Annals of Pure and Applied Logic 169 (2018) 835–849

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

On the class of flat stable theories ☆

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ARTICLE INFO

Article history:
Received 22 March 2018
Received in revised form 17 April 2018
Accepted 17 April 2018

MSC:
03C45
03C60
05D99
20F99

Keywords:
Stable theory
Strong
Weight
Regular types

ABSTRACT

A new notion of independence relation is given and associated to it, the class of flat theories, a subclass of strong stable theories including the superstable ones is introduced. More precisely, after introducing this independence relation, flat theories are defined as an appropriate version of superstability. It is shown that in a flat theory every type has finite weight and therefore flat theories are strong. Furthermore, it is shown that under reasonable conditions any type is non-orthogonal to a regular one. Concerning groups in flat theories, it is shown that type-definable groups behave like superstable ones, since they satisfy the same chain condition on definable subgroups and also admit a normal series of definable subgroup with semi-regular quotients.

1. Introduction

The notions of forking, orthogonality and regular types, among others, play a fundamental role in understanding the structure of stable theories. These were not only essential to carry out the classification programme, inside stable theories, but also have turned out to be relevant for the developments of geometric stability theory.

A stationary type is *regular* if it is orthogonal to all its forking extensions; recall that two stationary types p and q are *orthogonal* if, for any set C over which both types are based and any realizations $a \models p|C$ and $b \models q|C$, we have that $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$. Minimal types are the simplest example of regular types, where forking means being algebraic. Similar to minimal ones, regular types carry a notion of geometry associated to their

☆ This paper corresponds to 1133 in Shelah's publication list. Both authors were partially supported by the European Research Council grant 338821. The first author was also partially supported by the project MTM2014-59178-P.
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https://doi.org/10.1016/j.apal.2018.04.005
0168-0072/

set of realizations, and hence a dimension. Their main feature is that any type can be coordinatizated by regular ones, as long as the theory contains enough regular types. Consequently, their associated geometries determine many properties of the theory.

Formally, the fact that a theory has enough regular types can be rephrased as follows: Every type is non-orthogonal to a regular one. This holds for superstable theories but this property is not exclusive of superstability. Therefore, one may try to find reasonable conditions beyond superstability which yield the existence of enough regular types. In this paper we pursue this line of investigation on an attempt to find some reasonable structure theory beyond superstability.

We introduce the class of flat theories, a subclass of stable theories which extends superstability, and analyze the existence of regular types in this context. More precisely, in Section 2 we define the notion of ω-forking,[1] which implies the usual notion of forking, and show that in a stable theory it satisfies the usual properties of independence (see Theorem 2.12), except algebraicity since it can be the trivial relation. Afterwards, a flat theory is defined as a stable theory where every type does not ω-fork over a finite set. Since non-forking implies non-ω-forking, it follows immediately that a superstable theory is flat. As in the superstable case, a notion of ordinal-valued rank among types, called U_ω-rank, is available and we point out some of its basic properties, such as the Lascar inequalities.

In the third section, a more careful analysis of flat theories is carried out. Roughly speaking, we see that any type has a non-ω-forking extension which is non-orthogonal to a regular type. Consequently, every type is close to be non-orthogonal to a regular one, see Theorem 3.9. In particular, if all forking extensions of a type are also ω-forking, then it is non-orthogonal to a regular type. This is Corollary 3.10. Nevertheless, we cannot ensure that in general every type is non-orthogonal to a regular one, but we show that flat theories are strong (Theorem 3.20) and consequently every type is non-orthogonal to a type of weight one. In fact, this holds locally for a flat type under the mere assumption that the theory is stable.

Finally, in the last section groups in flat theories are analyzed. We show that any type-definable group in a flat theory looks like a superstable one, in the sense that they satisfy the same descending chain condition on definable subgroups and also admit a semi-regular decomposition. It should be noted that, while the notion of p-semi-regularity (also p-simplicity) originated in [7, Chapter V], here semi-regularity corresponds to a reformulation due to Hrushovski. Hence, in Theorem 4.5, by a semi-regular decomposition we mean that every such flat group admits a finite series of normal subgroups such that any generic type of each quotient is domination-equivalent to suitable finite product of some regular type.

2. A new independence relation

From now on, we work inside the monster model of a complete stable first-order theory, and we assume that the reader is familiarized with the general theory of stability theory.

2.1. Skew dividing and ω-forking

We introduce the notion of skew k-dividing and k-forking for a natural number $k \geq 1$.

Definition 2.1. Let $\pi(\bar{x})$ be a partial type. It is said to skew k-divide over A if there is an A-indiscernible sequence $(\bar{b}_n)_{n<\omega}$ and a formula $\varphi(\bar{x}; \bar{y}_0, \ldots, \bar{y}_{k-1})$ such that

$$\pi(\bar{x}) \vdash \varphi(\bar{x}; \bar{b}_0, \bar{b}_2, \ldots, \bar{b}_{2(k-1)}) \text{ and } \pi(\bar{x}) \vdash \neg\varphi(\bar{x}; \bar{b}_{i_0}, \ldots, \bar{b}_{i_{k-1}})$$

for any $i_0 < \ldots < i_{k-1} < 2k$ with $(i_0, i_1, \ldots, i_{k-1}) \neq (0, 2, \ldots, 2(k-1))$.

[1] Originally, called gorking by the second author.

In fact, in the definition of skew dividing we may allow formulas with parameters.

Remark 2.2. A partial type $\pi(\bar{x})$ skew k-divides over A if and only if there are a formula $\varphi(\bar{x};\bar{y}_0,\ldots,\bar{y}_{k-1},\bar{z})$, a tuple $\bar{c}$, and an $A\bar{c}$-indiscernible sequence $(\bar{b}_n)_{n<\omega}$ such that

$$\pi(\bar{x}) \vdash \varphi(\bar{x};\bar{b}_0,\bar{b}_2,\ldots,\bar{b}_{2(k-1)},\bar{c}) \text{ and } \pi(\bar{x}) \vdash \neg\varphi(\bar{x};\bar{b}_{i_0},\ldots,\bar{b}_{i_{k-1}},\bar{c})$$

for any $i_0 < \ldots < i_{k-1} < 2k$ with $(i_0,i_1,\ldots,i_{k-1}) \neq (0,2,\ldots,2(k-1))$.

Proof. Left to right is obvious by the definition of skew k-dividing. To prove the other direction, assume that the condition holds for $\varphi(\bar{x};\bar{y}_0,\ldots,\bar{y}_{k-1},\bar{z})$, a tuple $\bar{c}$, and a sequence $(\bar{b}_n)_{n<\omega}$. Set $\bar{y}'_i = \bar{y}_i\bar{z}$ and $\bar{b}'_n = \bar{b}_n\bar{c}$. Then the formula $\psi(\bar{x};\bar{y}'_0,\ldots,\bar{y}'_{k-1})$ defined as $\varphi(\bar{x};\bar{y}_0,\ldots,\bar{y}_{k-1},\bar{z})$ and the sequence $(\bar{b}'_n)_{n<\omega}$ witness that $\pi(\bar{x})$ skew k-divides over A. □

Definition 2.3. A partial type $\pi(\bar{x})$ is said to k-fork over A if it implies a finite disjunction of formulas, each of them skew k-dividing over A.

In other words, the set of formulas that k-fork over A is nothing else than the ideal generated by the formulas that skew k-divide over A. Furthermore, note that both notions are preserved under automorphisms of the ambient model.

Remark 2.4. The following holds:

(1) If $\pi_1(\bar{x}) \vdash \pi_2(\bar{x})$ and $\pi_2(\bar{x})$ skew k_2-divides over A_2, then $\pi_1(\bar{x})$ skew k_1-divides over A_1 for any $k_1 \leq k_2$ and $A_1 \subseteq A_2$. The same holds for k-forking.
(2) If $\pi(\bar{x})$ skew k-divides over A, then so does some finite subset $\pi_0(\bar{x})$ of $\pi(\bar{x})$. Similarly for k-forking.
(3) If a partial type $\pi(\bar{x})$ skew k-divides over A, then so does it over $\mathrm{acl}(A)$.
(4) Extension property. If a partial type $\pi(\bar{x})$ over B does not k-fork over A, then there is $p(\bar{x}) \in S(B)$ extending $\pi(\bar{x})$ which does not k-fork over A.

