Let $H$ be the range of $t$. Let, for the moment, $G$ be a $\mathbb{P}_{i_\alpha,\alpha}$-generic filter with $q \in G$. Now in $M[G]$ we have the value $L_\beta$ of $\dot{L}_\beta$ and $H \subset L_\beta$. We can also let $E$ denote the value of $\dot{E}[G]$. Recall that for each $s \in H^{<\omega}$, $E_s$ denotes the set of $\ell \in E$ such that there is some $T \in \mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha})$ with $s = \text{stem}(T)$ and $T \Vdash \ell \notin \dot{y}$. We have shown in Fact 6 that there is a $T \in \mathbb{L}(\mathfrak{D}^\alpha_{i_\alpha}) \cap M$ such that $E_s \in \mathcal{E}_1$ for all $s \in T$ above stem$(T)$. This means that there is an $\ell \in b \cap E$ such that $\ell \in E_s$ for each of the finitely many suitable $s$. For each $s$, choose $T_s \subset T$ witnessing $\ell \in E_s$. As before, and since there are only finitely many $s$ involved, we can assume that $\dot{T}_s = (\dot{D}^{<\omega})_s$ for some $H \subset \dot{D} \in \dot{\mathfrak{D}}^\alpha_{i_\alpha} \cap M$ and we then define an extension $q$ of $q$ so that $q'(\beta) = (\dot{D}^{<\omega})_{t_\alpha}$ ensures that $(\dot{L}_\beta^{<\omega})_s < \dot{T}_s$ for each $s$. Note that for such a condition $q'$ we have that $q' \cup \{(\alpha, (\dot{L}_\beta)^{<}\omega)\}$ forces that $\ell \notin \dot{y}$. But then it should be clear that $q'$ forces $\ell \notin \dot{x}[\dot{L}_\beta]$. This contradicts that $q$ forces $\ell \notin \dot{b} \cap (\dot{E} \cap (\omega \setminus (\dot{x}[\dot{L}_\beta])))$. □

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