A COMBINATORIAL THEOREM AND ENDOMORPHISM RINGS OF ABELIAN GROUPS II

Saharon Shelah (*)

University of Jerusalem

§0 Introduction

This paper was originally part of [Sh 8] . It was separated for technical reasons and partly extendend, particularly in §§5,6. However we do not require knowledge of the first part.

Let us first deal with the combinatorics. In [Sh 2], [Sh 5] we pointed out that combinatorial proofs from [Sh 1], chap.VIII, should be useful for proving the existence of many non-isomorphic structures as rigid indecomposable systems. We applied this in [Sh 3] for separable

(*) The author would like to thank the United States Israel Binational Science Foundation for partially supporting this research.

p-groups illustrating the impossibility of a characterization of such groups by reasonable invariants. In [Sh 2] we built a rigid Boolean algebra in every $\lambda > \aleph_0$; see also [Sh 7] for more results and details. The main idea of the following proof is taken from [Sh 1], chap.VIII, Th. 2.6. We will continue with the combinatorics of [Sh 4], which has been utilized by Dugas and Göbel in [DG 1], [DG 2] and by Göbel and Shelah in [GS]. The nicest feature of these proofs was the fact that they were carried out in ZFC. Their main drawbacks were:

(i) The algebraic objects had strong limit singular cardinal numbers of not small cofinality.

(ii) The combinatorics was not separated from the proof; so analogous proofs have to repeat it.

(iii) The combinatorics contained things specific for modules, so that it is not immediately applicable to other structures.

The combinatorics in this paper is designed to overcome these drawbacks without using extra axioms of set theory. In section 1 we deal with the combinatorics for λ with uncountable cofinality. This is accompanied with explanations for the case of the endomorphism rings of separable (abelian) p-groups. This is, in fact, repetitions of [Sh 8]. In section 2 we deal with the combinatorics for λ with cofinality $\aleph_0$ and end with conclusions for all λ. In section 6 we point out some improvements.

Let us turn to abelian group theory. The existence of indecomposable and even endo-rigid groups was stressed in Fuchs [Fu] ; see there for previous history. Fuchs [Fu], with some help of Corner,

Sh:227

proved the existence of indecomposable torsion-free abelian groups in every cardinal less than the first strongly inaccessible cardinal. Later Fuchs replaced the bound by the first measurable cardinal and Shelah [Sh 3] proved the existence of such groups in every cardinal. Eklof and Mekler [EM] proved, assuming V=L and λ regular, not weakly compact, the existence of strongly λ-free indecomposable groups of power λ. They used Jensen's work on L, more specifically the diamond on non reflecting (=sparse) stationary subsets S of $\{\delta<\lambda : \mathrm{cf}\,\delta = \aleph_0\}$. The main algebraic fact they used was as follows.

(*) If $G = H^1 \oplus H^2$ and $G = \bigcup G_n$, $G_n \subseteq G_{n+1}$, with G_n and G_{n+1}/G_n free (abelian) groups, then for some group G' extending G, G'/G_n is free for each n; but the decomposition of G does not "extend" to one of G' or even one of $G' \oplus G''$ (G'' free).

Dugas improved [EM], replacing indecomposable by endo-rigid. Hence his algebraic tool was like (*), replacing $H^1 \oplus H^2$ by an endomorphism of G. Then Shelah [Sh 6] proved the existence of strongly λ-free endo-rigid abelian groups of power λ for $\lambda = \aleph_1$ under the hypothesis $2^{\aleph_0} < 2^{\aleph_1}$ or more generally for λ satisfying ∇_λ. The set theory used rested on Devlin and Shelah [DS]. Note that $\exists\lambda\ \nabla_\lambda$ is not probable in ZFC. The main algebraic fact needed was as follows.

(**) If $G = \bigcup G_n$ with G_n and G_{n+1}/G_n free, $a,b \in G$ and $b \notin a\mathbb{Z}$, then there are groups H^1, H^2 extending G with H^i/G_n free, and there are no endomorphisms h_i of H^i such that $h_i(G) \subseteq G$, $h_1 \restriction G = h_2 \restriction G$ and $h_i(a) = b$.

Earlier Corner [C] dealt with a stronger problem, asking which rings can be represented as End(G). He proved that every reduced torsion-free countable ring R is representable. Dugas and Göbel [DG], following [Sh 6], used ∇_λ and removed Corner's countability restriction and added the very natural condition that "the p-adic integers cannot be embedded into the additive group of R". The combinatorics was as in [Sh 6], but the groups G in (**) were replaced by R-modules. The algebra rests on the notion <u>cotorsion-free</u>. Later Dugas and Göbel [DG 2] proved this result in ZFC, but as they use the method of [Sh 4] they obtain R-modules of strong limit λ with cf $\lambda > |R|$. More details on the history can be found in [Fu], [DG], [DG 1], [DG 2]. The following papers are now based on the combinatorial result developed in the following sections: Corner and Göbel [CG], Göbel and Shelah [GS 1], [GS 2] and [Sh 10].

In section 5 we will apply the combinatorial proposition to obtain the following

<u>0.1. Theorem</u>: Let R be a ring whose additive group R^+ is cotorsion-free, i.e. R^+ is reduced and has no subgroups isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or to the p-adic integers. For $\lambda = \lambda^{\aleph_0} > |R|$ there is an abelian group G of cardinality λ whose endomorphism ring is isomorphic to R and as an R-module it is $\aleph_1$-free.

We can relax the demands on R^+ and may require that G extends a suitable group G_0 such that R is realized by End(G) modulo a suitable ideal of "small" endomorphisms.

Let us turn to p-groups. Here we merely complete [Sh 8]; see the history there. We refer to Pierce [P] and Fuchs [Fu] for End(G) of a separable p-group G, small endomorphisms and $E_s(G)$, cf. also [Sh 4] and see Dugas and Göbel [DG 1] for the representation of all suitable rings R as $End(G)/E_s(G)$ in the case of strong limits λ with $cf\lambda > |R|$. In [Sh 8] we proved the following theorem for λ of cofinality $> \aleph_0$ and in section 3 we will complete this for any $\lambda \geq |R|^{\aleph_0}$.

<u>0.2. Theorem</u>: Let R be a ring whose additive group is the completion of a direct sum of copies of the p-adic integers. If $\lambda^{\aleph_0} \geq |R|$ then there exists a separable p-group G with basic subgroup of cardinality λ and $R \cong End(G)/E_s(G)$. As usually we get $End(G) = E_s(G) \oplus R$.

In section 4 we will show the necessity of the cardinality restriction. If $2^{\aleph_0} \leq \lambda < \lambda^{\aleph_0}$ and G has essential power λ, then $End(G)/E_s(G)$ has power 2^{λ}. For $\lambda < 2^{\aleph_0}$ the problem is independent.

We should like to fulheartedly thank Rüdiger Göbel and Luigi Salce for taking care of the typing of this paper.

<u>Notation</u>: Groups are always abelian. The letters G,H and sometimes K,L,M are reserved for abelian groups, or modules. Let R be a ring and R^+ its additive group. We fix h for homomorphism and f,g for general functions. Let $\mathbb{Z}$ denote the integers, $\mathbb{Q}$ the rationals, I_p the p-adic numbers, where p is a prime, $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}_{p^\infty}$ the quasi-cyclic divisible p-group. We use λ,μ,κ for infinite cardinals, n,m,k,l (and

sometimes i,j) for integers and $i,j,\alpha,\beta,\gamma,\delta,\xi,\zeta$ for ordinals (δ usually limit). Further ω is the first infinite ordinal. Let $A \subseteq^* B$ denote that A-B is finite.

§1 The combinatorial principle

1.1 Context: Let $\lambda > \kappa$ be fixed infinite cardinals. We shall deal with the case $\mathrm{cf}\,\lambda > \aleph_0$, $\lambda^{\aleph_0} = \lambda^{\kappa}$ and usually $\kappa = \aleph_0$. Let L be a set of function symbols, each with $\leq \kappa$ places, of power $\leq \lambda$. Let $\mathfrak{M}$ be the L-algebra freely generated by $\underline{\underline{T}} \stackrel{\mathrm{def}}{=} {}^{\kappa >}\lambda = \{\eta : \eta$ a sequence of length $< \kappa$ of ordinals $< \lambda\}$. We could replace $\underline{\underline{T}}$ by a set of urelements and let $\mathfrak{M}$ be the family of sets hereditarily of cardinality $\leq \kappa$ built from those urelements. For $\eta \in \underline{\underline{T}} \cup {}^{\omega}\lambda$ let $\mathrm{orco}(\eta) = \{\eta(n): n < \omega\}$, for a sequence $\bar{\eta} = \langle \eta_i : i < \beta \rangle$ let $\mathrm{orco}(\bar{\eta}) = \bigcup_{i<\beta} \mathrm{orco}(\eta_i)$, for $a = \tau(\bar{\eta}) \in \mathfrak{M}$ let $\mathrm{orco}(a) = \mathrm{orco}(\bar{\eta})$, and $\mathrm{orco}(\langle a_i : i < \beta \rangle) = \bigcup_{i<\beta} \mathrm{orco}(a_i)$, and similarly for a set. Now $\underline{\underline{T}}$ is naturally a tree and we consider the members of ${}^{\omega}\lambda$ as its branches.

1.2. Explanation: We shall explain here how this is used for the construction of a separable reduced p-group with a predetermined ring of endomorphisms modulo the small endomorphisms.

So let R be a ring with R^{+} the p-adic completion of a direct sum of copies of the p-adic integers I_p^{+}. Let B be $\bigoplus_{\eta \in \underline{\underline{T}}} R x_\eta$ with $p^{l(\eta)+1} x_\eta = 0$, i.e. B is an R-module freely generated by $x_\eta (\eta \in \underline{\underline{T}})$ except that $p^{l(\eta)+1} x_\eta = 0$. Let $\hat{B}$ be the torsion-completion of B; we can represent its elements as $\sum_{\eta \in \underline{\underline{T}}} p^{l(\eta) \dot{-} m} a_\eta x_\eta$, where $m < \omega$, each a_η belongs to R, $\{\eta : p^{l(\eta) \dot{-} m} a_\eta x_\eta \neq 0,\ l(\eta) \leq k\}$ is finite for each k, and $n \dot{-} m = \max\{0, n-m\}$ with the natural equality and addition.

Identify x_η with η and the sum above with appopriate members of $\mathfrak{M}$, hence $\hat{B}$ is a subset of $\mathfrak{M}$. Note that each $y \in \hat{B}$ is a countable sum of $p^{l(\eta) \dot{-} m} a_\eta x_\eta$, hence it depends on only countably many members of $\underset{=}{T}$.

Our desired group G will be an R-submodule of $\hat{B}$ containing B. So there is a natural embedding of R into End(G) and we identify $a \in R$ with the endomorphism $x \mapsto ax$. As we want $R = End(G)/E_s(G)$, we will need for every endomorphism h of G some $a \in R$ such that h-a is small. Remember that h is small iff h maps $\hat{B}$ into B. We shall try to "kill" the other endomorphisms by the right choice of G.

1.3. Definition

1) Let L_n be fixed vocabularies (=signatures), $|L_n| \le \kappa$, $L_n \subseteq L_{n+1}$, (with each predicate function symbol finitary for simplicity), $P_n \in L_{n+1} - L_n$ monadic predicates.

2) Let J_n be the family of sets (or sequences) of the form $\{\langle f_l, N_l\rangle : l \le n\}$ satisfying

a) $f_l : {}^{l \ge}\kappa \to \underset{=}{T}$ is a tree embedding, i.e.

(i) f_l is length preserving, i.e. $f_l(\eta)$ has the same length as $\eta \in {}^{l \ge}\kappa$;

(ii) f_l is order preserving, i.e. for $\eta, \nu \in {}^{l \ge}\kappa$, $\eta < \nu$ iff $f_l(\eta) < f_l(\nu)$.

b) f_{l+1} extends f_l (when $l+1 \le n$).

c) N_l is an L'_l-model of power $\le \kappa$, $N_l \subseteq \mathfrak{M}$, where $L'_l \subseteq L_l$.

d) $L'_{l+1} \cap L_l = L'_l$ and $N_{l+1} \restriction L'_l$ extends N_l.

e) If $P_m \in L'_{m+1}$ and $m<l\leq n$, then $P_m^{N_l} = |N_m|$.

f) Rang $(f_l) - \bigcup_{m<l}$ Rang (f_m) is included in $|N_l| - \bigcup_{m<l} |N_m|$.

3) Let J_ω be the family of pairs (f,N) sets (or sequences $\{(f_l,N_l) : l<\omega\}$ such that $\{(f_l,N_l) : l<n\}$ belongs to J_n for $n<\omega$.

