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A NONCONSERVATIVITY RESULT ON GLOBAL CHOICE

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The theory ZF(aa) was introduced in Kaufmann [5] and independently (in a slightly different form) in Kakuda [4], as an extension of ZF by way of a quantifier aaα ('for almost all α'). Intuitively, aa$\alpha\,\phi$ asserts that $\phi(\alpha)$ holds on a closed unbounded class of ordinals, though other interpretations are possible. The axioms of ZF(aa) consist of the ZF schemas together with a version of the schemas for the logic L(aa) from [1]; we need note only the 'diagonal intersection schema'

$$(\forall x)(\mathrm{aa}\alpha)\phi \rightarrow (\mathrm{aa}\alpha)(\forall x \in R_\alpha)\phi,$$

where as usual R_α is the set of all sets of rank less than α.

Although ZF(aa) is already stronger than ZF (for example it proves the existence of natural models of ZF, cf. [5]), its strength is greatly increased by adding the following 'determinacy' schema

[DET] $\quad \mathrm{aa}\alpha\,\phi \vee \mathrm{aa}\alpha\,\neg\phi,$

where ϕ may have any number of free variables. In fact it was shown in [5, §5] that ZF(aa)+DET is equiconsistent with

$$\mathrm{ZF}+\{(\exists\kappa)(\kappa \text{ is } n\text{-ineffable}) : n<\omega\}.$$

(n-ineffable cardinals are studied in Baumgartner [2].) Part of the proof shows that

$$\mathrm{ZF(aa)}+\mathrm{DET}+(\exists x)(V=\mathrm{OD}(x)) \vdash (\exists\kappa)(\kappa \text{ is } n\text{-ineffable}),$$

for all $n<\omega$. Here the statement $(\exists x)(V=\mathrm{OD}(x))$ is really a formal statement of global choice. The question then naturally arises whether a global choice hypothesis is necessary, and this was left open in [5, 5.10]. In light of the conservativity of a form of global choice over ZFC (cf. Felgner [3], for example),

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it is perhaps surprising that this question has a negative answer, as we prove:

Theorem. *The statement* $AC\wedge\neg(\exists\mu)$ *(μ is a Mahlo cardinal) is relatively consistent with* ZF(aa) + DET.

Since ineffable cardinals are Mahlo, this answers the question. We leave open, however, whether $\text{ZF(aa)}+\text{DET}\vdash(\exists\kappa)$ (κ is strongly inaccessible), although we know from [5, 4.10 and 5.6] that

$$\text{ZF(aa)}+\text{DET}\vdash(\exists\kappa)(\kappa \text{ is } n\text{-ineffable in } L), \quad \text{for all } n<\omega.$$

The consistency of $\text{ZF(aa)}+\text{DET}+\text{AC}+\neg\exists\kappa(\kappa$ is Mahlo) was first proved by Shelah assuming the consistency of $\text{ZF}+(\exists\kappa)$ (κ is a measurable cardinal). The details needed for the present relative consistency result were worked out by Kaufmann.

Proof of the Theorem. Assume ZF(aa) + DET is consistent and fix $n\in\omega-\{0\}$. By Proposition 4.10 of [5], the theory $T_0=\text{ZF}+V=L+$"κ is n-ineffable" is consistent. It is convenient to let T be the result of adding the schema "$R_\lambda <_m V$" (all $m<\omega$) plus "$\kappa<\lambda$" to T_0; then T is consistent, by the Reflection Theorem. We will work in T to produce a model of the theory

$$[\text{ZF(aa)}+\text{DET}+\text{AC}]\cap F^n+\neg(\exists\mu)(\mu \text{ is Mahlo}),$$

where F^n is defined as follows. First let $\text{qr}(\phi)$ be the aa-quantifier rank of ϕ, i.e.

$$\text{qr}(\phi)=0 \quad \text{for atomic } \phi,$$
$$\text{qr}(\neg\phi)=\text{qr}((\exists x)\phi)=\text{qr}(\phi),$$
$$\text{qr}(\phi\wedge\psi)=\max(\text{qr}(\phi),\text{qr}(\psi)), \quad \text{and}$$
$$\text{qr}((\text{aa}\alpha)\phi)=\text{qr}(\phi)+1.$$

Then $F^n=\{\phi:\text{qr}(\phi)\leq n\}$. In particular, F^0 is the language of ZF.

