

# A BANACH SPACE WITH FEW OPERATORS

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## ABSTRACT

Assuming the axiom (of set theory)  $V = L$  (explained below), we construct a Banach space with density character  $\aleph_1$  such that every (linear bounded) operator  $T$  from  $B$  to  $B$  has the form  $a\mathbf{I} + T_1$ , where  $\mathbf{I}$  is the identity, and  $T_1$  has a separable range. The axiom  $V = L$  means that all the sets in the universe are in the class  $L$  of sets constructible from ordinals; in a sense this is the minimal universe. In fact, we make use of just one consequence of this axiom,  $\diamond_{\aleph_1}$ , proved by Jensen, which is widely used by mathematical logicians.

NOTATION. Let  $i, j, \alpha, \beta, \gamma, \delta$  be ordinals,  $\omega$  the first infinite ordinal,  $\omega_1$  the first uncountable ordinal. Let  $k, l, m, n, p$  be natural numbers, and let  $a, b, c, d$  be reals, and  $x, y, z$  elements of a (vector, or norm, or Banach) space.

THE MAIN THEOREM. *Assume the axiom  $V = L$  holds. Then there is a Banach space  $\bar{Z}$ , and an element of the space  $z_i$  ( $i < \omega_1$ ), such that:*

(1)  *$\text{span}\{z_i : i < \omega_1\}$  is dense in  $\bar{Z}$ ,  $\|z_i\| = 1$ , and there are projections  $P_\alpha$  ( $\alpha < \omega_1$ ) of norm 1 of  $\bar{Z}$  into itself,  $P_\beta(z_i) = 0$  for  $i \geq \beta$ ,  $P_\beta(z_i) = z_i$  for  $i < \beta$ . So the density character of  $\bar{Z}$  is  $\omega_1$  and it has a basis  $\{z_i : i < \omega_1\}$ .*

(2) *If  $T : B \rightarrow B$  is (linear, bounded) operator, then for some real  $a$ ,  $Tz_i = az_i$  for all but countably many  $i$ 's. So  $T - a\mathbf{I}$  is an operator with a separable range.*

REMARKS. (1) We can prove similar theorems for higher cardinals, i.e., if  $\diamond(\{\delta < \lambda^+ : \text{cf}\delta = \lambda\})$ , we can construct a space with density character  $\lambda^+$  such that every operator  $T$  of the space is  $a\mathbf{I} + T_1$ ;  $T_1$  has range with density character  $\lambda$ .

(2) We can choose our space so that for every uncountable set of  $z_i$ 's, there is a countable set which generates an  $l_\infty$ -Banach space and an  $l_1$ -Banach space.

## The construction

STAGE A. Let  $\{z_i : i < \omega_1\}$  generate freely a vector space  $H$  over  $Q$  (the rationals). For a set  $I$  of ordinals let  $H_I$  and also  $H(I)$  denote  $\text{span}\{z_i : i \in I\}$

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(= the subvector space spanned by  $z_i, i \in I$ ). As an ordinal  $i$  is  $\{j : j < i\}$ ,  $H_i$  is the vector space spanned by  $\{z_j : j < i\}$ .

Let  $I_i^m$  ( $m < \omega, i < \omega_1$ ) be finite subsets of  $i$ , increasing with  $m$ , and  $i = \bigcup_m I_i^m$ . For subsets  $A_1, A_2, \dots$  of  $H$ ,  $\langle A_1, A_2, \dots \rangle_H$  is the span of  $A_1 \cup A_2 \cup \dots$ . We usually omit  $H$  and write  $y$  instead of  $\{y\}$ .

STAGE B. A subset  $I$  of  $\omega_1$  is called closed if for each limit ordinal  $i < \omega_1$  which satisfies  $(\forall j < i) (\exists \alpha) (j < \alpha < i \wedge \alpha \in I)$  belong to  $I$ .  $I$  is unbounded if  $(\forall i < \omega_1) (\exists j < \omega_1) (i < j \wedge j \in I)$ . A set of  $I \subseteq \omega_1$  is called *stationary* if it has a non-empty intersection with every closed unbounded subset of  $\omega_1$ .

STAGE C. By Jensen [1], if  $V = L$  then there are sets  $D_i$ , functions  $f_i (i < \omega_1)$  and  $r_i \in \{0, 1\}$  such that

- (i)  $f_i$  is a two-place function from  $H_i$  into the reals,  $D_i$  a subset of  $i$ .
- (ii) For every subset  $D$  of  $\omega_1$  and two-place function from  $H$  into the reals, and  $r \in \{0, 1\}$ ,  $\{i < \omega_1 : D \cap i = D_i, f|H_i = f_i, r_i = r\}$  is a stationary subset of  $\omega_1$ .

From now on  $f_i$  are as above.

STAGE D. In a norm space  $Z$ , for  $z \in Z, X \subseteq Z$ , we say  $z$  is *good* over  $X$  if  $(\forall x \in X) \|z + x\| \geq \|z\|, \|x\|$  and  $\|z\| = 1$ .