4) Let J'_ω be the family of (f,N) such that for some $\{(f_l,N_l) : l<\omega\}$ $f = \bigcup_{l<\omega} f_l$, $N = \bigcup N_n$ (i.e. $|N| = \bigcup_{n<\omega} |N_n|$, $L(N) = \bigcup_n L(N_n)$ and $N \restriction L(N_n) = \bigcup_{n\leq m<\omega} N_m \restriction L(N_n)$).

5) For any $(f,N) \in J'_\omega$ let (f_n,N_n) be as above (it is easy to show that (f_n,N_n) is uniquely determined, - notice d), e) in 2)).

6) Let $J'_n = \{(f_n,N_n)$: for some (f_l,N_l) $(l<n)$, $\{(f_l,N_l) : l\leq n\} \in J_n\}$, and we adopt the conventions of 4).

7) Usually we identify J_i and J'_i (for $i\leq\omega$).

8) A branch of Rang(f) or of f (for f as in 3)) is just $\eta \in {}^\omega\lambda$ such that for every $n<\omega$, $\eta \restriction n \in$ Rang(f).

1.4. Explanation of our strategy

We will obtain $W = \{(f^\alpha,N^\alpha) : \alpha<\alpha^*\}$, so that every branch η of f^α converges to some $\zeta(\alpha)$, $\zeta(\alpha) \in \mathrm{cf}\,\lambda$ non-decreasing. We have a free object generated by $\underline{\underline{T}}$ (B in our case) and by induction on α we define elements a_α and structures B_α (p-groups in our case) increasing continuously such that $B_{\alpha+1}$ extends B_α and $a_\alpha \in B_\alpha$. As usual $B_{\alpha+1}$ is "generated" by B_α and a_α, and a_α is in the completion of B_0. Every element will "depend" on few ($\leq\kappa$) members of $\underline{\underline{T}}$, and a_α is specially chosen: The set $Y_\alpha \subseteq \underline{\underline{T}}$ on which a_α "depends" is $Y^0_\alpha \cup Y^1_\alpha$ where Y^0_α is bounded below $\zeta(\alpha)$ (i.e. $Y^0_\alpha \subseteq {}^{\omega>}\zeta$ for some $\zeta<\zeta(\alpha)$) and Y^1_α is a branch of f^{α} (or something similar). See more in 1.8.

<u>1.5.</u> <u>Definition of the game</u>:

We define, for $W \in J_{\omega}$ a game $\underline{\underline{Gm}}(W) = \underline{\underline{Gm}}_{\lambda,\kappa}(W)$, which lasts ω moves:

In the n-th move player I chooses f_n, a tree-embedding of ${}^{n\geq}\kappa$ into ${}^{n\geq}\lambda$, extending $\bigcup_{l<n} f_l$ such that $\mathrm{Rang}(f_n) - \bigcup_{l<n} \mathrm{Rang}(f_l)$ is disjoint to $\bigcup_{l<n} |N_l|$. Then player II chooses N_n such that $\{(f_l, N_l): l \leq n\} \in J_n$. At the end player I wins if $(\bigcup_{n<\omega} f_n, \bigcup_{n<\omega} N_n) \in W$.

<u>1.6.</u> <u>Remark</u>: We shall be interested in W such that player I wins the game, but W is thin. Sometimes we need a strengthening of the second player in two respects: he can force (in the n-th move) $\mathrm{Rang}(f_{n+1}) - \mathrm{Rang}(f_n)$ to be outside a "small" set, and in the zero move he can determine an arbitrary initial segment of the play.

<u>1.7.</u> <u>Definition</u>: We define, for $W \subseteq J_{\omega}$, a game $\underline{\underline{G}}'m(W)$ which lasts ω moves (but in the context of §2 we make a small change). In the zero move player I chooses f_0, a tree embedding of ${}^{0\geq}\kappa$ into ${}^{0\leq}\lambda$ and player II chooses $k<\omega$ and $\{(f_l, N_l): l \leq k\} \in J_k$. In the n-th move $(n>0)$ player I chooses f_{k+n}, a tree embedding of ${}^{(k+n)\geq}\kappa$ into ${}^{(k+n)\geq}\lambda$, with Rang $(f_{k+n}) - \bigcup_{l<k+n}$ Rang (f_l) disjoint to $\bigcup_{l<k+n} N_l \cup \bigcup_{l<n} X_l$ and player II chooses N_{k+n} such that $\{(f_l, N_l): l \leq k+n\} \in J_{k+n}$ and $X_n \subseteq \underline{\underline{T}}$, $|X_n| < \lambda$.

<u>1.8.</u> <u>Remark</u>: What do we want from W? Adding an element for each (f,N)

we want to "kill" every undesirable endomorphism. For this W has to encounter every possible endomorphism, and this will follow from "W a barrier". For this $W = J_\omega$ is good enough, but we also want W to be thin enough so that various demands will have small interactions. For this serves disjointness and some further restrictions.

1.9. Definition:

1) We call $W \subseteq J_\omega$ a strong barrier if player I wins in $\underline{\underline{G}}m(W)$ and even $\underline{\underline{G}}'m(W)$; which just means he has a winning strategy.

2) We call W a barrier, if player II does not win in $\underline{\underline{G}}m(W)$ and even does not win in $\underline{\underline{G}}'m(W)$.

3) We call W disjoint if for any distinct $(f^l, N^l) \in W$ $(l=1,2)$ f^1 and f^2 have no common branch.

1.10. Explanation: What is the aim of W being a barrier or disjoint? Suppose h will be an undesirable endomorphism of G. If W is a barrier, for some $(f,N) \in W$ $N \restriction L_0 = (|N|, h \restriction |N|)$ and $(|N|, h \restriction |N|)$ is a "good approximation" of h. This is true as otherwise we can describe a winning strategy for II in $\underline{\underline{G}}m(W)$. If for each such (f^*, N^*) there is no $y \in G$ satisfying the equations $h(a_\alpha)$ should satisfy, then $h \restriction |N|$ cannot extend to an endomorphism of G. This follows already for $W = J_\omega$. But we want a tight control over the elements in G, this is done using disjointness, (1.4) and more. In order to derive the existence of a strong disjoint barrier, we first define a strategy for player I and only then define W. We note

1.11. Observation:

1) If $\lambda^\kappa = \lambda$ then there is a one-to-one function cd from J_ω onto λ.

2) If $\lambda^\kappa = \lambda^{\aleph_0}$, then there are functions cd_n from J_n into λ such that

a) if $m<n$ we can compute $cd_m(\langle (f_l, N_l): l<m \rangle)$ from $cd_n(\langle (f_l, N_l): l<n \rangle)$,

b) if $\langle (f^0_l, N^0_l): l<\omega \rangle \neq \langle (f^1_l, N^1_l): l<\omega \rangle$ are from J_ω, then for every large enough n $cd_n(\langle (f^0_l, N^0_l): l<n \rangle) \neq cd_n(\langle (f^1_l, N^1_l): l<n \rangle)$,

c) if $\lambda^\kappa = \lambda$ then cd_n is one-to-one.

3) There is a function $pr: \lambda \longrightarrow \lambda$ which is onto and for every $\alpha < \lambda$ there are λ many $\beta < \lambda$ with $pr(\beta) = \alpha$.

Remark: We shall use the functions cd_n only when $cf \lambda > \aleph_0$.

Proof: We should only note that $|J_\alpha| = \lambda^\kappa$ for $\alpha \leq \omega$.

1.12. Lemma: If $\lambda^\kappa = \lambda^{\aleph_0}$, then there is a strong disjoint barrier W.

Proof: First we define the winning strategy for player I and later W. In the strategy we code the play. Suppose $\lambda^{\aleph_0} = \lambda$ hence $\lambda^\kappa = \lambda$ for the moment. For $n=0$ player I has no choice. Let $n>0$ and $\langle (f_l, N_l): l<n \rangle$ be the play so far. Then player I defines his move f_n, a tree embedding from ${}^{n \geq}\kappa$ into $\underline{\underline{T}}$ such that it extends $\bigcup_{l<n} f_l$. For $\eta \in {}^n\kappa$ let $f_n(\eta) = f_{n-1}(\eta \restriction (n-1))^\frown \langle \gamma_\eta \rangle$ such that

(i) $f_n(\eta) \notin \bigcup_{l<n} X_l$;

(ii) $pr(\gamma_\eta) = cd_n(\langle (f_l, N_l): l<n \rangle)$;

(iii) $\eta \neq \nu \in {}^n\kappa$ implies $\gamma_\eta \neq \gamma_\nu$.

This is possible as by 1.11 (3), condition (ii) has λ many solutions, whereas $|X_l| < \lambda$ for $l<n$ and for (iii), define γ_η by induction on n for some well-ordering on ${}^n\kappa$; so $\leq \kappa$ many ordinals are excluded. Now let $W = \{<\bigcup_l f_l, \bigcup_l N_l> : <(f_l, N_l) : l<\omega> \text{ is a play of } \underline{\underline{G}}'_m \text{ in which}$ player I uses the strategy defined above$\}$. Trivially W is a barrier. Why is it disjoint? If η is a branch of f for $(f,N) \in W$, then by (ii) above we can reconstruct the play from η.

<u>1.13.</u> <u>The existence lemma</u>:

1) Suppose $\lambda^{\aleph_0} = \lambda^\kappa$, cf $\lambda > \kappa$ and $C^* \subseteq \lambda$ closed unbounded. Then there is $W = \{(f^\alpha, N^\alpha) : \alpha < \alpha^*\} \subseteq J_\omega$ and a function $\zeta : \alpha^* \to C^*$ such that

a) W is a strong disjoint barrier.

b) For $\alpha < \beta < \alpha^*$, $\zeta(\alpha) \leq \zeta(\beta)$.

c) $\mathrm{cf}(\zeta(\alpha)) = \aleph_0$ for $\alpha < \alpha^*$.

d) Every branch of $\mathrm{Rang}(f^\alpha)$ is an increasing sequence converging to $\zeta(\alpha)$.

e) For every $n < \omega$ for some $\xi < \zeta(\alpha)$, $\mathrm{orco}(N^\alpha_m) \subseteq \xi$.

f) If $\alpha + \kappa^{\aleph_0} \leq \beta < \alpha^*$ and η is a branch of $\mathrm{Rang}(f^\beta)$, then $\eta \upharpoonright k \notin N^\alpha$ for some $k < \omega$.

g) If $\lambda = \lambda^\kappa$ we can demand: if η is a branch of $\mathrm{Rang}(f^\alpha)$ and $\eta \upharpoonright k \in N^\beta$ for all $k < \omega$ (where $\alpha, \beta < \alpha^*$), then $N^\alpha \subseteq N^\beta$.

2) We can demand also:

h) For every stationary $S \subseteq \{\delta < \lambda \mid \mathrm{cf}\, \delta = \aleph_0\}$, $\{(f^\alpha, N^\alpha) : \alpha < \alpha^*, \zeta(\alpha) \in S\}$ is a disjoint barrier.

1.14. Remark: By (e) the ordinal $\zeta(\alpha)$ is the "infinity" of N^{α} and by (d) the branches of Rang(f^{α}) converge to infinity. Conditions (f) and (g) strenghten disjointness.

Proof: 1) Again we first define the strategy of player I, using (1.11) and the functions $cd_n (n<\omega)$ from there. For n=0 player I has a unique choice. So suppose n>0 and $\langle (f_l, N_l): l<n \rangle$ is the play so far. We have to define f_n extending f_{n-1}. Let $\gamma_\eta < \lambda$ for $\eta \in {}^n\kappa$, be such that (i)-(iv) below hold and then let $f_n(\eta) = f_{n-1}(\eta \restriction (n-1)^\frown \langle \gamma_\eta \rangle)$ for $\eta \in {}^n\kappa$.

The requirements are

(i) $f_n(\eta) \notin \bigcup_{l<n} X_l$;

(ii) $pr(\gamma_n) = cd_n(\langle (f_l, N_l): l<n \rangle)$;

(iii) if $\eta \neq \nu \in {}^n\kappa$, then $\gamma_\eta \neq \gamma_\nu$;

(iv) $\gamma > \sup(orco|N_{n-1}|)$, moreover there is a member of C* in the interval.

Note that $\sup(orco|N_{n-1}|) < \lambda$ as we have assumed $cf\,\lambda > \kappa$ and $orco|N_{n-1}|$ is a set of power $\leq \kappa$ (as $\|N_n\| \leq \kappa$, definition of $\mathfrak{M}$ and of orco (see 1.1)). The requirement (ii) has λ solutions, (i), (iv) exclude less than λ of them, and (iii) requires that we have ${}^n\kappa$ distinct ordinals satisfying (i), (ii), (iv). So we can carry on the definition, and then let W be as in the proof of 1.2. Clearly W is a strong disjoint barrier and (a) holds.

