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There is no bound on sizes of indecomposable Banach spaces

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ABSTRACT

Assuming the generalized continuum hypothesis we construct arbitrarily big indecomposable Banach spaces. i.e., such that whenever they are decomposed as $X \oplus Y$, then one of the closed subspaces X or Y must be finite dimensional. It requires alternative techniques compared to those which were initiated by Gowers and Maurey or Argyros with the coauthors. This is because hereditarily indecomposable Banach spaces always embed into ℓ_∞ and so their density and cardinality is bounded by the continuum and because dual Banach spaces of densities bigger than continuum are decomposable by a result due to Heinrich and Mankiewicz. The obtained Banach spaces are of the form $C(K)$ for some compact connected Hausdorff space and have few operators in the sense that every linear bounded operator T on $C(K)$ for every $f \in C(K)$ satisfies $T(f) = gf + S(f)$ where $g \in C(K)$ and S is weakly compact or equivalently strictly singular. In

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particular, the spaces carry the structure of a Banach algebra and in the complex case even the structure of a C^*-algebra.

1. Introduction

The research in the classical period of the isomorphic theory of Banach spaces led to questions of Lindenstrauss ([26]) and Johnson and Lindenstrauss ([18]), respectively, which can be phrased as follows:

(A) *Is it true that every infinite dimensional Banach space has a complemented infinite dimensional and infinite codimensional subspace?*
(B) *Is it true that every infinite dimensional Banach space has a complemented infinite dimensional subspace of density* $\leq$ *continuum?*

Recall that a linear closed subspace Y of a Banach space X is complemented in X if there is another closed linear subspace $Z \subseteq X$ such that $Y \cap Z = \{0\}$ and $Y + Z = X$. Y is complemented in X if and only if there is a bounded linear projection from X onto Y ([37]).

The first, spectacular negative solution to question (A) (such spaces are called indecomposable Banach spaces) was obtained by Gowers and Maurey in [14], where they constructed an infinite dimensional separable Banach space which has even a stronger property of being hereditarily indecomposable, i.e., each of its infinite dimensional closed subspaces is indecomposable. Being hereditary indecomposable is tightly related to having few operators in the sense that every operator on the space is a strictly singular perturbation of a multiple of identity (see [11] for exact description of the relation in both the real and the complex case). Every operator on a hereditarily indecomposable Banach space may even be a compact perturbation of a multiple of identity as recently proved by Argyros and Haydon ([1]). Many constructions of indecomposable Banach spaces followed the paper of Gowers and Maurey, however most of them, including non-separable ones, were hereditarily indecomposable, which as proved e.g., in [2] or [33], must embed in ℓ_∞ which limits their density character or cardinality to the continuum. This led to the following question of S. Argyros:

(C) *Is there an upper bound for densities of indecomposable Banach spaces?*

Assuming various additional properties of a Banach space the positive answer to question (B) and so to question (C) has been obtained by many authors, for a survey of this type of results see [33]. As many hereditarily indecomposable spaces are dual Banach spaces (see [2]) most relevant for us is the result of Heinrich and Mankiewicz [16], which

says that dual Banach spaces of density bigger than continuum are decomposable. Also several new upper bounds for densities of Banach spaces with some rigidity concerning basic sequences were recently obtained by P. Dodos, J. Lopez-Abad, S. Todorcevic ([7], [27], [29], [28]).

In the meantime a different kind of indecomposable Banach spaces was introduced in [21] by the first named author, namely, spaces of continuous functions[4] with few operators, or with few* operators in the sense of the following:

Definition 1.1. Let K be a compact Hausdorff space and let $T : C(K) \to C(K)$ be a bounded linear operator on $C(K)$.

(1) T is called a weak multiplier if $T^* = gI + S$ where $g\colon K \to \mathbb{R}$ is a Borel bounded function and S is a weakly compact operator on $C(K)^*$,
(2) T is called a weak multiplication if $T = gI + S$ where $g \in C(K)$ and S is a weakly compact operator on $C(K)$,
(3) The Banach space $C(K)$ has few operators (few* operators) if every linear bounded operator on $C(K)$ is a weak multiplication (weak multiplier),
(4) A point $x \in K$ is called a butterfly point if and only if there are disjoint open $U, V \subseteq K$ such that $\overline{U} \cap \overline{V} = \{x\}$.

We have the following:

Theorem 1.2 *(2.5., 2.7., 2.8 [21]; 13 [23]). Suppose that K is compact Hausdorff.*

- *If $C(K)$ has few operators and K is connected, then $C(K)$ is indecomposable,*
- *If $C(K)$ has few* operators and $K \setminus F$ is connected for any finite $F \subseteq K$, then $C(K)$ is indecomposable,*
- *If $C(K)$ has few* operators and K has no butterfly points, then $C(K)$ has few operators.*

The first constructions of an indecomposable Banach space as above (with few* operators in ZFC and with few operators under CH, both Ks separable) of density continuum appeared in [21] and some improvements followed, among others, in [32] (with few operators in ZFC for K nonseparable) and [36] (with few operators in ZFC for K separable), for a survey see [23]. In [22] and [24] the first consistent examples of Banach spaces giving the negative answer to question (B) and (A) respectively were presented. They were Banach spaces of the form $C(K)$ with few operators, however the (forcing) method was limited to the density 2^{ω_1}. Note that by the classification of separable Banach spaces of the form $C(K)$ due to Milutin, Bessaga and Pełczyński ([37]) indecomposable $C(K)$s

[4] By $C(K)$ we understand the Banach space of all real-valued continuous functions on a compact Hausdorff space K with the supremum norm.

must be nonseparable. On the other hand it is consistently possible to obtain indecomposable $C(K)$s, with few operators of densities strictly smaller than continuum ([10]). It should be also added that the classes of strictly singular and weakly compact operators coincide for $C(K)$ spaces ([31]).

The main result of this paper is to give the negative answer to question (C) and strengthen (compared to [22] and [24]) the negative answer to question (B) and provide new examples relevant to question (A) by proving:

Theorem 1.3. *Assume the generalized continuum hypothesis. For every cardinal κ there is an indecomposable Banach space of density bigger than κ. In particular it has no infinite dimensional complemented subspace of density smaller than κ. The spaces are (real Banach algebras) of the form $C(K)$ with few operators where K is compact Hausdorff and connected.*

Proof. Use Theorems 1.2, 2.5, and 5.3. □

The methods of the paper consist of a fusion of the techniques of constructing spaces of continuous functions with few operators developed by the first named author and other authors, in particular by I. Schlackow ([36]) and the techniques of S. Shelah developed in [38] and [39] for constructing endo-rigid[5] Boolean algebras, i.e. Boolean algebras with no other endomorphisms than finite perturbations of the identity on an ideal.

Both of these methods are related to rigidity of a compact K. For a compact K we introduce the following notions:

(a) K is piecewise strongly rigid, if for every continuous $\phi : K \to K$ there is a partition $U_1 \cup ... \cup U_k = K$ of K into pairwise disjoint clopen sets $U_1, ..., U_k$ for some $k \in \mathbb{N}$ such that $\phi \restriction U_i$ is either constant or the identity,
(b) K is strongly rigid, if every continuous $\phi : K \to K$ is either constant or the identity,
(c) K satisfies the weak* rigidity condition, if for every $\phi : K \to M(K)$ where $M(K)$ is space of Radon measures on K with the weak* topology (induced from $C(K)$) the set $\{\tau(x)|(K \setminus \{x\}) \mid x \in K\}$ is relatively weakly compact in the weak topology on $M(K)$.

