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SOME THEOREMS ON TRANSVERSALS

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1. INTRODUCTION

Let $\mathscr{F} = \langle F_i \mid i \in I \rangle$ be a set system with index set I. We write $|\mathscr{F}|$ to denote the cardinality of the system, i.e. $|\mathscr{F}| = |I|$. We call $\mathscr{F}$ a (κ, λ)-system, and write $\mathscr{F} \in S(\kappa, \lambda)$ if $|\mathscr{F}| = \kappa$ and $|F_i| = \lambda$ for every index $i \in I$. $S(\kappa, < \lambda)$ is defined is an analogous way. $\mathscr{F}_0$ is a *sub-system* of $\mathscr{F}$, $\mathscr{F}_0 \subset \mathscr{F}$, if $\mathscr{F}_0 = \langle F_i \mid i \in I_0 \rangle$ and $I_0 \subset I$. A transversal of $\mathscr{F}$ is a function with domain I such that $f(i) \in F_i$, $(i \in I)$ and $f(i) \neq f(j)$ if $i \neq j$. We denote by Trans $(\mathscr{F})$ the set of all transversals of $\mathscr{F}$. If $f \in$ Trans $(\mathscr{F})$ we call the system of elements $\langle f(i) \mid i \in I \rangle$ a *system of distinct representatives* of $\mathscr{F}$, and we call the set of elements $\{f(i) \mid i \in I\}$ a *transversal set* of $\mathscr{F}$.

A fundamental theorem of transversal theory asserts that, if either of the finiteness conditions

(1.1) $\qquad |\mathscr{F}| < \aleph_0$

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or

(1.2) $\quad |F_i| < \aleph_0, \qquad (i \in I)$

holds, then a necessary and sufficient condition for the existence of a transversal of the system $\mathscr{F} = \langle F_i \mid i \in I \rangle$ is that

(1.3) $\quad |\mathscr{F}(K)| \geqslant |K|$ *for every finite set* $K \subset I$,

where

$$\mathscr{F}(K) = \bigcup_{i \in K} F_i .$$

This result was proved by P. Hall [1] in the case that (1.1) holds (and also by D. König [2] in a different form), and by Marshall Hall [3] in the case that (1.2) holds. Combining these, an equivalent formulation of Marshall Hall's theorem is that, if (1.2) holds then

(1.4) $\quad \mathrm{Trans}(\mathscr{F}) \neq \phi \Leftrightarrow (\forall \mathscr{F}_0 \subset \mathscr{F})(|\mathscr{F}_0| < \aleph_0 \Rightarrow \mathrm{Trans}(\mathscr{F}_0) \neq \phi)$.

We are interested in possible extensions of this result to systems having infinite members.

For an infinite cardinal number μ we write $\mathscr{F} \in T(\mu)$ if and only if

$$\mathrm{Trans}(\mathscr{F}') \neq \phi \textit{ whenever } \mathscr{F}' \subset \mathscr{F} \textit{ and } |\mathscr{F}'| < \mu .$$

Using this notation, Marshall Hall's theorem (1.4) asserts that, for any cardinal κ,

$$\mathscr{F} \in S(\kappa, <\aleph_0) \wedge \mathscr{F} \in T(\aleph_0) \Rightarrow \mathrm{Trans}(\mathscr{F}) \neq \phi .$$

Our question then is whether there are other triples (κ, λ, μ) such that

(1.5) $\quad \mathscr{F} \in S(\kappa, <\lambda) \wedge \mathscr{F} \in T(\mu) \Rightarrow \mathrm{Trans}(\mathscr{F}) \neq \phi$.

A special case of (1.5) $(\kappa = \mu = \aleph_2,\ \lambda = \aleph_1)$ is stated as an unsolved problem in [4], and it appears again in this form (Problem 42C) in the collection of problems [5]. Erdős and Hajnal attribute this formulation of the question to W. Gustin. In a more recent paper [6] (which

(1.6) $\mathscr{F} \in S(\aleph_2, \aleph_0) \wedge \mathscr{F} \in T(\aleph_2) \wedge \mathrm{Trans}(\mathscr{F}) = \phi$.

The hypothesis V = L is not in fact needed here. It can be verified (essentially as in the proof of Theorem 1) that the system* $\mathscr{F} = \langle F_{\alpha\beta} \mid \omega \leqslant \alpha < \omega_1 \leqslant \beta < \omega_2 \rangle$ satisfies (1.6), where

$$F_{\alpha\beta} = \alpha \times \{\alpha, \beta\} = \bigcup_{\nu < \alpha} \{\langle \nu, \alpha \rangle, \langle \nu, \beta \rangle\} .$$

We will prove the following theorem and corollary which is more general. κ^+ denotes the successor cardinal of κ.

