

WHEN A FIRST ORDER T HAS LIMIT MODELS

BY

SAHARON SHELAH (Jerusalem and Piscataway, NJ)

Abstract. We sort out to a large extent when a (first order complete theory) T has a superlimit model in a cardinal λ . Also we deal with related notions of being limit.

Annotated content

0. Introduction. We give background and basic definitions. We then present existence results for stable T which have models that are saturated or close to being saturated.

1. On countable superstable non- \aleph_0 -stable. Consistently $2^{\aleph_1} \geq \aleph_2$ and some such (complete first order) T has a superlimit (non-saturated) model of cardinality \aleph_1 . This shows that we cannot prove a non-existence result fully complementary to the results in 0.9.

2. A strictly stable consistent example. Consistently $\aleph_1 < 2^{\aleph_0}$ and some countable stable not superstable T has a (non-saturated) model of cardinality \aleph_1 which satisfies some relatives of being superlimit.

3. On the non-existence of limit models. The proofs here are in ZFC. If T is unstable, it has no superlimit models of cardinality λ when $\lambda \geq \aleph_1 + |T|$. For unsuperstable T we have similar results but with “few” exceptional cardinals λ on which we do not know: $\lambda < \aleph_0$ which are $< \beth_\omega$. Moreover, if T is superstable and $\lambda \geq |T| + 2^{|T|}$ then T has a superlimit model of cardinality λ iff $|D(T)| \leq \lambda$ iff T has a saturated model. Lastly, we get weaker results on weaker relatives of superlimit.

0. Introduction

0A. Background and content. Recall that ([15, Ch. III]) if T is (first order complete and) superstable then for $\lambda \geq 2^{|T|}$, T has a saturated model M of cardinality λ and moreover

(*) if $\langle M_\alpha : \alpha < \delta \rangle$ is \prec -increasing, δ a limit ordinal $< \lambda^+$ and $\alpha < \delta \Rightarrow M_\alpha \cong M$ then $\bigcup \{M_\alpha : \alpha < \delta\}$ is isomorphic to M .

When investigating categoricity of an a.e.c. (abstract elementary class) $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$, the following property turns out to be central: M is a $\leq_{\mathfrak{k}}$ -universal model of cardinality λ with the property (*) above (called superlimit), possibly with additional parameter $\kappa = \text{cf}(\kappa) \leq \lambda$ (or stationary $S \subseteq \lambda^+$);

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we also consider some relatives of the “superlimit” notion, mainly limit, weakly limit and strongly limit. Those notions were suggested for a.e.c. in [13, 3.1]; see also the revised version [3, 3.3] and [19], or here in 0.7. But though coming from investigating non-elementary classes, they are meaningful for elementary classes and here we try to investigate them for elementary classes.

Recall that for a first order complete T , we know $\{\lambda : T \text{ has a saturated model of } T \text{ of cardinality } \lambda\}$, namely, it is $\{\lambda : \lambda^{<\lambda} \geq |D(T)| \text{ or } T \text{ is stable in } \lambda\}$; for the definitions of $D(T)$ and other notions see 0B below. What if we replace saturated by superlimit (or some relative)? Let $EC_\lambda(T)$ be the class of models M of T of cardinality λ .

If there is a saturated $M \in EC_\lambda(T)$ we have considerable knowledge on the existence of a limit model for the cardinal λ , by [15], as mentioned in [3, 3.6] (see 0.9(1),(2)). E.g. for superstable T in $\lambda \geq 2^{|T|}$ there is a superlimit model (the saturated one). It seems a natural question on [3, 3.6] whether it exhausts the possibilities of $(\lambda, *)$ -superlimit and (λ, κ) -superlimit models for elementary classes. Clearly the cases of the existence of such models of a (first order complete) theory T where there are no saturated (or special) models are rare, because even the weakest version of Definition [13, 3.1] = [3, 3.3] or here Definition 0.7 for λ implies that T has a universal model of cardinality λ , which is rare (see Kojman–Shelah [2] which includes earlier history and recently Džamonja [1]).

So the main question seems to be whether there are such cases at all. We naturally look at some of the previous cases of consistency of the existence of a universal model (for $\lambda < \lambda^{<\lambda}$), i.e., those for $\lambda = \aleph_1$.

E.g. a sufficient condition for some versions is the existence of $T' \supseteq T$ of cardinality λ such that $PC(T', T)$ is categorical in λ (see 0.4(3)). By [12] we have consistency results for such T_1 so naturally we first deal with the consistency results from [12]. In §1 we deal with the case of the countable superstable T_0 from [12] which is not \aleph_0 -stable. By [12] consistently $\aleph_1 < 2^{\aleph_0}$ and for some $T'_0 \supseteq T_0$ of cardinality \aleph_1 , $PC(T'_0, T_0)$ is categorical in \aleph_1 . We use this to get the consistency of “ T_0 has a superlimit model of cardinality \aleph_1 and $\aleph_1 < 2^{\aleph_0}$ ”.

In §2 for some stable non-superstable countable T_1 we have a parallel but weaker result. We reconsider the old consistency results of “some $PC(T'_1, T_1), |T'_1| = \aleph_1 > |T_1|$, is categorical in \aleph_1 ” from [12]. From this we deduce that in this universe, T_1 has a strongly (\aleph_1, \aleph_0) -limit model.

It is a reasonable thought that we can similarly have a consistency result for the theory of linear orders, but this is still unclear.

In §3 we show that if T has a superlimit model in $\lambda \geq |T| + \aleph_1$ then T is stable and T is superstable except possibly under some severe restrictions on

the cardinal λ (i.e., $\lambda < \beth_\omega$ and $\lambda < \lambda^{\aleph_0}$). We then prove some restrictions on the existence of some (weaker) relatives.

Summing up our results on the strongest notion, superlimit, by 1.1 + 3.1 we have:

CONCLUSION 0.1. *Assume $\lambda \geq |T| + \beth_\omega$. Then T has a superlimit model of cardinality λ iff T is superstable and $\lambda \geq |D(T)|$.*

In subsequent work we shall show that for some unstable T (e.g. the theory of linear orders), if $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa)$, then T has a medium (λ, κ) -limit model, whereas if T has the independence property, even weak (λ, κ) -limit models do not exist; see [5] and more in [6], [20], [4], [9].

0B. Basic definitions

NOTATION 0.2. Let T denote a complete first order theory which has infinite models but T_1, T' etc. are not necessarily complete.

If M, N denote models, then $|M|$ is the universe of M and $\|M\|$ its cardinality and $M \prec N$ means M is an elementary submodel of N .

Let $\tau_T = \tau(T)$, $\tau_M = \tau(M)$ be the vocabularies of T , M respectively.

Let $M \models \text{“}\varphi[\bar{a}]^{\text{(stat)}}$ ” mean that the model M satisfies $\varphi[\bar{a}]$ if the statement *stat* is true (or is 1 rather than 0).

DEFINITION 0.3. For $\bar{a} \in {}^\omega |M|$ and $B \subseteq M$ let

$$\text{tp}(\bar{a}, B, M) = \{\varphi(\bar{x}, \bar{b}) : \varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M), \bar{b} \in {}^{\text{lg}(\bar{y})} B \text{ and } M \models \varphi[\bar{a}, \bar{b}]\}.$$

Let

$$D(T) = \{\text{tp}(\bar{a}, \emptyset, M) : M \text{ a model of } T \text{ and } \bar{a} \text{ a finite sequence from } M\}.$$

If $A \subseteq M$ then

$$\mathbf{S}^m(A, M) = \{\text{tp}(\bar{a}, A, N) : M \prec N \text{ and } \bar{a} \in {}^m N\};$$

if $m = 1$ we may omit it.

A model M is λ -saturated when: if $A \subseteq M$, $|A| < \lambda$ and $p \in \mathbf{S}(A, M)$ then p is realized by some $a \in M$, i.e. $p \subseteq \text{tp}(a, A, M)$; if $\lambda = \|M\|$ we may omit it.

