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# ON THE NUMBER OF NON-ALMOST ISOMORPHIC MODELS OF $T$ IN A POWER

SAHARON SHELAH

**Let $T$ be a first order theory. Two models are almost isomorphic if they are elementarily equivalent in the language $L_{\infty,\omega}$. We investigate the number of non almost-isomorphic models of $T$ of power $\lambda$ as a function of $\lambda$, $I(T, \lambda)$. We prove $\mu > \lambda \geqq |T|$, $I(T, \lambda) \leqq \lambda$ implies $I(T, \mu) \leqq I(T, \lambda)$. In fact, we generalize the downward Skolem-Lowenheim theorem for infinitary languages. Th. (1, 4, 5).**

Let $L$ be a set of predicates with finite number of places and sufficient number of variables. (We assume there are no function symbols in $L$ for simplicity only.) $|L|$ will denote the number of predicates in $L$ plus $\aleph_0$. Models will be denoted by $M, N$. The set of elements of $M$ will be $|M|$, the cardinality of a set $A$ by $|A|$ and so the cardinality of $M$ by $||M||$. Unless specified otherwise, every model is an $L$-model. Cardinals will be denoted by $\lambda, \mu, \kappa, \chi$ ordinals $i, j, \alpha, \beta$. $T$ will denote a theory, i.e., set of sentences. We define $\mu^{(\lambda)} = \sum_{\kappa<\lambda} \mu^\kappa$. For cardinals $\lambda, \mu$ we define the language $L(\lambda, \mu)$ i.e., a set of formulas. This set is defined as the well known first-order language where we adjoin to its operations conjunction and disjunction on a set of $<\lambda$ formulas (i.e., $\bigwedge_{i \in I} \phi_i$, where $|I| < \lambda$) and existential or universal quantifications over a sequence of $<\mu$ variables. $L^*(\lambda, \mu)$ will be defined as $L(\lambda, \mu)$ where in addition we permit quantification of the form

$$[(\forall \bar{x}^1)(\exists \bar{y}^1) \cdots (\forall \bar{x}^n)(\exists \bar{y}^n) \cdots]_{n<\omega}$$

if

$$|\{x_0^1, x_1^1, \cdots, y_0^1, y_1^1, \cdots, x_0^n \cdots\}| < \mu .$$

$RL^*(\lambda, \mu)$ will denote the subset of $L^*(\lambda, \mu)$ consisting of the formulas $\Phi$ of $L^*(\lambda, \mu)$ such that for every subformula $\phi$ of $\Phi$, if $\phi = [(\forall \bar{x}^1)(\exists \bar{y}^1) \cdots]\psi$, then $\models \phi \leftrightarrow \neg[(\exists \bar{x}^1)(\forall \bar{y}^1) \cdots] \neg \psi$. Clearly $RL^*(\lambda, \mu) \supset L(\lambda, \mu)$. $K$ will denote any of those languages. Satisfaction (i.e., if $\phi = \phi(\bar{x})$, and $\bar{a}$ is a sequence from $|M|$, then $M \models \phi[\bar{a}]$) is defined naturally. (See Hanf [2] and Henkin [3].) The only nontotally trivial case is

$$\psi(\bar{z}) = [(\forall \bar{x}^0)(\exists \bar{y}^0)(\forall \bar{x}^1)(\exists \bar{y}^1) \cdots]\phi(\bar{z}, \bar{x}^0, \bar{x}^1, \cdots, \bar{y}^0, \bar{y}^1 \cdots) .$$

$M \models \psi[\bar{a}]$ if and only if there are functions $f_i^n(\bar{x}^0, \cdots, \bar{x}^n)$ such that for every sequence $\bar{a}^0, \bar{a}^1, \cdots$ from $M$, $M \models \phi[\bar{a}, \bar{a}^0, \bar{a}^1, \cdots, \bar{b}^0, \bar{b}^1, \cdots]$ where $\bar{b}^n = \langle \cdots, f_i^n(\bar{a}^0, \bar{a}^1, \cdots, \bar{a}^n), \cdots \rangle$. For a sentence $\psi$, $\models \psi$ if for every $M$, $M \models \psi$. (Such languages were first defined in Henkin [3].)