Theorem 1. *Let* κ, λ *be infinite cardinal numbers,* κ *regular and* $\lambda > \aleph_0$. *If there is a system* $\mathscr{F}$ *such that*

(1.7) $\mathscr{F} \in S(\kappa, < \lambda) \wedge \mathscr{F} \in T(\kappa) \wedge \mathrm{Trans}(\mathscr{F}) = \phi$,

then there is a system $\mathscr{F}_1$ *such that*

(1.8) $\mathscr{F}_1 \in S(\kappa^+, < \lambda) \wedge \mathscr{F}_1 \in T(\kappa^+) \wedge \mathrm{Trans}(\mathscr{F}_1) = \phi$.

If, in addition, $\mathscr{F} \in S(\kappa, \lambda_1)$ *where* $\lambda_1 \geqslant \aleph_0$, *then* $\mathscr{F}_1 \in S(\kappa^+, \lambda_1)$.

Corollary. *For* $\alpha \geqslant 0$ *and* $1 \leqslant n < \omega$, *there is a system* $\mathscr{F}$ *such that*

$$\mathscr{F} \in S(\aleph_{\alpha+n}, \aleph_\alpha) \wedge \mathscr{F} \in T(\aleph_{\alpha+n}) \wedge \mathrm{Trans}(\mathscr{F}) = \phi .$$

The corollary is an immediate deduction from the theorem. Consider the system $\mathscr{F} = \langle \xi \mid \omega_\alpha \leqslant \xi < \omega_{\alpha+1} \rangle$ where, as usual, the ordinal number ξ is the set $\{\eta \mid \eta < \xi\}$ of all smaller ordinals. Clearly, $\mathscr{F} \in S(\aleph_{\alpha+1}, \aleph_\alpha)$ and $\mathscr{F} \in T(\aleph_{\alpha+1})$. Also, by a theorem of Alexandroff and Urysohn [7] on regressive functions, we have $\mathrm{Trans}(\mathscr{F}) = \phi$. The corollary now follows from the theorem by induction on n.

*This is a modification of an example communicated to us by J. Truss, Leeds University, England.

We do not know if the assumed regularity of κ is necessary for the validity of Theorem 1. The simplest open question is whether there is a system $\mathcal{F}$ which satisfies

$$\mathcal{F} \in S(\kappa, \aleph_0) \wedge \mathcal{F} \in T(\kappa) \wedge \mathrm{Trans}(\mathcal{F}) = \phi$$

when $\kappa = \aleph_\omega$ or $\aleph_{\omega+1}$.* Hajnal pointed out to us that the remark in [6] regarding Problem 42C applies more generally, and that Jensen's result actually leads to the following theorem.

Theorem 2. *If* $V = L$ *and* κ *is a regular cardinal which is not weakly compact and* $\kappa > \lambda \geq \aleph_0$, *then there is an* $\mathcal{F}$ *which satisfies*

(1.9) $\quad \mathcal{F} \in S(\kappa, \lambda) \wedge \mathcal{F} \in T(\kappa) \wedge \mathrm{Trans}(\mathcal{F}) = \phi$,

The condition that κ not be weakly compact in Theorem 2 is essential. It is easy to prove the following.

Theorem 3. *If* κ *is weakly compact, then* $\mathcal{F} \in S(\kappa, < \kappa) \wedge \mathcal{F} \in T(\kappa) \Rightarrow \mathrm{Trans}(\mathcal{F}) \neq \phi$.

By way of contrast with the negative results in Theorems 1 and 2 we will establish the following positive Hall-type theorem. A special case of this has been used in [8] to settle a conjecture of Nash-Williams.

Theorem 4. *Let* λ *be an infinite cardinal number and suppose that* $\mathcal{F} = \langle F_i \mid i \in I_0 \cup I_1 \rangle$ *is a set system with*

(i) $I_0 \cap I_1 = \phi, \ |I_1| \leq \lambda$,

(ii) $|F_i| < \aleph_0, \ (i \in I_0)$,

(iii) $|F_i| \leq \lambda, \ (i \in I_1)$.

Then a necessary and sufficient condition for the existence of a transversal of $\mathcal{F}$ *is that*

(1.10) $\quad \mathrm{Trans}(\mathcal{F}') \neq \phi$ *whenever* $\mathcal{F}' \subset \mathcal{F}$ *and* $|\mathcal{F}| \leq \lambda$.