If  $z_0, \dots, z_k \in Z, X \subseteq Z$  we say  $(z_0, \dots, z_k)$  is good over  $X$  if  $\|z_i\| = 1$  and for any reals  $a_i$  and  $x \in X$

$$\left\| \sum_{i=0}^k a_i z_i + x \right\| \geq \left\| \sum_{i=0}^k a_i z_i \right\|, \|x\|.$$

Note that (a)  $(z_0)$  is good over  $X$  iff  $z_0$  is good over  $X$ ; (b) if  $(z_0, \dots, z_k)$  is good over  $X$  then so is every sequence from  $\langle z_0, \dots, z_k \rangle$ .

STAGE E. Suppose  $Y, Z$  are norm spaces,  $Y \cap Z = X$ , and let  $W$  be a vector space such that  $Y, Z$  are subspaces of it, and  $W = Y + Z$  (as vector spaces). We can define a norm on  $W$  which extends the norms on  $Y$  and  $Z$ , and get a norm space, as follows:

$$\|w\| = \inf \{ \|y\| + \|z\| : y \in Y; z \in Z, w = y + z \}.$$

In this case the unit ball of  $W$  is the convex hull of the unit balls of  $Y$  and  $Z$ . We call this  $N_1$ -amalgamation. Note that

- (a) if  $y \in Y$  is good over  $X$ , it will be good over  $Z$ ; and
- (b) if also  $z \in Z$  is good over  $X$  then  $\|y + z\| = 2$ .

STAGE F. Suppose that in Stage E,

$$Y = \langle X, y_0, \dots, y_k \rangle, \quad Z = \langle X, z_0, \dots, z_l \rangle,$$

$(y_0, \dots, y_k), (z_0, \dots, z_l)$  are good over  $X$ . Then there is another way to define a norm on  $W$  extending the norms on  $Y$  and  $Z$ : for  $x \in X$

$$\left\| \sum_{n=0}^k b_n y_n + \sum_{n=0}^l c_n z_n + x \right\| = \max \left\{ \left\| \sum_{n=0}^k b_n y_n + x \right\|, \left\| \sum_{n=0}^l c_n z_n + x \right\| \right\}.$$

We call this  $N_\infty$ -amalgamation (unlike  $N_1$ -amalgamation, it apparently does not depend only on  $Y$  and  $Z$ , but also on  $\text{span}\{y_0, \dots, y_k\}$  and  $\text{span}\{z_0, \dots, z_k\}$ ).

Note that

- (a)  $(z_0, \dots, z_l)$  is good over  $Y$ ,
- (b) for  $n \leq l, m \leq k, z_n + y_m$  is good over  $X$  and in particular  $\|z_n + y_m\| = 1$ ,
- (c) if  $l(1) < l$ , and we first amalgamate  $X, \langle X, y_0, \dots, y_k \rangle, \langle X, z_0, \dots, z_{l(1)} \rangle$  in the above-mentioned way and then amalgamate  $X' = \langle X, z_0, \dots, z_{l(1)} \rangle, \langle X', y_0, \dots, y_k \rangle, \langle X', z_{l(1)+1}, \dots, z_l \rangle$ , we get the same norm.

STAGE G. We shall define by induction on  $i < \omega_1$  norm spaces  $Z_i$ , increasing with  $i$ , such that  $Z_i$  as a vector space is  $H_i$ , and for some  $i$ 's, infinite sets  $S_i \subset \omega$  and elements  $y_i^m, y_i^m \in H_i$  (for  $m < \omega$ ) when  $r_i = 0$ , and  $y_{i,l}^m$  ( $m < \omega, 1 \leq l \leq p(m, i)$ ) when  $r_i = 1$ , such that (not distinguishing strictly between subspaces of  $H_i$  and of  $Z_i$ )

(\*) if  $\gamma \leq \alpha_0 < \alpha_1 < \dots < \alpha_k \leq i, \omega \leq i, k$  a natural number,  $r_\gamma = 0, y_\gamma^0$  is defined, then for infinitely many  $m \in S_\gamma$

(I) the amalgamation of the triple

$$H(I_\gamma^m), \langle H(I_\gamma^m), y_\gamma^m \rangle, \langle H(I_\gamma^m), z_{\alpha_0}, \dots, z_{\alpha_k} \rangle$$

is by the  $N_\infty$ -amalgamation, i.e., for  $x \in H(I_\gamma^m)$

$$\left\| ay_i^m + \sum_{l=0}^k b_l z_{\alpha_l} + x \right\| = \max \left\{ \left\| ay_i^m + x \right\|, \left\| \sum_{l=0}^k b_l z_{\alpha_l} + x \right\| \right\};$$

(II) the amalgamation of the triple

$$\langle H(I_\gamma^m), y_\gamma^m \rangle, \langle H(I_\gamma^m), y_\gamma^m, y_\gamma^m \rangle, \langle H(I_\gamma^m), y_\gamma^m, z_{\alpha_0}, \dots, z_{\alpha_k} \rangle$$

is by the  $N_1$ -amalgamation.

So in particular

(\*')  $z_\gamma$  is good over  $H_\gamma$ , and if  $\gamma \leq \alpha_0 \leq \alpha_1 \dots \leq \alpha_k$  then  $(z_{\alpha_0}, \dots, z_{\alpha_k})$  is good over  $H_\gamma$ .