A model M is *special* when letting $\lambda = \|M\|$, there is an increasing sequence $\langle \lambda_i : i < \text{cf}(\lambda) \rangle$ of cardinals with limit λ and a \prec -increasing sequence $\langle M_i : i < \text{cf}(\lambda) \rangle$ of models with union M such that M_{i+1} is λ_i -saturated of cardinality λ_{i+1} for $i < \text{cf}(\lambda)$.

DEFINITION 0.4. For any T let

$$\begin{aligned} \text{EC}(T) &= \{M : M \text{ is a } \tau_T\text{-model of } T\}, \\ \text{EC}_\lambda(T) &= \{M \in \text{EC}(T) : M \text{ is of cardinality } \lambda\}. \end{aligned}$$

For $T \subseteq T'$ let

$$\begin{aligned} \text{PC}(T', T) &= \{M \upharpoonright \tau_T : M \text{ is model of } T'\}, \\ \text{PC}_\lambda(T', T) &= \{M \in \text{PC}(T', T) : M \text{ is of cardinality } \lambda\}. \end{aligned}$$

We say M is λ -*universal* for T_1 when it is a model of T_1 and every $N \in \text{EC}_\lambda(T)$ can be elementarily embedded into M ; if $T_1 = \text{Th}(M)$ we may omit it.

We say $M \in \text{EC}(T)$ is *universal* when it is λ -universal for $\lambda = \|M\|$.

We are here mainly interested in

DEFINITION 0.5. Given T and $M \in \text{EC}_\lambda(T)$ we say that M is a *superlimit* or λ -*superlimit model* when: M is universal and if $\delta < \lambda^+$ is a limit ordinal, $\langle M_\alpha : \alpha \leq \delta \rangle$ is \prec -increasing continuous, and M_α is isomorphic to M for every $\alpha < \delta$, then M_δ is isomorphic to M .

REMARK 0.6. Concerning the following definition we shall use strongly limit in 2.14(1), medium limit in 2.14(2).

DEFINITION 0.7. Let λ be a cardinal $\geq |T|$. For parts (3)–(7) below, but not (8), to simplify the presentation we assume the axiom of global choice and that \mathbf{F} is a class function; alternatively restrict yourself to models with universe an ordinal $\in [\lambda, \lambda^+)$.

(1) For non-empty $\Theta \subseteq \{\mu : \aleph_0 \leq \mu < \lambda \text{ and } \mu \text{ is regular}\}$ and $M \in \text{EC}_\lambda(T)$ we say that M is (λ, Θ) -*superlimit* when: M is universal and

if $\langle M_i : i \leq \mu \rangle$ is \prec -increasing, $M_i \cong M$ for $i < \mu$ and $\mu \in \Theta$, then $\bigcup\{M_i : i < \mu\} \cong M$.

(2) If Θ is a singleton, say $\Theta = \{\theta\}$, we may say that M is (λ, θ) -*superlimit*.

(3) Let $S \subseteq \lambda^+$ be stationary. A model $M \in \text{EC}_\lambda(T)$ is called S -*strongly limit* or (λ, S) -*strongly limit* when for some function $\mathbf{F} : \text{EC}_\lambda(T) \rightarrow \text{EC}_\lambda(T)$ we have:

- (a) for $N \in \text{EC}_\lambda(T)$ we have $N \prec \mathbf{F}(N)$,
- (b) if $\delta \in S$ is a limit ordinal and $\langle M_i : i < \delta \rangle$ is a \prec -increasing continuous sequence ⁽¹⁾ in $\text{EC}_\lambda(T)$ and $i < \delta \Rightarrow \mathbf{F}(M_{i+1}) \prec M_{i+2}$, then $M \cong \bigcup\{M_i : i < \delta\}$.

(4) Let $S \subseteq \lambda^+$ be stationary. $M \in \text{EC}_\lambda(T)$ is called S -*limit* or (λ, S) -*limit* if for some function $\mathbf{F} : \text{EC}_\lambda(T) \rightarrow \text{EC}_\lambda(T)$ we have:

- (a) for every $N \in \text{EC}_\lambda(T)$ we have $N \prec \mathbf{F}(N)$,

⁽¹⁾ No loss if we add $M_{i+1} \cong M$, so this simplifies the demand on \mathbf{F} , i.e., only $\mathbf{F}(M')$ for $M' \cong M$ is required.

- (b) if $\langle M_i : i < \lambda^+ \rangle$ is a \prec -increasing continuous sequence of members of $\text{EC}_\lambda(T)$ such that $\mathbf{F}(M_{i+1}) \prec M_{i+2}$ for $i < \lambda^+$ then for some closed unbounded $(^2)$ subset C of λ^+ ,

$$[\delta \in S \cap C \Rightarrow M_\delta \cong M].$$

(5) We define $(^3)$ “ S -weakly limit”, “ S -medium limit” like “ S -limit”, “ S -strongly limit” respectively by demanding that the domain of \mathbf{F} is the family of \prec -increasing continuous sequences of members of $\text{EC}_\lambda(T)$ of length $< \lambda^+$ and replacing “ $\mathbf{F}(M_{i+1}) \prec M_{i+2}$ ” by “ $M_{i+1} \prec \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}$ ”.

(6) If $S = \lambda^+$ then we may omit S (in (3)–(5)).

(7) For non-empty $\Theta \subseteq \{\mu : \mu \leq \lambda \text{ and } \mu \text{ is regular}\}$, M is (λ, Θ) -strongly limit $(^4)$ if M is $\{\delta < \lambda^+ : \text{cf}(\delta) \in \Theta\}$ -strongly limit. Similarly for the other notions. If we do not write λ we mean $\lambda = \|M\|$.

(8) We say that $M \in K_\lambda$ is *invariantly strong limit* when in (3), \mathbf{F} is just a subset of $\{(M, N)/\cong : M \prec N \text{ are from } \text{EC}_\lambda(T)\}$ and in (3)(b) we replace “ $\mathbf{F}(M_{i+1}) \prec M_{i+2}$ ” by “ $(\exists N)(M_{i+1} \prec N \prec M_{i+2} \wedge ((M, N)/\cong) \in \mathbf{F})$ ”. But abusing notation we still write $N = \mathbf{F}(M)$ instead of $((M, N)/\cong) \in \mathbf{F}$. Similarly with the other notions, so we use the isomorphism type of $\bar{M} \hat{\ } \langle N \rangle$ for “weakly limit” and “medium limit”.

(9) In the definitions above we may say “ \mathbf{F} witnesses M is ...”

OBSERVATION 0.8. (1) Assume $\mathbf{F}_1, \mathbf{F}_2$ are as above and $\mathbf{F}_1(N) \prec \mathbf{F}_2(N)$ (or $\mathbf{F}_1(\bar{N}) \prec \mathbf{F}_2(\bar{N})$) whenever defined. If \mathbf{F}_1 is a witness then so is \mathbf{F}_2 .

(2) All versions of limit models imply being a universal model in $\text{EC}_\lambda(T)$.

(3) (The obvious implications diagram) For non-empty $\Theta \subseteq \{\theta : \theta \text{ is regular } \leq \lambda\}$ and stationary $S_1 \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) \in \Theta\}$:

$$\text{superlimit} = (\lambda, \{\mu : \mu \leq \lambda \text{ regular}\})\text{-superlimit}$$

$$\downarrow$$

$$(\lambda, \Theta)\text{-superlimit}$$

$$\downarrow$$

$$S_1\text{-strongly limit}$$

$$\downarrow$$

$$S_1\text{-medium limit} \quad S_1\text{-limit}$$

$$\downarrow$$

$$\downarrow$$

$$S_1\text{-weakly limit}$$

$(^2)$ Alternatively, we can use as a parameter a filter on λ^+ extending the co-bounded filter.

$(^3)$ Note that M is (λ, S) -strongly limit iff M is $(\{\lambda, \text{cf}(\delta) : \delta \in S\})$ -strongly limit.

$(^4)$ In [3] we replace “limit” by “limit $^-$ ” if “ $\mathbf{F}(M_{i+1}) \prec M_{i+2}$ ”, “ $M_{i+1} \prec \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}$ ” are replaced by “ $\mathbf{F}(M_i) \prec M_{i+1}$ ”, “ $M_i \prec \mathbf{F}(\langle M_j : j \leq i \rangle) \prec M_{i+1}$ ” respectively. But $(\text{EC}(T), \prec)$ has amalgamation.