If $\Gamma$ is a set of formulas (for example one of the languages defined above), $M$ is a $\Gamma$ elementary submodel of $N$, if the set of elements of $M$, $|M|$ is included in the set of elements of $N$, $|N|$, and for every formula $\phi(\bar{x})$, $\phi(\bar{x}) \in \Gamma$, and sequence $\bar{a}$ from $|M|$, $M \models \phi[\bar{a}]$ if and only if $N \models \phi[\bar{a}]$, $M$, $N$ are $\Gamma$-elementarily equivalent if for every sentence $\phi \in \Gamma$, $M \models \phi$ if and only if $N \models \phi$.

THEOREM 1. *Let $\lambda > \mu$, $\lambda$ regular and $T$ be a theory in $RL^*(\lambda, \mu)$ [i.e., $T \subset RL^*(\lambda, \mu)$] and $\Gamma$ be the set of subformulas of the formulas in $T$. Then for every model $M$ we can add $< \lambda + |T|^+$ functions of $< \mu$ places such that: If $A \subset M$, and $A$ is closed under those functions, then there exists a $\Gamma$-elementary submodel $N$ of $M$, $|N| = A$. So if $\kappa \geqq \lambda + |T|$ (or $\kappa \geqq$ the number of those functions) and $\kappa^{(\mu)} = \kappa$, and $T$ has a model of power $\geqq \kappa$, then $T$ has a model of power $\kappa$.*

*Proof.* This theorem is proved in [9], and is a straight-forward generalization of a theorem of Hanf in [2].

DEFINITION 1.

$$L(\infty, \mu) = \bigcup_{\lambda} L(\lambda, \mu),\ L^*(\infty, \mu) = \bigcup_{\lambda} L^*(\lambda, \mu),$$
$$RL^*(\infty, \mu) = \bigcup_{\lambda} RL^*(\lambda, \mu)\ .$$

DEFINITION 2. (1) $M$ and $N$ are $\mu$-almost isomorphic, $M \sim_\mu N$ if $M$, $N$ are $L(\infty, \mu)$-elementarily equivalent. We say $M$ and $N$ are almost isomorphic if $M \sim_{\aleph_0} N$, and we write $M \sim N$.

(2) $I(T, \lambda, \mu)$, is the number of non-$\mu$-almost-isomorphic models of $T$ of power $\lambda$. We assume always $\lambda$ is $\geqq$ then $|T|$.

See footnote 1.

THEOREM 2. *If $T$ is a theory in $RL^*(\lambda, \mu)$, $\mu = \aleph_0$ or $\mu = \mu_1^+$, $\kappa \geq \chi = \chi^{(\mu)} + \lambda + |T|$ and $I(T, \chi, \mu) \leqq \chi$ then $I(T, \kappa, \mu) \leqq I(T, \chi, \mu)$.*

The proof is broken into a series of lemmas.

REMARKS. (1) It is not hard to show that if $T \subset L(\lambda, \aleph_0)$, $I(T, \chi, \aleph_0) \leqq \chi$, then for every $\kappa_1, \kappa_2 \geqq \beth_{(2^{\lambda+\chi})^+}$, $I(T, \kappa_1, \aleph_0) = I(T, \kappa_2, \aleph_0)$. (See Makkai [7] and Eklof [15].)

[1] The results here appear in the notices [10] Th. 5 [11] Th. 3. The lemma has other uses: see [12] Th. 2.5 and Remark (4): in [11] their consequences are better formulated. In Th. 2 we can replace $T \subset RT^*(\lambda, \mu)$ by $T \subset RL^*(\lambda^+, \mu)$ and similarly in other cases.

(2) Let $\lambda = \mu = \aleph_0$ and suppose $|T| \leqq \kappa_0$. Then as the class of such theories is a set, there is a number $\kappa = HAI_{\kappa_0}$ (Hanf number of almost isomorphism) such that: for all $T$, $|T| \leqq \kappa_0$, $I(T, \kappa, \aleph_0) \leqq \kappa$ if and only if there is a $\chi$, $I(T, \chi, \aleph_0) \leqq \chi$, and $\kappa$ is the first such cardinality. (The existence of such numbers for a wide class of cases was proved by Hanf in [2].)

*Question* 1. What is $HAI_{\kappa_0}$? (Clearly if $\lambda \rightarrow (\kappa_0^+)^{<\omega}_{2^{\kappa_0}}$ then $HAI_{\kappa_0} < \lambda$).

(3) It is known that $M \sim N$, $\aleph_0 = \|M\| = \|N\|$ implies that $M$, $N$ are isomorphic (see Scott [8]).