Proof. We only prove (1) for skew dividing, the rest is standard. Assume that $\varphi = \varphi(\bar{x};\bar{y}_0,\ldots,\bar{y}_{k_2-1})$ and $(\bar{b}_\alpha)_{\alpha<\omega}$ witness that $\pi_2(\bar{x})$ skew k_2-divides over A_2. Now, set $\bar{z} = \bar{y}_{k_1}\ldots\bar{y}_{k_2-1}$ and let $\psi(\bar{x};\bar{y}_0,\ldots,\bar{y}_{k_1-1},\bar{z}) = \varphi$. Then the result follows from Remark 2.2 by enlarging the sequence $(\bar{b}_\alpha)_{\alpha<\omega}$ to $(\bar{b}_\alpha)_{\alpha<\omega+\omega}$ and taking $\bar{c} = \bar{b}_\omega\ldots\bar{b}_{\omega+k_2-k_1+1}$. □

Lemma 2.5. *A partial type $\pi(\bar{x})$ does not fork over A if and only if it does not 1-fork over A.*

Proof. It is clear that a global type is Lascar invariant over A if and only if it does not 1-fork over A. Thus, the statement follows as non-forking and non-1-forking satisfy the extension property. □

Nevertheless, for $k > 1$ forking and k-forking does not agree in general.

Example 2.6. Consider the first-order theory of an infinite set and let $\phi(x;y)$ be the formula $x = y$. For any element a, we have that the partial type $\{\phi(x;a)\}$ forks over $\emptyset$, but it does not 2-fork.

Lemma 2.7. *If the type $\mathrm{tp}(\bar{a}/B)$ does not skew k-divide over A, then for any A-indiscernible sequence I contained in B, there is some $J \subseteq I$ with $|J| < k$ such that $I \setminus J$ is an indiscernible set over $AJ\bar{a}$.*

Proof. Inductively on $n \leq k$, we obtain a strictly increasing sequence of natural numbers $(k_n)_{n\leq k}$ with $k_0 = 0$ for which there is a subsequence $J_n = (\bar{b}_m)_{m\in(k_n,k_{n+1})}$ of I without repetitions and a formula $\phi_n(\bar{x}; \bar{y}_0, \ldots, \bar{y}_{k_{n+1}-1}, \bar{z})$ such that:

- $\bullet_1$ there is some finite tuple $\bar{c}_n$ in AI such that $\phi_n(\bar{a}; \bar{b}_0, \ldots, \bar{b}_{k_{n+1}-1}, \bar{c}_n)$ holds,
- $\bullet_2$ there is some finite subset I_n of I, containing I_{n-1}, such that $\neg\phi_n(\bar{a}; \bar{b}_0, \ldots, \bar{b}_{k_n-1}, \bar{y}'_{k_n}, \ldots, \bar{y}'_{k_{n+1}-1}, \bar{c}_n)$ also holds for any $\bar{b}'_{k_n}, \ldots, \bar{b}'_{k_{n+1}-1}$ in $I \setminus I_n$, and
- $\bullet_3$ k_{n+1} is minimal with these properties.

Let Δ_n be the closure of ϕ_n under permuting the variables $\bar{y}_0, \ldots, \bar{y}_{k_{n+1}-1}$, and let Δ be the union of all these Δ_n. As $\mathrm{tp}(\bar{a}/AI)$ does not k-fork over A, there is some $n_* \leq k$ for which we cannot keep doing the construction for n_*. If $|J_{<n_*}| < k$, then as the truth value of any formula over $A\bar{a}J_{<n_*}$ is constant in a cofinal segment of $I \setminus J_{<n_*}$, the choice of n_* yields that $I \setminus J_{<n_*}$ is indiscernible over $A\bar{a}J_{<n_*}$, as desired.

Assume now that $|J_{<n_*}| \geq k$. Thus, by construction we have that

$\otimes_1$ $k_{n_*} \geq k$ and so $n_* \geq 1$.

Now, take some $\bar{b}^*_0, \ldots, \bar{b}^*_{2k_{n_*}-1} \in I \setminus (I_{<n_*} \cup J_{<n_*})$ without repetitions and set $\bar{b}_* = \bar{b}^*_{k_{n_*}} \ldots \bar{b}^*_{2k_{n_*}-1}$ and $\bar{b}^1_i = \bar{b}_i$ and $\bar{b}^2_i = \bar{b}^*_i$ for $i < k_{n_*}$.

Put $\bar{z} = \bar{z}_0 \ldots \bar{z}_{n_*}$ and also $\bar{c}_* = \bar{c}_0 \ldots \bar{c}_{n_*}$. Then let $\psi(\bar{x}; \bar{y}_0, \ldots, \bar{y}_{k_{n_*+1}-1}, \bar{z}, \bar{b}_*)$ denote the conjunction of the finite partial type $\mathrm{tp}_\Delta(\bar{a}\bar{b}_0 \ldots \bar{b}_{k_{n_*}-1}\bar{c}_*/\bar{b}_*)$.

Notice that $\psi(\bar{a}; \bar{b}^1_0, \ldots, \bar{b}^1_{k_{n_*}-1}, \bar{c}_*, \bar{b}_*)$ holds by construction. Thus, the set Λ of functions $\eta : \{0, \ldots, k_{n_*} - 1\} \longrightarrow \{1, 2\}$ such that $\psi(\bar{a}; \bar{b}^{\eta(0)}_0, \ldots, \bar{b}^{\eta(k_{n_*}-1)}_{k_{n_*}-1}, \bar{c}_*, \bar{b}_*)$ holds is non-empty. Moreover, note that $\bar{b}^1_0, \bar{b}^2_0, \ldots, \bar{b}^1_{k_{n_*}-1}, \bar{b}^2_{k_{n_*}-1}$ cannot witness that $\mathrm{tp}(\bar{a}/AI)$ k-forks over $A\bar{b}_*$, as $\mathrm{tp}(\bar{a}/AI)$ does not k-fork over A. Thus, there is some $\eta \in \Lambda$ such that $u_\eta = \{l < k_{n_*} : \eta(l) = 2\}$ is non-empty.

Fix some $\eta \in \Lambda$ with $u_\eta \neq \emptyset$, and let $n_{**} < n_*$ be minimal with the property that $u_\eta \cap [k_{n_{**}}, k_{n_{**}+1}) \neq \emptyset$. Since the formula ψ is symmetric on $\bar{y}_0, \ldots, \bar{y}_{k_{n_*+1}-1}$ by construction, we can rearrange the variables corresponding to the indices of u_η so that $u_\eta \cap [k_{n_{**}}, k_{n_{**}+1}) = [k_*, k_{n_{**}+1})$ for some $k_* \in [k_{n_{**}}, k_{n_{**}+1})$. Now, set $\bar{d}_0 = \bar{b}_0 \ldots \bar{b}_{k_{n_{**}}-1}$, $\bar{d}_1 = \bar{b}_{k_{n_{**}}} \ldots \bar{b}_{k_*-1}$, $\bar{d}^1_2 = \bar{b}^1_{k_*} \ldots \bar{b}^1_{k_{n_{**}+1}-1}$ and $\bar{d}^2_2 = \bar{b}^{\eta(0)}_{k_*} \ldots \bar{b}^{\eta(k_{n_{**}+1}-1)}_{k_{n_{**}+1}-1}$. Hence, by $\bullet_1$ we have that

$\otimes_2$ $\phi_{n_{**}}(\bar{a}; \bar{d}_0, \bar{d}_1, \bar{d}^1_2, \bar{c}_n)$ holds.

On the other hand, the choice of ψ and $\eta \in \Lambda$ yield that

$\otimes_3$ $\phi_{n_{**}}(\bar{a}; \bar{d}_0, \bar{d}_1, \bar{d}^1_2, \bar{c}_n)$ holds if and only if so does $\phi_{n_{**}}(\bar{a}; \bar{d}_0, \bar{d}_1, \bar{d}^2_2, \bar{c}_n)$.

Hence, by $\otimes_2$ and $\otimes_3$ we get that

$\otimes_4$ $\phi_{n_{**}}(\bar{a}; \bar{d}_0, \bar{d}_1, \bar{d}^2_2, \bar{c}_n)$ holds.

Observe that since $n_{**} < n_*$, we have that $I_{n_{**}}$ is contained in $I_{<n_*}$ and so $\bar{d}^2_2$ is formed with elements from $I \setminus I_{n_{**}}$. Thus, by $\bullet_2$ we have that

$\otimes_5$ $\neg\phi_{n_{**}}(\bar{a}; \bar{d}_0, \bar{b}'_{k_{n_{**}}}, \ldots, \bar{b}'_{k_*-1}, \bar{d}^2_2, \bar{c}_{n_{**}})$ holds for any pairwise distinct elements $\bar{b}'_{k_{n_{**}}}, \ldots, \bar{b}'_{k_*-1}$ of $I \setminus I_{n_{**}}$.