Assuming that K has no butterfly points condition (a) for the Stone space $K_{\mathcal{A}}$ of a Boolean algebra $\mathcal{A}$ is equivalent for the algebra $\mathcal{A}$ to be endo-rigid. Condition (c) is equivalent for $C(K)$ to have few* operators (Theorem 23 of [23]). For K connected (c) implies (b) and (b) is equivalent to (a) (cf. [35]). However (c) and (a) are not equivalent either in connected or totally disconnected situation. The classical space satisfying (b) and not (c) is the Cook continuum ([6]) and arbitrarily big spaces constructed by

[5] Formally a Boolean algebra $\mathcal{A}$ is called endo-rigid if for every endomorphism $h : \mathcal{A} \to \mathcal{A}$ the quotient algebra $\mathcal{A}/exker(h)$ is a finite Boolean algebra, where $exker(h) = \{a \vee b \in \mathcal{A} : a \in ker(h), b \in fix(h)\}$, $fix(h) = \{a \in \mathcal{A} : h(b) = b \text{ for all } b \leq a\}$, $ker(h) = \{a \in \mathcal{A} : h(a) = 0\}$.

Trnkova ([40]). The former space is a metrizable continuum, so by the Milutin theorem the corresponding $C(K)$ has as many operators as $C([0,1])$ and that is why it fails (c). A totally disconnected space satisfying (a) and not (c) is the Stone space of a Boolean algebra $\mathcal{A}$ minimally generated in the sense of Koppelberg ([19]) and endo-rigid. As proved by Borodulin–Nadzieja in [5], the Banach space $C(K_{\mathcal{A}})$ is not Grothendieck, but $C(K)$s which have few* operators must be Grothendieck (Theorem 2.4 [21]).

Both types of constructions of endo-rigid Boolean algebras and rigid Banach spaces $C(K)$ can be traced back to the papers [30] of Monk and [15] of Haydon respectively which surprisingly present practically the same constructions focusing on these different topics.

So our construction needs a stronger property than the constructions from [38] and [39]. The usual constructions of $C(K)$ spaces with few or few* operators ([21], [32], [36], [3]) consisted of obtaining the above weak* topological rigidity (c), and hence few* operators, by constructing K with asymmetric distribution of separations, for example, in the sense that given a sequence $\{x_n : n \in \mathbb{N}\} \subseteq E$ for some dense $E \subseteq K$ and a sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of K such that $x_n \notin U_n$ we have $\overline{\{x_n : n \in M\}} \cap \overline{\{x_n : n \in \mathbb{N} \setminus M\}} \neq \emptyset$ while $\overline{\bigcup\{U_n : n \in M\}} \cap \overline{\bigcup\{U_n : n \in \mathbb{N} \setminus M\}} = \emptyset$ for some infinite and coinfinite $M \subseteq \mathbb{N}$ (see [23] Theorems 24 and 25). It is clear that this method puts an upper bound of the density of $C(K)$ which is related to the number of all separable compact nonhomeomorphic Hausdorff spaces. In this paper we formulate a new asymmetry condition depending on additional parameters which incorporates the ideas of [38] and [39] in the context of weak* rigidity and connected spaces:

Definition 1.4. Let κ be a cardinal, K be a compact Hausdorff space with a open basis $\mathcal{B}$ and let $d_\alpha : K \to [0,1]$ be continuous for every $\alpha < \kappa$. Let $d_{1,\alpha} = d_\alpha$ and $d_{-1,\alpha} = 1 - d_\alpha$. $C(K)$ is said to have asymmetric distribution of separations in the direction of $\mathcal{D} = (d_\alpha : \alpha < \kappa)$ if and only if

Given

(i) $(f_n)_{n\in\mathbb{N}} \subseteq C(K)$ such that $f_n \cdot f_m = 0$ for all distinct $n, m \in \mathbb{N}$, $f_n : K \to [0,1]$ continuous and
(ii) a pairwise disjoint $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}$ such that $supp(f_n) \cap U_m = \emptyset$ for all $n, m \in \mathbb{N}$,
(iii) $\nu_n^\xi \in \{-1, 1\}$ for all $n \in \mathbb{N}$ and $\xi \in \kappa$,
(iv) $\{ U_n^\xi \mid n \in \mathbb{N},\ \xi \in \kappa \} \subseteq \mathcal{B}$ such that $U_n^\xi \subseteq U_n$ for every $n \in \mathbb{N}$ and $\xi \in \kappa$;

There exist an increasing sequence $(\eta_n)_{n\in\mathbb{N}} \subseteq \kappa$ and an infinite, coinfinite $M \subseteq \mathbb{N}$ such that

(a) the supremum $\bigvee_{n\in M} \left(f_n \cdot d_{\eta_n, \nu_n^{\eta_n}} \right)$ exists in $C(K)$,
(b) $\overline{\bigcup_{n\in M} U_n^{\eta_n}} \cap \overline{\bigcup_{n\in\mathbb{N}\setminus M} U_n^{\eta_n}} \neq \emptyset$.

Section 2 of the paper is devoted to proving the following theorem (recall that a topological space has c.c.c. if it does not contain an uncountable family of pairwise disjoint nonempty open sets):

Theorem 2.5. *Suppose that K and $\mathcal{D}$ are as in Definition 1.4. If $C(K)$ has asymmetric distribution of separations in the direction of $\mathcal{D}$ and K is c.c.c., then $C(K)$ has few* operators.*

This is done by modifying proofs of previously considered asymmetric conditions and using some stronger extraction principles (Lemma 2.1, and Lemma 2.2) also proved in this section. Section 3 is devoted to the reformulation of the existing theory concerning the inverse limit constructions ensuring asymmetric distribution of separations in previously considered senses. While the conditions from Definition 1.4 can be rephrased in the Banach algebra language (although the final result, Theorem 1.3 concerns only the Banach space structure), the proof techniques concerning separations and the connectedness involve the topological arguments in K. So the main object in Section 3 is the concrete representation $\nabla\mathcal{F}$ of the Gelfand space of the Banach algebra $[\mathcal{F}]$ generated by subsets $\mathcal{F} \subseteq C(L)$ for some extremally disconnected L.

In section 4 we introduce a concrete type of an inverse limit of compact spaces construction which on the level of the space of continuous functions is called a ladder family (Definition 4.1). The lemmas from Section 3 are used there to prove that if $\mathcal{F} \subseteq C(K)$ is a ladder family, then $C(\nabla\mathcal{F})$ is connected, has no butterfly points and provides a fertile environment for both the existence of suprema and nonseparated pairwise disjoint sequences of open sets in $\nabla\mathcal{F}$ needed to obtain the properties from Definition 1.4.