*Shelah has since proved this is false for $\kappa = \aleph_\omega$ (see his paper in Volume 3 of these proceedings). More generally, he has now proved that if $\mathrm{cf}\, \kappa < \kappa$ and $\lambda < \kappa$, then $\mathcal{F} \in S(\kappa, \lambda) \wedge \mathcal{F} \in T(\kappa) \Rightarrow \mathrm{Trans}(\mathcal{F}) \neq \phi$.

If A is a set of ordinals, then $\sup A$ denotes that the least ξ such that $\alpha \leqslant \xi$ for all $\alpha \in A$. B is a *cofinal* subset of A if $\sup B = \sup A$. A is *closed* if $\sup B \in A$ whenever $B \subset A$ and $\sup B < \sup A$. S is a *stationary* subset of A, $S \in \mathrm{Stat}(A)$, if and only if $S \cap B \neq \phi$ for every closed, cofinal subset B of A. The function f on A is *regressive* if $f(\xi) < \xi$ for all $\xi \in A - \{0\}$. The cofinality of ξ, $\mathrm{cf}(\xi)$ is the least ordinal α for which there is a function $g\colon \alpha \to \xi$ such that $\sup \{g(\sigma) \mid \sigma < \alpha\} = \xi$.

We use the following well-known facts. Let κ be a regular cardinal, $\kappa > \mu \geqslant \omega$.

1. If $S \in \mathrm{Stat}(\kappa)$ and f is regressive on S, then f is not $1-1$; in fact there is $\theta < \kappa$ such that $|f^{-1}(\theta)| = \kappa$;

2. $\{\xi \in \kappa \mid \mathrm{cf}(\xi) = \mu\}$ is a stationary subset of κ (see [9]).

3. PROOF OF THEOREM 1

We may assume that the system $\mathscr{F}$ which satisfies the hypothesis (1.7) is indexed by κ, i.e. $\mathscr{F} = \langle F_\nu \mid \nu < \kappa\rangle$. Let $C = \{\rho \mid \kappa \leqslant \rho < \kappa^+, \mathrm{cf}(\rho) = \kappa\}$. For each $\rho \in C$ there is an increasing sequence of ordinal numbers $\beta(\rho, \sigma)$, $(\sigma < \kappa)$ such that

$$\rho = \lim_{\sigma < \kappa} \beta(\rho, \sigma) .$$

Put

$$G(\rho, \sigma) = (\{\rho\} \times F_\sigma) \cup \{\beta(\rho, \sigma)\} \qquad (\rho \in C_\wedge \sigma < \kappa) .$$

We will prove that (1.8) holds with

$$\mathscr{F}_1 = \langle G(\rho, \sigma) \mid \rho \in C_\wedge \sigma < \kappa\rangle .$$

Clearly, $|\mathscr{F}_1| = \kappa |C| = \kappa^+$ (here we use the fact that κ is regular; if κ is singular we would have $C = \phi$). Also

$$|G(\rho, \sigma)| = |F_\sigma| + 1 < \lambda \qquad (\rho \in C,\ \sigma < \kappa)\,,$$

and

$$|G(\rho, \sigma)| = |F_\sigma| \ \text{ if } \ F_\sigma \ \text{ is infinite.}$$

It remains to show that

(3.1) $\in T(\kappa)$,

and

(3.2) $\mathrm{Trans}\,(\mathscr{F}_1) = \phi$.

In order to prove (3.1) it will be enough to prove that

(3.3) $\mathrm{Trans}\,(\mathscr{F}_1(\alpha)) \neq \phi$,

where $\mathscr{F}_1(\alpha) = \langle G(\rho, \sigma) \mid \rho \in C \wedge \rho < \alpha \wedge \sigma < \kappa \rangle$ and $\kappa^2 \leqslant \alpha < \kappa^+$. For, if $\mathscr{F}' \subset \mathscr{F}_1$ and $|\mathscr{F}'| \leqslant \kappa$, then $\mathscr{F}' \subset \mathscr{F}_1(\alpha)$ for some α with $\kappa^2 \leqslant \alpha < \kappa^+$.