LEMMA 0.9. *Let T be a first order complete theory.*

(1) *If λ is regular and M a saturated model of T of cardinality λ , then M is (λ, λ) -superlimit.*

(2) *If T is stable, and M is a saturated model of T of cardinality $\lambda \geq \aleph_1 + |T|$ and $\Theta = \{\mu : \kappa(T) \leq \mu \leq \lambda \text{ and } \mu \text{ is regular}\}$, then M is (λ, Θ) -superlimit (for $\kappa(T)$, see [15, III, §3]).*

(3) *If T is stable in λ and $\kappa = \text{cf}(\kappa) \leq \lambda$ then T has an invariantly strongly (λ, κ) -limit model.*

REMARK 0.10. Concerning 0.9(2), note that by [15] if λ is singular or just $\lambda < \lambda^{<\lambda}$ and T has a saturated model of cardinality λ then T is stable (even stable in λ) and $\text{cf}(\lambda) \geq \kappa(T)$.

Proof. (1) Let M_i be a λ -saturated model of T of cardinality λ for $i < \lambda$ with $\langle M_i : i < \lambda \rangle$ \prec -increasing and set $M_\lambda = \bigcup_{i < \lambda} M_i$. Now for every $A \subseteq M_\lambda$ of cardinality $< \lambda$ there is $i < \lambda$ such that $A \subseteq M_i$, so every $p \in \mathbf{S}(A, M_\lambda)$ is realized in M_i , hence in M_λ ; so clearly M_λ is λ -saturated. Remembering the uniqueness of a λ -saturated model of T of cardinality λ we finish.

(2) Use [15, III, 3.11]: if M_i is a λ -saturated model of T with $\langle M_i : i < \delta \rangle$ increasing and $\text{cf}(\delta) \geq \kappa(T)$ then $\bigcup_{i < \delta} M_i$ is λ -saturated.

(3) Let $\mathbf{K}_{\lambda, \kappa} = \{\bar{M} : \bar{M} = \langle M_i : i < \kappa \rangle \text{ is } \prec\text{-increasing continuous, } M_i \in \text{EC}_\lambda(T) \text{ and } (M_{i+2}, c)_{c \in M_{i+1}} \text{ is saturated for every } i < \kappa\}$. Clearly $\bar{M}, \bar{N} \in \mathbf{K}_{\lambda, \kappa} \Rightarrow M_\kappa \cong N_\kappa$. Also for every $M \in \text{EC}_\lambda(T)$ there is N such that $M \prec N$ and $(N, c)_{c \in M}$ is saturated, as also $\text{Th}((M, c)_{c \in M})$ is stable in λ ; so there is an invariant $\mathbf{F} : \text{EC}_\lambda(T) \rightarrow \text{EC}_\lambda(T)$ such that $M \prec \mathbf{F}(M)$ and $(\mathbf{F}(M), c)_{c \in M}$ is saturated; such \mathbf{F} witnesses the desired conclusion. $\blacksquare_{0.9}$

DEFINITION 0.11. For a regular uncountable cardinal λ let

$$\check{I}[\lambda] = \{S \subseteq \lambda : \text{some pair } (E, \bar{a}) \text{ witnesses } S \in \check{I}[\lambda], \text{ see below}\}.$$

We say that (E, \bar{u}) is a *witness* for $S \in \check{I}[\lambda]$ iff:

- E is a club of the regular cardinal λ ,
- $\bar{u} = \langle u_\alpha : \alpha < \lambda \rangle$, $u_\alpha \subseteq \alpha$ and $\beta \in u_\alpha \Rightarrow u_\beta = \beta \cap u_\alpha$,
- for every $\delta \in E \cap S$, u_δ is an unbounded subset of δ of order-type $\text{cf}(\delta)$ (and δ is a limit ordinal).

By [16, §1] we have

CLAIM 0.12. *If $\kappa^+ < \lambda$ and κ, λ are regular then some stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ belongs to $\check{I}[\lambda]$.*

By [11] we have

CLAIM 0.13. *If $\lambda = \mu^+$, $\theta = \text{cf}(\theta) \leq \text{cf}(\mu)$ and $\alpha < \mu \Rightarrow |\alpha|^{<\theta} \leq \mu$ then $S_\theta^\lambda \in \check{I}[\lambda]$.*

1. On superstable non- \aleph_0 -stable T . We first note that superstable T tend to have superlimit models.

CLAIM 1.1. *Assume T is superstable and $\lambda \geq |T| + 2^{\aleph_0}$. Then T has a superlimit model of cardinality λ iff T has a saturated model of cardinality λ iff T has a universal model of cardinality λ iff $\lambda \geq |D(T)|$.*

Proof. By [15, III, §5] we know that T is stable in λ iff $\lambda \geq |D(T)|$. Now if $|T| \leq \lambda < |D(T)|$ trivially there is no universal model of T of cardinality λ , hence no saturated model and no superlimit model, etc., recalling 0.8(2). If $\lambda \geq |D(T)|$, then T is stable in λ , hence has a saturated model of cardinality λ by [15, III] (hence universal) and the class of λ -saturated models of T is closed under increasing elementary chains by [15, III], so we are done. ■_{1.1}

The following are the prototypical theories which we shall consider.

DEFINITION 1.2.

$$\begin{aligned} T_0 &= \text{Th}(\omega 2, E_n^0)_{n < \omega} & \text{where } \eta E_n^0 \nu &\Leftrightarrow \eta \upharpoonright n = \nu \upharpoonright n, \\ T_1 &= \text{Th}(\omega(\omega_1), E_n^1)_{n < \omega} & \text{where } \eta E_n^1 \nu &\Leftrightarrow \eta \upharpoonright n = \nu \upharpoonright n, \\ T_2 &= \text{Th}(\mathbb{R}, <). \end{aligned}$$

Recall

OBSERVATION 1.3.

- (0) T_ℓ is a countable complete first order theory for $\ell = 0, 1, 2$.
- (1) T_0 is superstable non- \aleph_0 -stable.
- (2) T_1 is strictly stable, that is, stable non-superstable.
- (3) T_2 is unstable.
- (4) T_ℓ has elimination of quantifiers for $\ell = 0, 1, 2$.

CLAIM 1.4. *It is consistent with ZFC that $\aleph_1 < 2^{\aleph_0}$ and some $M \in \text{EC}_{\aleph_1}(T_0)$ is a superlimit model.*

Proof. By [12], for notational simplicity we start with $\mathbf{V} = \mathbf{L}$.

So T_0 is defined in 1.2 and it is the T from Theorem [12, 1.1]. Let S be the set of $\eta \in (\omega 2)^{\mathbf{L}}$. We define T' (called T_1 there) as the following theory:

- ⊗₁ (i) for each n the sentence saying E_n is an equivalence relation with 2^n equivalence classes, each E_n equivalence class divided into two by E_{n+1} , E_{n+1} refines E_n , E_0 is trivial,
- (ii) the sentences saying that
 - (α) for every x_0 , the function $z \mapsto F(x_0, z)$ is one-to-one and
 - (β) $x_0 E_n F(x_0, z)$ for each $n < \omega$,
- (iii) $E_n(c_\eta, c_\nu)^{\text{if } (\eta \upharpoonright n = \nu \upharpoonright n)}$ for $\eta, \nu \in S$.

In [12] it is proved that in some forcing ⁽⁵⁾ extension $\mathbf{L}^{\mathbb{P}}$ of \mathbf{L} , \mathbb{P} an \aleph_2 -c.c. proper forcing of cardinality \aleph_2 , and in $\mathbf{V} = \mathbf{L}^{\mathbb{P}}$, the class $\text{PC}(T', T_0) = \{M \upharpoonright \tau_{T_0} : M \text{ is a } \tau\text{-model of } T'\}$ is categorical in \aleph_1 .