We define a function ζ from W to λ : $\zeta((f,N)) = \sup(orco(Rang(f)))$. By (iv) above, $\zeta((f,N)) \in C^*$, and for every branch η of Rang(f),

$\sup(\mathrm{orco}(\eta)) = \zeta((f,N))$. Now we define by induction on i $(<|W|^+)$ for each ordinal $\zeta \in C^*$ a set W_i^ζ, $W_i^\zeta \subseteq W^\zeta \stackrel{\mathrm{def}}{=} \{(f,N): (f,N) \in W, \zeta((f,N)) = \zeta\}$ such that:

(α) $W_0^\zeta = \emptyset$;

(β) W_i^ζ is increasing continuous (in i);

(γ) $W_{i+1}^\zeta - W_i^\zeta$ has cardinality $\leq \kappa^{\aleph_0}$;

(δ) If $(f,N) \in W_{i+1}^\zeta$, $(f',N') \in W^\zeta$, η a branch of $\mathrm{Rang}(f')$ and $\{\eta \restriction k: k < \omega\} \subseteq N$, then $(f',N') \in W_{i+1}^\zeta$;

(ε) If $W \neq W_i^\zeta$, then $W_{i+1}^\zeta \neq W_i^\zeta$.

This is straightforward; only (iv) requires the following observation: $|N|$ contains at most $\kappa^{\aleph_0}$ branches of $\underline{\underline{T}}$. By ($\varepsilon$), $W^\zeta = \bigcup_i W_i^\zeta$.

To finish the proof of 1.13 (1), choose for every ζ, i a well-ordering $<^*_{\zeta,i}$ of W_i^ζ of order type $\leq \kappa^{\aleph_0}$. Define a well-ordering $<^*$ of W such that $\zeta((f^0,N^0)) < \zeta((f^1,N^1))$ implies $(f^0,N^0) <^* (f^1,N^1)$ and $(f^0,N^0) \in W_i^\zeta$, $(f^1,N^1) \in W_j^\zeta$, $i<j$ implies $(f^0,N^0) <^* (f^1,N^1)$ and if $(f^0,N^0), (f^1,N^1) \in W_i^\zeta$, $(f^0,N^0) <^* (f^1,N^1)$ if and only if $(f^0,N^0) <^*_{\zeta,i} (f^1,N^1)$. Now let $\{(f^\alpha,N^\alpha): \alpha < \alpha^*\}$ be a list of the members of W such that for $\alpha<\beta<\alpha^*$, $(f^\alpha,N^\alpha) <^* (f^\beta,N^\beta)$, and let $\zeta(\alpha) \stackrel{\mathrm{def}}{=} \zeta((f^\alpha,N^\alpha))$. Now we have already cheeked (a), and (b) is trivially satisfied. We have already observed that a branch η of $\mathrm{Rang}(f^\alpha)$ converges to $\zeta(\alpha) = \zeta((f^\alpha,N^\alpha))$, hence (c) and (d) hold. The demand (iv) above guarantees (e), and the condition (δ) (and the choice of $<^*_{\zeta,i}$ to have order type $\leq \kappa^{\aleph_0}$) ensures (f). For (g) use (1.11)(2)(c). In case 2 the same construction works. It can also easily be proved by taking elementary submodels M_η of $(H(\lambda),\in)$ which contains all relevant information, λ large enough, $M_n \in M_{n+1}$, $\sup(\lambda \cap \bigcup_n M_n) \in S$.

1.15 Remark: We may also want to build $2^{(\lambda^{\aleph_0})}$ objects of power $\lambda^{\aleph_0}$, each one like G, with no homomorphisms from one to the other, except the necessary ones. This can be done alternatively as follows.

1) Together with G we also build G' extending G and elements $a_i \in G'$ $(i < \lambda^{\aleph_0})$ and let for $A \subseteq \lambda^{\aleph_0}$, $G_A = \langle B \cup \{a_i : i \in A\}\rangle$. We then try to guarantee that $A \nsubseteq B$ implies that there are only necessary homomorphisms from G_A to G_B. This clearly suffices.

2) For each $A \subseteq^{\omega} \lambda$ we build G^A. We use W not only to approximate endomorphisms of G^A, but also look for N^{α} which is a submodel of (G^A, G^B, h) where $A \neq B \subseteq^{\omega} \lambda$, h a homomorphism from G^A to G^B. For (N^{α}, f^{α}) we try to add an element y to G^A and omit the corresponding type from G^B which prevents h to map y into G^B. Note that N^{α} "knows" $A \cap N^{\alpha}$, $B \cap N^{\alpha}$, but A,B themselves.

Sh:227

§2 The combinatorial principle for λ of cofinality $\aleph_0$

If we want to get a p-group (or similar algebraic objects) of density character λ, the combinatorics of §1 does not help us. Here we shall deal with this case. We also formulate a conclusion which holds for every λ, $\lambda^{\aleph_0} = \lambda^{\kappa}$ thus enables us to give a uniform proof.

2.1. Context: As we want to deal not only with the main case, $\lambda > \aleph_0 = \mathrm{cf}\lambda$, but also with $\lambda = \aleph_0$ we will have two possibilities

(1) $\lambda = \aleph_0 = \lambda_n$ and $\lambda_n^* = n!$ for each n.

(2) $\lambda > \aleph_0$, $\mathrm{cf}\,\lambda = \aleph_0$, $\kappa < \lambda$, $\lambda = \sum_{n<\omega} \lambda_n$, $\lambda_n = \lambda^*$ regular and $\kappa < \lambda_n < \lambda_{n+1} < \lambda$.

In both cases let D be a non-principal ultrafilter on ω, $\underset{=}{T} = \bigcup_{n<\omega} \prod_{m<n} \lambda_m$ and let L, $\mathfrak{M}$ be as in 1.1.

In the definition of $\underset{=}{G}'m(W)$ we make a change and demand $|X_l| < \lambda^*_{k+l-1}$ and stipulate $\lambda^*_{0-1} = 1$.

2.2. Definition: For f,g $\in {}^{\omega}\mathrm{Ord}$ (i.e. a function from ω to the class of ordinals) let $f <_D g$ iff $\{n: f(n)<g(n)\} \in D$ (and similarly $\leq_D$).

2.3. Claim: There are a regular cardinal μ, $\lambda \leq \mu \leq \lambda^{\aleph_0}$ and functions g_ξ in $\prod_{n<\omega} \lambda_n$ for $\xi < \mu$ such that:

(a) for $\xi < \gamma < \mu$, $g_\xi <_D g_\gamma$;

(b) for every $g \in \prod_{n<\omega} \lambda_n$ for some ξ, $g <_D g_\xi$;

(c) $g_\xi(n)$ is divisible by λ^*_{n-1}.

<u>Proof</u>: It is well known that the ultraproduct $\prod_{n<\omega}(\lambda_n,<)/D$ is a linear order, and let $\langle g_\xi : \xi<\mu\rangle$ be an increasing unbounded sequence (μ regular). Now μ is at most the power of the ultraproduct which is $\lambda^{\aleph_0}$, and as D is non-principal easily $\mu>\lambda$. Taking care of (c) is easy.

2.4. <u>Remark</u>: If $\lambda>2^{\aleph_0}$ we can choose D the filter of co-bounded subsets of ω, and (2.3) still holds; see [Sh 9].

2.5. <u>Remark</u>: The g_ξ are needed to slice $\prod_{n<\omega}\lambda_n/D$ similar to the range of the function ζ in Th. 1.13.

2.6. <u>Notation</u>: We identify any set $a \subseteq \mathfrak{M}$ with the function $\chi_a \in {}^\omega\mathrm{Ord}$, $\chi_a(n) = \sup(\lambda_n \cap \mathrm{orco}(a))$ using $<_D$.

2.7. <u>Observation</u>: 1) If $\lambda^{\aleph_0}=\lambda^\kappa$ there are functions cd_n from J_n to λ^*_{n-1} such that:

(a) If $m \le n$ then we can compute $cd_m(\langle(f_l,N_l): l<m\rangle)$ from $cd_n(\langle(f_l,N_l): l<n\rangle)$.

(b) If $\langle(f^0_l,N^0_l): l<\omega\rangle \neq \langle(f^1_l,N^1_l): l<\omega\rangle$ are in J_ω, then $cd_n(\langle(f^0_l,N^0_l): l<\omega\rangle \neq cd_n(\langle(f^1_l,N^1_l): l<\omega\rangle)$ for every large enough n.

(2) There are functions pr'_n from λ_n to λ^*_{n-1} such that for every $\alpha<\lambda_n$ divisible by λ^*_{n-1}, $\gamma<\lambda^*_{n-1}$ there are λ_n ordinals β satisfying $\alpha<\beta<\alpha+\lambda^*_n$, $pr'_n(\beta)=\gamma$.

Sh:227

<u>2.8.</u> <u>The existence theorem</u>: Suppose $\lambda^{\aleph_0} = \lambda^{\kappa}$.

1) Then there are $W=\{(f^{\alpha},N^{\alpha}): \alpha < \alpha^*\} \subseteq J_{\omega}$ and a function $\zeta: \alpha^* \longrightarrow \mu$ such that:

a) W is a disjoint barrier.

b) For $\alpha<\beta<\mu$, $\zeta(\alpha) \leq \zeta(\beta)$.

c) $cf(\zeta(\alpha)) = \aleph_0$ for every $\alpha<\alpha^*$.

d) For every branch η of $Rang(f^{\alpha})$, $\eta <_D g_{\zeta(\alpha)}$ but for every $\xi < \zeta(\alpha)$ $g_{\xi} <_D \eta$.

e) For every $n<\omega$ for some $\xi < \zeta(\alpha)$, $orco(|N^{\alpha}_{m}|) <_D g_{\xi}$.

f) If $\zeta(\alpha) = \zeta(\beta)$, $\alpha + \kappa^{\aleph_0} \leq \beta$, η a branch of $Rang(f^{\beta})$, then for some k, $\eta \restriction k \notin N_{\alpha}$.

2) For every stationary set $S \subseteq \{\delta<\mu: cf\,\delta = \aleph_0\}$ $\{(f^{\alpha},N^{\alpha}): \alpha<\alpha^*, \zeta(\alpha) \in S\}$ is a disjoint barrier.

<u>Proof</u>: We first define for every ordinal $\zeta<\mu$ of cofinality $\aleph_0$ a subset W^{ζ} of J_{ζ}. W^{ζ} is the set of $(f,N) \in J_{\omega}$ satisfying:

(i) for $\eta \in {}^{n+1}\kappa$, $pr'_{m}(f_{m}(\eta)(n) = cd_{m}(\langle (f_{l},N_{l}): l \leq n \rangle)$.

(ii) conditions (d) and (e) of 2.8. (1) hold with ζ taking the place of $\zeta(\alpha)$.

Now we let $W \stackrel{def}{=} \bigcup_{\zeta<\mu} W^{\zeta}$. The choice of the list $\{(f^{\alpha},N^{\alpha}): \alpha<\alpha^*\}$ of W and the function is just as in the proof of 1.13, except for the proof of one half of (a): W is a barrier, and to this the rest of the proof is dedicated. For notational simplicity we shall deal with the game $\underline{\underline{Gm}}(W)$ only. Suppose St* is a winning strategy for player II in $\underline{\underline{Gm}}(W)$. Let ϑ be a large enough regular cardinal. We can choose elementary submodels M_n of $(\underline{H}(\lambda), \in)$, such that St*, $\mathfrak{M}$, J_ω, $\langle g^*_\xi : \xi<\mu\rangle$ belong to each M_n, $\{i: i\le\kappa\} \subseteq |M_n|$, $M_n \in M_{n+1}$ and $\|M_n\| = \kappa$. Let $\zeta(n)=\sup(|M_n| \cap \mu)$ and $\zeta = \bigcup_{n<\omega} \zeta(n)$. As $M_n \in M_{n+1}$ (and $\|M_n\| = \kappa<\lambda<\mu$, μ regular) clearly $\zeta(n) \in M_{n+1}$, hence $\zeta(n)<\zeta(n+1)$. Also the function f^n, $Dom(f^n) = \omega$, $f^n(k) = \sup(|M_n| \cap \lambda_k)$ belongs to M_{n+1}; and as by 2.3 for some ξ, $f^n <_D g_\xi$, there is such $\xi \in M_{n+1}$, hence (as $\xi < \zeta(n+1)$, $g_\xi \le_D g_{\zeta(n+1)}$) clearly $f^n <_D g_{\zeta(n+1)}$.

Now we shall define a play $\langle (f_l, N_l): l<\omega\rangle$ of the game $\underline{\underline{G}}m$. We shall define f_n, N_n by induction on n so that

(*) $\langle (f_m, N_m): m<n\rangle$ form an initial segment of a play of $\underline{\underline{G}}m(W)$ in which player II uses the strategy St*, and it belongs to M_n.