Let α be fixed, $\kappa^2 \leqslant \alpha < \kappa^+$. Then

$$C(\alpha) = \{\rho \in C \mid \rho < \alpha\} = \{\rho_\tau \mid \tau < \kappa\}_{\neq}\,,$$

i.e. $\rho_\sigma \neq \rho_\tau$ if $\sigma < \tau < \kappa$. We shall define ordinals $\sigma_\tau < \kappa$ for $\tau < \kappa$ so that the κ sets

$$B_\tau = \{\beta(\rho_\tau, \sigma) \mid \sigma_\tau \leqslant \sigma < \kappa\} \qquad (\tau < \kappa)$$

are pairwise disjoint. Let $\tau_0 < \kappa$ and suppose that σ_τ has been defined for $\tau < \tau_0$. For each $\tau < \tau_0$ there is $\xi_\tau < \kappa$ such that

(3.4) $B_\tau \cap \{\beta(\rho_{\tau_0}, \sigma) \mid \xi_\tau \leqslant \sigma < \kappa\} = \phi$.

If $\rho_\tau < \rho_{\tau_0}$, then (3.4) holds with any choice for $\xi_\tau < \kappa$ such that $\beta(\rho_{\tau_0}, \xi_\tau) > \rho_\tau$. If, on the other hand, $\rho_\tau > \rho_{\tau_0}$, then the existence of ξ_τ such that (3.4) holds follows from the fact that $\mathrm{cf}\,(\rho_{\tau_0}) = \kappa$ and $|\{\beta \in B_\tau \mid \beta < \rho_{\tau_0}\}| < \kappa$. Hence, there are ordinals $\xi_\tau < \kappa$, $(\tau < \tau_0)$ such that (3.4) holds. Now put

For each $\tau < \kappa$ the sub family $\langle F_\nu \mid \nu < \tau \rangle$ of $\mathcal{F}$ has a transversal, i.e. there is a $1-1$ function f_τ on τ such that

$$f_\tau(\nu) \in F_\nu \qquad (\nu < \tau < \kappa)\,.$$

Now define a function g on $C(\alpha) \times \kappa$ by putting

$$g(\rho_\tau, \sigma) = \begin{cases} \langle \rho_\tau, f_\tau(\sigma) \rangle & \text{if } \sigma < \sigma_\tau\,, \\ \beta(\rho_\tau, \sigma) & \text{if } \sigma_\tau \leqslant \sigma < \kappa\,. \end{cases}$$

Clearly, $g(\rho_\tau, \sigma) \in G(\rho_\tau, \sigma)$, $(\sigma, \tau < \kappa)$ and g is $1-1$ since f is and the sets B_τ $(\tau < \kappa)$ are pairwise disjoint. Therefore, $g \in \text{Trans}\,(\mathcal{F}_1(\alpha))$. This proves (3.3) and hence (3.1).

We now prove (3.2). Suppose, on the contrary, that $\mathcal{F}_1$ has a transversal. Then there is a $1-1$ function h on $C \times \kappa$ such that $h(\rho, \sigma) \in G(\rho, \sigma)$. Suppose that for some $\rho \in C$ we have

$$h(\rho, \sigma) \neq \beta(\rho, \sigma) \qquad (\forall \sigma < \kappa)\,.$$

Then

$$h(\rho, \sigma) = \langle \rho, g(\sigma) \rangle \qquad (\sigma < \kappa)\,,$$

where g is a $1-1$ function on κ such that $g(\sigma) \in F_\sigma$. This contradicts the hypothesis that $\text{Trans}\,(\mathcal{F}) = \phi$. Hence, for each $\rho < \kappa$ there is $\sigma(\rho) < \kappa$ such that

$$h(\rho, \sigma(\rho)) = \beta(\rho, \sigma(\rho)) = \theta(\rho)\,.$$

Then $\theta(\rho) < \rho$ for $\rho \in C$ and, since C is a stationary subset of κ^+ (see 2), it follows that there are $\rho_1, \rho_2 \in C$ such that $\rho_1 \neq \rho_2$ and $\theta(\rho_1) = \theta(\rho_2)$. This contradicts our assumption that h is $1-1$. Therefore, (3.2) holds.

4. PROOF OF THEOREM 2

It follows from a theorem of Jensen [10] that, if $V = L$ and κ is a regular cardinal which is not weakly compact, then there is a set $A \subset \kappa$ such that

(i) $A \in \mathrm{Stat}(\kappa)$,

(ii) $A \cap \xi \notin \mathrm{Stat}(\xi)$, $(\xi < \kappa)$,

(iii) $\alpha \in A \Rightarrow \mathrm{cf}(\alpha) = \omega$.