Therefore, by $\otimes_4$, $\otimes_5$ and setting $\bar{c}'_{n_{**}} = \bar{d}_2^2\bar{c}_{n_{**}}$, we contradict the minimality of $k_{n_{**}+1}$ given by $\bullet_3$ since $k_* < k_{n_{**}+1}$. This finishes the proof. $\square$

Proposition 2.8. *Let $\bar{a}$ be a finite tuple, and let A be a subset of an $(|A| + |T|)^+$-saturated model M. Then, the following are equivalent:*

(1) *The type $\mathrm{tp}(\bar{a}/M)$ does not k-fork over A.*
(2) *For any A-indiscernible sequence I contained in M, there is some $J \subseteq I$ with $|J| < k$ such that $I \setminus J$ is an indiscernible set over $AJ\bar{a}$.*

Moreover, the above properties implies the following:

(3) *For any A-independent sequence I contained in M, there is some $J \subseteq I$ with $|J| < k$ such that $I \setminus J$ is independent from $AJ\bar{a}$ over A.*

Proof. (1) ⇒ (2) is the lemma above. To show (2) ⇒ (1), suppose that $\mathrm{tp}(\bar{a}/M)$ k-forks over A. Thus there is some formula $\psi(\bar{x}) \in \mathrm{tp}(\bar{a}/M)$ that k-forks over A. That is, the formula $\psi(\bar{x})$ implies a finite disjunction of formulas that skew k-divide over A. Note that by saturation of M each of these formulas can be taken with parameters over M. Thus we can find a formula $\phi(\bar{x}; \bar{y}_0, \dots, \bar{y}_{k-1})$ and an A-indiscernible sequence $(\bar{b}_n)_{n<\omega}$ witnessing this. Notice again by saturation that we may take $(\bar{b}_n)_{n<\omega}$ inside M. By (2), there is a finite subset J of ω with $|J| < k$ such that $I \setminus J$ is indiscernible over $A\bar{a}J$. Thus, there is some $l < k$ such that $2l \notin J$ and so $\phi(\bar{a}; \bar{b}_{i_0}, \dots, \bar{b}_{i_{k-1}})$ holds by indiscernibility taking $i_j = 2j$ for $j \neq l$ and $i_l = 2l + 1$, a contradiction.

Finally we see that (2) ⇒ (3). Assume that $I = (\bar{a}_s)_{s<\alpha}$ is an A-independent sequence and consider a Morley sequence $(\bar{a}_{s,t})_{t<\alpha}$ in $p_s = \mathrm{stp}(\bar{a}_s/A)$ with $\bar{a}_s = \bar{a}_{s,s}$ in a way that the array $(\bar{a}_{s,t})_{s,t<\alpha}$ is an independent set over A. By saturation, we may take this array inside M. Set $\bar{b}_t = (\bar{a}_{s,t})_{s<\alpha}$ and note that it realizes the stationary type $\bigotimes_{s<\alpha} p_s$. Consequently, as $(\bar{b}_t)_{t<\alpha}$ is A-independent, we obtain that it is an A-indiscernible sequence. Hence, by (2) there exists some $J \subseteq \alpha$ with $|J| < k$ such that $(\bar{b}_t)_{t\notin J}$ is indiscernible over $A\bar{a} \cup \{\bar{b}_t\}_{t\in J}$. Whence, since $(\bar{b}_t)_{t\notin J}$ is Morley sequence in $\bigotimes_{s<\alpha} p_s$, we have that $\bar{a} \cup \{\bar{b}_t\}_{t\in J}$ is independent from $(\bar{b}_t)_{t\notin J}$ over A, and so $I \setminus \{\bar{a}_s\}_{s\in J}$ is independent from $A\bar{a} \cup \{\bar{a}_t\}_{t\in J}$, as desired. $\square$

Remark 2.9. In view of Remark 2.4(4) and Proposition 2.8 we could have defined k-forking as follows: A partial type $\pi(\bar{x})$ does not k-fork over A if it can be extended to a complete type $p(\bar{x})$ over an $(|A| + |T|^+)$-saturated model M such that for any $\bar{a} \models p$ and any A-indiscernible sequence I contained in M, there is some $J \subseteq I$ with $|J| < k$ such that $I \setminus J$ is an indiscernible set over $AJ\bar{a}$.

Lemma 2.10. *If $\mathrm{tp}(\bar{a}_1/B)$ does not k_1-fork over $A \subseteq B$ and $\mathrm{tp}(\bar{a}_2/B\bar{a}_1)$ does not k_2-fork over $A\bar{a}_1$, then $\mathrm{tp}(\bar{a}_1\bar{a}_2/B)$ does not $(k_1 + k_2)$-fork over A.*

Proof. Consider an $(|A| + |T|)^+$-saturated model M extending B. By extension, *i.e.* Remark 2.4(4), there is some $\bar{a}'_1 \models \mathrm{tp}(\bar{a}_1/B)$ such that $\mathrm{tp}(\bar{a}'_1/M)$ does not k_1-fork over A. Let $\bar{a}'_2$ be such that $\bar{a}_1\bar{a}_2 \equiv_B \bar{a}'_1\bar{a}'_2$ and note that $\mathrm{tp}(\bar{a}'_2/B, \bar{a}'_1)$ does not k_2-fork over $A\bar{a}'_1$ by invariance. Hence, again by extension there is some $\bar{a}''_2 \equiv_{B\bar{a}'_1} \bar{a}'_2$ such that $\mathrm{tp}(\bar{a}''_2/M, \bar{a}'_1)$ does not k_2-fork over $A\bar{a}'_1$. Now, given an A-indiscernible sequence I contained in M, applying twice Lemma 2.7 we find two disjoint subsets J_1 and J_2 of I with $|J_1| < k_1$ and $|J_2| < k_2$ such that $I \setminus (J_1 \cup J_2)$ is indiscernible over $AJ_1J_2\bar{a}'_1\bar{a}''_2$. Hence by Proposition 2.8, we get that the type $\mathrm{tp}(\bar{a}'_1\bar{a}''_2/M)$ does not $(k_1 + k_2)$-fork over A and neither does $\mathrm{tp}(\bar{a}_1\bar{a}_2/M)$ by invariance. $\square$

In the light of the previous result we introduce the following notion.

Definition 2.11. A partial type $\pi(\bar{x})$ ω-forks over A if it k-forks over A for every natural number k. We write $\bar{a} \downarrow^{\omega}_{A} B$ whenever $\mathrm{tp}(\bar{a}/AB)$ does not ω-fork over A.

This notion satisfies the usual axioms of a ternary independence relation.

Theorem 2.12. *The ternary relation* $\downarrow^{\omega}$ *defined among imaginary sets satisfies the following properties:*

(1) *Invariance:* $\downarrow^{\omega}$ *is invariant under* $\mathrm{Aut}(\mathfrak{M})$.
(2) *Finite character:* $\bar{a} \downarrow^{\omega}_{A} B$ *if and only if* $\bar{a}' \downarrow^{\omega}_{A} B'$ *for any finite tuple* $\bar{a}' \subseteq \bar{a}$ *and any finite set* $B' \subseteq B$.
(3) *Transitivity: If* $\bar{a} \downarrow^{\omega}_{A\bar{b}} B$ *and* $\bar{b} \downarrow^{\omega}_{A} B$, *then* $\bar{a}\bar{b} \downarrow^{\omega}_{A} B$.
(4) *Base monotonicity: If* $\bar{a} \downarrow^{\omega}_{A} BC$, *then* $\bar{a} \downarrow^{\omega}_{AB} C$.
(5) *Extension: If* $\bar{a} \downarrow^{\omega}_{A} B$, *then for any* C *there exists some* $\bar{a}' \equiv_{AB} \bar{a}$ *with* $\bar{a}' \downarrow^{\omega}_{A} BC$.
(6) *Local character: For every finite tuple* $\bar{a}$ *and any set* B *there is some* $A \subseteq B$ *with* $|A| < |T|^+$ *such that* $\bar{a} \downarrow^{\omega}_{A} B$.
(7) *Symmetry:* $\bar{a} \downarrow^{\omega}_{A} \bar{b}$ *if and only if* $\bar{b} \downarrow^{\omega}_{A} \bar{a}$.

Proof. Invariance, finite character and base monotonicity are straightforward from the definition. Extension follows from Remark 2.4(4), and transitivity from Lemma 2.10. Furthermore, notice that the relation $\downarrow^{\omega}$ satisfies local character by stability, Lemma 2.5 and Remark 2.4(1).