However, letting M^* be any model from $\text{PC}(T', T_0)$ of cardinality \aleph_1 , it is easy to see that (in $\mathbf{V} = \mathbf{L}^{\mathbb{P}}$):

⊗₂ the following conditions on M are equivalent:

- (a) M is isomorphic to M^* ,
- (b) $M \in \text{PC}(T', T_0)$,
- (c) (α) M is a model of T_0 of cardinality \aleph_1 ,
 (β) M^* can be elementarily embedded into M ,
 (γ) for every $a \in M$ the set $\bigcap \{a/E_n^M : n < \omega\}$ has cardinality \aleph_1 .

But

- ⊗₃ every model M_1 of T of cardinality $\leq \aleph_1$ has a proper elementary extension to a model satisfying (c), i.e., (α)–(γ) of ⊗₂ above,
- ⊗₄ if $\langle M_\alpha : \alpha < \delta \rangle$ is an increasing chain of models satisfying (c) of ⊗₂ and $\delta < \omega_2$ then also $\bigcup \{M_\alpha : \alpha < \delta\}$ does.

Altogether we are done. ■_{1.4}

Naturally we ask

QUESTION 1.5. What occurs to T_0 for $\lambda > \aleph_1$ but $\lambda < 2^{\aleph_0}$?

QUESTION 1.6. Does the theory T_2 of linear order consistently have an (\aleph_1, \aleph_0) -superlimit (or only strongly limit) model? (but see §3).

QUESTION 1.7. What is the answer for T when T is countable superstable non- \aleph_0 -stable and $D(T)$ is countable for $\aleph_1 < 2^{\aleph_0}$ and $\aleph_2 < 2^{\aleph_0}$?

By the above for some such T , in some universe, for \aleph_1 the answer is yes, there is a superlimit model.

2. A strictly stable consistent example. We now look at models of T_1 (redefined below) in cardinality \aleph_1 ; recall

DEFINITION 2.1. $T_1 = \text{Th}(\omega(\omega_1), E_n)_{n < \omega}$ where $E_n = \{(\eta, \nu) : \eta, \nu \in \omega(\omega_1) \text{ and } \eta \upharpoonright n = \nu \upharpoonright n\}$.

REMARK 2.2. Note that T_1 has elimination of quantifiers. Moreover, if $\lambda = \sum \{\lambda_n : n < \omega\}$ and $\lambda_n = \lambda_n^{\aleph_0}$, then T_1 has a (λ, \aleph_0) -superlimit model in λ (see 2.15).

DEFINITION/CLAIM 2.3. Any model of T_1 of cardinality λ is isomorphic to $M_{A,h} := (\{(\eta, \varepsilon) : \eta \in A, \varepsilon < h(\eta)\}, E_n)_{n < \omega}$ for some $A \subseteq {}^\omega \lambda$ and $h :$

⁽⁵⁾ We can replace \mathbf{L} by any \mathbf{V}_0 which satisfies $2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2$.

${}^\omega\lambda \rightarrow (\text{Car} \cap \lambda^+) \setminus \{0\}$ where $(\eta_1, \varepsilon_1)E_n(\eta_2, \varepsilon_2) \Leftrightarrow \eta_1 \upharpoonright n = \eta_2 \upharpoonright n$; pedantically we should write $E_n^{M_{A,h}} = E_n \upharpoonright |M_{A,n}|$.

We write M_A for $M_{A,h}$ when A is as above and $h : A \rightarrow \{|A|\}$, so constantly $|A|$ when A is infinite.

For $A \subseteq {}^\omega\lambda$ and h as above the model $M_{A,h}$ is a model of T_1 iff A is non-empty and $(\forall \eta \in A)(\forall n < \omega)(\exists^{\aleph_0} \nu \in A)(\nu \upharpoonright n = \eta \upharpoonright n \wedge \nu(n) \neq \eta(n))$.

Above $M_{A,h}$ has cardinality λ iff $\sum \{h(\eta) : \eta \in A\} = \lambda$.

DEFINITION 2.4. We say that A is a (T_1, λ) -witness when:

- $A \subseteq {}^\omega\lambda$ has cardinality λ ,
- if $B_1, B_2 \subseteq {}^\omega\lambda$ are (T_1, A) -big (see below) of cardinality λ then $(B_1 \cup {}^{\omega>\lambda}, \triangleleft)$ is isomorphic to $(B_2 \cup {}^{\omega>\lambda}, \triangleleft)$.

A set $B \subseteq {}^\omega\lambda$ is called (T_1, A) -big when it is (λ, λ) - (T_1, A) -big; see below.

B is (μ, λ) - (T_1, A) -big means: $B \subseteq {}^\omega\lambda, |B| = |A| = \mu$ and for every $\eta \in {}^{\omega>\lambda}$ there is an isomorphism f from $({}^{\omega \geq \lambda}, \triangleleft)$ onto $(\{\eta \hat{\nu} : \nu \in {}^{\omega \geq \lambda}\}, \triangleleft)$ mapping A into $\{\nu : \eta \hat{\nu} \in B\}$.

$A \subseteq {}^\omega(\omega_1)$ is \aleph_1 -suitable when:

- $|A| = \aleph_1$,
- for a club of $\delta < \omega_1, A \cap {}^\omega\delta$ is everywhere non-meagre in the space ${}^\omega\delta$, i.e., for every $\eta \in {}^{\omega>\delta}$ the set $\{\nu \in A \cap {}^\omega\delta : \eta \triangleleft \nu\}$ is a non-meagre subset of ${}^\omega\delta$ (that is what is really used in [12]).

CLAIM 2.5. *It is consistent with ZFC that $2^{\aleph_0} > \aleph_1$ + there is a (T_1, \aleph_1) -witness; moreover every \aleph_1 -suitable set is a (T_1, \aleph_1) -witness.*

Proof. By [12, §2]. ■_{2.5}

REMARK 2.6. The witness does not give rise to an (\aleph_1, \aleph_0) -limit model as for the union of any “fast enough” \triangleleft -increasing ω -chain of members of $\text{EC}_{\aleph_1}(T_1)$, the relevant sets are meagre.

DEFINITION 2.7. Let A be a (T_1, λ) -witness. We define $K_{T_1, A}^1$ as the family of $M = (|M|, <^M, P_\alpha^M)_{\alpha \leq \omega}$ such that:

- (α) $(|M|, <^M)$ is a tree with $\omega + 1$ levels,
- (β) P_α^M is the α th level; let $P_{<\omega}^M = \bigcup \{P_n^M : n < \omega\}$,
- (γ) M is isomorphic to M_B^1 for some $B \subseteq {}^\omega\lambda$ of cardinality λ where M_B^1 is defined by $|M_B^1| = ({}^{\omega>\lambda}) \cup B, P_n^{M_B^1} = {}^n\lambda, P_\omega^{M_B^1} = B$ and $<^{M_B^1} = \triangleleft \upharpoonright |M_B^1|$, i.e., being an initial segment,
- (δ) moreover B is such that some f satisfies:
 - $f : {}^{\omega>\lambda} \rightarrow \omega$ and $f(\langle \rangle) = 0$ for simplicity,
 - $\eta \triangleleft \nu \in {}^{\omega>\lambda} \Rightarrow f(\eta) \leq f(\nu)$,
 - if $\eta \in B$ then $\langle f(\eta \upharpoonright n) : n < \omega \rangle$ is eventually constant,

- if $\eta \in {}^\omega \lambda$ then $\{\nu \in {}^\omega \lambda : \eta \hat{\ } \nu \in B \text{ and } m < \omega \Rightarrow f(\eta \hat{\ } (\nu \upharpoonright m)) = f(\eta)\}$ is (T_1, A) -big,
- for $\eta \in {}^\omega \lambda$ and $n \in [f(\eta), \omega)$ for λ ordinals $\alpha < \lambda$, we have $f(\eta \hat{\ } \langle \alpha \rangle) = n$.