For n=0 player I has a unique choice for f_0, and clearly $f_0 \in M_0$. As St* $\in M_0$ clearly $N_0 \in M_0$.

So suppose $\langle (f_m, N_m): m\le n\rangle$ satisfies (*). Let $k^0_n=\mathrm{Max}\{l: l\le n+1$ and $l=0$ or $g_{\zeta(0)}(n),\dots,g_{\zeta(l)}(n)<g_\zeta\}$ (k^0_n is well-defined as the set is finite and non-empty). We shall define f_{n+1} such that for $\eta \in {}^{n+1}\kappa$, $f_{n+1}(\eta) = f_n(\eta\restriction n)\widehat{\ }\langle\gamma_\eta\rangle$, where $g_{\zeta(k_n)} < \gamma_\eta < g_{\zeta(k_n)} + \lambda^*_{n-1} < \lambda_n$, (i) above holds and $\gamma_\eta \ne \gamma_\nu$ if $\eta\ne\nu$. By 2.7. this is possible. Moreover we can choose $f_{n+1} \in M_{n+2}$ as $M_{n+1} \in M_{n+2}$ (and $\langle (f_l, N_l): l\le n\rangle$, f_{n+1} belongs to M_{n+2}) also N_{n+1} belongs to M_{n+2}.

So $\langle (f_l, N_l): l<\omega\rangle \in J_\omega$ is the result of a play of $\underline{\underline{G}}m(W)$ in which player II uses his strategy St*. However we shall show that he loses the play, i.e. $\langle (f_l, N_l): l<\omega\rangle \in W$, thus getting the desired contradiction.

In fact $\langle (f_l, N_l): l<\omega\rangle \in W^\zeta$; the least trivial part is why condition (d) holds. Now $\eta <_D g_{\zeta(\alpha)}$ as for each branch η of $\mathrm{Rang}(\bigcup f_m)$, for every n, $\eta(n) < g_{\zeta(k(m))}(n) + \lambda^*_{m-1}$; now if $k(n) > 0$ $g_{\zeta(k(m))}(n) + \lambda^*_{m-1} \leq g_{\zeta}(n)$ (see 2.3) and $\{n: \eta(n) < g_{\zeta}(n)\} \in D$ as required.

On the other hand for each $m < \omega$, $A_m = \{n < \omega: n > m$ and $g_{\zeta(m)}(n) < g_{\zeta}(n)\} \in D$, hence $\bigcap_{m \leq 1} A_m \in D$, and for each $n \in \bigcap_{m \leq 1} A_m$, $g_{\zeta(1)}(n) \leq g_{\zeta(k_n)}(n)$ but $\eta(n) > g_{\zeta(k_n)}(n)$, hence $\{n: g_{\zeta(1)} \leq \eta(n)\} \supseteq \bigcap_{m \leq 1} A_m$, hence $g_{\zeta(1)} <_D \eta$.

2) The proof is similar except that we can demand $\zeta \in S$.

2.9. Remark: 1) In 2.8 (1)(d) we can demand $(\forall n)\, \eta(n) < g_{\zeta(\alpha)}(n)$ as w.l.o.g. $g_{\zeta}(n) > 0$ for every ζ and n, and in the proof when $k_n = 0$ use 0 instead $g_{\zeta(k^0_m)}(n)$.

2) In the proof we could have chosen an infinite $A^* \subseteq \omega$, $A^* \notin D$, and restrict (i) to $n \in A^*$. In this case we can demand only $|X_1| < \lambda_{k+1-1}$ in the present variant of the definition of $\underline{\underline{G}}\mathrm{m}'(W)$.

In fact we can conclude from 1.13, (2)-(8) an assertion, which is the one we shall use in §3, thus getting a uniform proof of Th. 3.5 for all λ. So here we are in the context common to §1 and §2.

2.10. Conclusion: Suppose $\lambda^\kappa = \lambda^{\aleph_0}$, $\lambda > \kappa$. Then there is $W = \{(f^\alpha, N^\alpha): \alpha < \alpha^*\}$ such that:

(a) W is a disjoint barrier (for λ, κ).

(b) For every $\alpha \leq \beta < \alpha^*$ and branch η of $\mathrm{Rang}(f^\beta)$ and $n < \omega$, for every large enough k, $\eta \upharpoonright k \notin N^\alpha_m$.

(c) If $\alpha + \kappa^{\aleph_0} \leq \beta < \alpha^*$ and η is a branch of $\mathrm{Rang}(f^\beta)$, then for every large enough k, $\eta \restriction k \notin N^\alpha$.

Proof: If (1.13) or (2.8) apply, this is immediate. The remaining case is $\aleph_0 < \mathrm{cf}\,\lambda \le \kappa$ (but the main case is anyhow $\kappa = \aleph_0$). For them note

2.11. Fact: 1) If $\kappa < \lambda^* \le \lambda$, $(\lambda^*)^{\aleph_0} = \lambda^\kappa$, we can repeat 1.13 (and everything else in §1) letting $\underline{\underline{T}}$, $\mathfrak{M}$ be defined using λ, by letting player II choose embeddings into $\bigcup_{n<\omega} {}^n(\lambda^*)$ (with the obvious changes).
2) The same holds for 2.8.

Sh:227

§3 On separable p-groups with predetermined endomorphism ring

We prove here that for suitable R, $R \cong \mathrm{End}(G)/E_s(G)$ for some G of density character λ, $|G| = \lambda^{\aleph_0}$.

It might be possible to predetermine $\dim p^n G[p] = \lambda'_n \leq \lambda$ with $\lambda = \lim\sup \lambda'_n$. Replace B in (3.5) by B′ using $p^{k(n)+1} x^n_\eta = 0$ for $p^{n+1} x^n_\eta = 0$ with some sequence k(n) such that $\lambda_n \leq \lambda'_{k(n)}$. Extend B′ to obtain the right G; but we have not checked the details.

3.1. Definition: A separable p-groups G is an abelian p-group such that every element belongs to a finite direct summand. We will deal with separable groups in §§3 and 4.

3.2. Definition: A map h from G into G_0 is called small if for every m and for every large enough n $h(p^n G[p^m]) = 0$ (where $p^n G[p^m] = \{p^n x : x \in G, p^{n+m} x = 0\}$).

3.3. Definition: For an abelian group G let End(G) be the ring of endomorphisms of G and let $E_s(G)$ be the set of all small endomorphisms of G.

Trivially

3.4. Lemma: $E_s(G)$ is an ideal of End(G).

<u>3.5. Theorem</u>: Let R be a ring such that R^+ is the p-adic completion of a free p-adic module. Suppose $\lambda \geq |R|$.

1) There is a separable p-group G with End(G) isomorphic to $R \oplus E_s(G)$ and $|G| = \lambda^{\aleph_0}$ and G has a basic subgroup of power λ.

2) There are groups $G_i (i < 2^{(\lambda^{\aleph_0})})$ as in (1) such that homomorphisms from G_i to G_j, $i \neq j$ are small.

<u>3.5.A. Remark</u>: We can replace $\lambda \geq |R|$ by $\lambda^{\aleph_0} > |R|$ (or even $\forall n\ \lambda^{\aleph_0} > |R^+/p^n R^+|$), $\lambda > \aleph_0$, without change in the proof. If $\lambda^{\aleph_0} = |R^+/p^n R^+|$ we get (1) and with more care, also (2).

<u>Proof</u>: By [Sh 5] we can restrict ourselves to the case $cf\,\lambda = \aleph_0$. We can choose regular $\lambda_n < \lambda$, $\aleph_0 < \lambda_n < \lambda_{n+1}$ such that $\lambda = \sum_{n<\omega} \lambda_n$. Let $\kappa = \aleph_0$. We shall use freely the notation of §2 in general and of (2.8) in particular.

<u>Stage A</u>: Let B be the R-module freely generated by $\{x_\eta : \eta \in \underline{\underline{T}}\}$ with $p^{l(\eta)+1} x_\eta = 0$. So every $b \in B$ is of the form $\sum_{\eta \in \underline{\underline{T}}} r_\eta x_\eta (r_\eta \in R)$ where $\{\eta : r_\eta \neq 0\}$ is finite. Let H be the torsion-completion of B so that any $b \in H$ is a formal infinite sum $\sum_{\eta \in \underline{\underline{T}}} r_\eta\, p^{l(\eta) \dot{-} m} x_\eta$ such that for every l, $\{\eta : \eta \in \underline{\underline{T}},\ l(\eta) \leq l \text{ and } r_\eta p^{l(\eta) \dot{-} m} x_\eta \neq 0\}$ is finite. This implies that $\underline{d}(b) \overset{\text{def}}{=} \{\eta \in \underline{\underline{T}} : r_\eta p^{l(\eta) \dot{-} m} x_\eta \neq 0\}$ is countable. We have $\sum_\eta r^1_\eta\, p^{l(\eta) \dot{-} m(1)} x_\eta = \sum_\eta r^2_\eta\, p^{l(\eta) \dot{-} m(2)} x_\eta$ iff for every η, $r^1_\eta\, p^{l(\eta) \dot{-} m(1)} - r^2_\eta\, p^{l(\eta) - m(2)}$ is divisible by $p^{l(\eta)+1}$, and we can define H as the set of those sums with the obvious addition (see [Fu]).

Sh:227

Note that r_η is not uniquely determined by b, but $r_\eta p^{l(\eta) \dot- m} x_\eta$ is. Note that H extends B and is torsion-complete. This means that $\sum_{n<\omega} y_n$ exists if $(\exists l)(\forall n)$ $p^l y_n = 0$ and for every l for every large enough n, y_n is divisible by p^l. In fact, if $y_n = \sum_{\eta \in \underline{T}} a^n_\eta x_\eta$, $\sum_{n<\omega} y_n = \sum_\eta (\sum_n a^n_\eta) x_\eta$ and $\sum_n a^n_\eta$ exists by the choice of R.

Note that

(A)(1) $\underline{d}(y+z) \subseteq \underline{d}(y) \cup \underline{d}(z)$.

(A)(2) $\underline{d}(\sum_{n<\omega} y_n) \subseteq \bigcup_{n<\omega} \underline{d}(y_n)$ when $\sum_{n<\omega} y_n$ is well-defined.

(A)(3) if $y \in H$ is divisible by p^n, then $l(\eta) \geq n-1$ for all $\eta \in \underline{d}(y)$.

(A)(4) if $y = \sum_{\eta \in \underline{T}} r_\eta x_\eta$, $r_\eta x_\eta$ is uniquely determined by η and we say "x_η appears in y as $r_\eta x_\eta$"; really $r_\eta + p^{l(\eta)+1} R$ is uniquely determined.

We shall build an R-module G, such that $B \leq G \leq H$. Let G^+ denote the additive group of G, and homomorphisms will be $\mathbb{Z}$-homomorphisms.

Stage B: Recall from [Fu]:

Fact: If $B \leq G \leq H$, then every endomorphism h of G^+ extends uniquely to an endomorphism of H^+. If h is small, the range of the extension is in G^+.

Stage C: The construction

We now define the R-module G. First identify members of H with members of $\mathfrak{M}$. If $A \subseteq H$, let SG(A) be the R-submodule of H generated by A. We define by induction on $\alpha < \alpha^*$ the following:

(1) The truth value of $\alpha \in J_0$, $\alpha \in J_1$, $\alpha \in J_2$ so that exactly one holds.

(2) For $\alpha \in J_0 \cup J_1$ we define elements $a_{\alpha,l}$ $(l<\omega)$, b_α of H.

(3) For $\alpha \in J_0 \cup J$ we fix a branch ν_α of $\mathrm{Rang}(f^\alpha)$ such that

(4) $a_{\alpha,m} = a^0_{\alpha,m} + a^1_{\alpha,m}$ (both in H) where $a^1_{\alpha,m} = \sum_{k \geq m} p^{k-m} x_{\nu_\alpha \restriction k}$ so that $pa^i_{\alpha,l+1} - a^i_{\alpha,l} \in B$.

(5) $a^0_{\alpha,l} \in N^\alpha_0$

(6) $\nu_\alpha \neq \nu_\beta$ for $\beta<\alpha$

For $J \subseteq J_1 \cap \alpha$ let $G^\alpha = SG(B \cup \{a_{\beta,l} : \beta<\alpha, \beta \in J_0 \cup J\})$ and let $G^{\alpha^*}_J = G_J$

(7) $b_\beta \notin G^\alpha_J$ for $\beta<\alpha$, when $\beta \in J_0$; and $a_{\beta,l} \in G^\alpha_J$ iff $\beta \in (J_0 \cup J) \cap \alpha$ when $\beta<\alpha$.