Finally, symmetry holds by [1, Theorem 2.5]. We offer a shorter proof using stability. By extension and finite character we can find an indiscernible sequence $(\bar{a}_i)_{i<|T|^+}$ in $\mathrm{tp}(\bar{a}/A, \bar{b})$ such that $\bar{a}_i \downarrow^{\omega}_{A} \bar{b}, (\bar{a}_j)_{j<i}$ for every $i < |T|^+$. In particular, we have that $\bar{a}_i \downarrow^{\omega}_{A} (\bar{a}_j)_{j<i}$. As any indiscernible sequence is an indiscernible set, we obtain inductively on i that $(\bar{a}_j)_{j<i} \downarrow^{\omega}_{A} \bar{a}_i$ by invariance, finite character and transitivity. Now, local character of $\downarrow^{\omega}$ implies the existence of some $i < |T|^+$ such that $\bar{b} \downarrow^{\omega}_{A,(\bar{a}_j)_{j<i}} \bar{a}_i$. Hence, we obtain that $\bar{b}, (\bar{a}_j)_{j<i} \downarrow^{\omega}_{A} \bar{a}_i$ by transitivity and so $\bar{b} \downarrow^{\omega}_{A} \bar{a}_i$ by finite character. Whence, we obtain the result by invariance. □

2.2. Flat theories

Next we introduce a subclass of stable theories which includes the superstable ones.

Definition 2.13. A stable theory is *flat* if for every finite tuple a and every set A, there exists a finite subset A_0 of A such that $a \downarrow^{\omega}_{A_0} A$.

It follows from the definition of flatness and ω-forking that any superstable theory is flat. Nevertheless, not every flat theory is superstable. The following exhibit can be seen as the archetypical example of flat non-superstable theory.

Example 2.14. Consider the first-order theory of countably many nested equivalence relations $\{E_i(x, y)\}_{i<\omega}$ such that $E_0(x, y)$ has infinitely many classes, and each E_i-class can be partitioned into infinitely many E_{i+1}-classes. This is a stable flat theory which is not superstable theory.

The importance of flatness is that the foundation rank associated to the binary relation of being an ω-forking extension among finitary complete types over sets takes ordinal values.

Definition 2.15. The U_{ω}-rank is the least function from the collection of all types (with parameters from the monster model) to the set of ordinals or ∞ satisfying for every ordinal α:

$$U_\omega(p) \geq \alpha + 1 \text{ if there is an } \omega\text{-forking extension } q \text{ of } p \text{ with } U_\omega(q) \geq \alpha.$$

As usual, to easer notation we write $U_\omega(a/A)$ for $U_\omega(tp(a/A))$.

The U_ω-rank is invariant under automorphism and clearly $U_\omega(p) \leq U(p)$ for any finitary complete type p. Since every type does not fork over a set of cardinality at most $|T|$, there are at most $2^{|T|}$ different U-ranks and so at most $2^{|T|}$ different U_ω-ranks. As these values form an initial segment of the ordinals, all of them are smaller than $(2^{|T|})^+$. Thus, it follows that every type of U_ω-rank ∞ has a forking extension of U_ω-rank ∞.

Proposition 2.16. *The following holds:*

(1) *If q extends p, then $U_\omega(p) \geq U_\omega(q)$. Moreover, if q is a non-ω-forking extension of p, then $U_\omega(p) = U_\omega(q)$.*
(2) *A theory is flat if and only if $U_\omega(p) < \infty$ for every finitary complete (real) type p.*

Proof. The proof is standard and it is left to the reader. □

Remark 2.17. It follows from the definition of U_ω-rank that a finitary complete type has U_ω-rank zero if and only if it has no ω-forking extensions. In particular, by the extension property we have that $U_\omega(a/A) = 0$ if and only if $a \mathop{\smile\!\!\!\!\!\!|}^{\omega}_{A} a$.

Recall that every ordinal α can be written in the Cantor normal form as a finite sum $\omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_k} \cdot n_k$ for ordinals $\alpha_1 > \ldots > \alpha_k$ and natural numbers $n_1, \ldots, n_k$. If additionally $\beta = \omega^{\alpha_1} \cdot m_1 + \ldots + \omega^{\alpha_k} \cdot m_k$, then the sum $\alpha \oplus \beta$, which is defined as $\omega^{\alpha_1} \cdot (n_1 + m_1) + \ldots + \omega^{\alpha_k} \cdot (n_k + m_k)$, is commutative. In fact, the sum $\oplus$ is the smallest symmetric strictly increasing function f among pairs of ordinals such that $f(\alpha, \beta + 1) = f(\alpha, \beta) + 1$.

The proof of the following result is standard, see for instance [3, Theorem 4].

Theorem 2.18 *(Lascar Inequalities). The following holds:*

(1) $U_\omega(a/Ab) + U_\omega(b/A) \leq U_\omega(ab/A) \leq U_\omega(a/Ab) \oplus U_\omega(b/A)$.
(2) *If $U_\omega(a/Ab) < \infty$ and $U_\omega(a/A) \geq U_\omega(a/Ab) \oplus \alpha$ for some ordinal α, then $U_\omega(b/A) \geq U_\omega(b/Aa) \oplus \alpha$.*
(3) *If $U_\omega(a/Ab) < \infty$ and $U_\omega(a/A) \geq U_\omega(a/Ab) + \omega^\alpha$ for some ordinal α, then $U_\omega(b/A) \geq U_\omega(b/Aa) + \omega^\alpha$.*
(4) *If $a \mathop{\smile\!\!\!\!\!\!|}^{\omega}_{A} b$, then $U_\omega(ab/A) = U_\omega(a/A) \oplus U_\omega(b/A)$.*

We finish this section by pointing out the existence of a link between forking and ω-forking via canonical bases and types of U_ω-rank zero.

Proposition 2.19. *If $a \mathop{\smile\!\!\!\!\!\!|}^{\omega}_{A} b$, then $U_\omega(cb(stp(a/Ab))/A) = 0$. Furthermore, the opposite holds assuming that $U_\omega(a/A) < \infty$.*

Proof. As non-ω-forking independence has finite character, notice that the type $tp(cb(stp(a/Ab))/A)$ has U_ω-rank zero if and only if $U_\omega(c/A) = 0$ for any finite tuple of $cb(stp(a/Ab))$. Now, suppose that $a \mathop{\smile\!\!\!\!\!\!|}^{\omega}_{A} b$ and let $\bar{a} = (a_i)_{i<\omega}$ be a Morley sequence in $stp(a/Ab)$. Thus one can easily see that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\omega}_{A} b$ using Theorem 2.12. Since any finite tuple c of $cb(stp(a/Ab))$ belongs to $dcl(\bar{a}) \cap acl(Ab)$, we then have $c \mathop{\smile\!\!\!\!\!\!|}^{\omega}_{A} c$ and so $U_\omega(c/A) = 0$.

For the opposite, assume that $U_\omega(a/A) < \infty$ and set $C = cb(stp(a/Ab))$. Thus, by the Lascar inequalities

$$U_\omega(a/A) \leq U_\omega(a/A, C) \oplus U_\omega(C/A) = U_\omega(a/A, C),$$

so $\mathrm{U}_\omega(a/A) = \mathrm{U}_\omega(a/A, C) < \infty$ and hence $a \mathop{\smile\!\!\!\!\!\mid}^{\omega}_{A} C$. Moreover, since $a \mathop{\smile\!\!\!\!\!\mid}_{C} Ab$ we have that $a \mathop{\smile\!\!\!\!\!\mid}^{\omega}_{CA} b$ and therefore $a \mathop{\smile\!\!\!\!\!\mid}^{\omega}_{A} b$ by transitivity, as desired. □

3. Searching for enough regular types

3.1. Types without ω-forking extensions

As we point out before, a type has U_ω-rank zero if and only if it has no ω-forking extensions. In this section we shall see that these types play a fundamental role towards the existence of enough regular types in flat theories.

Let $\mathbb{P}$ be an $\emptyset$-invariant family of partial types. A stationary type $p \in S(A)$ is *foreign* to $\mathbb{P}$ if for all sets $B \supseteq A$ and all realizations a of $p|B$ we have that $a \mathop{\smile\!\!\!\!\!\mid}_{B} c$ for any c such that $\mathrm{tp}(c/B)$ extends some member of $\mathbb{P}$. The type p is *(almost)* $\mathbb{P}$*-internal* if there exists some $B \supseteq A$, a realization $a \models p|B$ and some tuple $\bar{b} = (b_1, \ldots, b_n)$ such that $\bar{a} \in \mathrm{dcl}(B, \bar{b})$ ($\bar{a} \in \mathrm{acl}(B, \bar{b})$, respectively) and each type $\mathrm{tp}(b_i/B)$ extends a member of $\mathbb{P}$. Finally, it is $\mathbb{P}$*-analyzable* in α steps if for some realization a of p there is a sequence $(a_i)_{i<\alpha}$ in $\mathrm{dcl}(A, a)$ such that each type $\mathrm{tp}(a_i/A, (a_j)_{j<i})$ is $\mathbb{P}$-internal, and $a \in \mathrm{acl}(A, (a_i)_{i<\alpha})$.