CLAIM 2.8 (The Global Axiom of Choice). *If A is a (T_1, \aleph_1) -witness then:*

- (a) $K_{T_1, A}^1 \neq \emptyset$,
- (b) any two members of $K_{T_1, A}^1$ are isomorphic,
- (c) there is a function \mathbf{F} from $K_{T_1, A}^1$ to itself (up to isomorphism, i.e., $(M, \mathbf{F}(M))$ is defined only up to isomorphism) satisfying $M \subseteq \mathbf{F}(M)$ such that $K_{T_1, A}^1$ is closed under increasing unions of sequences $\langle M_n : n < \omega \rangle$ such that $\mathbf{F}(M_n) \subseteq M_{n+1}$.

Proof. (a): Trivial.

(b): By the definition of “ A is a (T_1, \aleph_1) -witness” and of $K_{T_1, A}^1$.

(c): We choose \mathbf{F} such that

- if $M \in K_{A, T_1}^1$ then $M \subseteq \mathbf{F}(M) \in K_{A, T_1}^1$ and for every $k < \omega$ and $a \in P_k^M$, the set $\{b \in P_{k+1}^{\mathbf{F}(M)} : a <_{\mathbf{F}(M)} b \text{ and } b \notin M\}$ has cardinality \aleph_1 .

Assume $M = \bigcup \{M_n : n < \omega\}$ where $\langle M_n : n < \omega \rangle$ is \subseteq -increasing, $M_n \in K_{A, T_1}^1$, $\mathbf{F}(M_n) \subseteq M_{n+1}$. Clearly M is as required at the beginning of Definition 2.7, that is, satisfies clauses (α) – (γ) there. To prove (δ) , we define $f : P_{< \omega}^M \rightarrow \omega$ by $f(a) = \text{Min}\{n : a \in M_n\}$. Pedantically, \mathbf{F} is defined only up to isomorphism. ■_{2.8}

CLAIM 2.9. *If A is a (T_1, λ) -witness then:*

- (a) $K_{T_1, A}^1 \neq \emptyset$,
- (b) any two members of $K_{T_1, A}^1$ are isomorphic,
- (c) if $M_n \in K_{T_1, A}^1$ and $n < \omega \Rightarrow M_n \subseteq M_{n+1}$ then $M := \bigcup \{M_n : n < \omega\} \in K_{T_1, A}^1$.

REMARK 2.10. If we omit clause (b), we can weaken the demand on the set A .

Proof. Assume $M = \bigcup \{M_n : n < \omega\}$, $M_n \subseteq M_{n+1}$, $M_n \in K_{T_1, A}^1$ and f_n witnesses $M_n \in K_{T_1, A}^1$. Clearly M satisfies clauses (α) – (γ) of Definition 2.7; we just have to find a witness f as in (δ) there.

For each $a \in M$ let $n(a) = \text{Min}\{n : a \in M_n\}$; clearly if $M \models “a < b < c”$ then $n(a) \leq n(b)$ and $n(a) = n(c) \Rightarrow n(a) = n(b)$. Let $g_n : M \rightarrow M$ be defined by: $g_n(a) = b$ iff $b \leq^M a$, $b \in M_n$ and b is \leq^M -maximal under those restrictions; clearly it is well defined. Now we define $f'_n : M_n \rightarrow \omega$ by induction on $n < \omega$ such that $m < n \Rightarrow f'_m \subseteq f'_n$, as follows.

If $n = 0$ let $f'_n = f_n$.

If $n = m + 1$ and $a \in M_n$ we let $f'_n(a)$ be $f'_m(a)$ if $a \in M_m$ and be $(f_n(a) - f_n(g_m(a))) + f'_m(g_m(a)) + 1$ if $a \in M_n \setminus M_m$. Clearly $f := \bigcup \{f'_n : n < \omega\}$ is a function from M to ω , $a \leq^M b \Rightarrow f(a) \leq f(b)$, and for any $a \in M$ the set $\{b \in M : a \leq^M b \text{ and } f(b) = f(a)\}$ is equal to $\{b \in M_{n(a)} : f_{n(a)}(a) = f_{n(a)}(b) \text{ and } a \leq^M b\}$. ■_{2.9}

DEFINITION 2.11. Let A be a (T_1, λ) -witness. We define $K_{T_1, A}^2$ as in Definition 2.7 but f is constantly zero.

CLAIM 2.12 (The Global Axiom of Choice). *If A is a (T_1, \aleph_1) -witness then:*

- (a) $K_{T_1, A}^2 \neq \emptyset$,
- (b) any two members of $K_{T_1, A}^2$ are isomorphic,
- (c) there is a function \mathbf{F} from $\bigcup \{K_{T_0, A}^2 : \alpha < \omega_1\}$ to $K_{T_1, A}^2$ which satisfies:
 - (α) if $\bar{M} = \langle M_i : i \leq \alpha + 1 \rangle$ is an \prec -increasing sequence of models of T then $M_{\alpha+1} \subseteq \mathbf{F}(\bar{M}) \in K_{T_1, A}^2$,
 - (β) when $\omega_1 = \sup\{\alpha : \mathbf{F}(\bar{M} \upharpoonright r(\alpha+2)) \subseteq M_{\alpha+2}\}$ and is a well defined embedding of M_α into $M_{\alpha+2}$, the union of any increasing ω_1 -sequence $\bar{M} = \langle M_\alpha : \alpha < \omega_1 \rangle$ of members of $K_{T_1, A}^2$ belongs to $K_{T_1, A}^2$.

REMARK 2.13. Instead of the global axiom of choice, we can restrict the models to have universe a subset of λ^+ (or just a set of ordinals).

Proof. (a): Easy.

(b): By the definition.

(c): Let $\langle \mathcal{U}_\varepsilon : \varepsilon < \omega_1 \rangle$ be an increasing sequence of subsets of ω_1 with union ω_1 such that $\varepsilon < \omega_1 \Rightarrow |\mathcal{U}_\varepsilon \setminus \bigcup_{\zeta < \varepsilon} \mathcal{U}_\zeta| = \aleph_1$. Let $M^* \in K_{T_1, A}^2$ be such that ${}^\omega(\omega_1) \subseteq |M^*| \subseteq {}^{\omega \geq}(\omega_1)$ and $M_\varepsilon^* := M^* \upharpoonright^{\omega \geq}(\mathcal{U}_\varepsilon)$ belongs to $K_{T_1, A}^2$ for every $\varepsilon < \omega_1$.

We choose a pair (\mathbf{F}, \mathbf{f}) of functions with domain $\{\bar{M} : \bar{M} \text{ an increasing sequence of members of } K_{T_1, A}^2 \text{ of length } < \omega_1\}$ such that:

- $\mathbf{F}(\bar{M})$ is an extension of $\bigcup \{M_i : i < \ell g(\bar{M})\}$ from $K_{T_1, A}^2$,
- $\mathbf{f}(\bar{M})$ is an embedding from $M_{\ell g(\bar{M})}^*$ into $\mathbf{F}(\bar{M})$,
- if $\bar{M}^\ell = \langle M_\alpha : \alpha < \alpha_\ell \rangle$ for $\ell = 1, 2$ and $\alpha_1 < \alpha_2$, $\bar{M}^1 = \bar{M}^2 \upharpoonright \alpha_1$ and $\mathbf{F}(\bar{M}^1) \subseteq M_{\alpha_1}$ then $\mathbf{f}(\bar{M}^1) \subseteq \mathbf{f}(\bar{M}^2)$,
- if $a \in \mathbf{F}(\bar{M})$ and $n < \omega$ then for some $b \in M_{\ell g(\bar{M})}^*$ we have $\mathbf{F}(\bar{M}) \models a E_n(\mathbf{f}(\bar{M})(b))$.

Now check. ■_{2.12}

CONCLUSION 2.14. *Assume there is a (T_1, \aleph_1) -witness (see Definition 2.4) for the first-order complete theory T_1 from 2.1. Then:*

- (1) T_1 has an (\aleph_1, \aleph_0) -strongly limit model.
- (2) T_1 has an (\aleph_1, \aleph_1) -medium limit model.
- (3) T_1 has an (\aleph_1, \aleph_0) -superlimit model.

Proof. (1) By 2.8 the reduction of problems on $(EC(T_1), \prec)$ to $K_{T_1, A}^1$ (which is easy) is exactly as in [12].