(8) If $N^\alpha = (|N^\alpha|, L, h,\dots)$, L a subgroup of $G_{J\cap\alpha} \cap |N^\alpha|$ for some $J \subseteq J_1 \cap \alpha$, h an endomorphism of L, and we can find ν_α, $a^i_{\alpha,l}$ $(i=0,1;\ l<\omega)$, b_α, $l(\alpha)<\omega$ satisfying (2)-(7) for $\alpha+1$ (stipulating $\alpha \in J_0$) such that for every endomorphism h′ of H extending h, $h'(a_{\alpha,l(\alpha)}) = b_\alpha$, then $\alpha \in J_0$

(9) If the hypothesis of (8) fails, but we can find a_α, $a_{\alpha,l}$, ν_α such that conditions (2)-(7) hold, then $\alpha \in J_1$; otherwise $\alpha \in J_2$.

<u>Remark</u>: Really $J_2 = \emptyset$

<u>Stage D</u>: <u>Claim</u>:

1) Every element x of G (where $J \subseteq J_1$) can be represented as $\sum_{l=1}^{k} r_l a_{\alpha_l,m} + b$, where $b \in B$ and $\alpha_k<\dots<\alpha_1$.

2) If $k>0$, $\zeta(\alpha_1) = \zeta(\alpha_k)$, then $\mathrm{Rang}(\nu_{\alpha_1}) \subseteq^* \underline{d}(x)$; moreover there is $m^*<\omega$ such that $\underline{d}(x) \cap \{\rho \in \underline{\underline{T}} : \nu_{\alpha_1} \restriction m^* \leq \rho\}$ is equal to

$\{\nu_{\alpha_1}\restriction i : m^*\le i<\omega\}$, and if $x=\sum_{\eta\in \underline{T}} r_\eta x_\eta$, then $\eta=\nu_{\alpha_1}\restriction l$, $l\ge m$ implies $r_\eta x_\eta = r_1 p^{l(\eta)-m} x_\eta$.

3) The representation in 1) is totally determined by m, hence k, $\langle\alpha_1,\dots,\alpha_k\rangle$ depend on x only, and if $\sum_{l=1}^{k} r'_l a_{\alpha_l,m(1)} + b'$ is another representation, then $r_l p^{m(2)-m} a_{\alpha_l,m(2)} = r'_l p^{m(2)-m(1)} a_{\alpha_l,m(1)}$ for every large enough m(2).

Remark: The claim explains the peculiar choice of the a_α : by having special domains for them we have specific severe restraints of the domain of any $x\in G$.

Proof: 1) By the definition of G we can represent x as $\sum_{l=1}^{k} r_l a_{\alpha_l,m_l} + b$, $b\in B$, $\alpha_l<\alpha^*$. Of course w.l.o.g. $\alpha_k<\dots<\alpha_1$. If k=0 we finish, otherwise let $m = \mathrm{Max}\{m_1,\dots m_k\}$. As $a_{\alpha,l} - pa_{\alpha,l+1}\in B$ we can easily transform this to $\sum_{l=1}^{k} r_l a_{\alpha_l,m} + b'$.

2) and 3) Easy.

Stage E: Claim: If h is an endomorphism of H mapping B into G_{J_1} such that for no $r\in R$ with h−r a small endomorphism of G_J, then for some $a^*_l\in H$, $pa^*_{l+1} - a^*_l\in B$ and for some l_1 $h(a^*_{l_1})\notin SG(G_{J_1}\cup\{a^*_l: l<\omega\})$.

Proof: The proof is by cases.

Case I: There is $l(*)<\omega$ such that for every $n<\omega$ there are $r\in R$, $\eta\in\underline{T}$ satisfying $l(\eta)\ge n$ and $\underline{d}(\,h(p^{l(\eta)-l(*)}rx_\eta))\not\subseteq\{\eta\}$. In this case we can easily choose by induction on $i<\omega$ $r_i \in R$, ν_i, $\eta_i \in \underline{T}$ and $n_i<\omega$ such that:

(i) $\nu_i \in \underline{d}(\ h(\ p^{l(\eta)-l(*)}\ r_i x_{\eta_i}))$

(ii) $l(\eta_i) > n_i > \mathrm{Max}\{i+l(\eta_j)+l(*)+l(\nu_j) : j<i\}$

(iii) $\nu_i \neq \eta_i$.

Now for every function $s \in {}^{\omega}\mathbb{Z}$ and $k<\omega$ we define $a_k^s = \sum_{k\le i<\omega} s(i)p^{l(\eta_i)-l(*)-k}\, r_i x_{\eta_i}$.

We shall prove that for some s, $\langle a_k^s : k<\omega\rangle$ satisfies the requirements on $\langle a_k^* : k<\omega\rangle$ in the claim. Note that $a_k^s - pa_{k+1}^s \in B$ for every k (and $a_k^s \in H$).

We now define by induction on n, K_n such that

(i) K_n is a countable subset of G_J,

(ii) $K_n \subseteq K_{n+1}$

(iii) for $i<\omega$, $r_i x_{\eta_i} \in K_0$

(iv) if $x \in K_n$ then $h(x) \in K_{n+1}$

(v) if $x,y \in K_n$, then $x+y \in K_{n+1}$, $x-y \in K_{n+1}$

(vi) If $\rho \in \underline{d}(y)$, $y,z \in K_m$, $\alpha<\alpha^*$, $r \in R$, $\rho < \nu_\alpha$ and $\underline{d}(z-ra_{\alpha,m}) \cap \{\eta \in \underline{T} : \rho<\eta\}=\emptyset$, then $ra_{\alpha,m} \in K_{m+1}$ (note that $ra_{\alpha,m}$ is uniquely determined). Let $K=\bigcup_{n<\omega} K_n$ and $W^* = \bigcup\{\underline{d}(y) : y \in K\}$; so clearly W^* is countable.

We want to prove that $h(a_0^s) \notin SG(G_J \cup \{a_i^s : i<\omega\})$ for some s. We suppose this does not hold for a given s and shall get restrictions on s, so that this will guide us in choosing an appropiate s.

As $a_i^s - pa_i^s \in G_J$ for some j_s, and $r^s \in R$, $h(a_0^s) - r^s a_{j_s}^s \in G_J$. Hence for every $y \in K \cap G_J$ $h(a_0^s) - r^s a_{j_s}^s + y \in G_J$, hence applying stage D(1)

(*) $h(a_0^s) - r^s a_{j_s}^s + y = r_1 a_{\alpha_1,m} + r_2 a_{\alpha_2,m} +\dots+ r_k a_{\alpha_k,m} + b$, where

$\alpha_k < \ldots < \alpha_1 < \alpha^*$, $b \in B$.

Of course r_1, α_1, m and b depend on s,y. Note that by stage D(2)(3) s and y determine k, α_l (l=1,..,k) uniquely, and essentially r_1. Let $\underline{S}^0_y = \{s \in {}^{\omega}\mathbb{Z}$: for y we get α_1 minimal}, $S^1_y = \{s \in {}^{\omega}\mathbb{Z} : h(a^s_0) = r^s a^s_{j_s} + r^s y \in B\}$. Our argument will rest on the computation of the domain. As for the left hand side

$$\underline{d}(h(a^s_0) - r^s a^s_{j_s} + y) \subseteq \underline{d}(h(a^s_0)) \cup \underline{d}(r^s a_{j_s}) \cup \underline{d}(y) \subseteq$$

$$\underline{d}(h(\sum_{i<\omega} s(i) p^{l(\eta_i) \dot{-} l(*)} r_i x_{\eta_i})) \cup \underline{d}(r^s a_{j_s}) \cup W^* \subseteq$$

$$\underline{d}(\sum_{i<\omega} s(i) p^{l(\eta_i) - l(*)} h(r_i x_{\eta_i})) \cup \{\eta_i : i<\omega\} \cup W^* \subseteq$$

$$\underline{d}(h(r_i x_{\eta_i})) \cup W^* \subseteq \bigcup_{i<\omega} \underline{d}(h(r_i x_{\eta_i})) \cup W^* \subseteq W^*.$$

Now we apply stage D(1) on the right hand side of (*). So there is $\rho \in \underline{\underline{T}}$ in the domain of the right hand side (hence of W*) such that:

$(**)_1$ $\underline{d}(r_1 a_{\alpha_1,m} + \ldots + r_k a_{\alpha_k,m} + b) \cap \{\eta \in \underline{\underline{T}} : \rho \le \eta\}$ is a branch (except the first $l(\rho)$ elements) (in fact $\{\nu_{\alpha_1} \restriction i : l(\rho) \le i < \omega\}$);

$(**)_2$ for some $r \in R$ every $\nu \in \{\eta \in \underline{\underline{T}} : \rho \le \eta\}$ appears in $r_1 a_{\alpha_1,m} + \ldots + r_k a_{\alpha_k,m} + b$ as $p^{i-m} r x_\nu$ or $0 x_\nu$.

We can substitute the left hand side of (*) and get $(**)'_1$, $(**)'_2$.

This is quite a strong restriction. From s (and remembering (i),(ii),(iii) above) we know much on $\underline{d}(h(a^s_0) - r_s a^s_{j_s})$, and so get a contradiction.

Now ${}^{\omega}\mathbb{Z}$ is a topological space having the Baire property. As W* is countable and the ρ above is necessarily in W*, it suffices to prove

$(+)_1$ for every $\rho \in W$, $y \in K$ the set of $s \in \underline{S}^0_y$ for which $(**)'_1$, $(**)'_2$ hold is meagre (= of the first category);

$(+)_2$ for every $y \in K$ the set $\underline{S}^1_y$ is meagre.

So let $\rho \in W$, $u \in \bigcup_m {}^m\mathbb{Z}$, $n<\omega$ and we should find a function $t \in \bigcup_m {}^m\mathbb{Z}$ extending u such that no $s \in {}^\omega\mathbb{Z}$ extends t.

If $y + \sum_{i\in \mathrm{Dom}\, s} s(i)p^{l(\eta_i)-l(*)} h(r_i x_{\eta_i})$ does not satisfy $(**)'_1$ or $(**)'_2$, then there is $k<\omega$ such that this is examplified even if we restrict ourselves to $\eta \in \underline{\underline{T}}$ of length $\leq k$. However $i>k+l(*)$ implies $l(\eta_i)\geq i>k+l(*)$ which implies that $p^{l(\eta_i)-l(*)} h(r_i x_{\eta_i})$ is divisible by p^{k+1} (in H). Hence every $\eta \in \underline{d}(p^{l(\eta_i)-l(*)} h(r_i x_{\eta_i}) \pm r_i x_{\eta_i})$ has lenght $>k$. So if $n' = \mathrm{Max}\{n, k+l(*)\}$ and $t(i)=0$ whenever $n\leq i<n'$, t extends s, then t is as required. This argument is a little inaccurate, because $\sum_{i\in \mathrm{Dom}\, s} r_s r_i x_{\eta_i}$ is represented in the left hand side of (*), but as this involves finitely many members of $\underline{\underline{T}}$ it can be correct trivially.

So we can assume that $y + \sum_{i\in \mathrm{Dom}\, s} s(i)p^{l(\eta_i)-l(*)} h(r_i x_{\eta_i})$ satisfies $(**)'_1$, $(**)'_2$ if we ignore $\{\eta_i : i \in \mathrm{Dom}\ s\}$. Moreover w.l.o.g. this holds for every $t \in \bigcup_{n<\omega} {}^n\mathbb{Z}$ extending s.

Now if we can find $s_a, s_b, s_c \in \underline{S}^0_y$ as exemplified by distinct $\alpha_{1,a}$, $\alpha_{2,b}$, $\alpha_{3,c}$, such that s_a, s_b, s_c extend s and $|\{\alpha_{1,a}, \alpha_{2,b}, \alpha_{3,c}\}| \geq 3$ let $s^* \in {}^\omega\mathbb{Z}$ be defined by $s^*(n) = s_a(n) - s_b(n) + s_c(n)$ and then $t = s^* \upharpoonright n$, n large enough, extends s as required. Otherwise only say $\alpha_{1,a}, \alpha_{2,b}$ appear and the contradiction is even easier.

So it remains to prove $(+)_2$, i.e. suppose some $\underline{S}^1_y$ is not meager and get a contradiction. As easily $\underline{d}(b) \subseteq W^*$, clearly $b \in K$, so w.l.o.g. $b=0$. Looking at "how x_{ν_i} may appear" this is trivial.

Let $h(rx_\eta) = a_{r,\eta} x_\eta + h_1(rx_\eta)$ $(a_{r,\eta} \in R)$. Note that $h_1(rx_\eta)$ is not necessarily an endomorphism but a function from G_J to G_J which is small.