In Section 5 we use the combinatorial principle $\diamondsuit(E^\kappa_\omega)$ which follows, by a result of Gregory, from the generalized continuum hypothesis for any regular uncountable cardinal κ to perform a particular construction of a ladder family. The character of $\diamondsuit(E^\kappa_\omega)$ as a prediction tool allows us to balance the amount of the suprema and nonseparated pairwise disjoint sequences of open sets to obtain the conditions from Definition 1.4. The main theorem of Section 5 completing the list of all ingredients needed to obtain Theorem 1.3 is:

Theorem 5.3. *Assume the generalized continuum hypothesis. Let κ be the successor of a cardinal of uncountable cofinality. There is a compact Hausdorff connected c.c.c. space K of weight κ without a butterfly point such that $C(K)$ has asymmetric distribution of separations in the direction of some $\mathcal{D} \subseteq C_I(K)$.*

We do not know if the hypothesis of the generalized continuum hypothesis can be removed from Theorem 1.3. In [39] Shelah's black boxes were used to avoid any additional set theoretic assumption in the construction of endo-rigid Boolean algebras. The Banach space construction seems more demanding in this context. The first and the third named authors would like to thank Gabriel Salazar for discussions concerning Shelah's black boxes.

The obtained spaces $C(K)$ have other usual properties of $C(K)$s with few operators proved in [21] such as having proper subspaces, in particular hyperplanes not isomorphic to the entire space, not being isomorphic to $C(L)$ for L totally disconnected etc. One could point out one property not mentioned in the literature that the space $C_{\mathbb{C}}(K)$ of complex valued functions of K is an indecomposable complex Banach space which additionally carries the structure of a commutative C^*-algebra.[6]

Making considerably less technical effort and following the ideas of this paper one could construct a totally disconnected K of arbitrarily big size such that $C(K)$ has few operators. This would already provide Banach spaces of densities κ, for arbitrarily big κ without complemented infinite dimensional subspaces of densities less than κ giving a strong negative answer to question (B). We opted for presenting just the connected example. Our K has one additional peculiar property: while it has no nontrivial convergent sequence (this would give rise to a complemented copy of c_0) for any pairwise disjoint sequence $(U_n)_{n\in\mathbb{N}}$ of nonempty open subsets of K there are only countably many sets $M \subseteq \mathbb{N}$ such that $\overline{\{U_n \mid n \in M\}} \cap \overline{\{U_n \mid n \in \mathbb{N} \setminus M\}} = \emptyset$. This follows from Lemma 4.5 and the construction.

The terminology and notation of the paper should be standard. In set theory we follow [17], [25], in topology [8], in Boolean algebras [20], [12], in Banach spaces [9], [37]. Important conventions include:

- $C_I(K) = \{f \in C(K) \mid f : K \to [0,1]\}$,
- $(f_n)_{n\in\mathbb{N}}$ are pairwise disjoint if $f_n \cdot f_m = 0$ for all distinct $n, m \in \mathbb{N}$,
- For $f : X \to Y$ and $U \subseteq X$ the restriction $f \restriction U : U \to Y$ satisfies $(f \restriction U)(x) = f(x)$ for all $x \in U$,
- GCH is the generalized continuum hypothesis i.e., the statement that 2^κ is the successor cardinal κ^+ for every infinite cardinal κ,
- $E^\kappa_\omega = \{\alpha \in \kappa \mid \mathrm{cf}(\alpha) = \omega\}$ denotes the set of ordinals smaller than κ of cofinality ω,
- $\mathrm{supp}(f)$ denotes $f^{-1}[\mathbb{R} \setminus \{0\}]$ for any real valued function f.

[6] To see this look at $C_{\mathbb{C}}(K)$ as $C(K) \oplus C(K)$ with the multiplication by a complex scalar defined as $(\alpha + i\beta)(f, g) = (\alpha f - \beta g, \beta f + \alpha g)$. A linear operator on $C_{\mathbb{C}}(K)$ can be identified with a 2×2 matrix A of operators on $C(K)$ such that $T(f, g) = A(f, g)$. The $\mathbb{C}$-linearity of T imposes the condition $iT(1, 0) = T(0, 1)$ which yields (by $i(f, g) = (-g, f)$):

$$T(f,g) = \begin{bmatrix} T_1 & -T_2 \\ T_2 & T_1 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$$

for some operators T_1, T_2 on $C(K)$. If $C(K)$ has few operators, this reduces to a sum of a matrix of weakly compact operators and an operator of multiplication by a complex function $M_h(f+ig) = (h_1+ih_2)(f+ig) = (h_1 f - h_2 g) + i(h_2 f + h_1 g)$ for some $h_1, h_2 \in C(K)$. Hence as in the real case every projection P on $C_{\mathbb{C}}(K)$ is of the form $hI + S$ for $h \in C_{\mathbb{C}}(K)$ and S strictly singular, and the condition $P^2 = P$ yields that $h^2 = h$ as no multiplication can be strictly singular for continuous functions on a K with no isolated points (K is connected). Hence $h(x) = 0$ or $h(x) = 1$ for each $x \in K$ and so $h = 1$ or $h = 0$ since K is connected. It follows that $P = I + S$ of $P = S$ where S is finite dimensional since S is a projection as well, and so $C_{\mathbb{C}}(K)$ is indecomposable indeed.

2. Few* operators from asymmetric distributions of separations

The purpose of this section is to prove Theorem 2.5. This amounts to applying the theory initiated in [21] and later developed in [22], [24], [32], [36], [23] in the new context of the sequence $\mathcal{D} = \{d_\alpha : \alpha < \kappa\}$. The first step is to prove a Rosenthal lemma[7] type extraction principle in the flavour of the approach from Chapter 4.3 of [36] (cf. Theorems 24 and 25 of [23]):

Lemma 2.1. *Let K be a compact Hausdorff space, $T : C(K) \to C(K)$ a bounded linear operator and let $\varepsilon > 0$. Let $(f_n)_{n\in\mathbb{N}} \subseteq C_I(K)$ be pairwise disjoint and let $(U_n)_{n\in\mathbb{N}}$ be a pairwise disjoint family of nonempty open subsets of K. Then there are an infinite $M \subseteq \mathbb{N}$ and nonempty open sets $V_n \subseteq U_n$ for $n \in \mathbb{N}$ such that for all $m \in M$ and for all sequences $(g_n)_{n\in\mathbb{N}} \subseteq C_I(K)$ satisfying $g_n \leq f_n$ for $n \in \mathbb{N}$ we have*

$$\sup_{x\in V_m} \sum_{n\in M\setminus\{m\}} |T(g_n)(x)| < \varepsilon.$$

Proof. Let us introduce an auxiliary notation: for sets $M \subseteq N \subseteq \mathbb{N}$ and two pairwise disjoint sequences $(V_n)_{n\in M}$, $(U_n)_{n\in N}$ of nonempty open subsets of K we write $(V_n)_{n\in M} \prec (U_n)_{n\in N}$ if $V_n \subseteq U_n$ for all $n \in M$.

By recursion on $k \in \mathbb{N}$ construct

- $n_1 < ... < n_k$ in $\mathbb{N}$,
- $(V_{n_1}, ..., V_{n_k}) \prec (U^0_{n_1}, ..., U^{k-1}_{n_k}) \prec (U_{n_1}, ..., U_{n_k})$,
- $\mathbb{N} = X_0 \supseteq X_1 \supseteq ... \supseteq X_k$ such that X_k is an infinite subset of $\mathbb{N} \setminus [1, n_k]$,
- $(U_n)_{n\in\mathbb{N}} = (U^0_n)_{n\in X_0} \succ (U^1_n)_{n\in X_1} \succ \cdots \succ (U^k_n)_{n\in X_k}$,

such that

$$|T(g_{n_k})(x)| \leq \frac{\varepsilon}{2^{k+1}} \tag{$*$}$$

holds for all $0 \leq g_{n_k} \leq f_{n_k}$ and for all $x \in U^k_n$ with $n \in X_k$. Moreover

$$\sum_{n\in X_k} |T(g_n)(x)| \leq \frac{\varepsilon}{2^k} \tag{$**$}$$

holds for all $x \in V_{n_k}$ and for all $0 < g_n \leq f_n$ with $n \in X_k$.