The following result, see [6, Corollary 7.4.6], plays an essential role in this section.

Fact 3.1. *If the type* $\mathrm{stp}(a/A)$ *is not foreign to* $\mathbb{P}$*, then there is some imaginary element* $a_0 \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A)$ *such that* $\mathrm{stp}(a_0/A)$ *is* $\mathbb{P}$*-internal.*

Let $\mathbb{P}_0$ denote the family of types of U_ω-rank zero. It is easy to see that any finitary complete type which is $\mathbb{P}_0$-analyzable in finitely many steps must have U_ω-rank zero by the Lascar inequalities. Consequently, we obtain the following:

Lemma 3.2. *If the type* $\mathrm{stp}(a/A)$ *is not foreign to* $\mathbb{P}_0$*, then there is some imaginary element* $a_0 \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A)$ *such that* $\mathrm{U}_\omega(a_0/A) = 0$.

Given a set A, set $\mathrm{cl}_{\mathbb{P}_0}(A)$ to be the set of all elements b such that $\mathrm{tp}(b/A)$ has U_ω-rank zero. By [4, Corollary 6] we obtain the following decomposition lemma, see also [3, Corollary 6]. For the sake of completeness we give a (direct) proof.

Lemma 3.3. *For any tuple* a *and any set* A*, the type* $\mathrm{stp}(a/A_0)$ *is foreign to* $\mathbb{P}_0$*, where* $A_0 = \mathrm{dcl}(A, a) \cap \mathrm{cl}_{\mathbb{P}_0}(A)$*. Moreover, it has the same* U_ω*-rank as* $\mathrm{tp}(a/A)$*.*

Proof. Suppose that $\mathrm{stp}(a/A_0)$ is not foreign to the family of types of U_ω-rank zero. Thus, there is some $a_0 \in \mathrm{dcl}(A_0, a) \setminus \mathrm{acl}(A_0)$ such that $\mathrm{tp}(a_0/A_0)$ is internal to the family of types of U_ω-rank zero. That is, there are is some $C \mathop{\smile\!\!\!\!\!\mid}_{A_0} a$ and some $b_1, \ldots, b_n$ with $\mathrm{U}_\omega(b_i/A_0C) = 0$ such that $a_0 \in \mathrm{dcl}(A_0C, b_1, \ldots, b_n)$. Hence we have that $\mathrm{U}_\omega(a_0/A_0) = 0$. On the other hand, notice that $a_0 \in \mathrm{dcl}(A, a)$ by definition of A_0 and moreover that $\mathrm{U}_\omega(A_0/A) = 0$ since any finite tuple of elements from A_0 has U_ω-rank zero over A again by Lascar inequalities. Thus

$$\mathrm{U}_\omega(a_0/A) \leq \mathrm{U}_\omega(a_0A_0/A) \leq \mathrm{U}_\omega(a_0/A_0) \oplus \mathrm{U}_\omega(A_0/A) = 0$$

and so $a_0 \in A_0$, a contradiction. Finally, the second part of the statement follows once more by the Lascar inequalities since $\mathrm{U}_\omega(A_0/A) = 0$. □

Definition 3.4. We say that a complete type is ω*-minimal* if every forking extension of it is also an ω-forking extension.

Lemma 3.5. *A non-forking extension of an ω-minimal type is again ω-minimal.*

Proof. To see this, let q be a non-forking extension of an ω-minimal type p with parameters over A. Assume that $q = p|B$ and consider a forking extension q' of q over a set B'. Let a be a realization of q'. Notice that $a \not\perp_B B'$ and $a \perp_A B$. Thus $a \not\perp_A BB'$ and so $a \not\perp^\omega_A BB'$ since $p = \mathrm{tp}(a/A)$ is ω-minimal. Moreover, we obtain that $a \not\perp^\omega_B B'$ by transitivity since $a \perp^\omega_A B$, yielding that q' is an ω-forking extension of $q = \mathrm{tp}(a/B)$. Thus, the type q is also ω-minimal. □

Remark 3.6. If an ω-minimal type p has ordinal U_ω-rank, then every forking extension of it has strictly smaller U_ω-rank. Hence, using the Lascar inequalities it is easy to see that any ω-minimal stationary type of monomial U_ω-rank is regular. Namely, if p is an ω-minimal stationary type with $\mathrm{U}_\omega(p) = \omega^\alpha$ but there is a forking extension p' of p which is non-orthogonal to p, then there is set A and realizations a of $p|A$ and a' of $p'|A$ with $a \not\perp_A a'$. However, this implies that $\mathrm{U}_\omega(a/Aa') < \mathrm{U}_\omega(a/A) = \omega^\alpha$ since $\mathrm{tp}(a/A) = p|A$ is ω-minimal and also that $\mathrm{U}_\omega(a'/A) = \mathrm{U}_\omega(p') < \mathrm{U}_\omega(p) = \omega^\alpha$, yielding that

$$\omega^\alpha = \mathrm{U}_\omega(a/A) \leq \mathrm{U}_\omega(a/Aa') \oplus \mathrm{U}_\omega(a'/A) < \omega^\alpha,$$

a contradiction.

Proposition 3.7. *A stationary type is ω-minimal if and only if it is foreign to $\mathbb{P}_0$.*

Proof. Assume first that p is ω-minimal but it is not foreign to the family of type of U_ω-rank zero. Thus, there is some set A, some realization a of $p|A$ and some tuple $\bar{b} = (\bar{b}_1, \ldots, b_n)$ with each $\mathrm{tp}(b_i/A)$ of U_ω-rank zero such that $a \not\perp_A \bar{b}$. As p is ω-minimal, then so is $\mathrm{tp}(a/A)$ and so $a \not\perp^\omega_A \bar{b}$. It then follows that $\bar{b} \not\perp^\omega_A a$ by symmetry and so $\mathrm{U}_\omega(\bar{b}/A) > 0$, a contradiction.

For the other direction, suppose towards a contradiction that $p = \mathrm{tp}(a/A)$ is foreign to $\mathbb{P}_0$ but there is some tuple b such that $a \not\perp_A b$ and $a \perp^\omega_A b$. We then have that $\mathrm{cb}(\mathrm{stp}(b/Aa))$ is not algebraic over A and so $\mathrm{cb}(\mathrm{stp}(b/Aa)) \not\perp_A a$. Consequently, there is some finite tuple $c \in \mathrm{cb}(\mathrm{stp}(b/Aa))$ such that $a \not\perp_A c$ and so $\mathrm{U}_\omega(c/A) > 0$, since $p = \mathrm{tp}(a/A)$ is foreign to $\mathbb{P}_0$. On the other hand, as $b \perp^\omega_A a$, Proposition 2.19 yields that $\mathrm{U}_\omega(c/A) = 0$, a contradiction. Therefore the type $\mathrm{tp}(a/A)$ is an ω-minimal extension of p. □

As a consequence of Lemma 3.3 and Proposition 3.7 we obtain the following:

Corollary 3.8. *Any stationary type $p = \mathrm{tp}(a/A)$ has an ω-minimal extension of the same U_ω-rank, namely the type $\mathrm{tp}(a/\mathrm{dcl}(Aa) \cap \mathrm{cl}_{\mathbb{P}_0}(A))$.*

The next result shows the existence of many regular types in a flat theory.

Theorem 3.9. *If the type p has rank $\mathrm{U}_\omega(p) = \beta + \omega^\alpha n$, with $n > 0$ and $\beta \geq \omega^{\alpha+1}$ or $\beta = 0$, then it has a non-ω-forking extension q which is not weakly orthogonal to an ω-minimal regular type of U_ω-rank ω^α.*

Proof. Let $p = \mathrm{tp}(a/A)$ and suppose that $\mathrm{U}_\omega(a/A) = \beta + \omega^\alpha n$ with $n > 0$ and $\beta \geq \omega^{\alpha+1}$ or $\beta = 0$. Let b be a tuple such that $\mathrm{U}_\omega(a/Ab) = \beta + \omega^\alpha(n-1)$ and set b' to be $\mathrm{cb}(\mathrm{stp}(a/Ab))$. Since $a \perp_{b'} Ab$, we have that $a \perp^\omega_{Ab'} Ab$ and so $\mathrm{U}_\omega(a/Ab') = \mathrm{U}_\omega(a/Ab)$. Thus, we may assume that $b' = b$.