(4) Ehrenfeucht in [1] defined a model to be rigid if it has no nontrivial automorphisms and tried to investigate what can be the class of cardinals in which a certain theory has a rigid model. He gives some examples, but does not prove any theorem of the form: If $T$ has a rigid model of one power, then it has a rigid model in another power.

DEFINITION. $M$ is $\mu$-rigid if there do not exist two different sequences of length $<\mu$, $\bar{a}$, $\bar{b}$, such that $(M, \bar{a}) \sim_\mu (M, \bar{b})$. ($(M, \bar{a})$ is the model $M$ when we adjoin the $a$'s as individual constants.) See footnote 2. Clearly

THEOREM. *If $\mu < \lambda$, and $M$ is $\mu$-rigid, then it is $\lambda$-rigid and also rigid. By a proof similar to that of Theorem 2, we can prove*:

THEOREM. *If a first-order theory $T$ has a $\mu$-rigid model of power $\lambda$, $|T| + \aleph_0 \leqq \kappa = \kappa^{(\mu)} \leqq \lambda$, $\mu = \mu_1^+$ or $\mu = \aleph_0$, then $T$ has a $\mu$-rigid model of power $\kappa$.*

*Proof of Theorem* 2.

DEFINITION 3. (1) Let $L_1$ be $L$ where we adjoin to it one two-place predicate $E$ and variables $y, y_0, y_1, \cdots$ we assume $E, y, y_0 \cdots \neq L$. We shall write $xEy$ instead $E(x, y)$.

(2) If $R \in L$ then $R^M$ will denote the relation of $M$ corresponding to $R$.

(3) Let $\{M_i : i \in I\}$ be a set of $L$-models and we define their sum $N = \bigoplus_{i \in I} M_i$, (or $\bigoplus\{M_i : i \in I\}$). For simplicity we assume that the sets $|M_i|$ are pairwise disjoint. $N$ will be an $L_1$-model $|N| = \bigcup_{i \in I} |M_i|$, $R^N = \bigcup_{i \in I} R^{M_i}$ for every $R \in L$, and $E^N = \{\langle a, b \rangle : (\exists i)[a, b \in |M_i|]\}$.

(4) For every formula $\phi$ of any language, we define by induction

[2] Barwise [14] suggests a similar definition and argues its naturality.

$\bar{\phi}$: if $\phi$ is atomic $\bar{\phi} = \phi$; $\overline{\neg\phi} = \neg\bar{\phi}$, $\overline{\phi \vee \psi} = \bar{\phi} \vee \bar{\psi}$, (likewise for the other connectives), $\overline{\exists(\exists\bar{x})\phi} = (\exists\bar{x})[\bar{\phi} \wedge \bigwedge_i x_i E y]$, (where $\bar{x} = \langle \cdots x_i \cdots \rangle$) $\overline{(\forall\bar{x})\phi} = (\forall\,\bar{x})[\bigwedge_i x_i E y \rightarrow \bar{\phi}]$, and

$$\overline{[(\forall\bar{x}^1)(\exists\bar{y}^1)\cdots]\phi} = [(\forall\bar{x}^1)(\exists\bar{y}^1)\cdots](\bigwedge_{i,n} x_i^n E y \rightarrow \bar{\phi} \wedge \bigwedge_{i,n} y_i^n E y)$$

if the language contains such formulas. Clearly for any language $K$, $\phi \in K \Rightarrow \bar{\phi} \in K$. Also, if $\phi$ is a sentence $(\forall y)\bar{\phi}$ is a sentence.

(5) Define

$$\bar{T} = \{(\forall y)\bar{\phi}\colon \phi \in T\} \cup \{(\forall x) x E x,\ (\forall x_0 x_1 x_2)(x_0 E x_1 \wedge x_0 E x_2 \rightarrow x_1 E x_2)\}\,.$$

LEMMA 3. *Each $M_i$ is an $L$-model of $T$ if and only if $\bigoplus_{i \in I} M_i$ is an $L_1$-model of $\bar{T}$.*

*Proof.* Immediate

DEFINITION 4.