We also demand

(\*\*) if  $\gamma \leq \alpha_0 < \alpha_1 < \dots < \alpha_k \leq i, \omega \leq i, k$  a natural number,  $r_\gamma = 1$ , then for infinitely many  $m < \omega$  the amalgamation of the triple

$$H(I_\gamma^m), \langle H(I_\gamma^m), y_{i,1}^m, \dots, y_{i,p(m,\gamma)}^m \rangle, \langle H(I_\gamma^m), z_{\alpha_0}, \dots, z_{\alpha_k} \rangle$$

is by  $N_\infty$ -amalgamation.

For  $i$  limit  $Z_i = \bigcup_{j < i} Z_j$ , for  $i < \omega$ ,  $\|\sum_{l < i} a_l z_l\| = \max_{l < \omega} |a_l|$ .

STAGE H. Now we do the induction step, so we suppose the norm on  $H_i$  is defined,  $i \geq \omega$ , and we call the norm space  $Z_i$ . In this stage we shall define  $y_i^m, y_i^r$  ( $m < \omega$ ) and  $S_i$ , and in the next stage we shall define the norm on  $H_{i+1}$ . Remember that  $f_i$  is a two place function from  $H_i$  to  $R$  given by the Jensen diamond (see Stage B).

If there is a (bounded) operator  $T$  on  $\bar{Z}_i$  such that for every  $x, y \in H_i$ ,  $f_i(x, y) = \|Tx - y\|$ , it is unique, and we call it  $T_i$ .

If  $T_i$  is not defined we do not define  $S_i, y_i^m, y_i^r$ . So suppose  $T_i$  is defined.

(a) If  $Y$  is a Banach space,  $T$  an operator on  $Y$ ,  $H \subseteq Y$  a subspace, then let

$$c(H, T, Y) = \sup\{d(Ty, \langle H, y \rangle) : y \in Y, y \text{ good over } H\},$$

where  $d(y_1, H_1)$  is the distance between  $y_1$  and  $H_1$ , i.e.,  $\inf\{\|y_1 - x\| : x \in H_1\}$ , and let

$$c_\varepsilon(H, T, Y) = \sup\{d(Ty, H) : d(Ty, \langle H, y \rangle) \geq c(H, T, Y) - \varepsilon$$

and  $y$  is good over  $H\}$ .

Note that  $c(H, T, Y) \leq \|T\|$  and it decreases with  $H$ .

Now if  $r_i = 0$ , choose  $y_i^m, y_i^r$  in  $H_i$  such that:

- (b) (i)  $d(Ty_i^m, \langle H(I_i^m), y_i^m \rangle) \geq c(H(I_i^m), T_i, \bar{Z}_i) - 1/m$ ,
- (ii)  $y_i^r$  is good over  $H(I_i^m)$ ,
- (iii)  $d(Ty_i^r, H(I_i^m)) \geq c_{1/m}(H(I_i^m), T_i, \bar{Z}_i) - 1/m$ ,
- (iv)  $\|Ty_i^m - y_i^r\| < 1/m$ .

Clearly  $c_{1/m}(H(I_i^m), T_i, \bar{Z}_i)$  is a real number of absolute value  $< \|T\|$ , hence there is an infinite set  $S_i \subseteq \omega$  such that

(c) for  $k < m \neq n$  in  $S_i$ ,  $1/k > |c_{1/m}(H(I_i^m), T_i, \bar{Z}_i) - c_{1/n}(H(I_i^m), T_i, \bar{Z}_i)|$ .

(d) If  $r_i = 1$  choose a  $p = p(m, i) < \omega$  and  $y_{i,1}^m \in \{z_\alpha : \max I_i^m < \alpha < i, \alpha \in D_i\}$  such that:

(i) for every  $x \in H(I_i^m)$ ,

$$\left\| \sum_{i=1}^{p(m)} a_i y_{i,1}^m + x \right\| = \sup \|a_i y_{i,1}^m + x\|$$

(notice each  $y_{i,1}^m$  is good over  $H(I_i^m)$ ),

(ii) if among the  $p$ 's satisfying (i) there is a maximal one, this will be our  $p$ ; otherwise choose  $p = m$ .

STAGE I. Now we have to define the norm on  $H_{i+1}$  (after we have defined it on  $H_i$ ), and define, if necessary,  $y_i^m, y_i^m (m < \omega)$  or  $y_{i1}^m$ .

We have to satisfy the requirements (\*) and (\*\*) from Stage G; when  $\alpha_k < i$  they are satisfied by the induction hypothesis. Clearly there are only countably many appropriate requirements, so we can find a list of them of length  $\omega$ , each appearing infinitely many times.

Let  $\{\beta_n : n < \omega\}$  be a list of  $i = \{j : j < i\}$ . Now we define by induction on  $n < \omega$  a finite set  $J_n \subseteq i$ , and a norm space  $Z_i^m$  which as a vector space is  $H(J_n \cup \{i\})$  (we shall not distinguish) such that

- (i)  $J_n \subseteq J_{n+1}$ ,
- (ii)  $Z_i^m$  is a subspace of  $Z_i^{n+1}$ ,
- (iii)  $i = \bigcup_{n < \omega} J_n$ ,
- (iv) in  $Z_i^m$ ,  $z_i$  is good over  $H(J_n)$ .