<u>Case II</u>: For some $l(*)$ for every $m>l(*)$ there are $\eta^a, \eta^b \in \underline{T}$, $l(\eta^a)$, $l(\eta^b)>m$ and $r^a, r^b \in R$ such that $p^{l(\eta^a)-l(*)} a_{r^a,\eta^a} x_{\eta^a} + p^{l(\eta^b)-l(*)} a_{r^b,\eta^b} x_{\eta^b}$ is not an R-multiple of $r^a x_{\eta^a} + r^b x_{\eta^b}$.

We can easily assume $\eta^a \neq \eta^b$ (as we can try a third candidate). The proof is like case I (using $r_i^a x_{\eta_i^a} + r_i^b x_{\eta_i^b}$ instead of $r_i x_{\eta_i}$)).

<u>Case III</u>: not case I nor case II. In this case easily for some $r \in R$, $h-r$ is small.

<u>Stage F</u>: <u>Claim</u>: If $h: G_\emptyset \longrightarrow G_{J_1}$ is a homomorphism, then for some $r \in R$, $h-r$ is small.

<u>Proof</u>: Suppose there is no such r. We can extend h uniquely to an endomorphism $\hat{h}$ of H. Let $\hat{h}(\sum_{\eta \in \underline{T}} r_\eta x_\eta) = \sum_{\eta \in \underline{T}} h(r_\eta x_\eta)$ and $l(*)$, $a_l^* \in B$ $(l<\omega)$ be from stage E.

Now we can define a strategy for player II in the play $\underline{Gm}(\{(f^\alpha, N^\alpha): \alpha < \alpha^*\})$ (see §2). He plays so that $|N_m| \cap H$ is closed under h, $a_l^* \in N_0$, $h^{N_m} = h \restriction |N_m|$, $L^{H_m} = |N_m| \cap G_{J_1}$. An (f^α, N^α) is a barrier for some α, (f^α, N^α) is the result of such a play. Let $l(\alpha) = l(*)$ and ν be a branch of $\mathrm{Rang}(f^\alpha)$ not in $\{\nu_\beta : \beta < \alpha\}$. We want to show that in (8) of stage C there are $a_{\alpha,l}^0$ $(l<\omega)$, b_α as required there ($a_{\alpha,m}^1$ is already defined from ν). Ignoring requirement (7) for a moment this clearly suffices, as then $\hat{h}(a_{\alpha,l(\alpha)})$ necessarily belongs to G_{J_1}, but it is b_α and $b_\alpha \notin G_{J_1}$.

First try $a_{\alpha,m}^0 = 0$, then the only thing that can go wrong is

$h(a^0_{\alpha,1(*)}) \in SG(G^\alpha_{J_1\cap\alpha} \cup \{a^0_{\alpha,m} : m<\omega\})$, i.e. for some m and $r_1 \in R$

(1) $h(a^1_{\alpha,1(*)}) - r_1 a^1_{\alpha,m} \in G^\alpha_{J_1\cap\alpha}$.

If this fails try $a^0_{\alpha,m} = a^*_m$ and again the only thing that can go wrong is $h(a^1_{\alpha,1(*)} + a^0_{\alpha,1(*)}) \in SG(G^\alpha_{J_1\cap\alpha} \cup \{a^0_{\alpha,m} : m<\omega\})$, so for some $r_2 \in R$

(2) $h(a^1_{\alpha,1(*)} + a^0_{\alpha,1(*)}) - r_2(a^1_{\alpha,1(*)} + a^0_{\alpha,1(*)}) \in G^\alpha_{J_1\cap\alpha}$.

Subtracting (2) from (1) we get

(3) $h(a^0_{\alpha,1(*)}) - r_2 a^0_{\alpha,1(*)} + (r_1-r_2)a^1_{\alpha,1(*)} \in G^\alpha_{J_1\cap\alpha}$.

As $a^0_{\alpha,1}$, $h(a^0_{\alpha,1}) \in N^\alpha_0$, computation of the domain (using stage D) leads to $\nu=\nu_\beta$ for some $\beta<\alpha$; contradiction.

We shall denote the successful try by $a^{0,\nu}_{\alpha,1}$, $a^{1,\nu}_{\alpha,1}$. As $\nu_\beta(\beta<\alpha)$ can be a branch of $\mathrm{Rang}(f^\alpha)$ only if $\beta<\alpha<\beta+2^{\aleph_0}$ (see (f) of Th. 2.8.) all except $<2^{\aleph_0}$ branches of $\mathrm{Rang}(f^\alpha)$ will do. We still have to deal with requirement (7) from stage C. We deal with it for each β.

If $\beta+2^{\aleph_0} \le \alpha$, comparison of domains leads to a contradiction.

As $|\{\beta: \beta<\alpha<\beta+2^{\aleph_0}\}|<2^{\aleph_0}$, it suffices to prove that for each such β, $b_\beta \in SG(G^\alpha_{J_1\cap\alpha} \cup \{a^{0,\nu}_{\alpha,1} + a^{1,\nu}_{\alpha,1} : 1<\omega\}$ for at most one ν (as then all branches of $\mathrm{Rang}(f^\alpha)$ except $<2^{\aleph_0}$ will do).

So suppose $\nu^1 \ne \nu^2$ are branches of $\mathrm{Rang}(f^\alpha)$ (but $\notin \{\nu_\gamma : \gamma<\alpha\}$) and for i=1,2:

$$b_\beta \in SG(G^\alpha_{J_1\cap\alpha} \cup \{a^{0,\nu^i}_{\alpha,1} + a^{1,\nu^i}_{\alpha,1} : 1<\omega\}).$$

So (for i=1,2) for some m: $\quad b_\beta - r^i(a^{0,\nu^i}_{\alpha,m} + a^1_{\alpha,m}) \in G^\alpha_{J_1\cap\alpha}$.

Subtracting we get

$$(r^1 a^{0,\nu^1}_{\alpha,m} - r^2 a^{0,\nu^2}_{\alpha,m}) + (r^1 a^{0,\nu^1}_{\alpha,m} - r^2 a^{0,\nu^2}_{\alpha,m}) \in G^\alpha_{J_1\cap\alpha} .$$

Computing domains we get a final contradiction.

Sh:227

Stage G: Proof of the Theorem

It is easy to show that $|J_1| = \lambda^{\aleph_0}$. Now by stage F each $G_J (J \subseteq J_1)$ has no "undesirable" endomorphisms. Let $\{J^\xi : \xi < 2^{\aleph_0}\}$ be a family of subsets of J_1 with $|J^\xi - J^\zeta| = \lambda^{\aleph_0}$ for $\xi \neq \zeta$. So it suffices to prove for $\xi \neq \zeta$ that every homomorphism h from G_{J^ξ} to G_{J^ζ} is small. But by stage F h-r is small for some $r \in R$ (and clearly r is unique). Let $\alpha \in J^\xi - J^\zeta$, and consider $h(a_{\alpha,1(\alpha)})$; for every m>1 $a_{\alpha,1(\alpha)} - p^{m-1(\alpha)} a_{\alpha,m} \in B$, and we know that $(h-r)(p^{m-1(\alpha)} a_{\alpha,m}) = 0$ for every large enough $m<\omega$. So $(h-r)(a_{\alpha,1(\alpha)}) = (h-r)(y_\alpha)$ for some $y_\alpha \in B$. As $|J^\xi - J^\zeta| > \lambda + |R| = |B|$, for some $\alpha \neq \beta \in J^\xi - J^\zeta$ $y_\alpha = y_\beta$. So $(h-r)(a_{\alpha,1(\alpha)} - a_{\beta,1(\beta)}) = 0$, hence $h(a_{\alpha,1(\alpha)} - a_{\beta,1(\beta)}) = r(a_{\alpha,1(\alpha)} - a_{\beta,1(\beta)})$. But by stage D $r(a_{\alpha,1(\alpha)} - a_{\beta,1(\beta)}) \in G_{J^\zeta}$ implies r=0, so h is small.

§4 The necessity of $|G|^{\aleph_0} = |G|$ in the groups we have constructed

In the previous section (and in [Sh 8] we have constructed an abelian p-group G with a prescribed ring $R = \mathrm{End}(G)/E_S(G)$. For an arbitrary λ we build such a group of power $(|R|+\lambda^{\aleph_0})$. The restriction $|G| \geq |R|$ is obvious. The stronger restriction $|G| \geq |R|^{\aleph_0}$ is also quite necessary as we shall show in (4.2).

4.1. Context: Let G be an abelian reduced p-group. So there are λ_n, x_i^n $(i < \lambda_n)$ such that $\{x_i^n : i < \lambda_n, n < \omega\}$ generate an abelian group freely except that $p^{n+1} x_i^n = 0$, $B \subseteq G$, B is dense in G. So renaming we can assume that G is contained in $\hat{B}$, the torsion-completion of B.

So for every $x \in S$, $x = \sum_{(n,i) \in S} a_i^{n,x} x_i^n$, where $S_x \subseteq \{(n,i): n < \omega,\ i < \lambda_n\}$ is countable, $a_i^{n,x} \in \mathbb{Z}$, and for every n $\{i : (n,i) \in S_x\}$ is finite.

It is known that $G = G^0 + G^1$ (direct sum), G^1 is bounded (i.e. $(\exists n)(\forall x \in G^1)$ $p^n x = 0$) and for G^0 the cardinals λ_n satisfies $(\forall n)(\exists^\infty m)$ $\lambda_n \leq \lambda_m$. It is known that ess pow(G) = Min{λ: for every n large enough, $\lambda_n < \lambda$}.

4.2. Theorem: If G is an abelian separable reduced p-group of ess pow(G) $= \lambda > 2^{\aleph_0}$, then $\mathrm{End}(G)/E_S(G)$ has power 2^λ.

Remark: The proof will give much more explicit information.

Sh:227

Proof: It is easy to show that for $G = G^0 + G^1$, G^1 bounded, the rings $\mathrm{End}(G)/E_S(G)$ and $\mathrm{End}(G^0)/E_S(G^0)$ are isomorphic and ess pow(G) = ess pow(G^0).

So we can assume G as in (4.1), and for every n there are infinitely many m with $\lambda_n < \lambda_m$. Let $\lambda = \sum_{n<\omega} \lambda_n$.

We know that every endomorphism of G is determined by its restriction to B. Now the number of functions from B to G is $\le |G|^{|B|} \le (\lambda^{\aleph_0})^{\lambda} = 2^{\lambda}$. So $\mathrm{End}(G)/E_S(G)$ has power $\le 2^{\lambda}$.

4.3. Fact: Suppose $A_n \subseteq \lambda_n$, $A = \bigcup_n \{n\} \times A_n$, and let $G_A = \bigoplus_{(n,i)\in A} \mathbb{Z} x_i^n$. A sufficient condition for G_A to be a direct summand of G is:

(*) For every $x \in G$, $S_x \cap A$ is finite (on S_x see 4.1)

Proof of the fact: We shall define a projection h from G onto G_A: $h(x) = h(\sum_{(n,i)\in S_x} a_i^{n,x} x_i^n) = \sum_{(n,i)\in A} a_i^{n,x} x_i^n$. This is well-defined (and the result is in G_A) by (*); and the checking is easy.

4.4. Fact: There are $A_n \subseteq \lambda_n$, $\sum |A_n| = \lambda$ such that $\bigcup_{n<\omega} \{n\} \times A_n$ satisfies (*).

This suffices for 4.2. We have already proved $|\mathrm{End}(G)/E_S(G)| \le 2^{\lambda}$. Clearly by 4.3 $|\mathrm{End}(G)/E_S(G)| \ge |\mathrm{End}(G_A)/E_S(G_A)|$ and clearly $|\mathrm{End}(G_A)/E_S(G_A)| \ge 2^{\lambda}$.

Proof of 4.4.: Choose $n(k) < \omega$ for $k < \omega$ such that $n(k) < n(k+1)$, $\lambda_{n(k)} \le \lambda_{n(k+1)}$, and $\lambda = \sum_k \lambda_{n(k)}$. Now we can choose by induction on k, for every $\eta \in \prod_{m\le k} \lambda_{n(m)}$ subsets A_η of $\lambda_{n(k)}$ such that:

(i) $|A_\eta| = \lambda_{n(1(\eta))}$

(ii) for $\eta \ne \nu \in \prod_{m\le k} \lambda_{n(m)}$, $A_\eta \cap A_\nu = \emptyset$.

This is easily done. Now for every $\eta \in \prod_{k<\omega} \lambda_{n(k)}$ let A_l^η be $A_{\eta \restriction (k+1)}$ if $l=n(k)$ and $\emptyset$ otherwise. Clearly $A_l^\eta \subseteq \lambda_l$, $\sum_l |A_l^\eta| = \sum_k \lambda_{n(k)} = \lambda$, hence it suffices to prove that for some $\eta \in \prod_{k<\omega} \lambda_{n(k)}$, $A^\eta = \bigcup_l \{l\} \times A_l^\eta$ satisfies (*).