(2) By 2.12.

(3) Like part (1) using Claim 2.9. ■_{2.14}

CLAIM 2.15. *If $\lambda = \sum\{\lambda_n : n < \omega\}$ and $\lambda_n = \lambda_n^{\aleph_0}$, then T_1 has a (λ, \aleph_0) -superlimit model in λ .*

Proof. Let M_n be the model M_{A_n, h_n} where $A_n = {}^\omega(\lambda_n)$ and $h_n : A_n \rightarrow \lambda_n^+$ is constantly λ_n . Clearly,

- (*)₁ M_n is a saturated model of T_1 of cardinality λ_n ,
- (*)₂ $M_n \prec M_{n+1}$,
- (*)₃ $M_\omega = \bigcup\{M_n : n < \omega\}$ is a special model of T_1 of cardinality λ .

The main point is:

- (*)₄ M_ω is (λ, \aleph_0) -superlimit model of T_1 .

[Why? Toward this assume:

- N_n is isomorphic to M_ω , say $f_n : M_\omega \rightarrow N_n$ is an isomorphism,
- $N_n \prec N_{n+1}$ for $n < \omega$.

Let $N_\omega = \bigcup\{N_n : n < \omega\}$ and we should prove $N_\omega \cong M_\omega$, so just N_ω is a special model of T_1 of cardinality λ suffice.

Let $N'_n = N_\omega \upharpoonright (\bigcup\{f_n(M_k) : k \leq n\})$. Clearly $N'_n \prec N'_{n+1} \prec N_\omega$ and $\bigcup\{N'_n : n < \omega\} = N_\omega$ and $\|N'_n\| = \lambda_n$. So it suffices to prove that N'_n is saturated and direct inspection shows this. ■_{2.15}

3. On non-existence of limit models. Naturally we assume that non-existence of superlimit models for unstable T is easier to prove. For other versions we need to look more. We first show that for $\lambda \geq |T| + \aleph_1$, if T is unstable then it does not have a superlimit model of cardinality λ , and if T is unsuperstable, we show this for “most” cardinals λ . On “ Φ proper for K_{or} or K_{tr}^ω ”, see [15, VII] or [7] or hopefully some day in [8, III]. We assume some knowledge of stability.

CLAIM 3.1. (1) *If T is unstable, $\lambda \geq |T| + \aleph_1$, then T has no superlimit model of cardinality λ .*

(2) *If T is stable non-superstable and $\lambda \geq |T| + \beth_\omega$ or $\lambda = \lambda^{\aleph_0} \geq |T|$ then T has no superlimit model of cardinality λ .*

REMARK 3.2. We assume some knowledge of EM models for linear orders I and members of K_{tr}^ω as index models (see, e.g., [15, VII]).

(2) We use the following definition in the proof, as well as a result from [17] or [18].

DEFINITION 3.3. For cardinals $\lambda > \kappa$ let $\lambda^{[\kappa]}$ be the minimal μ such that for some, equivalently for every set A of cardinality λ there is $\mathcal{P}_A \subseteq [A]^{\leq \kappa} = \{B \subseteq A : |B| \leq \kappa\}$ of cardinality λ such that any $B \in [\lambda]^{\leq \kappa}$ is the union of $< \kappa$ members of \mathcal{P}_A .

Proof of Claim 3.1. (1) Towards a contradiction assume M^* is a superlimit model of T of cardinality λ . As T is unstable we can find m and $\varphi(\bar{x}, \bar{y})$ such that

- $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\tau(T)}$ linearly orders some infinite $\mathbf{I} \subseteq {}^m M, M \models T$ so $\ell g(\bar{x}) = \ell g(\bar{y}) = m$.

We can find a Φ which is proper for linear orders ([15, VII]) and $F_\ell (\ell < m)$ such that $F_\ell \in \tau_\Phi \setminus \tau_T$ is a unary function symbol for $\ell < m, \tau_T \subseteq \tau(\Phi)$ and for every linear order I , $\text{EM}(I, \Phi)$ has Skolem functions and its τ_T -reduct $\text{EM}_{\tau(T)}(I, \Phi)$ is a model of T of cardinality $|T| + |I|$ and $\tau(\Phi)$ is of cardinality $|T| + \aleph_0$ and $\langle a_s : s \in I \rangle$ is the skeleton of $\text{EM}(I, \Phi)$, that is, it is an indiscernible sequence in $\text{EM}(I, \Phi)$ and $\text{EM}(I, \Phi)$ is the Skolem hull of $\{a_s : s \in I\}$, and letting $\bar{a}_s = \langle F_\ell(a_s) : \ell < m \rangle$ in $\text{EM}(I, \Phi)$ we have $\text{EM}_{\tau(T)}(I, \Phi) \models \varphi[\bar{a}_s, \bar{a}_t]^{\text{if}(s < t)}$ for $s, t \in I$.

Next we can find Φ_n (for $n < \omega$) such that:

- (a) Φ_n is proper for linear orders and $\Phi_0 = \Phi$,
- (b) $\text{EM}_{\tau(\Phi)}(I, \Phi_n) \prec \text{EM}_{\tau(\Phi)}(I, \Phi_{n+1})$ for every linear order I and $n < \omega$; moreover
- (b)⁺ $\tau(\Phi_n) \subseteq \tau(\Phi_{n+1})$ and $\text{EM}(I, \Phi_n) \prec \text{EM}_{\tau(\Phi_n)}(I, \Phi_{n+1})$ for every $n < \omega$ and linear order I ,
- (c) if $|I| \leq n$ then $\text{EM}_{\tau(\Phi)}(I, \Phi_n) = \text{EM}_{\tau(\Phi)}(I, \Phi_{n+1})$ and $\text{EM}_{\tau(T)}(I, \Phi_n) \cong M^*$,
- (d) $|\tau(\Phi_n)| = \lambda$.

This is easy. Let Φ_ω be the limit of $\langle \Phi_n : n < \omega \rangle$, i.e. $\tau(\Phi_\omega) = \bigcup \{\tau(\Phi_n) : n < \omega\}$ and if $k < \omega$ then $\text{EM}_{\tau(\Phi_k)}(I, \Phi_\omega) = \bigcup \{\text{EM}_{\tau(\Phi_k)}(I, \Phi_n) : n \in [k, \omega)\}$. So as M^* is a superlimit model, for any linear order I of cardinality λ , $\text{EM}_{\tau(T)}(I, \Phi_\omega)$ is the direct limit of $\langle \text{EM}_{\tau(T)}(J, \Phi_n) : J \subseteq I \text{ finite} \rangle$, each isomorphic to M^* , so as we have assumed that M^* is a superlimit model it follows that $\text{EM}_{\tau(T)}(I, \Phi_\omega)$ is isomorphic to M^* . But by [14, III] or [7] which may eventually be [8, III] there are 2^λ many pairwise non-isomorphic models of this form varying I on the linear orders of cardinality λ , contradiction.

(2) First assume $\lambda = \aleph_0$. Let $\tau \subseteq \tau_T$ be countable such that $T' = T \cap \mathbb{L}(\tau)$ is not superstable. Clearly if M^* is a (λ, \aleph_0) -limit model then

$M^* \upharpoonright \tau'$ is not \aleph_1 -saturated. [Why? As in [10, Ch. VI, §6], but we shall give full details: there are $N_* \models T$, $p = \{\varphi_n(\lambda, \bar{a}_n) : n < \omega\}$ a type in N_* , $\bar{a}_n \triangleleft \bar{a}_{n+1}$, $\bar{a}_\langle \rangle$ empty and $\varphi_{n+1}(x, \bar{a}_{n+1})$ forks over \bar{a}_n . Let $\mathbf{F}(M)$ be such that if $n < \omega$ and $\bar{b}_n \subseteq M$ realizes $\text{tp}(\bar{a}_n, \emptyset, N_*)$ then for some \bar{b}_{n+1} from $\mathbf{F}(M)$ realizing $\text{tp}(\bar{a}_{n+1}, \emptyset, N_*)$, the type $\text{tp}(\bar{b}_{n+1}, M, \mathbf{F}(M))$ does not fork over b_n .] But if $\kappa = \text{cf}(\kappa) \in [\aleph_1, \lambda]$ and M^* is a (λ, κ) -limit then $M^* \upharpoonright \tau'$ is \aleph_1 -saturated, contradiction.