For  $n = 0$  let  $H_0$  be the empty set, and the norm  $Z_i^0$  is  $\|az_i\| = |a|$ .

Suppose we have defined  $Z_i^n$  for  $n$ , and let us define  $Z_i^{n+1}$ . Let  $\langle k, \gamma, \alpha_0, \dots, \alpha_{k-1} \rangle$  be the  $n$ -th in the list of cases of (\*) and (\*\*) from Stage G. Assume for now that  $r_\gamma = 0$  (the case  $r_\gamma = 1$  is just simpler). If  $\{\alpha_0, \dots, \alpha_{k-1}\} \not\subseteq J_n$ , we let  $J_{n+1} = J_n \cup \{\beta_n\}$ , and we define the norm of  $Z_i^{n+1}$  by  $N_1$ -amalgamation of  $H(J_n)$ ,  $Z_i^n$ ,  $H(J_{n+1})$  (see Stage E).

Now if  $\{\alpha_0, \dots, \alpha_{k-1}\} \subseteq J_n$ , let  $J_n - \gamma = \{\beta_0, \dots, \beta_l\}$  (as  $\gamma \leq \alpha_0 < \dots$  necessarily  $\{\alpha_0, \dots, \alpha_{k-1}\} \subseteq \{\beta_0, \dots, \beta_l\}$ ). By the induction hypothesis, (\*) of Stage G holds for  $\gamma \leq \beta_0 \leq \dots \leq \beta_l$  hence there is an  $m \in S_\gamma$  satisfying

- (1)  $J_n \cap \gamma \subseteq I_\gamma^m$  (possible as (\*) says "for infinitely many  $m$ 's" and  $\gamma = \bigcup_m I_\gamma^m$ ,  $I_\gamma^m$  increase with  $m$ , and  $J_n$  is finite),
- (2) the amalgamation of the triple

$$H(I_\gamma^m), \langle H(I_\gamma^m), y_\gamma^m \rangle, \langle H(I_\gamma^m), z_{\beta_0}, \dots \rangle$$

is an  $N_\infty$ -amalgamation,

- (3) the amalgamation of the triple

$$\langle H(I_\gamma^m), y_\gamma^m \rangle, \langle H(I_\gamma^m), y_\gamma^m, y_\gamma^m \rangle, \langle H(I_\gamma^m), y_\gamma^m, z_{\beta_0}, \dots \rangle$$

is an  $N_1$ -amalgamation.

We choose a finite  $J_{n+1}$  such that  $J_n \subseteq J_{n+1} \subseteq i$ ,  $\beta_n \in J_{n+1}$  and  $y_\gamma^m, y_\gamma^m \in H(J_{n+1})$  (this is trivial). Now we define  $Z_i^{n+1}$  by successive amalgamation.

( $\alpha$ ) We make an  $N_1$ -amalgamation of the triple  $H(J_n)$ ,  $Z_i^n$ ,  $H(J_n \cup I_\gamma^m)$ :  $z_i$  is good (in it) over  $H(J_n \cup I_\gamma^m)$  by (a) of Stage E.

( $\beta$ ) We make an  $N_\infty$ -amalgamation of  $H(J_n \cup I_\gamma^m)$ ,  $H(J_n \cup I_\gamma^m \cup \{i\})$  (defined in  $\alpha$ ), and  $\langle H(J_n \cup I_\gamma^m), y_\gamma^m \rangle$  (possible as  $z_i$  is good over  $H(J_n \cup I_\gamma^m)$  by ( $\alpha$ ) and  $y_\gamma^m$

is good over  $H(J_n \cup I_\alpha^m)$  by the choice of  $m$  to satisfy (2) and (a) of Stage F). By (a) of Stage F,  $z_i$  is good over  $\langle H(J_n \cup I_\gamma^n), y_\gamma^n \rangle$  in the amalgamated space we have just defined.

( $\gamma$ ) We make the  $N_i$ -amalgamation of

$$\langle H(J_n \cup I_\gamma^m), y_\gamma^m \rangle, H(J_{n+1}), \langle H(J_n \cup I_\gamma^m \cup \{i\}), y_\gamma^m \rangle$$

(with the norm defined in ( $\beta$ )) and call it  $Z_i^{n+1}$ . By (a) of Stage E  $z_i$  is good over  $H(J_{n+1})$  in  $Z_i^{n+1}$ .

It is easy to check that (I) and (II) of (\*) hold for  $\gamma, \alpha_0, \dots, \alpha_k$  and  $m$  (by (c) of Stage F).

So  $Z_i^n$  is defined for every  $i$ , and let  $Z_{i+1} = \bigcup_{n < \omega} Z_i^n$ . Clearly  $Z_{i+1}$  as a vector space is  $H_{i+1}$  (as  $\beta_n \in J_{n+1}$ ). Each requirement  $\gamma \cong \alpha_0 < \dots < \alpha_k = i$  appears in our list infinitely many times so for every  $n$  big enough  $\{\alpha_0, \dots, \alpha_k\} \subseteq J_n$ , so clearly (\*) holds for  $i + 1$ .