As n_0 is undefined, the above is vacuously true for $k = 0$. So, suppose we have the above objects for $k \geq 0$ and let us construct the corresponding objects for $k + 1$. Note that $(*)$ and $(**)$ are worded in such a way that given X_k we need to find $n_{k+1} \in X_k$ and an infinite $X_{k+1} \subseteq X_k \setminus [1, n_{k+1}]$ such that $(*)$ and $(**)$ are satisfied for $k + 1$ in

[7] See Lemma 1.1 of [34].

place of k. That is, the previous $(n_1, ..., n_k)$ and $(V_{n_1}, ..., V_{n_k})$ play no role when we pass to $(*)$ and $(**)$ for $k+1$. First we will take care of $(*)$.

Suppose that there is no n_{k+1}, X_{k+1} and U^{k+1}_n for $n \in X_{k+1}$ such that $(*)$ holds, that is, for all $n' \in X_k$, all infinite $X \subseteq X_k \setminus [1, n']$, and all $(U'_n)_{n\in X} \prec (U^k_n)_{n\in X_k}$ there exist a number $n'' \in X$, and $0 \leq g_{n',n''} \leq f_{n'}$, and an element $x_{n''} \in U'_{n''}$ such that

$$|T(g_{n',n''})(x_{n''})| > \frac{\varepsilon}{2^{k+2}}. \tag{***}$$

We will derive contradiction from this hypothesis. Let $l \in \mathbb{N}$ be such that $\frac{l\varepsilon}{2^{k+2}} > \|T\|$. Applying the above recursively on $i \leq 2l$ we can construct (note that the index $k+1$ below is fixed and indicates only that we are in the $(k+1)$-th stage of the recursive construction):

- an increasing $(n^i_{k+1})_{i\leq 2l} \subseteq X_k$ with $n^1_{k+1} > n_k$,
- an infinite X^i_{k+1} such that $X^{i+1}_{k+1} \subseteq X^i_{k+1} \subseteq X_k \setminus [1, n^i_{k+1}]$,
- nonempty open W^i_ns for $n \in X^i_{k+1}$ such that

$$(W^{i+1}_n)_{n\in X^{i+1}_{k+1}} \prec (W^i_n)_{n\in X^i_{k+1}} \prec (U^k_n)_{n\in X_k},$$

- $0 \leq g_{n^i_{k+1},n} \leq f_{n^i_{k+1}}$ for $n \in X^i_{k+1}$ such that

$$|T(g_{n^i_{k+1},n})(x)| > \frac{\varepsilon}{2^{k+2}}$$

for all $x \in W^i_n$, and all $n \in X^i_{k+1}$.

To move from i to $i+1$ we set $n^{i+1}_{k+1} = \min X^i_{k+1}$ and use repeatedly the above hypothesis for each $j \in \mathbb{N}$ with $n' = n^{i+1}_{k+1}$, $X = X^i_{k+1} \cap [n''_{j-1}, \infty)$, and $(U'_n)_{n\in X} = (W^i_n)_{n\in X}$ to obtain $n''_j \in X$, $g_{n',n''_j} \leq f_{n'}$, and $x_{n''_j} \in U'_{n''_j}$. Then $X^{i+1}_{k+1} = (n''_j)_{j\in\mathbb{N}}$ and we use the continuity of $|T(g_{n',n''_j})|$ to conclude that if it is bigger than $\frac{\varepsilon}{2^{k+2}}$ at point $x_{n''_j}$, then it is bigger than $\frac{\varepsilon}{2^{k+2}}$ at some neighbourhood $W^i_{n''_j}$ of that point.

Arriving at $i = 2l$ we set $m = \min X^{2l}_{k+1}$, pick $x_0 \in W^{2l}_m$ and fix a finite set $F \subseteq [1, 2l]$ of cardinality not less than l such that all numbers $T(g_{n^i_{k+1},m})(x_0)$ have the same sign for $i \in F$. Then we have

$$|T(\sum_{i\in F} g_{n^i_{k+1},m})(x_0)| > \frac{l\varepsilon}{2^{k+2}} \geq ||T||.$$

This is a contradiction since the norm of $\sum_{i\in F} g_{n^i_{k+1},m}$ is less than or equal to one.

Hence our hypothesis was false, that is, there is $n_{k+1} \in X_k$ and an infinite $X'_{k+1} \subseteq X_k \setminus [1, n_{k+1}]$ such that for some nonempty $U^{k+1}_n \subseteq U^k_n$ with $n \in X'_{k+1}$ the condition $(***)$ holds. That is $(*)$ holds for $k+1$ in place of k.

Now we will choose a nonempty $V_{n_{k+1}} \subseteq U^k_{n_{k+1}}$ and an infinite $X_{k+1} \subseteq X'_{k+1}$ such that (∗∗) holds for $k+1$ instead of k. Let

$$s = \sup_{x \in U^k_{n_{k+1}}} \{ \sum_{n \in X'_{k+1}} |T(g_n)(x)| \mid 0 \leq g_n \leq f_n \}.$$

Note that $s \leq 2\|T\|$ as for the supremum we can consider finite sums of numbers with constant sign, which by the linearity of T are reduced to values of the operator T on vectors of norm less than or equal to one. Choose $x_0 \in U^k_{n_{k+1}}$ such that

$$s - \sup\{ \sum_{n \in X'_{k+1}} |T(g_n)(x_0)| \mid 0 \leq g_n \leq f_n) \} < \frac{\varepsilon}{2^{k+3}},$$

and then a finite $F \subseteq X'_{k+1}$ and $0 \leq g_n \leq f_n$ for $n \in F$ such that

$$s - \sum_{n \in F} |T(g_n)(x_0)| < \frac{\varepsilon}{2^{k+2}}.$$

Now, note that by the continuity of the functions $T(g_n)$ for $n \in F$ at x_0 there is a nonempty neighbourhood of x_0 of the form $V_{n_{k+1}}$ for $V_{n_{k+1}} \subseteq U^k_{n_{k+1}}$ where the above inequality holds. Put $X_{k+1} = X'_{k+1} \setminus F$ and note that by the choice of s and F we have (∗∗) with $k+1$ in the place of k.