The Lascar inequalities yield that $\mathrm{U}_\omega(b/A) \geq \omega^\alpha$, and so we can find some set B with $\mathrm{U}_\omega(b/B) = \omega^\alpha$. Moreover, note that we may take B containing A in a way that $B \perp_{Ab} a$ and $B = \mathrm{acl}(B)$. Now, by Corollary 3.8, we know that $\mathrm{tp}(b/B_0)$ is ω-minimal and has U_ω-rank ω^α, where $B_0 = \mathrm{dcl}(Bb) \cap \mathrm{cl}_{\mathbb{P}_0}(B)$. Furthermore, since $B_0 \subseteq \mathrm{dcl}(Bb)$ we have that $B_0 \perp_{Ab} a$ and so

$$\mathrm{U}_\omega(a/B_0, b) = \mathrm{U}_\omega(a/A, b) = \beta + \omega^\alpha(n-1)$$

and

$$b = \mathrm{cb}(\mathrm{stp}(a/Ab)) = \mathrm{cb}(\mathrm{stp}(a/B_0b)).$$

Observe that since $\omega^\alpha = \mathrm{U}_\omega(b/B_0)$, the type $\mathrm{tp}(b/B_0)$ cannot be algebraic and so $a \not\perp_{B_0} b$. As $\mathrm{tp}(b/B_0)$ is ω-minimal we then have $\mathrm{U}_\omega(b/B_0, a) < \mathrm{U}_\omega(b/B_0) = \omega^\alpha$ and hence $\omega^\alpha = \mathrm{U}_\omega(b/B_0) \geq \mathrm{U}_\omega(b/B_0a) + \omega^\alpha$. Whence

$$\mathrm{U}_\omega(a/B_0) \geq \mathrm{U}_\omega(a/B_0b) + \omega^\alpha = \beta + \omega^\alpha(n-1) + \omega^\alpha = \mathrm{U}_\omega(a/A)$$

by the Lascar inequalities and so $a \perp^\omega_A B_0$. Since $\mathrm{tp}(b/B_0)$ is ω-minimal of monomial U_ω-rank, it is regular by Remark 3.6. This finishes the proof. □

Corollary 3.10. *If p is foreign to $\mathbb{P}_0$ and $\mathrm{U}_\omega(p) = \beta + \omega^\alpha n$, with $n > 0$ and $\beta \geq \omega^{\alpha+1}$ or $\beta = 0$, then p is non-orthogonal to an ω-minimal regular type of U_ω-rank ω^α.*

Proof. By Theorem 3.9 there exists a non-ω-forking extension q of p which is not weakly orthogonal to an ω-minimal regular type q' of U_ω-rank ω^α. Since p is also ω-minimal by Proposition 3.7, the type q is indeed a non-forking extension of p and so p is not orthogonal to q'. □

Remark 3.11. So far all results given in this section follow from the fact that the ω-forking independence is an independence relation (in the sense of Theorem 2.12) with a well-behaved notion of rank. In other words, if in a stable theory we have an independence relation $\perp^*$ then one can define the corresponding notions of U_*-rank, $*$-minimality, $*$-flatness and all results of this section adapt to this context.

3.2. Hereditarily triviality

In this subsection we will show that types which are not foreign to $\mathbb{P}_0$ must have finite weight. For this, we introduce the following notion.

Let λ denote an arbitrary cardinal.

Definition 3.12. A partial type π over A is *hereditarily λ-trivial* if for any a realizing π, any set $B \supseteq A$ and any independent sequence I over B, there is some $J \subseteq I$ with $|J| < \lambda$ for which $aJ \perp_B I \setminus J$.

Observe that any hereditarily λ-trivial complete type has weight strictly smaller than λ. However, in Exercise 3.17 [7, Chapter V] it is given an example of a finite weight type p which is not hereditarily $w(p)$-trivial.

Example 3.13. Consider an infinite vector space over a finite field, and let I be a linearly independent set. Fix a finite set $J \subseteq I$ with $|J| > 1$ and let $a = \sum_{x \in J} x$. Then there is no finite subset J' of I with $|J'| \leq w(a) = 1$ such that $I \setminus J'$ is independent from $J'a$.

Now, we show some basic lemmas on hereditarily trivial types.

Lemma 3.14. *Assume $a \perp_A B$ with $A \subseteq B$. If $\mathrm{tp}(a/B)$ is hereditarily λ-trivial, then so is $\mathrm{tp}(a/A)$.*

Proof. Let I be an independent sequence over $C \supseteq A$, and consider a set B' such that $B' \equiv_{Aa} B$ and $B' \perp_{Aa} CI$. Thus $B' \perp_A CI$ by transitivity and invariance, and so the sequence I is independent over $C \cup B'$. As $\mathrm{tp}(a/B)$ is hereditarily λ-trivial, so is $\mathrm{tp}(a/B')$ and hence there exists some subset J of I with

$|J| < \lambda$ such that $I \setminus J \mathop{\smile\!\!\!\!\!\!\mid}_{B'C} aJ$. On the other hand, as $B' \mathop{\smile\!\!\!\!\!\!\mid}_A CI$ we have that $B' \mathop{\smile\!\!\!\!\!\!\mid}_C I$ and so the sequence $I \setminus J$ is independent from aJ over C by transitivity, as desired. □

Lemma 3.15. *If* $\mathrm{tp}(a/A)$ *is hereditarily* λ_1*-trivial, and* $\mathrm{tp}(b/A, a)$ *is hereditarily* λ_2*-trivial, then* $\mathrm{tp}(ab/A)$ *is hereditarily* $(\lambda_1 + \lambda_2)$*-trivial.*

Proof. Consider an independent sequence I over a set $B \supseteq A$. As $\mathrm{tp}(a/A)$ is hereditarily λ_1-trivial, there exists some $J_1 \subseteq I$ with $|J_1| < \lambda_1$ and $J_1 a \mathop{\smile\!\!\!\!\!\!\mid}_B I \setminus J_1$. Thus, the sequence $I \setminus J_1$ is independent over BaJ_1. Since $\mathrm{tp}(b/A, a)$ is hereditarily λ_2-trivial, we can find a subset $J_2 \subseteq I \setminus J_1$ such that $J_2 b \mathop{\smile\!\!\!\!\!\!\mid}_{BaJ_1} I \setminus (J_1 \cup J_2)$ and $|J_2| < \lambda_2$. Therefore, we get $J_1 J_2 ab \mathop{\smile\!\!\!\!\!\!\mid}_B I \setminus (J_1 \cup J_2)$ by transitivity. Hence, the type $\mathrm{tp}(ab/A)$ is hereditarily $(\lambda_1 + \lambda_2)$-trivial, as desired. □

As a consequence we obtain:

Proposition 3.16. *Suppose that* λ *is infinite. A finitary type analyzable in the family of hereditarily* λ*-trivial types is itself hereditarily* λ*-trivial.*

Proof. Firstly we show that a finitary type $p = \mathrm{tp}(a/A)$ that is internal to a family of hereditarily λ-trivial types is itself hereditarily λ-trivial. To do so, suppose that there is some set B with $a \mathop{\smile\!\!\!\!\!\!\mid}_A B$ and some tuple $\bar{b} = (b_1, \dots, b_n)$ with $\mathrm{tp}(b_i/B)$ hereditarily λ-trivial such that $a \in \mathrm{dcl}(B, \bar{b})$. It follows from the definition that $\mathrm{tp}(b_i/B, b_{<i})$ is also hereditarily λ-trivial and so is $\mathrm{tp}(\bar{b}/B)$ by Lemma 3.15. Again it follows easily from the definition that $\mathrm{tp}(a/B)$ is also hereditarily λ-trivial, and then so is $\mathrm{tp}(a/A)$ by Lemma 3.14.

Now, suppose that $p = \mathrm{tp}(a/A)$ is analyzable in a family of hereditarily λ-trivial types. By definition, there is a sequence $(a_i)_{i<\alpha}$ in $\mathrm{dcl}(A, a)$ such that each type $\mathrm{tp}(a_i/A, (a_j)_{j<i})$ is internal to the given family of hereditarily λ-trivial types, and $a \in \mathrm{acl}(A, (a_i)_{i<\alpha})$. We have just seen in the paragraph above that each $\mathrm{tp}(a_i/A, (a_j)_{j<i})$ is hereditarily λ-trivial. Hence, as a is a finite tuple, we have that α is indeed a natural number and so applying α many times Lemma 3.15 we obtain that $\mathrm{tp}((a_i)_{i<\alpha}/A)$ is also hereditarily λ-trivial. Whence, the type $\mathrm{tp}(a/A)$ is hereditarily λ-trivial as well since $a \in \mathrm{acl}(A, (a_i)_{i<\alpha})$. □

For an infinite cardinal λ, let $\mathbb{P}_{\mathrm{ht},\lambda}$ be the family of all hereditarily λ-trivial types. It follows from the result above that given a set A, the set $\mathrm{cl}_{\mathbb{P}_{\mathrm{ht},\lambda}}(A)$ of all tuples b such that $\mathrm{tp}(b/A)$ is hereditarily λ-trivial is a closure operator. Alternatively, this can be easily seen using Lemma 3.14 and 3.15.