STAGE J. We have defined  $Z_i$  for  $i < \omega_1$ . Let  $Z = \bigcup_{i < \omega_1} Z_i$  (so as a vector space it is  $H$ ), and  $\bar{Z}$ , its completion, is the Banach space which exemplifies our theorem.

So let  $T$  be an operator on  $Z$ , and we shall prove it is as mentioned in the theorem, i.e., for some  $a$ , for every large enough  $i$ ,  $Tz_i = az_i$ . We define a two place function  $f$  from  $H$  into  $R$ :

$$f(x, y) = \|Tz - y\|.$$

By Stage B

$$I = \{i < \omega_1 : f \upharpoonright H_i = f_i, r_i = 0\}$$

is a stationary subset of  $\omega_1$  (see Stage B).

STAGE K. For each finite-dimensional subspace  $G$  of  $Z$  and  $m < \omega$  there is  $y_G^m \in Z$  good over  $G$  such that

$$d(Ty_G^m, G, y_G^m) \geq c(G, T, \bar{Z})(1 - 1/m) \quad d(T_G^m y, H) \geq c_{1/m}(G, T, \bar{Z}) - 1/m.$$

For each  $x \in Z$  there is  $i(x) < \omega_1$  such that  $x, Tx \in \bar{Z}_{i(x)}$ . Now for each  $\alpha < \omega_1$ ,  $A_\alpha = \{i(x) : x \in H_\alpha \text{ or } x = y_G^m \text{ for some finite-dimensional } G \subseteq H_\alpha, m < \omega\}$  is countable, hence  $i(\alpha) = \sup A_\alpha < \omega_1$ . Now  $A = \{j < \omega_1 : (\forall \alpha < j) i(\alpha) < j\}$  is a closed unbounded subset of  $\omega_1$  (closed-trivially by the definition, unbounded because  $i(\alpha)$  increases with  $\alpha$ , so if  $j_0 = j, j_{n+1} = i(j_n)$ , then  $j_0 \leq \bigcup_n j_n < \omega_1$  and  $\bigcup_n j_n$  is in this set). As  $I$  is stationary (see Stage B for definition, and Stage J for the fact) there is  $\bar{\gamma} \in A \cap I$  ( $I$  from Stage J). Clearly  $T$  maps  $Z_\gamma$  into  $\bar{Z}_\gamma$ , hence it maps  $\bar{Z}_\gamma$  into  $\bar{Z}_\gamma$ , and

$$c(H(I_\gamma^m), T, \bar{Z}) = c(H(I_\gamma^m), T_\gamma, \bar{Z}_\gamma) \quad \text{and}$$

$$c_{1/m}(H(I_\gamma^m), T, \bar{Z}) = c_{1/m}(H(I_\gamma^m), T_\gamma, \bar{Z}_\gamma)$$

(as  $\gamma \in A$ ) and  $T_\gamma = T|Z_\gamma$  (as  $\gamma \in I$ ).

STAGE L. Now we shall prove that for every  $i > \gamma$ ,  $Tz_i \in \langle Z_\gamma, z_i \rangle$  ( $\gamma$  is as chosen at the end of Stage K, and will remain fixed).

For this it suffices to prove that for any real  $\varepsilon > 0$ ,  $d(Tz_i, \langle H(I_\gamma^m), z_i \rangle) \leq 5\varepsilon$  for some  $m < \omega$ . So let  $\varepsilon > 0$  be given. Now  $Tz_i$  is in the closure of  $Z = \text{span}\{z_\alpha : \alpha < \omega_1\}$ , so for some  $l(0) < \omega$  and  $a_l \in R$ , and distinct  $\beta(l) < \omega_1$  (for  $l \leq l(0)$ ):

$$(a) \quad \|Tz_i - \sum_{l \leq l(0)} a_l z_{\beta(l)}\| < \varepsilon.$$

So we can choose  $k < \omega$ , and  $\alpha_0 < \dots < \alpha_k < \omega_1$ ,  $\gamma \leq \alpha_0$  such that

$$\{i, \beta(0), \dots, \beta(l(0))\} - \gamma \subseteq \{\alpha_0, \dots, \alpha_k\}.$$

Now by (\*) (from Stage G), for infinitely many  $m \in S_\gamma$ , I and II from (\*) hold (for our  $k, \gamma, \alpha_0, \dots, \alpha_k$ ). So we can choose some  $m$  for which  $\{\beta(0), \dots, \beta(l(0))\} \cap \gamma \subseteq I_\gamma^m$  and  $1/m < \varepsilon$ . Clearly

$$(b) \quad z_i = \text{def} \sum_{l \leq l(0)} a_l z_{\beta(l)} \in H(I_\gamma^m \cup \{\alpha_0, \dots, \alpha_k\})$$

and by I of (\*) and Stage F

$$(c) \quad z_\gamma + y_\gamma^m \text{ is good over } H_i^m.$$

Now we shall write a series of inequalities which will prove  $d(Tz_i, \langle H(I_\gamma^m), z_i \rangle) \leq 5\varepsilon$ ; for notational convenience let  $x$  range over  $H(I_\gamma^m)$ , and  $a, b$  range over  $R$ .