Working in T, for all $\alpha_0<\cdots<\alpha_{n-1}<\kappa$, let

$$S_{\alpha_0\cdots\alpha_{n-1}}=\{\phi(x_0,\ldots,x_{n-1},y,\bar a)\in F^0: R_\lambda\models\phi(\alpha_0,\ldots,\alpha_{n-1},\mathbb{P},\bar a),\ \bar a\in R_{\alpha_0}\},$$

where $\mathbb{P}$ is the partial order defined below.

Applying the definition of n-ineffability (and a slight argument as in the proof of [5, 5.3]), we obtain a stationary set $X\subseteq\kappa$ which has the following indiscernibility property:

(*) Let $[X]^n=\{\bar\alpha\in X^n:\alpha_0<\cdots<\alpha_{n-1}\}$. Then for $\bar\alpha,\bar\beta\in[X]^n$ and all $\phi\in F^0$ with parameters in $R_{\alpha_0\cap\beta_0}$, $R_\lambda\models\phi(\bar\alpha,\mathbb{P})\leftrightarrow\phi(\bar\beta,\mathbb{P})$.

Our plan is now to use a form of Easton forcing to 'destroy' every Mahlo cardinal below κ, and yet use X to define an interpretation of 'aa' so that with this interpretation, $R_\kappa^{V[G]}$ is still a model of ZF(aa) + DET. Formally let $S=\{\alpha<\kappa:\alpha$

is Mahlo}, and for $\alpha \in S$ let $\mathbb{P}(\alpha) = \{X \subseteq \alpha : X$ is a closed bounded set of non-inaccessibles}. Now let $\mathbb{P}$ consist of all sequences $\langle p_\alpha : \alpha \in S \rangle$ with the property that $p_\alpha \in \mathbb{P}_\alpha$ for all $\alpha \in S$ and if $\beta \leq \kappa$ is regular, and if we set $\text{supp}(p) = \{\alpha \in S : p_\alpha \neq \emptyset\}$, then $\text{supp}(p) \cap \beta$ is bounded below β. $\mathbb{P}$ is ordered by coordinate-wise end extension: that is, $p \leq q$ iff for all $\alpha \in S$, $q_\alpha = p_\alpha \cap (\max(q_\alpha)+1)$. Our model of the given extension of $\text{ZF(aa)} \cap F^n$ is found in the forcing extension $V[G]$: let $R_\kappa^+ = (R_\kappa^{V[G]}, \in, X)$ where we define

$$R_\kappa^+ \models (\text{aa}\alpha)\phi(\alpha, \bar{a}) \quad \text{iff} \quad R_\kappa^+ \models (\forall \gamma)(\exists \alpha \in X - \gamma)\phi(\alpha, \bar{a}).$$

To show that R_κ^+ is the desired model we prove a sequence of claims.

Claim 1. $R_\kappa^+ \models \neg(\exists \alpha)(\alpha$ *is Mahlo*).

Proof. This is clear from the choice of $\mathbb{P}$.

Claim 2. $\mathbb{P}$ *has the* κ*-c.c.*

Proof. This is a standard pressing-down argument; a similar proof appears in (for example) Kunen [6, Lemma VIII.4.4 and Exercise VIII.J4]. Suppose A is an antichain of power κ; we may enumerate A as $\langle a_\alpha : \alpha \in S \rangle$ where S is the set of inaccessible cardinals below κ. Define a function $f : S \to \kappa$ by: $f(\alpha) = \sup(\text{supp}(a_\alpha) \cap \alpha)$. Notice $f(\alpha) < \alpha$ by definition of $\mathbb{P}$. By Födor's Lemma we may choose $S' \subseteq S$ such that $|S'| = \kappa$ and $f[S'] = \{\gamma\}$ for some $\gamma < \kappa$. Since κ is inaccessible, there exists $S'' \subseteq S'$ such that $|S''| = \kappa$ and for all $\alpha, \beta \in S''$, $a_\alpha \cup a_\beta \in \mathbb{P}$ and $a_\alpha \cup a_\beta \leq a_\alpha, a_\beta$, a contradiction.