$$\psi_\alpha^n = \psi_\alpha^n(\bar{x}^0, \bar{x}^1, \cdots, \bar{x}^n, \bar{y}^0, \cdots, \bar{y}^n) = \bigwedge \{R(x_{j_1}^{i_1}, \cdots, x_{j_k}^{i_k} \cdots) \leftrightarrow R(y_{j_1}^{i_1}, \cdots, y_{j_k}^{i_k} \cdots)\colon i_1, \cdots, i_k \cdots \in \{0, \cdots, n\},\ R \in L,\ j_1, \cdots, j_k \cdots < \alpha\}$$

where

$$\bar{x}^n = \langle \cdots x_i^n \cdots \rangle_{i<\alpha},\ \bar{y}^n = \langle \cdots y_i^n \cdots \rangle_{i<\alpha}\,.$$

Also let

$$\Phi_\alpha^m = [\bigwedge_{\substack{i<\alpha\\ 2n<m}} x_i^{2n} E x \wedge \bigwedge_{\substack{i<\alpha\\ 2n+1<m}} y_i^{2n+1} E y] \rightarrow [\bigwedge_{\substack{i<\alpha\\ 2n+1<m}} x_i^{2n+1} E x \wedge \bigwedge_{\substack{i<\alpha\\ 2n<m}} y_i^{2n} E y \wedge \bigwedge_{n<m} \psi_\alpha^n(\bar{x}^0, \cdots, \bar{x}^n, \bar{y}^0, \cdots, \bar{y}^n)]\colon$$

$$\phi_\alpha^\omega = \bigwedge_{m<\omega} \Phi_\alpha^m = \phi_\alpha^\omega(x, y, \bar{x}^0, \bar{y}^0, \bar{x}^1, \bar{y}^1, \cdots)\,.$$

For even $n$

$$\phi_\alpha^n = \phi_\alpha^n(x, y, \bar{x}^0, \bar{y}^0, \cdots, \bar{x}^{n-1}, \bar{y}^{n-1}) = [(\forall\bar{x}^n)(\exists\bar{y}^n)(\forall\bar{y}^{n+1})(\exists\bar{y}^{n+1})\cdots]\phi_\alpha^\omega\,.$$

For odd $n$

$$\phi_\alpha^n(x, y, \bar{x}^0, \bar{y}^0, \cdots, \bar{x}^{n-1}, \bar{y}^{n-1}) = [(\forall\bar{y}^n)(\exists\bar{x}^n)(\forall\bar{x}^{n+1})(\exists\bar{y}^{n+1})(\forall\bar{y}^{n+2})\cdots]\phi_\alpha^\omega\,.$$

LEMMA 4. *If*

$$a \in |M|,\ b \in |N|,\ M, N \in \{M_i\colon i \in I\},\ M^* = \bigoplus_{i \in I} M_i\,,$$

*and $\mu = \kappa^+$ or $\mu = \aleph_0$, and $\kappa$ is finite, then $M \sim_\mu N$ if and only if $M^* \models \phi_\kappa^0[a, b]$.*

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REMARK. Keisler in [5] used sentences similar to $\phi_\alpha^n$. These sentences can be seen as asserting something about an appropriate game (between a player choosing $\bar{x}^0, y^1, x^2, \cdots$ and a player choosing $\bar{y}^0, \bar{x}^1, \cdots$). In this presentation a similar theorem appears in Karp [4].

*Added in proof.* See also Benda [13].

*Proof.*

*Part A-* Suppose $M \sim_\mu N$.

For every two sequences $\bar{a}, \bar{b}$ of elements of $M$, either there is a formula $\phi_{\bar{a},\bar{b}}(\bar{x})$ of $L(\infty, \mu)$ such that $M \models \phi_{\bar{a},\bar{b}}[\bar{a}]$, $M \models \neg\, \phi_{\bar{a},\bar{b}}[\bar{b}]$, or there is no such $\phi$ and in this case, we let $\phi_{\bar{a},\bar{b}}(\bar{x}) = (x_0 = x_0)$.