The case $\lambda \geq |T| + \beth_\omega$ is more complicated (the assumption $\lambda \geq \beth_\omega$ is to enable us to use [17] or see [18] for a simpler proof; we can use weaker but less transparent assumptions; maybe $\lambda \geq 2^{\aleph_0}$ suffices).

As T is stable non-superstable by [15] for some $\bar{\Delta}$:

- ⊗₁ for any μ there are M and $\langle a_{\eta, \alpha} : \eta \in {}^\omega \mu \text{ and } \alpha < \mu \rangle$ such that
 - (a) M is a model of T ,
 - (b) $\mathbf{I}_\eta = \{a_{\eta, \alpha} : \alpha < \mu\} \subseteq M$ is an indiscernible set (and $\alpha < \beta < \mu \Rightarrow a_{\eta, \alpha} \neq a_{\eta, \beta}$),
 - (c) $\bar{\Delta} = \langle \Delta_n : n < \omega \rangle$ and $\Delta_n \subseteq \mathbb{L}_{\tau(T)}$ infinite,
 - (d) for $\eta, \nu \in {}^\omega \mu$ we have $\text{Av}_{\Delta_n}(M, \mathbf{I}_\eta) = \text{Av}_{\Delta_n}(M, \mathbf{I}_\nu)$ iff $\eta \upharpoonright n = \nu \upharpoonright n$.

Hence by [15, VIII] (or see [7] assuming M^* is a universal model of T of cardinality λ):

- ⊗_{2.1} there is $\bar{\Phi}$ such that:
 - (a) $\bar{\Phi}$ is proper for K_{tr}^ω , $\tau_T \subseteq \tau(\bar{\Phi})$, $|\tau(\bar{\Phi})| = \lambda \geq |T| + \aleph_0$,
 - (b) for $I \subseteq {}^\omega \geq \lambda$, $\text{EM}_{\tau(\bar{\Phi})}(I, \bar{\Phi})$ is a model of T and $I \subseteq J \Rightarrow \text{EM}(I, \bar{\Phi}) \prec \text{EM}(J, \bar{\Phi})$,
 - (c) for some two-place function symbol F if for $I \in K_{\text{tr}}^\omega$ and $\eta \in P_\omega^I$, I a subtree of ${}^\omega \geq \lambda$, for transparency we let $\mathbf{I}_{I, \eta} = \{F(a_\eta, a_\nu) : \nu \in I\}$, then $\langle \mathbf{I}_{I, \eta} : \eta \in P_\omega^I \rangle$ are as in ⊗₁(b), (d).

Also

- ⊗_{2.2} if $\bar{\Phi}_1$ satisfies (a)–(c) of ⊗_{2.1} and M is a universal model of T then there is $\bar{\Phi}_2^*$ satisfying (a)–(c) of ⊗_{2.1} and $\bar{\Phi}_1 \leq \bar{\Phi}_2^*$ (see ⊗_{2.3}(a)) and for every finitely generated $J \in K_{\text{tr}}^\omega$ (see ⊗_{2.3}(b)) there is $M' \cong M$ such that $\text{EM}_{\tau(T)}(J, \bar{\Phi}_1) \prec M' \prec \text{EM}_{\tau(T)}(J, \bar{\Phi}_2^*)$,
- ⊗_{2.3} (a) we say $\bar{\Phi}_1 \leq \bar{\Phi}_2$ when $\tau(\bar{\Phi}_1) \subseteq \tau(\bar{\Phi}_2)$ and $J \in K_{\text{tr}}^\omega \Rightarrow \text{EM}(J, \bar{\Phi}_1) \prec \text{EM}_{\tau(\bar{\Phi}_1)}(J, \bar{\Phi}_2)$,
- (b) we say that $J \subseteq I$ is *finitely generated* if it has the form $\{\eta_\ell : \ell < n\} \cup \{\rho : \text{for some } n, \ell \text{ we have } \rho \in P_n^I \text{ and } \rho \prec^I \eta_\ell\}$ for some $\eta_0, \dots, \eta_{n-1} \in P_\omega^I$,
- ⊗_{2.4} if $M_* \in \text{EC}_\lambda(T)$ is superlimit (or just weakly S -limit, with $S \subseteq \lambda^+$ stationary) then there is $\bar{\Phi}$ as in ⊗_{2.1} above such that $\text{EM}_{\tau(T)}(J, \bar{\Phi}) \cong M_*$ for every finitely generated $J \in K_{\text{tr}}^\omega$,

⊗_{2.5} we fix Φ as in ⊗_{2.4} for $M_* \in \text{EC}_\lambda(T)$ superlimit.

Hence (mainly by clause (b) of ⊗_{2.1} and ⊗_{2.4} as in the proof of part (1))

⊗₃ if $I \in K_{\text{tr}}^\omega$ has cardinality $\leq \lambda$ then $\text{EM}_{\tau(\Phi)}(I, \Phi)$ is isomorphic to M^* .

Now by [17], we can find regular uncountable $\kappa < \beth_\omega$ such that $\lambda = \lambda^{[\kappa]}$ (see Definition 3.3).

Let $S = \{\delta < \kappa : \text{cf}(\delta) = \aleph_0\}$ and $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ be such that η_δ is an increasing sequence of length ω with limit δ .

For a model M of T let $\text{OB}_{\bar{\eta}}(M) = \{\bar{\mathbf{a}} : \bar{\mathbf{a}} = \langle a_{\eta_\delta, \alpha} : \delta \in W \text{ and } \alpha < \kappa \rangle, W \subseteq S \text{ and in } M \text{ they are as in } \otimes_1(b), (d)\}$. For $\bar{\mathbf{a}} \in \text{OB}_{\bar{\eta}}(M)$ let $W[\bar{\mathbf{a}}]$ be W as above and let

$$\begin{aligned} \Xi(\bar{\mathbf{a}}, M) &= \{\eta \in {}^\omega \kappa : \text{there is an indiscernible set} \\ &\quad \mathbf{I} = \{a_\alpha : \alpha < \kappa\} \text{ in } M \text{ such that for every } n, \\ &\quad \text{for some } \delta \in W[\bar{\mathbf{a}}], \eta \upharpoonright n = \eta_\delta \upharpoonright n \text{ and} \\ &\quad \text{Av}_{\Delta_n}(M, \mathbf{I}) = \text{Av}_{\Delta_n}(M, \{a_{\eta_\delta, \alpha} : \alpha < \kappa\})\}. \end{aligned}$$

Clearly:

- ⊗₄ (a) if $M \prec N$ then $\text{OB}_{\bar{\eta}}(M) \subseteq \text{OB}_{\bar{\eta}}(N)$,
 (b) if $M \prec N$ and $\bar{\mathbf{a}} \in \text{OB}_{\bar{\eta}}(M)$ then $\Xi(\bar{\mathbf{a}}, M) \subseteq \Xi(\bar{\mathbf{a}}, N)$.

Now by the choice of κ it should be clear that:

- ⊗₅ if $M \models T$ is of cardinality λ then we can find an elementary extension N of M of cardinality λ such that for every $\bar{\mathbf{a}} \in \text{OB}_{\bar{\eta}}(M)$ with $W[\bar{\mathbf{a}}]$ a stationary subset of κ , for some stationary $W' \subseteq W[\bar{\mathbf{a}}]$ the set $\Xi[\bar{\mathbf{a}}, N]$ includes $\{\eta \in {}^\omega \kappa : (\forall n)(\exists \delta \in W')(\eta \upharpoonright n = \eta_\delta \upharpoonright n)\}$ (moreover we can even find $\varepsilon^* < \kappa$ and $W_\varepsilon \subseteq W$ for $\varepsilon < \varepsilon^*$ satisfying $W[\bar{\mathbf{a}}] = \bigcup \{W_\varepsilon : \varepsilon < \varepsilon^*\}$),
 ⊗₆ we find $M \in \text{EC}_\lambda(T)$ isomorphic to M^* such that for every $\bar{\mathbf{a}} \in \text{OB}_{\bar{\eta}}(M)$ with $W[\bar{\mathbf{a}}]$ a stationary subset of κ , we can find a stationary subset W' of $W[\bar{\mathbf{a}}]$ such that the set $\Xi[\bar{\mathbf{a}}, M]$ includes $\{\eta \in {}^\omega \mu : (\forall n)(\exists \delta \in W')(\eta \upharpoonright n = \eta_\delta \upharpoonright n)\}$.