$$(d) \quad c(H(I_\gamma^m), T_i, \bar{Z}_i) = \quad \text{[as } \gamma \in A, \text{ see Stage K]}$$

$$c(H(I_\gamma^m), T, \bar{Z}) \cong \quad \text{[by } c\text{'s definition, and (c) above]}$$

$$d(T(z_i + y_\gamma^m), \langle H_\gamma^m, z_i + y_\gamma^m \rangle) \cong \quad \text{[by } d\text{'s definition]}$$

$$\inf_{a,x} \|T(z_i + y_\gamma^m) + a(z_i + y_\gamma^m) + x\| \cong \quad \text{[as } \|Tz_i - z_i\| < \varepsilon, Ty_\gamma^m = T_i y_\gamma^m \text{ and}$$

$$\|T_i y_\gamma^m - y_\gamma^m\| \leq 1/m \text{ as mentioned in (b)}$$

of stage H]

$$\inf_{a,x} \|z_i + y_\gamma^m + az_i + ay_\gamma^m + x\| - 1/m - \varepsilon \cong \quad \text{[by II of (*)]}$$

$$\inf_{a,b,x,x_1} (\|y_\gamma^m + ay_\gamma^m + x + (by_\gamma^m + x_1)\| +$$

$$\begin{aligned}
& + \|z_i + az_i - (by_\gamma^m + x_1)\| - 1/m - \varepsilon) = \inf_{a,b,x_1,x_2} (\|y_\gamma^m + ay_\gamma^m + by_\gamma^m + x_1\| + \\
& + \|z_i + az_i - by_\gamma^m + x_2\| - 1/m - \varepsilon) \cong \\
& \inf_{a,b,x_1,x_2} \|y_\gamma^m + ay_\gamma^m + by_\gamma^m + x_1\| + \inf_{a,b,x_1,x_2} \|z_i + az_i - by_\gamma^m + x_2\| - 1/m - \varepsilon \cong \\
& \quad \text{[as } \|Ty_\gamma^m - y_\gamma^m\| < 1/m, \|Tz_i - z_i\| < \varepsilon] \\
& \inf_{a,b,x_1} \|Ty_\gamma^m + ay_\gamma^m + by_\gamma^m + x_1\| \\
& - 1/m + \inf_{a,b,x_2} \|Tz_i + az_i - by_\gamma^m + x_2\| - \varepsilon - 1/m - \varepsilon \cong \quad \text{[by } d\text{'s definition]} \\
& d(Ty_\gamma^m, \langle H_\gamma^m, y_\gamma^m \rangle) + \inf_{a,b,x} \|Tz_i + az_i - by_\gamma^m + x\| - 2/m - 2\varepsilon \cong \\
& \quad \text{[by (b) of Stage H]} \\
& c(H(I_\gamma^m), T_i, \bar{Z}_i) - 1/m + \inf_{a,b,x} \|Tz_i + az_i - by_\gamma^m + x\| - 2/m - 2\varepsilon.
\end{aligned}$$

Comparing the first and last elements we see that

$$(e) \inf_{a,b,x} \|Tz_i + az_i - by_\gamma^m + x\| \cong 3/m + 2\varepsilon.$$

Now by the choice of  $m$

$$(f) 1/m < \varepsilon.$$

Combining we get  $d(Tz_i, \langle H_\gamma^{m+1}, z_i \rangle) \cong d(Tz_i, \langle H_\gamma^m, y_\gamma^m \rangle) \cong 3/m + 2\varepsilon < 5\varepsilon$ .

STAGE M. For each  $\beta < \omega_1$  we define an operator  $P_\beta$  on  $\bar{Z}$ :  $P_\beta(z_i) = 0$  for  $i \geq \beta$ , and  $P_\beta(z_i) = z_i$  for  $i < \beta$ . It is easy to check that:

- (a)  $P_\beta$  is well defined and is a projection with norm 1 onto  $Z_\beta$ ;
- (b) for  $\beta < \alpha$ ,  $P_\beta P_\alpha = P_\alpha P_\beta = P_\beta$ .
- (c) If  $P_\alpha(x) \neq 0$ ,  $\alpha$  limit, then for some  $\beta < \alpha$ ,  $P_\beta(x) \neq 0$ .

STAGE N. Let  $T, \gamma$  be as in Stage L. So for every  $i \geq \gamma$ ,  $Tz_i \in \langle \bar{Z}, z_i \rangle$ , so  $Tz_i = d_i z_i + x_i^0$ ,  $x_i^0 \in \bar{Z}_\gamma$ .