For $\alpha \in A$, let B_α be a set of ordinals of order type ω such that $\sup(B_\alpha) = \alpha$. Let B be any set of power λ disjoint from $\bigcup_{\alpha \in A} B_\alpha$. We will show that the (κ, λ)-system $\mathscr{F} = \langle B_\alpha \cup B \mid \alpha \in A\rangle$ satisfies (1.9).

Suppose that $\mathscr{F}$ has a transversal f. Let $A' = \{\alpha \in A \mid f(\alpha) \notin B\}$ then $A' \in \mathrm{Stat}(\kappa)$ and f is regressive and $1-1$ on A'. This is impossible and hence $\mathrm{Trans}(\mathscr{F}) = \phi$. To show that $\mathscr{F} \in T(\kappa)$ it will be enough to show that the system $\langle B_\alpha \mid \alpha \in A \cap \xi\rangle$ has a transversal for $\xi < \kappa$. We will actually, by transfinite induction on $\xi < \kappa$, prove the following slightly stronger statement R_ξ: *If* D_α *is a set of ordinals of type* ω *such that* $\sup(D_\alpha) = \alpha$, $(\alpha \in A \cap \xi)$, *then* $\mathrm{Trans}(\langle D_\alpha \mid \alpha \in A \cap \xi\rangle) \neq \phi$.

Let $\xi_0 < \kappa$ and assume that R_ξ holds for $\xi < \xi_0$. If $\xi_0 = \eta + 1$, then $A \cap \xi_0 = A \cap \eta$ and so R_{ξ_0} holds. Now assume that ξ_0 is a limit ordinal. By (ii) there is a closed cofinal subset C of ξ_0 such that $C \cap A = \phi$. Let $C = \{\nu_\sigma \mid \sigma < \rho\}$, where $\nu_0 < \nu_1 < \ldots < \xi_0$. We can assume that $\nu_0 = 0$ since $0 \notin A$. For $\alpha \in A \cap \xi_0$ there is $\sigma = \sigma(\alpha) < \rho$ such that $\nu_\sigma < \alpha < \nu_{\sigma+1}$. Put $E_\alpha = D_\alpha \cap [\nu_\sigma, \nu_{\sigma+1})$. By the induction hypothesis, the system $G_\sigma = \langle E_\alpha \mid \alpha \in A \cap \xi_0 \wedge \sigma(\alpha) = \sigma\rangle$ has a transversal $(\sigma < \rho)$. Moreover, the systems G_σ, $(\sigma < \rho)$ are pairwise strongly disjoint and hence $\langle D_\alpha \mid \alpha \in A \cap \xi_0\rangle$ also has a transversal. This shows that R_{ξ_0} holds and the proof is complete.

ment and (ii), $A(z) = \{x \in A \mid x \leqslant z\}$ is well-ordered by $\leqslant$ for all $z \in A$. The *order*, $O(z)$, *of* $z \in A$ is the ordinal number which is the type of $(A(z), \leqslant)$. The *order* of the tree is $\bigcup_{z \in A} O(z)$. A *branch* is a set $B \subset A$ which is well-ordered by $\leqslant$ and is such that $x \leqslant y \in B \Rightarrow x \in B$. The cardinal κ is weakly compact if it has the *tree property*, i.e. whenever $(A, \leqslant)$ is a tree of order κ having fewer than κ elements of order ξ for all $\xi < \kappa$, then there is a branch of order κ. (Erdős and Tarski [11] proved that if κ has the tree property then $\kappa \to (\kappa, \kappa)^2$, i.e. any graph on κ either contains a complete subgraph of order κ or an edge-free set of order κ. Hanf proved the converse (see [12]). This fact easily implies the following lemma which is stronger than Theorem 3. We cannot find precisely this statement in the literature although equivalents are known; it is expressed in the style of Rado's selection lemma [13] and we give the simple proof.

Lemma. *Let* κ *be weakly compact and let* $\langle F_\nu \mid \nu < \kappa \rangle$ *be a* $(\kappa, <\kappa)$*-system. Suppose that, for each* $\xi < \kappa$; f_ξ *is a function with domain* ξ *such that* $f_\xi(\nu) \in F_\nu$, $(\nu < \xi)$. *Then there is a function* f *defined on* κ *such that*

$$(\forall \xi < \kappa)(\exists \eta < \kappa)(f \restriction \xi = f_\eta \restriction \xi) .$$

Remark. If $f_\xi \in \text{Trans}(\langle F_\nu \mid \nu < \xi \rangle)$, $(\xi < \kappa)$, then clearly $f \in \text{Trans}(\langle F_\nu \mid \nu < \xi \rangle)$.