As the number of $\eta \in \prod_k \lambda_{n(k)} = \lambda^{\aleph_0}$ is $> |G| + 2^{\aleph_0}$, it suffices to prove:

(**) for every $x \in G$ the number of $\eta \in \prod_k \lambda_{n(k)}$ for which $S_x \cap A^\eta$ is infinite, is $\le 2^{\aleph_0}$.

This is easy: for suppose $\eta_i \in \prod_k \lambda_{n(k)}$ are distinct, for $i<(2^{\aleph_0})^+$, the number of possible $S_x \cap A^{\eta_i}$ is $\le 2^{\aleph_0}$ (= the number of subsets of S_x). Hence for some $i \ne j$ $S_x \cap A^{\eta_i} = S_x \cap A^{\eta_j}$ is infinite but $S_x \cap (\{l\} \times \lambda_l)$ is finite for each l. Hence for no n is $A^{\eta_i} \cap A^{\eta_j} \subseteq \bigcup_{l\le n} (\{l\} \times \lambda_l)$. But for some n $\eta_i(n) \ne \eta_j(n)$, hence easily $A^{\eta_i} \cap A^{\eta_j} \subseteq \bigcup_{l<n} \{l\} \times \lambda_l$; contradiction.

4.5. Lemma: 1) Assume $MA + 2^{\aleph_0} > \lambda$. If G is a separable (abelian) p-group, $\lambda = \text{ess pow}(G)$, then $\text{End}(G)/E_s(G)$ has power 2^λ.

2) Assume V (=the universe of set theory) is a generic extension of V' by adding λ many Cohen reals, $\lambda > \aleph_0$. Assume R is a ring, R^+ the completion of a direct sum of copies of $\underline{\underline{I}}_p^+$, $|R^+/p^n R^+| = \lambda$. Then in V there is a separable p-group G, $|G| = \lambda$, $\text{End}(G) \cong E_s(G) + R$.

Sh:227

Proof: We define a forcing notion P: $P = \{(A,B): B,A \subseteq \{x_i^n : i<\lambda_n, n<\omega\}$, $B \cap A = \emptyset$, A finite and for some finite $Y \subseteq G$, $B = \bigcup_{y \in Y} \underline{d}(y) - \omega\}$; order natural. P satisfies the countable chain condition: if $(A_\alpha, B_\alpha) \in P$ for $\alpha < \omega_1$ w.l.o.g. for some $n(*)$ $A_\alpha \subseteq \{x_i^n : i<\lambda_n, n<n(*)\}$. Note that $A_\alpha \cap \{x_i^n : i<\lambda, n<n(*)\}$ is finite for each α, the rest is by the Δ-system lemma.

So by MA we can easily get $A_n \subseteq \lambda_n$, $|A_n| = \lambda_n$ satisfying (*) of Fact 4.3, and we finish as in the proof of 4.2.

2) Left to the reader (provided that he knows what Cohen reals are).

4.6. Remark: We may wonder when we can have $|R| > \lambda^{\aleph_0}$. Now

(*) there are left ideals I_α $(\alpha<\lambda)$ of R such that

(i) $\bigcap_{\alpha<\lambda} I_\alpha = \{0\}$

(ii) if $\langle x_\alpha : \alpha < \lambda \rangle$ is a sequence of members of R satisfying $(x_\alpha + I_\alpha) \cap (x_\beta + I_\beta) \neq \emptyset$ for every $\alpha, \beta < \lambda$, then $\bigcap_{\alpha<\lambda} (x_\alpha + I_\alpha) \neq \emptyset$.

This seems necessary (if $R = \mathrm{End}(G)/E_s(G)$, $B \subseteq G$ is basic, $B = \{x_i : i<\lambda\}$, let $I_i = \{r \in R : r\, x_i = 0\}$) and sufficient (proof as in (1)) with B being the R-module freely generated by $\{x_\eta^i : i<\lambda, \eta \in \underline{\underline{T}}\}$ except $p^{l(\eta)+1} x_\eta^i = 0$, $r x_\eta^i = 0$ $(r \in I_i)$.

§5 Abelian groups with predetermined ring of endomorphisms

What can we say about End(G)=R ?. Clearly R is a ring with identity 1 and G is an R-module. If $n\cdot 1=0$, then $n\cdot G=0$ and G is a direct sum of cyclic groups. So we may discard this case. Trivially G has a divisible sub-group $D\neq 0$ iff R^+ has a divisible subgroup $\neq 0$. In this case every homomorphism h of G into D extends to an endomorphism of G, so we cannot control End(G). If G is uncountable then End(G) has cardinality $2^{|G|}$.

Hence we assume that G is reduced, i.e. G has no nonzero divisible subgroup. Define the $\mathbb{Z}$-adic metric d on it:

$$d(x,y) = \mathrm{Min}\{2^{n} : n! \text{ divides } x-y\}.$$

We can now define the completion G^c of G.

5.1. Theorem: Suppose R is a ring with 1, characteristic 0 such that R^+ is reduced. Suppose also G_0 an R-module with G_0^+ reduced. Suppose further that $\lambda^{\aleph_0}\geq|G_0|+|R|$, cf $\lambda>\aleph_0$.

1) There is an R-module K extending G_0 such that:

(a) K has cardinality $\lambda^{\aleph_0}$.

(b) K/G_0 is an $\aleph_1$-free R-module.

(c) If $h\in\mathrm{End}(K)$, we find $r\in R$ such that $h-r=h'$ is inessential, i.e. in this context $\mathrm{Rang}(h')\subseteq SG(G_0\cup A\cup G[q])$ for some finite $A\subseteq G$, $q\in\mathbb{Z}$.

2) If R^+, G_0 are cotorsion-free (i.e. in addition R^+, G_0 have no subgroups isomorphic to the additive groups of p-adic integers),

then h−r=0 in (c).

Proof:

Stage A: Let B be the free R-module generated by $G_0 \cup \{x_\eta : \eta \in \underline{\underline{T}}\}$ except the equations which hold in G_0. Let H be the completion of B; so every $y \in B$ has the form $y = \sum_{\eta \in \underline{\underline{T}}} r_\eta x_\eta + \sum_{n<\omega} g_n$, $g_n \in G_0$, $r_\eta \in R^c$ for all but finitely many $\eta \in \underline{\underline{T}}$ and $n<\omega$, $(n!)$ divides r_η and $(n!)$ divides g_n (in G_0). Note that $\underline{d}(y) = \{\eta \in \underline{\underline{T}} : r_\eta \neq 0\}$ is countable but $\underline{d}(y) \cap \prod_{n<m} \lambda_n$ may be infinite. However (A)(1),(2),(3),(4) (from (3.5)) still hold (replacing p^n by $n!$).

We can define $\underline{d}_n(y) = \{\eta \in \underline{\underline{T}} : r_\eta$ is divisible by $n\}$ and use it similarly.

Stage B: As in (3.5).

Stage C: The construction

We identify H with a subset of $\mathfrak{M}$. We define by induction on $\alpha<\alpha^*$:

1) The truth value of $\alpha \in J_0$.

2) For $\alpha \in J_0$ we define $a_{\alpha,l} \in H$ for $l<\omega$.

3) For $\alpha \in J$, a branch ν_α of Range(f^α) such that, for $\alpha \in J_0$

4) $a_{\alpha,m} = a^0_{\alpha,m} + a^1_{\alpha,m}$ (both in H), $a^1_{\alpha,m} = \sum_{k<\omega} \left(\prod_{m \le i \le k} i! \right) x_{\nu_\alpha \restriction k}$ and $(l!)\, a^i_{\alpha,l+1} - a^i_{\alpha,l} \in B$

5) $a^0_{\alpha,m} \in N^\alpha_0$

6) $\nu_\alpha \neq \nu_\beta$ for $\beta<\alpha$

Let $G^\alpha = SG(B \cup \{a_{\beta,m} : \beta<\alpha,\ m<\omega\})$.

Sh:227

7) $b_\beta \notin G^\alpha$ for $\beta<\alpha$.

8) If $N^\alpha = (|N^\alpha|, L, h, \ldots)$, L a subgroup of G^α, h an endomorphism of L, and we cannot find ν^α, $a^i_{\alpha,m}$ $(i=0,1;\ m<\omega)$, b_α, $l(\alpha)<\omega$ satisfying (2)-(7) for $\alpha+1$ (stipulating $\alpha \in J_0$) such that for every endomorphism h' of a group G', $G^\alpha \subseteq G' \subseteq H$, extending h, $h'(a_{\alpha,l(\alpha)}) = b_\alpha$, then $\alpha \in J_0$ and for every endomorphism h' of a group G', $G_\alpha \subseteq G' \subseteq H$, extending h, $h'(a_{\alpha,l(\alpha)}) = b_\alpha$.

Stage D: As in (3.5). We can note there that $G^\alpha \cap G_0^c = G_0$.

Stage E: Claim: If h is an endomorphism of $G=G^{\alpha^*}$ such that for no $r \in R$ $h-r$ is inessential, then for some $a^*_l \in H$, $la^*_{l+1} - a^*_l \in B$ and $h(a^*_l) \notin SG(G \cup \{a^*_l : l<\omega\})$.

Proof: It is done by cases:

Case I: For every finite $W \subseteq \underline{T}$ and $l<\omega$ $(l>0)$ for some $\eta, \nu \in \underline{T} - W$ and $r \in R$, $\eta \neq \nu$ and ν appears in $h(lrx_\eta)$. This is handled like case I of stage E in (3.5) (but here, in order to guarantee that ν appears in the infinite sum, we split the sum into two parts - in the first it appears, but the coefficient is not divisible by some k, in the second it appears with coefficient divisible by k).

So for some finite W* and $l^* \in \{1,2,3,\ldots\}$ for every $r \in R$, $\eta \in \underline{T} - W$, for some $a_{r,\eta} \in R$ $h(l^*rx_\eta) = a_{r,\eta} x_\eta \in SG(G_0 \cup \{x_\eta : \eta \in W^*\})$.

<u>Case II</u>: Not case I, but for every finite $W \subseteq \underline{\underline{T}}$ and $k<\omega$ for some $\eta^a, \eta^b \in \underline{\underline{T}} - W \cup W^*$, $r^a \in R$, $r^b \in R$ the following holds: $1r^a \neq 1r^b$.
As in (3.5).

<u>Case III</u>: Neither case I nor case II.

Let W^{**}, 1^{**} exemplify the failure of case II (and w.l.o.g. $W^* \subseteq W^{**}$). Clearly there is $r^* \in R^c$, such that for every $\eta \in \underline{\underline{T}} - W^* \cup W^{**}$ and $r \in R$, $1^{**}1^*a_{r,\eta} = 1^{**}r^*$. So for $\eta \in \underline{\underline{T}} - W^{**}$ $h(1^{**}1^*rx_\eta) - 1^{**}r^*rx_\eta \in SG(G_0 \cup \{x_\eta : \eta \in W^*\})$, so choosing $r=1$ ($\eta \in \underline{\underline{T}} - W^{**}$) we see that 1^* divides r^* and the result is in R, so let $r^*=1^*r^{**}$. We can conclude that for some $\eta \in \underline{\underline{T}} - W^* \cup W^{**}$, $1^{**}1^*(h(rx_\eta)-r^{**}(rx_\eta)) \in SG(G_0 \cup \{x_\eta : \eta \in W^*\})$.
So $(h-r^{**})(rx_\eta) \in SG(G_0 \cup \{x_\eta : \eta \in W^*\} \cup G[1^{**}1^*])$.

We still have to prove that $h'(SG(G_0 \cup \{x_\eta : \eta \in W^*\} \subseteq SG(G_0 \cup A \cup G(k))$ for some finite A and k, where $h'=h-r^{**}$. We can choose by induction on n finite subsets $W_n \subseteq \underline{\underline{T}}$, $U_n \subseteq \alpha^*$ and $g_n \in SG(G_0 \cup \{x_\eta : \eta \in W^*\})$ such that $W^{**} \subseteq W_n \subseteq W_{n+1}$, $U_n \subseteq U_{n+1}$ and

$$h((n!)g_n) \notin SG(G_0 \cup \{x_\eta : \eta \in W_n\} \cup \{a_{\alpha,1} : \alpha \in U_n, 1<|U_n|\}).$$

We then get a contradiction as in case I of stage E in (3.5).

<u>Stage F</u>: <u>Claim</u>: For every endomorphism h of G, for some $r \in R$, $h-r$ is inessential.

As in (3.5).

<u>Stage G</u>: K/G_0 is an $\aleph_1$-free R-module. Easy.

Proof of theorem 5.1:

1) As in (3.5).