Similarly as in Lemma 3.3 (or by [4, Corollary 6]) we get the existence of types foreign to $\mathbb{P}_{\mathrm{ht},\lambda}$.

Corollary 3.17. *For any tuple* a *and any set* A*, the type* $\mathrm{tp}(a/A_0)$ *is foreign to the family of hereditarily* λ*-trivial types, where* $A_0 = \mathrm{dcl}(A, a) \cap \mathrm{cl}_{\mathbb{P}_{\mathrm{ht},\lambda}}(A)$.

Now, we focus our attention to the family of hereditarily ω-trivial types, which contains $\mathbb{P}_0$ as it is shown in the next lemma.

Lemma 3.18. *Let* $\bar{a}$ *be a possibly infinite tuple and* a *a finite tuple such that* $\bar{a}$ *is contained in* $\mathrm{acl}(a)$*. If a type* $p = \mathrm{tp}(\bar{a}/A)$ *has* U_ω*-rank zero, then it is hereditarily* k*-trivial for some natural number* k*.*

Proof. Suppose that $p = \mathrm{tp}(\bar{a}/A)$ has U_ω-rank 0, where $\bar{a} \subseteq \mathrm{acl}(a)$ and a is a finite tuple. Let B and I be as in the definition of hereditarily ω-trivial and consider an $(|A| + |T|)^+$-saturated model M containing B, I and a. Since

$$\mathrm{U}_\omega(\bar{a}/M) \leq \mathrm{U}_\omega(\bar{a}/A) = 0$$

we have that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\omega}_{A} M$. Now, let a' be a finite tuple such that $a' \equiv_{A\bar{a}} a$ with $a' \mathop{\smile\!\!\!\!\!\!|}_{A\bar{a}} M$ and so $a' \mathop{\smile\!\!\!\!\!\!|}^{\omega}_{A} M$ by transitivity. It then follows that $\mathrm{tp}(a'/M)$ does not k-fork over B for some natural number k by definition. Hence, Proposition 2.8 yields the existence of a subset J of I with $|J| < k$ such that $I \setminus J$ is independent from Ja' over B. Whence, as $\bar{a} \subseteq \mathrm{acl}(a')$ by invariance, we get that $I \setminus J$ is independent from $J\bar{a}$ over B and so $p = \mathrm{tp}(\bar{a}/A)$ is hereditarily k-trivial. □

Proposition 3.19. *If a type is not foreign to $\mathbb{P}_{\mathrm{ht},\omega}$, then it dominates an hereditarily ω-trivial type and it is non-orthogonal to a type of weight one.*

Proof. Let $p = \mathrm{tp}(a/A)$ be a type which is not foreign to $\mathbb{P}_{\mathrm{ht},\omega}$. Thus, there exists some element a_0 of $\mathrm{dcl}(A, a) \setminus \mathrm{acl}(A)$ such that $\mathrm{tp}(a_0/A)$ is internal to $\mathbb{P}_{\mathrm{ht},\omega}$ by Fact 3.1. Hence, by Proposition 3.16 the latter type is indeed hereditarily ω-trivial and clearly it is dominated by $\mathrm{tp}(a/A)$.

Now, as $\mathrm{tp}(a_0/A)$ has finite weight it is non-orthogonal to a type of weight one by a result of Hyttinen, see [2, Proposition 5.6.6]. Hence, the type $\mathrm{tp}(a/A)$ is also non-orthogonal to a type of weight one. □

3.3. Flatness and finite weight

Now, we are ready to prove that a flat theory is strong, *i.e.* every type has finite weight. In fact, we obtain a local version of this.

Theorem 3.20. *A finitary type p with $\mathrm{U}_\omega(p) < \infty$ has finite weight and therefore it is non-orthogonal to a type of weight one.*

Proof. We proceed by induction on the U_ω-rank of the type. The case of U_ω-rank 0 follows by Lemma 3.18 and the fact that an hereditarily trivial type has finite weight.

Now, let $p \in S(\emptyset)$ be a finitary type and assume that $\mathrm{U}_\omega(p) = \beta + \omega^\alpha \cdot n$ with $n > 0$ and $\beta \geq \omega^{\alpha+1}$ or $\beta = 0$. Let a be a realization of p and set $A = \mathrm{dcl}(a) \cap \mathrm{cl}_{\mathbb{P}_0}(\emptyset)$. By Lemma 3.3 the type $\mathrm{stp}(a/A)$ is foreign to $\mathbb{P}_0$ and $\mathrm{U}_\omega(a/A) = \mathrm{U}_\omega(p)$. Thus, applying Corollary 3.10 we can find an ω-minimal regular type q which is non-orthogonal to $\mathrm{tp}(a/A)$. Let C and b be such that $a \mathop{\smile\!\!\!\!\!\!|}_{A} C$ and $b \models q|C$ with $a \not\mathop{\smile\!\!\!\!\!\!|}_{C} b$. Note that the latter implies the existence of some imaginary element

$$a_0 \in \mathrm{dcl}(\mathrm{cb}(b/C, a)) \setminus \mathrm{acl}(C).$$

Thus $a_0 \in \mathrm{acl}(C, a)$ and also $a_0 \in \mathrm{dcl}(b_0, \dots, b_m)$ for some initial segment $b_0, \dots, b_m$ of a Morley sequence in $\mathrm{stp}(b/C, a)$. Hence, we then have that

$$w(a_0/C) \leq w(b_0, \dots, b_m/C) \leq m,$$

since $\mathrm{tp}(b/C) = q|C$ is regular and so of weight 1.

Since $a \mathop{\smile\!\!\!\!\!\!|}_{A} C$, the type $\mathrm{tp}(a/C)$ is also foreign to $\mathbb{P}_0$ and thus $\mathrm{U}_\omega(a_0/C) > 0$, as $a \not\mathop{\smile\!\!\!\!\!\!|}_{C} a_0$ by the choice of a_0. Hence, by the Lascar inequality

$$\mathrm{U}_\omega(a/C, a_0) + \mathrm{U}_\omega(a_0/C) \leq \mathrm{U}_\omega(a, a_0/C) = \mathrm{U}_\omega(a/C)$$

we then have that $\mathrm{U}_\omega(a/C, a_0) < \mathrm{U}_\omega(a/C)$. Therefore, putting altogether we get

$$w(a/A) = w(a/C) = w(a, a_0/C) \leq w(a/C, a_0) + w(a_0/C) < \omega,$$

since by induction the type $\mathrm{tp}(a/C, a_0)$ has finite weight.

Finally, notice that $\mathrm{tp}(A)$ has U_ω-rank zero, since $A \subseteq \mathrm{cl}_{\mathbb{P}_0}(\emptyset)$. As a is finite and $A \subseteq \mathrm{dcl}(a)$, using Lemma 3.18 we then see that $\mathrm{tp}(A)$ has finite weight, which yields that $\mathrm{tp}(a)$ has also finite weight, since

$$w(a) = w(a, A) \leq w(a/A) + w(A).$$

This finishes the first part of the statement. For the second, it suffices to notice as before that a type of finite weight is non-orthogonal to a type of weight one. □

As an immediate consequence we obtain:

Corollary 3.21. *Any flat theory is strong.*

In the light of these results, it seems reasonable to ask the following:

Question 1. In a flat theory, is every type non-orthogonal to a regular type? Or, is every type hereditarily 1-trivial type non-orthogonal to a regular type?

4. Flat groups

In this final section we describe the structure of type-definable groups in flat theories. It turns out that this resemblances to the structure of a superstable group, since in the framework of groups we can find enough regular types. More precisely, we recover [5, Corollary 5.3] where a superstable group G is shown to admit a normal series of definable subgroups

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_m \trianglerighteq \{1\}$$

such that each group G_i/G_{i+1} is p_i-semi-regular for some regular type p_i. We refer the reader to [6, Chapter 7] for the general theory of p-simplicity and semi-regularity; which was originally introduced (in a different way) in [7, Chapter V]. We recall some of the basic definitions.