Proof of Lemma. Let $A = \{f_\xi \restriction \mu \mid \mu \leqslant \xi < \kappa\}$. Then the partially ordered set $(A, \subseteq)$ is a tree of order κ. Since κ is strongly inaccessible, there are fewer than κ choice functions of $F \restriction \xi$, $(\xi < \kappa)$ and so the tree has fewer than κ elements of order ξ, $(\xi < \kappa)$. Hence there is a branch B of order κ. Let $f = \bigcup B$. For each $\xi < \kappa$ we have $f \restriction \xi \in A$ and hence $f \restriction \xi = f_\eta \restriction \xi$ for some $\eta < \kappa$.

6. PROOF OF THEOREM 4

The necessity of (1.10) is obvious, we have to prove the sufficiency.

Let S be any set. We shall define a set $S^* \supset S$ in the following way. For $B \subset\subset S$, let

$$G_S(B) = \{K \mid K \subset\subset I_0 \wedge S \cap \mathscr{F}(K) = B \wedge$$
$$\wedge \mid \mathscr{F}(K) \setminus B \mid < \mid K \mid \wedge (\forall i \in K)(F_i \not\subset S)\} .$$

If $G_S(B) = \phi$, put $H_S(B) = B$; if $G_S(B) \neq \phi$, select $K \in G_S(B)$ and put $H_S(B) = \mathscr{F}(K)$. Now define

$$S^* = \bigcup_{B \subset\subset S} H_S(B) .$$

Since $H_S(B) \supset B$, we have that $S^* \supset S$. Also, if S is an infinite set, then $|S^*| = |S|$.

Now put $A_0 = \mathscr{F}(I_1)$, $A_{n+1} = A_n^*$, $(n < \omega)$, $\bar{A} = \bigcup_{n<\omega} A_n$. Then $|\bar{A}| \leqslant \lambda$. Put

$$I_3 = \{i \in I \mid F_i \subset \bar{A}\} , \qquad I_4 = I \setminus I_3 .$$

Then $I_1 \subset I_3$ and $I_4 \subset I_0$. The hypothesis implies that any finite subfamily of $\mathscr{F}$ has a transversal and therefore

$$|\{i \in I \mid F_i = F_{i_0}\}| \leqslant |F_{i_0}| \qquad (i_0 \in I_0) .$$

It follows from this that $|I_0 \cap I_3| \leqslant \lambda$ and hence $|I_3| \leqslant \lambda$. Therefore, by assumption, there is a transversal f of $\mathscr{F}_3 = \langle F_i \mid i \in I_3 \rangle$. We will show that f can be extended to a transversal of $\mathscr{F}$, i.e. there is a transversal of $\mathscr{F}_4 = \langle F_i \mid i \in I_4 \rangle$ whose range is disjoint from the set $T = \{f(i) \mid i \in I_3\}$.

Suppose this is false. Then, since the members of $\mathscr{F}_4$ are finite sets, it follows from (1.3) that there is a finite set $K \subset I_4$ such that

$$|\mathscr{F}(K) \setminus T| < |K| .$$

Let $B = \mathscr{F}(K) \cap \bar{A}$. Then

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(6.1) $\quad |\mathscr{F}(K) \setminus B| < |K|$.

Also, since B is a finite set, there is an integer n_0 such that $B \subset\subset A_n$ $(n_0 \leqslant n < \omega)$. Let $n_0 \leqslant n < \omega$. By (6.1) and the fact that K is a finite subset of I_4 it follows that

$$K \in G_{A_n}(B) \neq \phi .$$

Therefore, there is $K_n \subset\subset I_0$ such that

$$A_n \cap \mathscr{F}(K_n) = B ,$$

(6.2) $\quad |\mathscr{F}(K_n) \setminus B| < |K_n|$,

(6.3) $\quad (\forall i \in K_n)(F_i \not\subset A_n)$,

(6.4) $\quad (\forall i \in K_n)(F_i \subset A_{n+1})$.

By (6.4), $K_n \subset I_3$ and therefore, by (6.2), there is $i_n \in K_n$ such that $f(i_n) \in B$. This defines i_n for $n_0 \leqslant n < \omega$. By (6.3) and (6.4) we see that $i_n \neq i_p$, $(n_0 \leqslant n < p < \omega)$. Therefore, since f is $1-1$,

$$|B| \geqslant |\{f(i_n) \mid n_0 \leqslant n < \omega\}| \geqslant \aleph_0 .$$

This contradiction proves the theorem.

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