[Why? We choose (M_i, N_i) for $i < \kappa^+$ such that:

- $M_i \in \text{EC}_\lambda(T)$ is \prec -increasing continuous,
- M_{i+1} is isomorphic to M^* ,
- $M_i \prec N_i \prec M_{i+1}$,
- (M_i, N_i) are like (M, N) in ⊗₅.

Now $M = \bigcup \{M_i : i < \kappa^+\}$ is as required. The model M is isomorphic to M^* as M^* is superlimit.]

Now the model from ⊗₆ is not isomorphic to $M' = \text{EM}_{\tau(T)}({}^\omega \lambda \cup \{\eta_\delta : \delta \in S\}, \Phi)$ where Φ is from ⊗_{2.1}. But $M' \cong M^*$ by ⊗₃.

Altogether we are done. ■_{3.1}

The following claim says in particular that if some not unreasonable pcf conjectures hold, the conclusion holds for every $\lambda \geq 2^{\aleph_0}$.

CLAIM 3.4. *Assume T is stable non-superstable, $\lambda \geq |T|$ and $\lambda \geq \kappa = \text{cf}(\kappa) > \aleph_0$.*

(1) *T has no (λ, κ) -superlimit model provided that $\kappa = \text{cf}(\kappa) > \aleph_0$, $\kappa^{\aleph_0} \leq \lambda$ and $\lambda = \mathbf{U}_D(\lambda) := \text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^\kappa \text{ and for every } f : \kappa \rightarrow \lambda \text{ for some } u \in \mathcal{P} \text{ we have } \{\alpha < \kappa : f(\alpha) \in u\} \in D^+\}$, where D is a normal filter on κ to which $\{\delta < \kappa : \text{cf}(\delta) = \aleph_0\}$ belongs.*

(2) *Similarly if $\lambda \geq 2^{\aleph_0}$ and letting $J_0 = \{u \subseteq \kappa : |u| \leq \aleph_0\}$, $J_1 = \{u \subseteq \kappa : u \cap S_{\aleph_0}^\kappa \text{ non-stationary}\}$ we have $\lambda = \mathbf{U}_{J_1, J_0}(\lambda) := \text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\aleph_0}$, and if $u \in J_1$ and $f : (\kappa \setminus u) \rightarrow \lambda$ then for some countable infinite $w \subseteq \kappa(u)$ and $v \in \mathcal{P}$, $\text{Rang}(f \upharpoonright w) \subseteq v\}$.*

Proof. Like 3.1. ■

CLAIM 3.5. (1) *Assume T is unstable and $\lambda \geq |T| + \beth_\omega$. Then for at most one regular $\kappa \leq \lambda$, T has a weakly (λ, κ) -limit model and even a weakly (λ, S) -limit model for some stationary $S \subseteq S_\kappa^\lambda$.*

(2) *Assume T is unsuperstable and $\lambda \geq |T| + \beth_\omega(\kappa_2)$ and $\kappa_1 = \aleph_0 < \kappa_2 = \text{cf}(\kappa_2)$. Then T has no model which is a weak (λ, S) -limit where $S \subseteq \lambda$ and $S \cap S_{\kappa_\ell}^\lambda$ is stationary for $\ell = 1, 2$.*

Proof. (1) Assume $\kappa_1 \neq \kappa_2$ form a counterexample. Let $\kappa < \beth_\omega$ be regular large enough such that $\lambda = \lambda^{[\kappa]}$ (see Definition 3.3) and $\kappa \notin \{\kappa_1, \kappa_2\}$. Let m and $\varphi(\bar{x}, \bar{y})$ be as in the proof of 3.1. Then

(*) if $M \in \text{EC}_\lambda(T)$ then there is N such that:

- (a) $N \in \text{EC}_\lambda(T)$,
- (b) $M \prec N$,
- (c) if $\bar{\mathbf{a}} = \langle \bar{a}_i : i < \kappa \rangle \in {}^\kappa({}^m M)$ for $\alpha < \kappa$ then for some $\mathcal{U} \in [\kappa]^\chi$, for every uniform ultrafilter D on κ to which \mathcal{U} belongs there is $\bar{a}_D \in {}^n N$ such that $\text{tp}(\bar{a}_D, N, N) = \text{Av}(D, \bar{\mathbf{a}}, M) = \{\psi(\bar{x}, \bar{c}) : \psi(\bar{x}, \bar{z}) \in \mathbb{L}(\tau_T), \bar{c} \in {}^{\ell g(\bar{z})} M \text{ and } \{\alpha < \kappa : N \models \psi[\bar{a}_{i_\alpha}, \bar{c}]\} \in D\}$.

Similarly

- ⊠₁ for every function \mathbf{F} with domain $\{\bar{M} : \bar{M} \text{ an } \prec\text{-increasing sequence of models of } T \text{ of length } < \lambda^+ \text{ each with universe } \in \lambda^+\}$ such that $M_i \prec \mathbf{F}(\bar{M})$ for $i < \ell g(\bar{M})$ and $\mathbf{F}(\bar{M})$ has universe $\in \lambda^+$ there is a sequence $\langle M_\varepsilon : \varepsilon < \lambda^+ \rangle$ obeying \mathbf{F} such that: for every $\varepsilon < \lambda^+$ and $\bar{\mathbf{a}} \in {}^\kappa({}^m(M_\varepsilon))$ for $\alpha < \kappa$, there is $\mathcal{U} \in [\kappa]^\kappa$ such that for every ultrafilter D on κ to which \mathcal{U} belongs, for every $\zeta \in (\varepsilon, \lambda^+)$ there is $\bar{\mathbf{a}}_{D, \zeta} \in {}^m(M_{\zeta+1})$ realizing $\text{Av}(D, \bar{\mathbf{a}}, M_\zeta)$ in $M_{\zeta+1}$.

Hence

\boxplus_2 for $\langle M_\alpha : \alpha < \lambda^+ \rangle$ as in \boxplus_1 , for every limit $\delta < \lambda^+$ of cofinality $\neq \kappa$ and every $\bar{\mathbf{a}} = \langle \bar{a}_i : i < \kappa \rangle \in {}^\kappa(m(M_\delta))$, there is $\mathcal{U} \in [\kappa]^\kappa$ such that for every ultrafilter D on κ to which \mathcal{U} belongs, there is a sequence $\langle \bar{b}_\varepsilon : \varepsilon < \text{cf}(\delta) \rangle \in {}^{\text{cf}(\delta)}(m(M_\delta))$ such that for every $\psi(\bar{x}, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{c} \in {}^{\text{lg}(\bar{z})}(M_\delta)$, and for every $\varepsilon < \text{cf}(\delta)$ large enough, $M_\delta \models \psi[\bar{b}_\varepsilon, \bar{c}]$ iff $\psi(\bar{x}, \bar{c}) \in \text{Av}(D, \bar{\mathbf{a}}, M_\delta)$.

The rest should be clear.

(2) Combine the above and the proof of 3.1(2). ■_{3.5}

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Saharon Shelah
Einstein Institute of Mathematics
Edmond J. Safra Campus
Givat Ram
The Hebrew University of Jerusalem
Jerusalem, 91904, Israel
and
Department of Mathematics
Hill Center – Busch Campus
Rutgers, The State University of New Jersey
110 Frelinghuysen Road
Piscataway, NJ 08854-8019, U.S.A.
E-mail: shelah@math.huji.ac.il
<http://shelah.logic.at>

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