Fix a regular type p. Recall that a stationary type q is said to be *hereditarily orthogonal* to p if p is orthogonal to any extension of q. A stationary type q is *p-simple* if for some set B with p and q based on B, there exist $c \models q|B$ and an independent sequence I of realizations of $p|B$ such that $\mathrm{stp}(c/B, I)$ is hereditarily orthogonal to p. The type q is p-semi-regular if it is p-simple and domination-equivalent to $p^{(n)}$. In fact, a p-simple type $q = \mathrm{stp}(a/A)$ is p-semi-regular if and only if $\mathrm{tp}(d/A)$ is not hereditarily orthogonal to p for every $d \in \mathrm{dcl}(A, a) \setminus \mathrm{acl}(A)$, see [6, Lemma 7.1.18] for a proof. Finally, concerning groups, we say that a group is *p-simple* if some (any) generic type is p-simple, and it is *p-semi-regular* group if some (any) generic is p-semi-regular.

The main facts concerning p-simplicity for groups, which we recall bellow, are shown by Hrushovski in [5], see also [6, Lemma 7.4.7] for a proof.

Fact 4.1. *Let G be a type-definable group and let $q \in S_G(\emptyset)$ be some generic type.*

(1) *If q is not foreign to an $\emptyset$-invariant family $\mathbb{P}$ of types, then there exists a relatively definable normal subgroup N of G of infinite index such that G/N is $\mathbb{P}$-internal.*
(2) *If q is non-orthogonal to a regular type p, then there exists a relatively definable normal subgroup N of G such that G/N is p-simple (even p-internal), and that a generic type of G/N is non-orthogonal to p.*

Before proceeding to analyze flat groups, we first see that the U_ω-rank behaves as the U-rank for groups. Note that a generic type $p \in S_G(A)$ has maximal U_ω-rank: If $\mathrm{tp}(h/A)$ is another type, then taking $g \models p|A, h$ we get

$$\begin{aligned} \mathrm{U}_\omega(h/A) &= \mathrm{U}_\omega(h/A,g) = \mathrm{U}_\omega(gh/A,g) \le \mathrm{U}_\omega(gh/A) \\ &= \mathrm{U}_\omega(gh/A,h) = \mathrm{U}_\omega(g/A,h) = \mathrm{U}_\omega(p), \end{aligned}$$

since $g \mathop{\smile\!\!\!\!\!\!|}^{\omega}_{A} h$, $h \mathop{\smile\!\!\!\!\!\!|}^{\omega}_{A} g$ and $gh \mathop{\smile\!\!\!\!\!\!|}^{\omega} A, h$. We then set $\mathrm{U}_\omega(G)$ to be the U_ω-rank of some (any) generic type; note that *a priori* a type of maximal U_ω-rank might not be generic. Similarly, we can define the U_ω-rank of a coset space to be the U_ω-rank of its generic type. More precisely, if g is generic of G over A and $E(x,y)$ is the equivalence relation $x^{-1}y \in H$ for some relatively definable subgroup H of G, then $\mathrm{tp}(g_E/A)$ is the generic for G/H and moreover note that $\mathrm{tp}(g/A, g_E)$ is a generic for the coset gH. Thus, since $\mathrm{U}_\omega(gH) = \mathrm{U}_\omega(H)$, using the Lascar inequalities, we get

$$\mathrm{U}_\omega(H) + \mathrm{U}_\omega(G/H) \le \mathrm{U}_\omega(G) \le \mathrm{U}_\omega(H) \oplus \mathrm{U}_\omega(G/H).$$

The following key fact is a generalization of Example 3.13.

Lemma 4.2. *A generic type of an infinite type-definable group is not hereditarily k-trivial for any natural number k. In particular, there is no hereditarily k-trivial partial type defining an infinite group.*

Proof. Let G be an infinite type-definable group and suppose, towards a contradiction, that the principal generic $p \in S_G(\emptyset)$ of G is hereditarily k-trivial. Now, let $(a_i)_{i<k+2}$ be an independent sequence of realizations of p and set $a = \prod_{i<k+1} a_i$. As a realizes p, by assumption there is some subset J with $|J| < k$ such that $(a_i)_{i\in J} a \mathop{\smile\!\!\!\!\!\!|} (a_i)_{i\notin J}$. Thus, the definition of a yields the existence of some $k \notin J$ such that $a_k \in \mathrm{dcl}(a, (a_i)_{i\in J})$ and so a_k is independent from itself. This implies that p is an algebraic type and so G is finite, a contradiction.

The second part of the statement follows from the fact that if G is type-defined by an hereditarily k-trivial type, then so is any generic. □

As a consequence, we then have by Lemma 3.18 that a group of U_ω-rank zero must be finite. More generally, we obtain:

Lemma 4.3. *Let G be a type-definable group and let H be a relatively definable subgroup of H. Then $\mathrm{U}_\omega(G) = \mathrm{U}_\omega(H) < \infty$ if and only if G/H is finite.*

Proof. It is enough to use the above Lascar inequalities for groups and notice that by Lemma 3.18 a type-definable group has U_ω-rank 0 if and only if it is finite. □

Corollary 4.4. *Let G be a type-definable group with $\mathrm{U}_\omega(G) < \infty$. Then, there is no infinite sequence of relatively definable subgroups, each having infinite index in its predecessor.*

Now, we can obtain the semi-regular decomposition for flat groups.

Theorem 4.5. *Let G be a type-definable group with $\mathrm{U}_\omega(G) < \infty$. Then, there exist finitely many regular types $p_0, \ldots, p_m$ and a series of relatively definable subgroups*

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_m \trianglerighteq \{1\}$$

such that each group G_i/G_{i+1} is p_i-internal and p_i-semi-regular.

Proof. We proceed by induction on the U_ω-rank. Assume that $\mathrm{U}_\omega(G) = \beta + \omega^\alpha \cdot n$ for some ordinal $\beta \ge \omega^{\alpha+1}$ or $\beta = 0$ and some $n \ge 0$. Moreover, since a group of U_ω-rank zero is finite we may assume that $n > 0$.

We first claim that a generic type of G is foreign to the set of types of U_ω-rank strictly smaller than ω^α. Otherwise, the previous fact yields the existence of a relatively definable normal subgroup N of G of infinite index such that G/N is internal to the family of types of U_ω-rank strictly smaller than ω^α. Thus, we then have that $U_\omega(G/N) < \omega^\alpha$ by the Lascar inequalities and so, the inequation

$$\beta + \omega^\alpha \cdot n = U_\omega(G) \leq U_\omega(N) \oplus U_\omega(G/N)$$

yields that $U_\omega(N) = U_\omega(G)$, a contradiction by Lemma 4.3 since G/N is infinite. Therefore, any generic type of G is foreign, and so orthogonal, to any type of U_ω-rank strictly smaller than ω^α. In particular, it is foreign to $\mathbb{P}_0$ and consequently, by Corollary 3.10, some generic type q of G is non-orthogonal to an ω-minimal regular type p of U_ω-rank ω^α. Thus, the second point of the previous fact yields the existence of a relatively definable normal subgroup H such that G/H is p-internal (so p-simple), and some generic type q' of G/H is non-orthogonal to p.

Assume p, q and q' are stationary over A. Since p is ω-minimal, any forking extension of p has smaller U_ω-rank and hence, the first part of the proof implies that q is orthogonal to any forking extension of p. Whence, the same is true of q' since q dominates q'. Consequently, a standard argument (see the proof of [6, Corollary 7.1.19]) yields that q' is p-semi-regular. Namely, as q' is p-internal, there is some set B containing A, some $c_1, \ldots, c_k$ realizing p and some $a \models q'|B$ such that $a \in \mathrm{dcl}(B, c_1, \ldots, c_k)$. Fix some $d \in \mathrm{dcl}(a, A) \setminus \mathrm{acl}(A)$, and note then that there exists some $m \leq k$ such that $d \mathop{\smile\!\!\!\!\!\!\mid}_A Bc_{<m}$ but $d \not\mathop{\smile\!\!\!\!\!\!\mid}_{B,c_{<m}} c_m$. Setting $r = \mathrm{tp}(c_m/B, c_{<m})$, we clearly have that $\mathrm{tp}(d/A)$ is non-orthogonal to r and then so is $q' = \mathrm{tp}(a/A)$, since $d \in \mathrm{dcl}(A, a)$. Thus, necessarily r must be a non-forking extension of p, as q' is orthogonal to any forking extension. This implies that r is a regular type, as so is p, and moreover that p and r are non-orthogonal. Hence, we then have $\mathrm{tp}(d/A)$ is non-orthogonal to p, yielding that q' is semi-regular by [6, Lemma 7.1.18], say. Therefore, we have shown that G/H is p-internal and p-semi-regular.

Finally, as G/H is infinite and so $U_\omega(H) < U_\omega(G)$ by Lemma 4.3, the inductive hypothesis applied to H yields the statement. □

To finish the paper, a question:

Question 2. Is there a flat non-superstable group?

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