For the next claim, set

$$\mathbb{P}_\alpha = \{p \in \mathbb{P} : \text{supp}(p) \subseteq \alpha\} \quad \text{and} \quad \mathbb{P}^\alpha = \{p \in \mathbb{P} : \text{supp}(p) \cap \alpha = \emptyset\}.$$

A standard trick in iterated forcing is to notice that $\mathbb{P} \cong \mathbb{P}_\alpha \times \mathbb{P}^\alpha$ for all $\alpha < \kappa$, and use nice properties of $\mathbb{P}_\alpha$ and $\mathbb{P}^\alpha$ to obtain results about $V[G]$. The following claim is typical. Recall that a partial order is λ-*distributive* if the intersection of fewer than λ dense open sets is dense, and that no such forcing adds bounded subsets of λ.

Claim 3. *For all cardinals* $\alpha < \kappa$, $\mathbb{P}^\alpha$ *is* α*-distributive.*

Proof. Suppose $\alpha < \kappa$ and $\langle D_i : i < \beta \rangle$ is a sequence of dense open subsets of $\mathbb{P}^\alpha$, where β is a cardinal below α; we show $\bigcap \{D_i : i < \beta\}$ is dense. Given $p \in \mathbb{P}$, define a sequence $\langle p_i : i \leq \beta \rangle$ by induction on i. Set $p_0 = p$, and choose $p_{i+1} \in D_i$ such that $p_{i+1} \leq p_i$ and such that for all $\gamma \in \text{supp}(p_i)$, $\beta < \sup((p_i)_\gamma)$. This guarantees that for limit i, if we set $p^\gamma = \bigcup \{(p_j)_\gamma : j < i\}$ for all $\gamma \in \bigcup \{\text{supp}(p_j) : j < i\}$, and then set $(p_i)_\gamma = p^\gamma \cup \{\sup(p^\gamma)\}$ for all such γ ($(p_i)_\gamma = \emptyset$ for all other γ), then $p_i \in \mathbb{P}^\alpha$.

Finally, $p_\beta \in D_i$ for all $i < \beta$ and $p_\beta \leq p$, so we're done.

Claim 4. *κ is inaccessible in $V[G]$.*

Proof. κ is regular in $V[G]$ because $\mathbb{P}$ is κ-c.c. (Claim 2). Fix cardinals $\alpha<\beta<\kappa$. We may write $G=G^\beta\times G_\beta$ where G^β is $\mathbb{P}^\beta$-generic over V and G_β is $\mathbb{P}_\beta$-generic over $V[G^\beta]$; see any treatment of product forcing, e.g. [6, VIII.1.3]. Since $\mathbb{P}^\beta$ is β-distributive (Claim 3), $V[G^\beta]$ contains no new subsets of α, so $2^\alpha<\kappa$ in $V[G^\beta]$. Since $|\mathbb{P}_\beta|<\kappa$ in $V[G^\beta]$, it is clear that 2^α remains less than κ in $V[G^\beta][G_\beta]=V[G]$.

Claim 5. *Suppose $A\subseteq\alpha<\kappa$ and $A\in V[G]$. Then $A\in V[G_\beta]$ for some $\beta<\kappa$.*

Proof. This is standard using the κ-c.c.; see for example Lemma VIII.5.14 of [6], which is similar.

For the remainder of the proof we return to the metatheory.