Let $\phi_{\bar{a}}(\bar{x}) = \bigwedge_{\bar{b}} \phi_{\bar{a},\bar{b}}(\bar{x}) \in L(\infty, \mu)$. Let $\overline{\phi_{\bar{a}}(\bar{x})} = \phi'_{\bar{a}}(y, \bar{x})$. Let $\alpha < \mu$. We define the functions

$$f_i^{2n}(\bar{x}^0, \bar{y}^0, \bar{y}^1, \bar{x}^1, \bar{x}^2, \cdots, \bar{y}^{2n-1}, \bar{x}^{2n-1}, \bar{x}^{2n}) ,$$
$$f_i^{2n+1}(\bar{x}^0, \bar{y}^0, \bar{y}^1, \bar{x}^1, \bar{x}^2, \cdots, \bar{x}^{2n}, \bar{y}^{2n}, \bar{y}^{2n+1})$$

for $i < \alpha$ such that: If $\bar{a}^0, \bar{b}^0, \bar{a}^1, \bar{b}^1 \cdots$ are sequences of length $\alpha$, $\bar{a}^{2n}$ a sequence of elements of $M$, and $\bar{b}^{2n+1}$ a sequence of elements of $N$, and for every $n$

$$\bar{b}^{2n} = \langle \cdots f_i^{2n}(\bar{a}^0, \bar{b}^0, \cdots, \bar{a}^{2n}) \cdots \rangle_{i<\alpha}$$
$$\bar{a}^{2n+1} = \langle \cdots f_i^{2n+1}(\bar{a}^0, \cdots, \bar{b}^{2n+1}) \cdots \rangle_{i<\alpha}$$

then $M^* \models \phi_\alpha^\omega[a, b, \bar{a}^0, \bar{b}^0, \cdots]$.

Suppose we have defined $f_i^n$ for $n < 2m$, and let us define $f_i^{2m}$ for $i < \alpha$. ($f_i^{2m+1}$ are defined similarly.)

If for some $n < 2m$, $i < \alpha$ $b_i^n \notin |N|$, or for some $i < \alpha$, $n \leqq 2m$ $a_i^n \notin |M|$, then $f_i^{2m}(\bar{a}^0, \cdots, a^{2m})$ is defined as an arbitrary element of $M^*$. Also if there exists a formula $\psi(\bar{z}^1, \cdots, \bar{z}^n) \in L(\infty, \mu)$ such that

$$M \models \psi[\bar{a}^0, \bar{a}^1, \cdots, \bar{a}^{2m-1}] N \models \neg\, \psi[\bar{b}^0, \cdots, \bar{b}^{2m-1}] ,$$

we define $f_i^{2m}(\bar{a}^0 f^0 \cdots \bar{a}^{2m})$ arbitrarily.

So assume none of the previous cases occur. Define $\bar{a}[n] = \bar{a}^0 \frown \bar{a}^1 \frown \cdots \frown \bar{a}^n$ (the concatenation of $\bar{a}_1, \cdots, \bar{a}^n$) and $\bar{b}[n] = \bar{b}^0 \frown \cdots \frown \bar{b}^n$. Clearly

$$M \models (\forall \bar{x})(\phi_{\bar{a}[2m-1]}(\bar{x}) \longrightarrow (\exists \bar{z})\phi_{\bar{a}[2m]}(\bar{x}, \bar{z})) .$$

As $M \sim_\mu N$, $N$ also satisfies the above sentence; so there exists $\bar{b}^{2m}$ such that for every $\phi \in L(\infty, \mu)$, $M \models \phi[\bar{a}^0, \cdots, \bar{a}^{2m}]$ if and only if $N \models \phi[\bar{b}^0, \cdots, \bar{\phi}^{2m}]$. Let $f_i^{2m}(\bar{a}^0, \bar{b}^0, \cdots, \bar{a}^{2m}) = \bar{b}_i^{2m}$.

Clearly this shows that $M^* \models \phi^0_\alpha[a, b]$ for every $\alpha < \mu$, and in particular for $\kappa$.

*Part B.* We now assume that $M^* \models \phi^0_1[a, b]$, and $\mu = \aleph_0$. The proof in the case $\mu = \kappa^+$ or $1 < \kappa < \aleph_0$ is similar. For simplicity, we shall not distinguish between $\bar{a} = \langle a_0 \rangle$ and $a_0$.

Two sequences, $\bar{a}$ from $M$ and $\bar{b}$ from $N$, of length $n$, $n < \omega$, will be called equivalent if $M^* \models \phi^n_1[a, b, \bar{a}, \bar{b}]$. If $n = 2m$, clearly for every $b^{n+1} \in |N|$ there exists $a^{n+1} \in |M|$ such that $\bar{a} \frown \langle a^{n+1} \rangle$ and $\bar{b} \frown \langle b^{n+1} \rangle$ are equivalent, and similarly for $n = 2m + 1$.