This completes the recursive construction. Note that $n_{k+1} \in X_k$ for each $k \in \mathbb{N}$. To verify the statement of the lemma let $M = (n_k)_{k \in \mathbb{N}}$ and choose a sequence $(g_n)_{n \in M} \subseteq C_I(K)$ with $g_n \leq f_n$ for all $n \in M$ and $m = n_k \in M$ and $x \in V_{n_k}$. Then

$$\sum_{n \in M \setminus \{m\}} |T(g_n)(x)| \leq \sum_{1 \leq i < k} |T(g_i)(x)| + \sum_{n \in X_k} |T(g_n)(x)|$$

as $M \setminus \{n_1, \ldots, n_k\} \subseteq X_k$. The first sum is not bigger than

$$\sum_{1 \leq i < k} \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2}(1 - 2^{-k+1}) \leq \varepsilon/2$$

by applying (∗) since $n_k \in X_i$ for each $i < k$ and $V_{n_k} \leq U^i_{n_k}$. On the other hand the second sum is not bigger than $\varepsilon/2$ by applying directly (∗∗). Hence we obtain the statement of the lemma. □

Another extraction principle which we will need is the following:

Lemma 2.2. *Suppose that V_n's for $n \in \mathbb{N}$ are pairwise disjoint open sets in a compact space K and $x_n \in K \setminus V_n$ are distinct. Suppose that $\varepsilon > 0$ and μ_n for $n \in \mathbb{N}$ is a Radon measure on K such that $|\mu_n|(V_n) > \varepsilon$ for all $n \in \mathbb{N}$. Then there are: an infinite $M \subseteq \mathbb{N}$, open $V'_n \subseteq V_n$ and a $\delta > 0$ such that for all $n \in M$ we have that $|\mu_n|(V'_n) > \delta$ and*

$$\overline{\bigcup_{n \in M} V'_n} \cap \{x_n : n \in M\} = \emptyset.$$

Proof. By going to a subset we may assume that $\{x_n : n \in \mathbb{N}\}$ forms a discrete subspace of K. By the regularity of the measures and by going to subsets of V_ns we may assume that $(\overline{V_n})_{n\in\mathbb{N}}$ is pairwise disjoint and $x_n \notin \overline{V_n}$ for each $n \in \mathbb{N}$. Consider a coloring $c : [\mathbb{N}]^2 \to \{0,1,2\}$ defined for distinct $n, m \in \mathbb{N}$ by $c(\{n,m\}) = 0$ if $x_n \in \overline{V}_m$ and $n < m$, $c(\{n,m\}) = 1$ if the previous condition does not hold and $x_m \in \overline{V}_n$ and $n < m$, and $c(\{n,m\}) = 2$ if $\{x_n, x_m\} \cap (\overline{V}_n \cup \overline{V}_m) = \emptyset$. Apply the Ramsey theorem for c obtaining an infinite subset of $\mathbb{N}$ which is homogeneous for c. However, a three element 0-homogeneous set or 1-homogeneous set would contradict the pairwise disjointness of $\overline{V}_n$s, so we have an infinite 2-homogeneous set. Hence, by going to a subset we may assume that $x_n \notin \overline{V}_m$ for any two $n, m \in \mathbb{N}$. Let U_n be an open neighbourhood of x_n such that $U_n \cap V_m = \emptyset$ for all $m \leq n$ in $\mathbb{N}$. We will consider two cases.

Case 1. There is $\delta > 0$ and a point $x \in K$ such that for each open neighbourhood W of x the set $\{n \in \mathbb{N} : |\mu_n|(V_n \cap W) > \delta\}$ is infinite.

As V_n's are pairwise disjoint and by the regularity of the measures by going from V_n to its subset we may assume that $x \notin \overline{V}_n$ for every $n \in \mathbb{N}$. Further removing at most one index we may assume that $x \notin \{x_n : n \in \mathbb{N}\}$.

Now recursively define a decreasing sequence $(W_k)_{k\in\mathbb{N}}$ of open neighbourhoods of x and a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that the following two conditions hold:

$$|\mu_{n_k}|\big(V_{n_k} \cap (W_k \setminus \overline{W}_{k+1})\big) > \delta \tag{1}$$

$$x_{n_k} \notin \overline{W_{k+1}}. \tag{2}$$

This is possible by the hypothesis of Case 1. Put $V'_{n_k} = V_{n_k} \cap (W_k \setminus \overline{W}_{k+1})$. It follows that

$$\overline{\bigcup_{k\in\mathbb{N}} V'_{n_k}} \setminus \bigcup_{k\in\mathbb{N}} \overline{V'_{n_k}} \subseteq \bigcap_{k\in\mathbb{N}} \overline{W}_k,$$

which is disjoint from $\{x_{n_k} : k \in \mathbb{N}\}$ by (2). Since $\{x_{n_k} : k \in \mathbb{N}\}$ is disjoint form $\bigcup_{n\in\mathbb{N}} \overline{V}_n$ by the argument before Case 1, we conclude the proof of the Lemma in this case for $M = \{n_k : k \in \mathbb{N}\}$.

Case 2. Case 1 does not hold.

Since the hypothesis of Case 1 fails, for every $n \in \mathbb{N}$ and for every $\delta' > 0$ there is an $m(n,\delta') \in \mathbb{N}$ and an open neighbourhood $W(n,\delta')$ of x_n such that

$$|\mu_k|(V_k \cap \overline{W(n,\delta')}) < \delta' \tag{1}$$

for all $k > m(n,\delta')$. Thus, one can choose recursively a strictly increasing sequence of integers $(k_n)_{n\in N}$ such that $k_n > m(k_j, \frac{\varepsilon}{2^{j+2}})$ for all $j < n$. Consider

$$V'_{k_n} = V_{k_n} - \bigcup\{\overline{W(k_j, \frac{\varepsilon}{2^{j+2}})} : j < n\}.$$

By (1) we have $|\mu_{k_n}|(V_{k_n} \cap \overline{W(k_j, \frac{\varepsilon}{2^{j+2}})}) < \frac{\varepsilon}{2^{j+2}}$ for $j < n$ and so

$$|\mu_{k_n}|(V'_{k_n}) > \varepsilon/2 = \delta. \tag{2}$$

Now, note that $W(k_j, \frac{\varepsilon}{2^{j+2}})$ is disjoint from V'_{k_n} for $n > j$ and hence $W(k_j, \frac{\varepsilon}{2^{j+2}}) \cap U_{k_j}$ is an open neighbourhood of x_{k_j} witnessing the fact that $x_{k_j} \notin \overline{\bigcup_{n\in\mathbb{N}} V'_{k_n}}$ (recall that U_ns are open neighbourhoods of x_ns such that $U_n \cap V_m = \emptyset$ for all $m \leq n$ in $\mathbb{N}$). So this proves the lemma for $M = \{n_k : k \in \mathbb{N}\}$ and $\delta = \varepsilon/2$. □

Now recall Definition 1.1 and the following characterization of weak multipliers:

Theorem 2.3 *([21] Definition 2.1, Theorem 2.2). Let K be a compact Hausdorff space and let $T : C(K) \to C(K)$ be a bounded linear operator. The following conditions are equivalent:*