Claim 6. *For all $\phi\in F^0$, the following is provable in T:*

$$(\forall\delta<\kappa)(\forall\bar a\in R_\kappa^{V[G_\delta]})(\exists\gamma<\kappa)(\forall\bar\alpha\in[X-\gamma]^n)$$
$$(\forall\bar\beta\in[X-\gamma]^n)[V[G_\delta]\models\phi(\bar a,\bar\alpha,\mathbb{P})\leftrightarrow\phi(\bar a,\bar\beta,\mathbb{P})].$$

Proof. Fix $\phi\in F^0$; we work in T. Fix $\bar a\in R_\kappa^{V[G_\delta]}$. In V, let

$$D=\{p\in\mathbb{P}_\delta:(\forall\gamma<\kappa)(\exists\bar\alpha\in[X-\gamma]^n)(p\text{ decides }\phi(\bar a,\bar\alpha,\mathbb{P}))\},$$

where we confuse elements of $V[G_\delta]$ with their names in the forcing language (for notational simplicity). We show D is dense. Fix $p\in\mathbb{P}_\delta$. For all $\bar\alpha\in[X]^n$ choose $p_{\bar\alpha}\leq p$ such that $p_{\bar\alpha}$ decides $\phi(\bar a,\bar\alpha,\mathbb{P})$. Since $|\mathbb{P}_\delta|<\kappa$ we may choose κ-many $\bar\alpha$ such that $p_{\bar\alpha}$ is constant and the set of all α_0 has supremum κ, and this $p_{\bar\alpha}$ belongs to D. Now it follows easily from the indiscernibility property (*) of X that for all $p\in D$, either

$$(\forall\bar\alpha\in[X]^n)(p,\bar a\in R_{\alpha_0}\to p\Vdash\phi(\bar a,\bar\alpha,\mathbb{P}))$$

or

$$(\forall\bar\alpha\in[X]^n\ (p,\bar a\in R_{\alpha_0}\to p\Vdash\neg\phi(\bar a,\bar\alpha,\mathbb{P})).$$

Since D is dense, the conclusion follows.

Claim 7. *Suppose $\phi\in F^m$, $m\leq n$. Then the following are theorems of T.*

(i) ϕ is equivalent in R_κ^+ to a formula of the form $(\text{aa}\alpha_0)\cdots(\text{aa}\alpha_{m-1})\theta$ where $\theta\in F^0$.

(ii) $$(\forall\bar a\in R_\kappa^+)(\exists\gamma<\kappa)(\forall\bar\alpha\in[X-\gamma]^{n-m})$$
$$(\forall\bar\beta\in[X-\gamma]^{n-m}(R_\kappa^+\models\phi(\bar a,\bar\alpha)\leftrightarrow\phi(\bar a,\bar\beta)).$$

Proof. First we prove (ii) for $m=0$, working within T. Given $\phi=\phi(\bar x,\bar y)\in F^0$ and $\bar a\in R_\kappa^+$, we choose $\beta<\kappa$ such that $\bar a\in V[G_\beta]$, by Claim 5. It is routine to check that $\mathbb{P}^\beta$ is homogeneous in $V[G_\beta]$, so for all $\bar{\alpha} \in [X]^n$, either $\mathbb{P}^\beta \Vdash [R_\kappa \models \phi(\bar{a}, \bar{\alpha})]$ or $\mathbb{P}^\beta \Vdash [R_\kappa \models \neg\phi(\bar{a}, \bar{\alpha})$, where here R_κ refers to R_κ in $V[G]$. So for $\psi = \phi$ or $\psi = \neg\phi$,

$$(\forall \gamma < \kappa)(\exists \bar{\alpha} \in [X-\gamma]^n)\mathbb{P}^\beta \Vdash [R_\kappa \models \psi(\bar{a}, \bar{\alpha})].$$

Claim 6 then gives the desired result, since the predicate "$\mathbb{P}^\beta \Vdash [R_\kappa \models \psi(\bar{a}, \bar{\alpha})]$" is definable from $\bar{a}$, β, $\bar{\alpha}$, and $\mathbb{P}$.