Let $\phi(\bar{x}) \in L(\infty, \mu)$, $\bar{x}$ a finite sequence of variables. We shall prove that if $\bar{a}$, $\bar{b}$ are equivalent then $M \models \phi[\bar{a}]$ if and only if $N \models \phi[\bar{b}]$. As subformulas of formulas with $< \aleph_0$ free variables have $< \aleph_0$ free variables we can prove it by induction. For atomic formulas it follows from the definition of $\phi^n_1$. For $\neg \phi$, $\phi \vee \psi$, it is immediate, and so also for the other connectives. For quantification it follows by the fact mentioned above after the definition of equivalent sequences.

So we have proved that if $\bar{a}$, $\bar{b}$ are equivalent sequences, $\phi(\bar{x}) \in L(\infty, \mu)$, then $M \models \phi[\bar{a}]$ if and only if $N \models \phi[\bar{b}]$. Since the sequences of length zero from $M$ and $N$ are equivalent (by our hypotheses $M^* \models \phi^0_1(a, b)$), we get our conclusion that $M \sim N$. This proves Lemma 4.

LEMMA 5. $\phi^0_\alpha(x, y) \in RL^*(\infty, \mu)$. See footnote 3.

*Proof.* It is easily seen that the only thing we have to prove is:

$$\models [(\forall \bar{x}^0)(\exists \bar{y}^0)(\forall y^1)(\exists x^1) \cdots] \bigwedge_{n<\omega} \phi^n_\alpha \leftrightarrow \neg [(\exists \bar{x}^0)(\forall \bar{y}^0)(\exists \bar{y}^1)(\forall x^1) \cdots] \bigvee_{n<\omega} \neg \phi^n_\alpha .$$

For simplicity, let $\alpha = 1$.

It is not hard to see that if $M \models [(\forall x^0)(\exists y^0) \cdots] \bigwedge_{n<\omega} \phi^n_1$, then $M \models \neg [(\exists x^0)(\forall y^0) \cdots] \bigvee_{n<\omega} \neg \phi^n_1$. (See, for example, Keisler [6].)

So suppose $M \models \neg [(\exists \bar{x}^0)(\forall y^0) \cdots] \bigvee_{n<\omega} \neg \phi^n_1$. It is not hard to see that for every $n < \omega$, and formula $\phi$

$$\models \neg [(\forall z_1)(\exists z_2)(\forall z_3) \cdots]\phi \leftrightarrow (\exists z_1) \neg [(\exists z_2)(\forall z_3) \cdots]\phi$$
$$\models (\exists z_1) \neg [(\exists z_2)(\forall z_3) \cdots]\phi \leftrightarrow (\exists z_1)(\forall z_2) \neg [(\forall z_3) \cdots]\phi , \qquad \text{etc.}$$

Now let us define functions $g_n(x^0, y^0, y^1, \cdots, x^i \cdots y^j \cdots)_{i,j<n}$. Let

$$\theta_n(x, y, x^0, y^0, x^1, y^1, \cdots, x^n, y^n) = \neg [\forall x^n)(\exists y^n)(\forall y^{n+1})(\exists x^{n+1}) \cdots] \bigvee_{n<\omega} \neg \phi^n_1 .$$

[3] This lemma is, in fact, a translation of a well known theorem from game theory.

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(This is for even $n$, the definition for odd $n$ is clear.) The functions will be such that if $a^0, \cdots, a^n \in |M|, b^0, \cdots, b^n \in |N|$, and for every $2m \leqq n b^{2m} = g_{2m}(a^0, b^0, \cdots)$, and for every $2m+1 \leqq n a^{2m+1} = g_{2m+1}(a^0, b^0, \cdots)$; then $M^* \models \theta_n[a, b, a^0, b^0 \cdots]$. The definition is self-evident. Let $a^0 \cdots a^n \cdots \in |M|, b^0 \cdots b^n \cdots \in |N|$ be such that for every $2m b^{2m} = g_{2m}(a^0, b^0 \cdots)$ and for every $2m+1$ $a^{2m+1} = g_{2m+1}(a^0, b^0 \cdots)$ and let $n < \omega$. As $M^* \models \theta_{n+1}[a, b, a^0, b^0 \cdots a^n, b^n]$, clearly $M^* \models \phi_1^n(a, b, a^0, b^0 \cdots a^n, b^n)$.