(1) T is a weak multiplier,
(2) for every pairwise disjoint sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_I(X)$ and every sequence $(x_n)_{n\in\mathbb{N}} \subseteq K$ such that $f_n(x_n) = 0$ for all $n \in \mathbb{N}$ we have

$$\lim_{n\to\infty} T(f_n)(x_n) = 0.$$

Lemma 2.4. *Let K be a compact Hausdorff space and let $T : C(K) \to C(K)$ be a bounded linear operator. If T is not a weak multiplier, then there exist $\varepsilon > 0$, a pairwise disjoint sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_I(K)$ and a pairwise disjoint sequence $(U_n)_{n\in\mathbb{N}}$ of nonempty open subsets of K such that*

$$\operatorname{supp}(f_n) \cap U_m = \emptyset \text{ for all } n, m \in \mathbb{N}$$

and

$$|T(f_n)| \restriction U_n > \varepsilon \text{ for all } n \in \mathbb{N}.$$

Proof. By Theorem 2.3(2) there is a pairwise disjoint sequence $(g_n)_{n\in\mathbb{N}} \subseteq C_I(K)$ and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq K$ such that $g_n(x_n) = 0$ for all $n \in \mathbb{N}$, and $|T(g_n)(x_n)| > \varepsilon'$ for some $\varepsilon' > 0$. Let $V_n = \operatorname{supp}(g_n)$ and $\mu_n = T^*(\delta_{x_n})$. Note that the hypothesis of Lemma 2.2 is satisfied for ε' instead of ε, so we may find appropriate $\delta > 0$, and infinite $M \subseteq \mathbb{N}$ and $V'_n \subseteq V_n$ for each $n \in \mathbb{N}$. Let U'_n be open neighbourhoods of x_n for $n \in M$ such that $\overline{\bigcup_{n\in M} V'_n} \cap U'_n = \emptyset$. Since $\{x_n : n \in \mathbb{N}\}$ may be assumed to be discrete by going to a subsequence, we may assume that U'_ns are pairwise disjoint. Now choose $f_n \in C_I(K)$ such that $\operatorname{supp}(f_n) \subseteq V'_n$ and that $|\int f_n d\mu_n| > \delta/2$; this can be done since $|\mu_n(V'_n)| > \delta$. It follows that $|T(f_n)(x_n)| > \delta/2$ for each $n \in \mathbb{N}$. Now find $U_n \subseteq U'_n$

such that $|T(f_n) \restriction U_n| > \delta/2$. By re-enumerating M and putting $\varepsilon = \delta/2$ we obtain the statement from the lemma. □

Theorem 2.5. *Suppose that K and $\mathcal{D}$ are as in Definition 1.4. If $C(K)$ has asymmetric distribution of separations in the direction of $\mathcal{D}$ and K is c.c.c., then $C(K)$ has few* operators.*

Proof. Arguing by contradiction, we suppose that $C(K)$ does not have few* operators. By Definition 1.1 there is a bounded linear operator $T\colon C(K) \to C(K)$ which is not a weak multiplier. Let κ and $\mathcal{B}$ be as in Definition 1.4. Then by Lemma 2.4 there are

- a pairwise disjoint sequence $(f_n)_{n\in\mathbb{N}}$ in $C_I(K)$,
- a pairwise disjoint sequence $(U_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$ such that

$$\operatorname{supp}(f_n) \cap U_m = \emptyset, \text{ for all } n, m \in \mathbb{N},$$

- $\varepsilon > 0$,

and for each $n \in \mathbb{N}$ we have

$$|T(f_n)| \restriction U_n > 2\varepsilon.$$

Now by applying Lemma 2.1 for $\varepsilon/3$ we may assume that for any $m \in \mathbb{N}$ and for any sequence $(g_n)_{n\in\mathbb{N}} \subseteq C_I(K)$ such that $g_n \leq f_n$ for all $n \in \mathbb{N}$ we have

$$\sup_{x\in U_m} \sum_{n\in M\setminus\{m\}} |T(g_n)(x)| < \varepsilon/3. \tag{$*$}$$

To make use of the asymmetric distribution of separations in the direction of $\mathcal{D}$ we need to construct the following:

- $\{(\nu_n^\xi)_{n\in\mathbb{N}} \mid \xi \in \kappa\} \subseteq \{\pm 1\}$,
- $\{\,(U_n^\xi)_{n\in\mathbb{N}} \mid \xi \in \kappa\,\} \subseteq \mathcal{B}$ satisfying $U_n^\xi \subseteq U_n$ for every $n \in \mathbb{N}$ and every $\xi \in \kappa$.

We will construct the above objects in such a way that for all $n \in \mathbb{N}$ and all $\xi \in \kappa$ we have $|T(f_n d_{\nu_n^\xi,\xi})| \restriction U_n^\xi > \varepsilon$. This is achieved in the following way. Fix $\xi \in \kappa$, $n \in \mathbb{N}$, and $x_n \in U_n$. Since $f_n = f_n d_{1,\xi} + f_n d_{-1,\xi}$, we have either

$$|T(f_n d_{1,\xi})(x_n)| > \varepsilon \text{ or } |T(f_n d_{-1,\xi})(x_n)| > \varepsilon.$$

We choose $\nu_n^\xi \in \{\pm 1\}$, the one for which the above holds and define $U_n^\xi \in \mathcal{B}$ to be an open neighbourhood of x_n included in U_n such that

$$|T(f_n d_{\nu_n^\xi,\xi})| \restriction U_n^\xi > \varepsilon. \tag{$**$}$$

This completes the construction of $\{(\nu_n^\xi)_{n\in\mathbb{N}} \mid \xi \in \kappa\}$ and $\{(U_n^\xi)_{n\in\mathbb{N}} \mid \xi \in \kappa\}$.

Let us fix an almost disjoint family $\{N_\alpha \mid \alpha < \omega_1\}$ of infinite subsets of $\mathbb{N}$. We will be considering the sets:

- $F_\alpha = (f_n)_{n\in N_\alpha}$,
- $U_\alpha = (U_n)_{n\in N_\alpha}$,
- $\mathcal{N}_\alpha = \{(\nu_n^\xi)_{n\in N_\alpha} \mid \xi \in \kappa\}$,
- $\mathcal{U}_\alpha = \{(U_n^\xi)_{n\in N_\alpha} \mid \xi \in \kappa\}$.

We use the hypothesis that $C(K)$ has asymmetric distribution of separations in the direction of $\mathcal{D}$ for F_α, U_α, $\mathcal{N}_\alpha$, and $\mathcal{U}_\alpha$ for each $\alpha < \omega_1$ obtaining increasing sequences $(\eta_n^\alpha)_{n\in\mathbb{N}} \subseteq \kappa$ and infinite sets $M_\alpha \subseteq N_\alpha$ such that

(1) $\bigvee_{n\in M_\alpha} f_n d_{\nu_n^{\eta_n^\alpha},\eta_n^\alpha}$ exists in $C(K)$,
(2) $\overline{\bigcup\{U_n^{\eta_n^\alpha} \mid n \in M_\alpha\}} \cap \overline{\bigcup\{U_n^{\eta_n^\alpha} \mid n \in N_\alpha \setminus M_\alpha\}} \neq \emptyset$.

Let us define $g_n^\alpha = f_n d_{\nu_n^{\eta_n^\alpha},\eta_n^\alpha}$ for all $n \in \mathbb{N}$, $\alpha < \omega_1$. If for some $\alpha < \omega_1$ we had

- for all $n \in M_\alpha$, $x \in U_n^{\eta_n^\alpha}$: $|T(\bigvee_{n\in M_\alpha} g_n^\alpha)(x)| \geq 2\varepsilon/3$ and
- for all $n \in N_\alpha \setminus M_\alpha$, $x \in U_n^{\eta_n^\alpha}$: $|T(\bigvee_{n\in M_\alpha} g_n^\alpha)(x)| \leq \varepsilon/3$

then, we would separate the sets in (2) contradicting the condition (2). Therefore the conjunction of the above statements are false for each $\alpha < \omega_1$. By going to an uncountable subset of ω_1 we may assume that there is $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for each $\alpha < \omega_1$ we have some $\emptyset \neq V_\alpha \subseteq U_{n_0}^{\eta_{n_0}^\alpha} \subseteq U_{n_0}$ such that either

(3) $n_0 \in M_\alpha$ and for all $x \in V_\alpha$ we have $|T(\bigvee_{n\in M_\alpha} g_n^\alpha)(x)| < 2\varepsilon/3 - \delta$ or
(4) $n_0 \in N_\alpha \setminus M_\alpha$ and for all $x \in V_\alpha$ we have $|T(\bigvee_{n\in M_\alpha} g_n^\alpha)(x)| > \varepsilon/3 + \delta$.