Next, we prove (ii) restricted to formulas ϕ of the form $\mathrm{aa} z_0 \cdots \mathrm{aa} z_{m-1} \theta(\bar{z}, \bar{x}, \bar{y})$, where $\theta \in F^0$ and $|\bar{y}| = n - m$. By the case $m = 0$ proved above, either

(1) $\quad R_\kappa^+ \models \mathrm{aa} \bar{y}\ \mathrm{aa} \bar{z}\ \theta(\bar{z}, \bar{a}, \bar{y})$, or

(2) $\quad R_\kappa^+ \models \mathrm{aa} \bar{y}\ \mathrm{aa} \bar{z}\ \neg\theta(\bar{z}, \bar{a}, \bar{y})$.

This suffices, since (2) implies $R_\kappa^+ \models \mathrm{aa} \bar{y}\ \neg \mathrm{aa} \bar{z}\ \theta(\bar{z}, \bar{a}, \bar{y})$.

It remains to prove (i); then (ii) follows from (i) together with the case of (ii) just shown. The proof is by induction on formulas. The atomic step is trivial, and the negation step follows from the case of (ii) proved above. The cases $\phi_1 \wedge \phi_2$ and $\mathrm{aa} z\, \phi$ are obvious, so it suffices to check that

$$R_\kappa^+ \models \forall x\ \mathrm{aa}\alpha_0 \cdots \mathrm{aa}\alpha_{m-1}\theta \leftrightarrow \mathrm{aa}\alpha_0 \cdots \mathrm{aa}\alpha_{m-1}\ \forall x \in R_{\alpha_0}\ \theta,$$

for all $\theta \in F^0$ with parameters in R_κ^+. The reverse direction is an easy exercise, so let us assume $R_\kappa^+ \models \forall x\ \mathrm{aa}\bar{\alpha}\ \theta$. It follows that in the *standard* interpretation of 'aa',

$$R_\kappa^+ \models \forall x\ \mathrm{aa}\bar{\alpha}\ (\bar{\alpha} \in [X]^m \to \theta)$$

and hence (also in the standard interpretation)

$$R_\kappa^+ \models \mathrm{aa}\alpha_0\ \forall x \in R_{\alpha_0}\ \mathrm{aa}\alpha_1 \cdots \mathrm{aa}\alpha_{m-1}\ (\bar{\alpha} \in [X]^m \to \theta),$$

by the closure of the closed unbounded filter under diagonal intersections. By the κ-completeness of this filter,

$$R_\kappa^+ \models \mathrm{aa}\alpha_0 \mathrm{aa}\alpha_1 \cdots \mathrm{aa}\alpha_{m-1}\ \forall x \in R_{\alpha_0}\ (\alpha \in [X]^m \to \theta),$$

again in the standard intepretation. Since X is stationary, we have

$$R_\kappa^+ \models (\forall \gamma)(\exists \bar{\alpha} \in [X-\gamma]^m)\ \forall x \in R_{\alpha_0}\ \theta.$$

Finally, the case $m = 0$ of (ii) implies

$$R_\kappa^+ \models (\exists \gamma)(\forall \bar{\alpha} \in [X-\gamma]^m)\ \forall x \in R_{\alpha_0}\ \theta,$$

and the proof of Claim 7 is complete.

Finally, we prove that for all ϕ in $[\mathrm{ZF(aa)} + \mathrm{DET} + \mathrm{AC}] \cap F^n$, $T \vdash (R_\kappa^+ \models \phi)$; this combined with Claim 1 proves the theorem. Since κ is inaccessible in $V[G]$ (Claim 4), the ZFC schemas hold in R_κ^+. The other axioms of ZF(aa) (cf. [5]) trivially hold in R_κ^+, except that the argument for the $\forall$ step in proving Claim 7(i) is appropriate for proving the validity of the diagonal intersection axiom in R_κ^+. That $R_\kappa^+ \models \mathrm{DET} \cap F^n$ is an easy exercise using Claim 7; see for example the proof of Claim 7(ii) restricted to formulas $\mathrm{aa}\bar{\alpha}\ \theta$ with $\theta \in F^0$.

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