We shall prove that for some  $\delta, \gamma \leq \delta < \omega_1$ , and for every  $i \geq \delta$ ,  $x_i^0 = 0$ . Suppose not, so  $A_1 = \{i < \omega_1 : i \geq \gamma, \|x_i^0\| \neq 0\}$  is uncountable. For each  $i \in A_1$  choose a minimal  $\beta_i \leq \gamma$  such that  $P_{\beta_i}(x_i^0) \neq 0$  (it exists as  $P_\gamma(x_i^0) = x_i^0$ , because  $x_i^0 \in \bar{Z}_\gamma$ ). By (c) of Stage M  $\beta_i$  is a successor ordinal, so for some  $\beta < \gamma$ ,  $A_2 = \{i \in A_1 : \beta_i = \beta + 1\}$  is uncountable. So for each  $i \in A_2$ , for some real  $d_i^1 \neq 0$ ,  $P_\beta(x_i^0) = d_i^1 z_\beta$ . So for some  $a > 0$  and  $s \in \{1, -1\}$ ,  $A_3 = \{i \in A_2 : sd_i^1 > a\}$  is uncountable. So for each  $i \in A_3$ ,  $P_\beta Tz_i = d_i^1 x_\beta$ ,  $sd_i^1 > a$ .



By Stage B,  $I' = \{i < \omega_1 : r_i = 1, f|H_i = f_i, A_3 \cap i = D_i\}$  is a stationary subset of  $\omega_1$ . Let

$A = \{i < \omega_1 : i \text{ is limit, } i > \gamma, \text{ and } A_3 \cap i \text{ is unbounded below } i, \text{ and in (d) of Stage H, if we ask } y_{i,l}^m \text{ in } \{z_\alpha : \max I_i^m < \alpha, \alpha \in A_3\} \text{ the value of } p = p(m, i) \text{ does not change}\}.$

As in Stage K, we can prove  $A$  is closed and unbounded so  $I \cap A \neq \emptyset$ , and choose in it an element  $\delta$ . Now for infinitely many  $m < \omega$ ,  $p(m, \delta) \geq m$ . Otherwise choose  $m_0 < \omega$  such that

(a)  $m \geq m_0 \Rightarrow p(m, \delta) < m$

and choose  $i \in A_2, i > \delta$ . By (\*\*) of Stage G, for some  $m > m_0$ ,  $H(I_\delta^m), \langle H(I_\delta^m), z_i \rangle, \langle H(I_\delta^m), y_{\delta,1}^m, \dots, y_{\delta,p(m)}^m \rangle$  have  $N_\alpha$ -amalgamation. Now checking (b) of Stage H, we see that  $z_i$  was an appropriate candidate for being  $y_{\delta,p(m,\delta)+1}^m$  hence  $p(m, \delta) = m$ , contradiction.

So for  $m, l, y_{\delta,l}^m \in \{z_\alpha : \alpha \in A_2\}$ , hence  $P_\beta Ty_{\delta,l}^m \in \{sbx_\beta : b > a\}$ . Now for every  $m$ , (see (\*\*) of Stage G)

$$\begin{aligned} \left\| \sum_{l=1}^{p(m,\delta)} y_{\delta,l}^m \right\| &= \max_l \|y_{\delta,l}^m\| = 1, \\ \left\| T \left( \sum_{l=1}^{p(m,\delta)} y_{\delta,l}^m \right) \right\| &\geq \left\| P_{\beta+1} T \left( \sum_{l=1}^{p(m,\delta)} y_{\delta,l}^m \right) \right\| \quad [\text{as } \|P_{\beta+1}\| = 1 \text{ by Stage M}] \\ &= \left\| \sum_{l=1}^{p(m,\delta)} P_{\beta+1} Ty_{\delta,l}^m \right\| \quad [\text{as } P_{\beta+1} Ty_{\delta,l}^m \in \{sbx_\beta : b > a\}] \\ &= \sum_{l=1}^{p(m,\delta)} \|P_{\beta+1} Ty_{\delta,l}^m\| \\ &\geq p(m, \delta)a \\ &\geq ma. \end{aligned}$$

Hence  $\|T\| \geq ma$ , as  $a > 0$ ,  $m$  ( $m < \omega$ ) arbitrarily large, we get a contradiction.

STAGE P. (we omit O as a stage). We now want to show that  $d_i$  ( $i < \omega_1$ ) is eventually constant. Otherwise there are distinct reals  $d^0, d^1$  such that

(a) for  $l = 1, 2$  and  $\alpha < \omega_1$ , and  $\varepsilon > 0$ , there is  $i, \alpha < i < \omega_1$ , and  $|d_i - d^l| < \varepsilon$ ; w.l.o.g.  $d^0 = 0, d^1 = 1$  (otherwise, we look at the operator  $1/(d^1 - d^0)(T - d^0\mathbf{I})$  ( $\mathbf{I}$ —the identity operator).

Let  $\varepsilon > 0$  be arbitrary,  $\varepsilon < 1/100$ . Choose  $\alpha < \beta < \delta$  ( $\geq \gamma$ ),  $|d_\alpha| < \varepsilon, |1 - d_\beta| < \varepsilon$ . By (\*) of Stage G, for  $k = 1, \alpha_0 = \alpha, \alpha_1 = \beta, i = \gamma$  we can find  $m(1) < m$  in  $S_\gamma$  such that (I) and (II) of (\*) holds for  $m$  and for  $m(1)$  and

$$1/m(1) < \varepsilon, \quad 12m(1) < m.$$

We now try to get a contradiction to the choice of  $y_\gamma^m$ . We repeat Stage L with  $z_\alpha$  for  $z_i$  so (b), (c), (d) holds ((a) is trivialized—we know better), but we want to deviate in the middle of (d):

$$c(H(I_\gamma^m), T_i, \bar{Z}_i) \cong \inf_{a, b, x_1, x_2} (\|y_\gamma^m + ay_\gamma^m + by_\gamma^m + x_1\| + \|Tz_\alpha + az_\alpha - by_\gamma^m + x_2\| - 1/m).$$