This gives for each $\alpha < \omega_1$ the following statement:

- $n_0 \in \mathbb{N} \setminus M'_\alpha$ and for all $x \in V_\alpha$ we have $|T(\bigvee_{n\in M'_\alpha} g_n^\alpha)(x)| > \varepsilon/3 + \delta$,

where $M'_\alpha = M_\alpha$ if (4) holds and $M'_\alpha = M_\alpha \setminus \{n_0\}$ if (3) holds. In the latter case the above condition follows from (3) and (**). Now consider $m \in \mathbb{N}$ such that $m\delta/2 > \|T\|$. Using the c.c.c. of the space K we may find (see Lemma 3.1) $\alpha_1 < \ldots < \alpha_m < \omega_1$ such that

$$V_{\alpha_1} \cap V_{\alpha_2} \cap \cdots \cap V_{\alpha_m} \neq \emptyset.$$

So let x_0 be a point in $V_{\alpha_1} \cap V_{\alpha_2} \cap \cdots \cap V_{\alpha_m} \neq \emptyset$.

Now take $k \in \mathbb{N}$ such that for each $1 \leq i < j \leq m$ we have

$$(M'_{\alpha_i} \setminus k) \cap (M'_{\alpha_j} \setminus k) = \emptyset.$$

By $(*)$ for each $1 \leq i \leq m$ we have that

$$\sum_{n \in M'_{\alpha_i} \cap k} |T(g_n^{\alpha_i})(x_0)| < \varepsilon/3$$

and so for each $1 \leq i \leq m$ we have

$$|T\Big(\bigvee_{n \in M'_{\alpha_i} \setminus k} g_n^{\alpha_i}\Big)(x_0)| > \delta.$$

For $m/2$ indices $1 \leq i \leq m$, say from a set $F \subseteq \{1, ..., m\}$ all the reals

$$T\Big(\bigvee_{n \in M'_{\alpha_i} \setminus k} g_n^{\alpha_i}\Big)(x_0)$$

have the same sign, and so

$$\|T\Big(\sum_{i \in F} \bigvee_{n \in M'_{\alpha_i} \setminus k} g_n^{\alpha_i}\Big)\| \geq |T\Big(\sum_{i \in F} \bigvee_{n \in M'_{\alpha_i} \setminus k} g_n^{\alpha_i}\Big)(x_0)| = \sum_{i \in F} |T\Big(\bigvee_{n \in M'_{\alpha_i} \setminus k} g_n^{\alpha_i}\Big)(x_0)|$$
$$> m\delta/2.$$

However

$$\|\sum_{i \in F} \bigvee_{n \in M'_{\alpha_i} \setminus k} g_n^{\alpha_i}\| \leq 1$$

as $(M'_{\alpha_i} \setminus k)$s are pairwise disjoint. This is impossible since $\|T\| < m\delta/2$. This completes the proof. □

3. Controlling separations and the connectedness

The purpose of this section is to generalize the methods developed in [21,22,24,36] necessary for controlling the constructions of $C(K)$ spaces with few operators. To be able to adopt the ideas coming from the constructions of endo-rigid Boolean algebras we need to operate in the language of Banach algebras of functions, on the other hand the above mentioned methods concerning separations and the connectedness involve the topological arguments in K. So in this section we work at the same time with a family $\mathcal{F}$ of functions in $C(L)$ for some extremally disconnected L, the Banach algebra $[\mathcal{F}]$ generated by it and a concrete representation $\nabla\mathcal{F}$ of the Gelfand space of the Banach algebra $[\mathcal{F}]$.

3.1. Boolean algebras, their Stone spaces and continuous functions on the Stone spaces

Recall that a topological space is called c.c.c. if it does not contain an uncountable collection of pairwise disjoint open sets. We have the following:

Lemma 3.1. *Suppose K is a compact Hausdorff space and that κ is a cardinal. Then*

- $\{0,1\}^{\kappa\times\mathbb{N}}$ *is c.c.c.*
- *If K is c.c.c., $m\in\mathbb{N}$ and $\{V_\xi : \xi<\omega_1\}$ is a collection of nonempty open subsets of K, then there are distinct $\xi_1,...,\xi_m$ such that $V_{\xi_1}\cap...\cap V_{\xi_m}\neq\emptyset$.*

Proof. The first condition follows from the Hewitt–Marczewski–Pondiczery Theorem (Theorem 2.3.17. of [8]). For the second condition prove it by induction on $m\in\mathbb{N}$. For $m=2$ it is the c.c.c. Given it for m, build recursively pairwise disjoint family of sets $F_\alpha\subseteq\omega_1$ of cardinality m for $\alpha<\omega_1$ such that $W_\alpha=\bigcap_{\xi\in F_\alpha}V_\xi\neq\emptyset$, now apply the c.c.c. for $(W_\alpha)_{\alpha<\omega_1}$. □

In this section we use the following notation:

- κ will denote an uncountable regular cardinal,
- If $\mathcal{A}$ is a Boolean algebra, $S(\mathcal{A})$ denotes the Stone space of $\mathcal{A}$, i.e., a compact Hausdorff totally disconnected space such that there is a Boolean isomorphism between $\mathcal{A}$ and the algebra of clopen subsets of $S(\mathcal{A})$,
- the clopen set of $S(\mathcal{A})$ corresponding to an element a of $\mathcal{A}$ will be denoted by $s_{\mathcal{A}}(a)$,
- $Fr(\kappa)$ denotes the free Boolean algebra generated by $(e_{\alpha,n})_{\alpha<\kappa,n\in\mathbb{N}}$,
- For $A\subseteq\kappa$, $Fr(A)$ denotes the subalgebra of $Fr(\kappa)$ generated by $(e_{\alpha,n})_{\alpha\in A,n\in\mathbb{N}}$,
- For $A\subseteq\kappa$, $\overline{Fr}(A)$ denotes the Boolean completion of $Fr(A)$,
- We will identify $\overline{Fr}(A)$ with a subalgebra of $\overline{Fr}(B)$ when $A\subseteq B\subseteq\kappa$,
- For $A\subseteq\kappa$, L_A denotes the Stone space of $\overline{Fr}(A)$,
- L_κ will be denoted by L,
- $I=[0,1]$,
- The supremum of a family $\mathcal{F}$ of functions will be denoted by $\bigvee\mathcal{F}$. In principle the supremum of the same family of functions can depend on the ambient lattice of functions, so we will need to add where the supremum is taken.

For the Stone duality or other dual terminology concerning Boolean algebras see [20] or [12], for Gleason spaces see [13]. The following proposition is the summary of standard facts concerning the above objects:

Proposition 3.2.