2) So suppose h is inessential and nonzero. By stage G there is a finitely generated R-submodule L of K, $L \cong \bigoplus_{i=1}^{n} R^+$, $L \cap G_0 = \{0\}$, and $n < \omega$, $\mathrm{Rang}(n\,h) \subseteq L + G_0$, but as G is torsion-free, we can disregard n. Using projection w.l.o.g. $\mathrm{Rang}(h) \subseteq G_0$ or $\mathrm{Rang}(h) \subseteq L'$, $L' \cong R^+$. Also $\mathrm{Rang}(h)$ is complete. (Otherwise we can prove the conclusion of stage E (if $\sum \prod_{i=1}^{n} (i!) h(a_n) \notin G$, $a_n \in G$, choose $\eta \notin \{\eta_i : i\}$, η an ω-branch of $\underline{\underline{T}}$, and try $a_n^* = \sum_{m \geq n} \prod_{i=n}^{m} i!(h(a_n) + x_{\eta \restriction m})$ and also try $a_n^* = \sum_{m \geq n} \prod_{i=1}^{m} i!1x_{\eta \restriction m})$ and then get a contradiction as in stage F).

§6 Revisiting the combinatorics

The combinatorics in §1 and §2 can be strengthened and modified in various ways, which may be useful in other contexts.

6.1. Claim: In the context of section 1 or 2 suppose χ is a cardinal satisfying $\lambda^{\chi} = \lambda$, $\chi \geq \kappa^{\aleph_0} + \kappa^+$ (e.g. $\chi = (\kappa^{\aleph_0})^+$).

Then we can prove Th. 1.13 or 2.8 when weaken (b) and strengthen (f) to

(b′) W is a disjoint barrier (not necessarily strong).

(f′) If $\alpha<\beta<\alpha^*$, η is a branch of Rang(f^{β}), then for some k, $\eta \restriction k \notin N^{\alpha}$.

Proof: Let $W^* = \{(f_*^{\alpha}, N_*^{\alpha}) : \alpha<\alpha^*\}$, ζ^* be what we get applying (1.13) (or (2.8)) for λ, χ (instead of (λ,κ). We consider only those α's for which N_*^{α} codes a tree $\{(f^{\alpha,\eta}, N^{\alpha,\eta}) : \eta \in {}^{\omega>}\chi\}$ such that:

(i) $N^{\alpha}_{*,n}$ codes $\{(f^{\alpha,\eta}, N^{\alpha,\eta}) : \eta \in {}^{n\geq}\chi\}$ and includes each $|N^{\alpha,\eta}|$.

(ii) $\langle (f^{\alpha,\eta\restriction 1}, N^{\alpha,\eta\restriction 1}) : 1<l(\eta)\rangle \in J_n$ for $\eta \in {}^{n}\chi$.

(iii) Rang($f^{\alpha,\eta}$) $\subseteq$ Rang(f^{α}).

(iv) if $\eta, \nu \in {}^{(n+1)}\chi$, $\eta(1) \neq \nu(1)$, then Rang($f^{\alpha,\eta}$) $\cap$ Rang($f^{\alpha,\nu}$) $\subseteq$ Rang($f^{\alpha,\eta\restriction 1}$).

We now define by induction on α (α as above) a member ρ_α of ${}^{\omega}\chi$ such that

(*) if $\beta<\alpha<\beta+\chi^{\aleph_0}$, then for every ω-branch η of Rang($\bigcup_{k<\omega} f^{\alpha,\rho_\alpha\restriction k}$), for some $m<\omega$, $\eta(0) \notin \bigcup_{k<\omega} N^{\beta,\rho_\beta\restriction k}$.

Why is this possible? For a given α there are less than $\aleph_0$ possible β's, and for each β the number of "unsuccessful" $\rho \in {}^{\omega}\chi$ is $\leq |\bigcup_{k<\omega} N^{\beta,\rho_\beta \restriction k}| = \kappa$. So the number of unsuccessful $\rho \in {}^{\omega}\chi$ is $\leq \kappa |\{\beta : \beta < \alpha < \beta + \chi^{\aleph_0}\}|$. As $\chi \geq \kappa^{\aleph_0} + \kappa^+$ we finish the proof of (*).

Now $W = \{ (\bigcup_{k<\omega} f^{\alpha,\rho_\alpha \restriction k}, \bigcup_{k<\omega} N^{\alpha,\rho_k \restriction k}) : \alpha < \alpha^*,\ \alpha$ as above$\}$ is the required barrier (using the same function ζ^*).

We use only some $\alpha < \alpha^*$, but this is a minor point.

Why is W a barrier? Suppose player II has a winning strategy of $\underset{=}{Gm}_{\lambda,\kappa}(W)$ and we can easily describe one for $\underset{=}{Gm}'_{\lambda,\kappa}(W^*)$.

<u>6.2. Remark</u>: 1) In (6.1), we may sometimes weaken the demand on χ to $\chi \geq (\kappa^+)^{\omega}$. We need that ${}^{\omega\geq}\chi$ is not $\bigcup\{T_i : i < \chi^{\aleph_0}\}$, each T_i is closed under initial segments and $|T_i \cap {}^{\omega>}\chi| \leq \kappa$.

<u>6.3. Concluding Remark</u>: 1) We can let player II determine f_n for, say, odd n, with no significant change.

2) We can make the games last ϑ moves $(\vartheta \neq \omega)$, which gives no significant change.

3) For strong limit singular λ, we can use the theorem from Rubin, Shelah [R Sh] to get similar theorems, weakening a little the "barrier" condition. (E.g. if $\mathrm{cf}\,\lambda = \aleph_0$, we know that for any model with universe of power λ and $\chi < \lambda$ operations, there is a Δ-system tree of sub-models. Now we can list all such trees $\{T_i : i < \lambda^{\aleph_0} = 2^{\lambda}\}$ and choose by induction a branch from each, so that they are as disjoint as possible. For $\mathrm{cf}\,\lambda > \aleph_0$, $\lambda = \sum_{i<\mathrm{cf}\lambda} \lambda_i$, λ_i increasing continuous,

$2^{\lambda_i} < \lambda_{i+1} < \lambda$, we deal with each λ_j, cf$\,\delta = \aleph_0$; see also [Sh 7] .

4) Note that some of the combinatorics of [Sh 4] is not used here: if we have enough elements, for some large subset of them their domain behaves as a Δ-system, with the same coefficient of the common parts, so the difference of any two has domain disjoint to the "heart", so we can make it to be disjoint to a predescribed set.

6.4. Definition: We define the game $\underline{\underline{Gm}}''(W)$ as $\underline{\underline{Gm}}'(W)$, but ($\vartheta$ a regular cardinal, $\underline{\underline{T}} = {}^{\vartheta>}\lambda$)

(i) the game lasts ϑ moves.

(ii) Dom(f_α) is any subset of ${}^{\vartheta>}\kappa$, closed under initial segments but with no ϑ-branch.

(iii) in odd stages α player I chooses f_α .

6.5. Theorem: Suppose $\vartheta \leq \kappa$, $\lambda^{<\vartheta} = \lambda^{\kappa}$. Then for some $W=\{(f^\alpha, N^\alpha): \alpha<\alpha^*\}$ and function ζ

1) If cf $\lambda > \vartheta > \aleph_0$, then

(a) W is disjoint, and in $\underline{\underline{Gm}}''(W)$ player II has a winning strategy.

(b) For $\alpha<\beta<\alpha^*$, $\zeta(\alpha) \leq \zeta(\beta)$.

(c) cf$(\zeta(\alpha)) = \vartheta$.

(d) Every branch η of f^α satisfies: $(\forall i<\vartheta)\ \eta(i)<\zeta(\alpha)$ and $\zeta(\alpha) = \bigcup_{i<\vartheta} \eta(i)$.

(e) for every $i<\vartheta$ for some $\xi<\zeta(\alpha)$, orco$(N_i^\alpha) \leq \xi$.

(f) If $\alpha+\kappa^\vartheta \leq \beta<\alpha^*$ and η is a ϑ-branch of Rang(f^β), then $\eta \restriction i \notin N^\alpha$ for some $i<\vartheta$.

(g) If $\lambda = \lambda^{\kappa}$ we can demand: if η is a ϑ-branch of Rang(f^{α}) and $\eta \restriction i \in N^{\beta}$ for every $i < \vartheta$ (where $\alpha, \beta < \alpha^*$), then $N^{\alpha} \subseteq N^{\beta}$.

2) As (1) is the parallel to (1.13)(1), so parallel to (1.13)(2), (2.8),(2.9),(6.1) holds.

On the proof: The point is that, if η is a ϑ-branch, for it to "code the play" it is enough that for a closed unbounded set of $i < \vartheta$, $\eta(i)$ code appropiate information on the first i moves. (When $\lambda < \lambda^{<\vartheta}$, remember we can split ϑ to ϑ disjoint stationary sets).

6.6. Remark: So clearly we could have divided the choices of the f_{α}'s between player I and II differently, as long as for each ϑ-branch η of $\bigcup_{i<\vartheta} f_i$, {i: player II chooses $\eta(i)$} belong to D, for some fixed filter D on ϑ.

6.7. Theorem: In (6.1) we can strengthen it by replacing (b) by

(b)" W is disjoint and player II has no winning strategy in $\underset{\approx}{\mathrm{Gm}}$"(W).

Point of the Proof: Unlike (6.5) we do not have a filter on ω, but we can try for each η all infinite subsets of ω as "the set of choices of player II".

Sh:227

References

[CG] A.L.S. Corner and R. Göbel, Prescribing endomorphism algebras - a uniform treatment, Proc. London Math. Soc. (to appear).

[C] A.L.S. Corner, Every countable reduced torsion free ring is an endomorphism ring, Proc. London Math. Soc. (3) 13 (1963), 687-710.

[DS] K. Devlin and S. Shelah, A weak form of the diamond which follows from $2^{\aleph_0} < 2^{\aleph_1}$, Israel J. Math. 29 (1978), 239-247.

[DG] M. Dugas and R. Göbel, Every cotorsion free ring is an endomorphism ring, Proc. London Math. Soc. (3) 45 (1982), 319-336.

[DG 1] ----------------------, On endomorphism rings of primary abelian groups, Math. Ann. 261 (1982), 359-385.

[DG 2] ----------------------, Every cotorsion free algebra is an endomorphism algebra, Math. Z. 181 (1982), 451-470.

[Fu] L. Fuchs, Infinite abelian groups, Academic Press, New York, vol.I 1970, vol.II 1973.

Sh:227

[EM] P.C. Eklof and A.M. Mekler, On constructing indecomposable groups in L, J. Algebra 49 (1977), 96-103.

[GS] R. Göbel and S. Shelah, Semi-rigid classes of torsion-free abelian groups, J. Algebra (to appear).

[GS 1] ------------------------, Modules over arbitrary domains I, Math. Z. (to appear).

[GS 2] ------------------------, Modules over arbitrary domains II, submitted to Fund. Math.

[P] P.S. Pierce, Homomorphisms of primary abelian group, pp. 214-310 in: Topics in abelian groups, Chicago, Scott and Foresman 1963, ed. J.M. Irwin and E.A. Walker.

[R Sh] M. Rubin and S. Shelah, Combinatorial problems on trees, partitions, Δ-systems and large free subsets, Annals of Pure and Applied Logic.

[Sh 1] S. Shelah, Classification Theory, North Holland Publ. Co Amsterdam, 1978.

[Sh 2] ---------, Why there are many non-isomorphic models for unsuperstable theories, Proc. of the International Congress of Math. Vancouver (1974), 553-557.

[Sh 3] ---------, Infinite abelian groups, Whitehead problem and some constructions, Israel J. Math. 18 (1974), 243-256.

[Sh 4] ---------, Existence of rigid like families of abelian p-groups, Model Theory and Algebra, ed. D. H. Saraceno and V. B. Weisfening, Lecture Notes in Math. Springer Verlag Vol. 498, Berlin 1975, 384-402.

[Sh 5] ---------, The lazy model theorist guide to stability, Logique et Analyse 71-72 (1975), 241-308.

[Sh 6] ---------, On endo-rigid strongly $\aleph_1$-free abelian groups in , Israel J. Math. 40 (1981), 291-295.

[Sh 7] ---------, Construction of many complicated uncountable structures and Boolean algebras, Israel J. Math. 45 (1983), 100-146.

[Sh 8] ---------, A combinatorial principle and endomorphism rings of p-groups, Proceedings of the 1980/1 Jerusalem Model Theory Year, Israel J. Math (to appear).

[Sh 9] ---------, On powers of singular cardinals, Notre Dame J. of

Sh:227

Formal Logic.

[Sh 10] ---------, Existence of endo-rigid Boolean algebras, appear in a Springer Lecture Notes' volume).