So for some  $a, b, x_1, x_2$  we get this infimum up to  $1/m$ , so

$$\begin{aligned} c(H(I_\gamma^m), T_i, \bar{Z}_i) + 2/m &\cong \|y_\gamma^m + ay_\gamma^m + by_\gamma^m + x_1\| + \|Tz_\alpha + az_\alpha - by_\gamma^m + x_2\| \cong \\ &\quad [\text{as } \|Ty_\gamma^m - y_\gamma^m\| < 1/m \text{ and } Tz_\alpha = d_\alpha z_\alpha] \\ \|Ty_\gamma^m + (a+b)y_\gamma^m + x_1\| + \|d_\alpha z_\alpha + az_\alpha - by_\gamma^m + x_2\| - 1/m &= \quad [\text{by I of } (*)] \\ \|Ty_\gamma^m + (a+b)y_\gamma^m + x_2\| + \max\{\|(d_\alpha + a)z_\alpha + x_2\|, \|-by_\gamma^m + x_2\|\} - 1/m \\ &\quad [\text{as } z_\alpha, y_\alpha^m \text{ are good over } H(I_\alpha^m)] \end{aligned}$$

$$\begin{aligned} &\cong \|Ty_\gamma^m + (a+b)y_\gamma^m + x_1\| + \max\{|d_\alpha + a|, |b|\} - 1/m \\ &\cong d(Ty_\gamma^m, \langle H(I_\gamma^m), y_\gamma^m \rangle) + \max\{|d_\alpha + a|, |b|\} - 1/m \\ &\cong c(H(I_\gamma^m), T_\gamma, \bar{Z}_\gamma) + \max\{|d_\alpha + a|, |b|\} - 2/m. \end{aligned}$$

We can conclude that

$$\begin{aligned} \text{(b)} \quad &|b|, |d_\alpha + a| < 4/m, \\ \text{(c)} \quad &\|Ty_\gamma^m + (a+b)y_\gamma^m + x_1\| \leq d(Ty_\gamma^m, \langle H(I_\gamma^m), y_\gamma^m \rangle) + 4/m \end{aligned}$$

(for (b) look at the first and last terms in our series of inequalities, for (c), if it fails use this in the passage from the fifth term to the sixth term, and we shall get a contradiction).

Combining (b) and (c) we get

$$\text{(d)} \quad \|Ty_\gamma^m - d_\alpha y_\gamma^m + x_1\| \leq d(Ty_\gamma^m, \langle H(I_\gamma^m), y_\gamma^m \rangle) + 12/m.$$

Now remember  $|d_\alpha| < \varepsilon$ ,  $1/m < \varepsilon$  hence

$$\text{(e)} \quad \|Ty_\gamma^m + x_1\| \leq d(Ty_\gamma^m, \langle H(I_\gamma^m), y_\gamma^m \rangle) + 13\varepsilon.$$

Similarly, for  $\beta$  instead  $\alpha$ , (d) holds, but  $|1 - d_\beta| < \varepsilon$  hence for some  $x'_1 \in H(I_\gamma^m)$

$$\text{(f)} \quad \|Ty_\gamma^m - y_\gamma^m + x'_1\| \leq d(Ty_\gamma^m, \langle H(I_\gamma^m), y_\gamma^m \rangle) + 13\varepsilon.$$

By the version of (d) for  $(\beta)$ , for  $y = y_\gamma^m + z_\beta$  and the choice of  $y_\gamma^m$  in Stage H

$$\text{(g)} \quad d(Ty, \langle H(I_\gamma^m), y \rangle) > c(H(I_\gamma^m), T_i, \bar{Z}_i) - 1/m \quad (1).$$

Now  $z_\beta$  is good over  $\langle H(I_\gamma^m), y_\gamma^m, Ty_\gamma^m \rangle$  hence

$$\begin{aligned}
 \text{(h)} \quad d(Ty, H(I_\gamma^m)) &= \inf_x \|Ty_\gamma^m + d_\beta z_\beta + x\| \\
 &= \inf_{a, x_1, x_2} [\|Ty_\gamma^m - ay_\gamma^m + x_1\| + \|d_\beta z_\beta - ay_\gamma^m + x_2\|] \\
 &\cong d(Ty_\gamma^m, \langle H(I_\gamma^m), y_\gamma^m \rangle) + 1 - \varepsilon \\
 &\cong d(Ty_\gamma^m, H(I_\gamma^m)) + 1 - 14\varepsilon
 \end{aligned}$$

[by (e)]

So  $y$  contradicts the definition of  $c_{1/m(1)}(H(I_\gamma^m), \bar{T}_\gamma, \bar{Z}_\gamma)$  and the choice of  $y_\gamma^m$ .

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$$\|ay + bz + x\| = \max \{\|ay + x\|, \|bz + x\|\}$$

(see Stage F).

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