So $M^* \models \bigwedge_{n<\omega} \phi_1^n(a, b, a^0, b^0, \cdots, a^n b^n)$, and hence $M^* \models \phi_1^\omega[a, b, a^0, b^0 \cdots]$. So $M^* \models \phi_1^0[a, b]$ (as this is true for every $a^0, b^1, a^2, b^3 \cdots$) and this is the desired conclusion.

LEMMA 6. *Let* $\mu = \kappa^+$ or $\mu = \aleph_0$, $\kappa = 1$, *$T$ a theory in $RL^*(\lambda, \mu)$, $\chi = \chi^{(\mu)} + \lambda + |T|$, and $I(T, \chi, \mu) \leqq \chi$. Then for every model $N$ of $T$ of power $>\chi$, there exists a model $M$ of $T$ of power $\chi$ such that $M \sim_\mu N$.*

REMARK. This clearly proves Theorem 2.

*Proof.* Let $\{M_i: i \in I\}$ be a maximal set of non-$\mu$-almost-isomorphic models of $T$ of power $\chi$, and let $N$ be a model of $T$ of power $>\chi$ such that for no $i \in I$, $N \sim_\mu M_i$.

Let $M^* = \bigoplus(\{N\}\{M_i: i \in I\})$. Clearly $M^*$ is a model of $T_1 = \bar{T} \cup \{(\forall x, y)[\neg xEy \rightarrow \neg \phi_\kappa^0(x, y)]\}$. Let $a \in |N|$, and $A = \{a\} \cup \bigcup \{|M_i|: i \in I\}$. Clearly, $|A| = \chi$.

Let $\Gamma$ be the set of subformulas of formulas $\in T_1$. By Theorem 1, it follows that $M^*$ has a $\Gamma$-elementary submodel $N^*$, $|N^*| \supset A$, $\chi = ||N^*|| =$ (the power of $N^*$), such that every equivalence class (of $E$) in $N^*$ has exactly $\chi$ elements. Clearly, $N^* = \bigoplus(\{N_1\} \cup \{M_i: i \in I\})$, and for every $i$, $N_1, M_i$ are models of $T$, and they are non-$\mu$-almost-isomorphic. So $N_1$ contradicts the definition of $\{M_i: i \in I\}$, thus proving Lemma 6.

This ends the proof of Theorem 2.

## REFERENCES

TM will denote the Proc. 1963 Berkeley Symposium on Theory of models, North Holland Publ. Co., 1965.

1. A. Ehrenfeucht, *Elementary Theories with Models Without Automorphisms*, TM, pp. 70–76.
2. W. Hanf, *Doctoral Dissertation*, University of California, 1962.
3. L. Henkin, *Some Remarks on Infinitely Long Formulas, Infinitistic Methods*, Warsaw, (1961), 167–183.

4. C. Karp, *Finite Quantifier Equivalence*, TM, pp. 407-412.
5. H. J. Keisler, *Some applications of infinitely long formulas*, J. Symbolic Logic **30** (1965), 339-349.
6. ———, *Formulas With Linearly Ordered Quantifiers*, Lecture Notes in Math. 72. The Syntex and Semantics of Infinitary Languages, (1968), 96-131.
7. M. Makkai, Notices of A.M.S., vol. **16** (1964), p. 322.
8. D. Scott, *Logic with denumerable long formulas and finite strings of quantifiers*, TM, pp. 329-341.
9. S. Shelah, Master's thesis written under the guidence of Professor H. Gaifman. The Hebrew Univ. Jerusalem 1967.
10. *On the number of the non-isomorphic models of a theory in a cardinality*, Notices of Amer. Math. Soc., **17** (1970), 576.
11. ———, *Some unconnected results in model theory*, Notices of the Amer. Math. Soc., **18** (1971), 576. April.
12. ———, *A combinatorial problem, stability and order for models and theories in infinitary languages*, to appear (Pacific J. Math.)
13. M. Benda, *Reduced products and nonstandard logics*, J. Symbolic Logic, **34** (1969), 424-436.
14. J. Barwise, *Back and forth thru infinitary logic*, in a forthcoming book edited by Morley.
15. P. C. Eklof, *On the existence of $L_{\infty,k}$-indiscernibles*, Proc. Amer. Math. Soc., **25** (1970), 798-800.

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THE HEBREW UNIVERSITY
AND
THE UNIVERSITY OF CALIFORNIA, LOS ANGELES