- *For any $A\subseteq\kappa$ the space L_A is the Gleason space of I^A,*
- *For any $A\subseteq\kappa$ the space L_A is extremally disconnected and c.c.c., which implies that bounded subsets of $C(L_A)$ have suprema in the lattice $C(L_A)$,*

- *For $A \subseteq \kappa$ there is a continuous surjective map $p_A : L \to L_A$ dual to the inclusion $\overline{Fr}(A) \subseteq \overline{Fr}(\kappa)$, in particular $p_A^{-1}[s_{\overline{Fr}(A)}(a)] = s_{\overline{Fr}(\kappa)}(a)$ for $a \in \overline{Fr}(A)$,*
- *For $A \subseteq \kappa$ there is an isometric inclusion $C(L_A) \subseteq C(L)$ induced by p_A.*

Lemma 3.3 *([24] Corollary 2.5). Suppose that A, B are disjoint subsets of κ and $t_A \in L_A$ and $t_B \in L_B$. There exists a point $t \in L$ such that $p_A(t) = t_A$ and $p_B(t) = t_B$.*

Definition 3.4. For all $\alpha \in \kappa$ we define a Cantor-like surjection $d_\alpha \in C_I(L)$ by

$$d_\alpha(x) = \sum_{n\in\mathbb{N}} \frac{\chi_{s_{\overline{Fr}(\kappa)}(e_{\alpha,n})}(x)}{2^n} \in I,$$

for all $x \in L$. We will use $d_{1,\alpha} = d_\alpha$ and $d_{-1,\alpha} = 1 - d_\alpha$. We will also use the notation $\mathcal{D} = \{d_\alpha : \alpha \in \kappa\}$.

Definition 3.5. We say that $f \in C(L)$ depends on a set $A \subseteq \kappa$, if $p_A(s) = p_A(t)$ implies $f(s) = f(t)$ for any $s, t \in L$. We say that $\mathcal{F} \subseteq C(L)$ depends on a set $A \subseteq \kappa$, if every $f \in \mathcal{F}$ depends on A.

Lemma 3.6. *Let $\alpha \in \kappa$, then*

(1) d_α depends on the set $\{\alpha\}$,
(2) $d_\alpha[L] = I$.

Proof. By Proposition 3.2 $\chi_{s_{\overline{Fr}(\kappa)}(e_{\alpha,n})} = \chi_{s_{\overline{Fr}(\{\alpha\})}(e_{\alpha,n})} \circ p_{\{\alpha\}}$. So, if $s, t \in L$ satisfy $p_{\{\alpha\}}(s) = p_{\{\alpha\}}(t)$, then $d_\alpha(s) = d_\alpha(t)$, as required. The second part follows from the fact that $e_{\alpha,n}$s as free generators are independent, which implies that for every $\sigma \in \{0,1\}^{\mathbb{N}}$ there is $t \in L$ such that $\chi_{s_{\overline{Fr}(\kappa)}(e_{\alpha,n})}(t) = \sigma(n)$ for every $n \in \mathbb{N}$. Now use the standard fact that the mapping ϕ from $\{0,1\}^{\mathbb{N}}$ into I given by $\phi(\sigma) = \sum_{n\in\mathbb{N}} \frac{\sigma(n)}{2^n}$ is surjective. □

Lemma 3.7 *([24] Lemma 2.10). Each $f \in C(L)$ depends on some countable $A \subseteq \kappa$.*

Lemma 3.8. *Suppose that X, Y are compact spaces, $\phi : X \to Y$ is a continuous surjection and that $f \in C(X)$ is such that $f(x_1) = f(x_2)$ whenever $\phi(x_1) = \phi(x_2)$. Then there is a continuous $g \in C(Y)$ such that $f = g \circ \phi$. In particular, for every $f \in C(L)$ which depends on some $A \subseteq \kappa$ there exist $g \in C(L_A)$ such that $f = g \circ p_A$.*

Proof. By the hypothesis one can well define $g : Y \to \mathbb{R}$ satisfying $f = g \circ \phi$. Since ϕ is a closed onto mapping (2.4.8. of [8]) it is a quotient map and so g is continuous (2.4.2. of [8]). □

Lemma 3.9. *Suppose that X, Y are compact spaces $\phi\colon X \to Y$ is continuous and $Z \subseteq X$, then $f[\overline{Z}] = \overline{f[Z]}$.*

Proof. Continuous mappings of compact spaces are closed, so ϕ is a closed function (see 1.4.C of [8]). □

3.2. Algebras of functions and their Gelfand spaces

Given $\mathcal{F} \subseteq C_I(L)$ we will consider the closed algebra over the reals containing constant functions generated by $\mathcal{F}$ in $C(L)$, that is the real unital Banach algebra generated by $\mathcal{F}$, we will denote it by $[\mathcal{F}]$. $[\mathcal{F}]_I$ will stand for $[\mathcal{F}] \cap C_I(L)$.

The role of the Stone space for Boolean algebras is played for commutative Banach algebras by the Gelfand space. We will work with the following concrete representation $\nabla\mathcal{F}$ of the Gelfand space of $[\mathcal{F}]$ (cf. [24]):

Definition 3.10 *([24]).* Let $\mathcal{F}, \mathcal{G}$ be families of elements of $C_I(L)$.

(1) Define $\Pi\mathcal{F}\colon L \to I^{\mathcal{F}}$ by a formula

$$\big((\Pi\mathcal{F})(x)\big)(f) = f(x)$$

for all $x \in L$ and for all $f \in \mathcal{F}$,
(2) the image $\Pi\mathcal{F}[L] \subseteq I^{\mathcal{F}}$ is denoted by $\nabla\mathcal{F}$,
(3) for $\mathcal{G} \subseteq \mathcal{F} \subseteq C_I(L)$ we define the natural projection

$$\pi_{\mathcal{G},\mathcal{F}}\colon \nabla\mathcal{F} \to \nabla\mathcal{G},$$

which is the restriction of the natural projection from $I^{\mathcal{F}}$ to $I^{\mathcal{G}}$,
(4) given $\mathcal{F} \subseteq C_I(L)$ we say that $f \in C(\nabla\mathcal{F})$ depends on a set $A \subseteq \kappa$ if $f \circ \Pi\mathcal{F}$ depends on A.

Proposition 3.11. *Suppose that $\mathcal{F} \subseteq C_I(L)$. Then there is an isometric isomorphism of real Banach algebras $T_{\mathcal{F}} : C(\nabla\mathcal{F}) \to [\mathcal{F}]$ induced by $\Pi\mathcal{F}$ such that $T_{\mathcal{F}}(\pi_{\{f\},\mathcal{F}}) = f$ for each $f \in \mathcal{F}$.*

Proof. The surjective continuous function $\Pi\mathcal{F} : L \to \nabla F$ induces an isometric isomorphic embedding of $C(\nabla\mathcal{F})$ into $C(L)$ simply by sending $g \in C(\nabla\mathcal{F})$ to $g \circ \Pi\mathcal{F}$. Then it is clear that $T_{\mathcal{F}}(\pi_{\{f\},\mathcal{F}}) = f$, so the image of $T_{\mathcal{F}}$ includes $[\mathcal{F}]$. It remains to show that it is included in $[\mathcal{F}]$. For this it is enough to show that $C(\nabla\mathcal{F})$ is generated as a unital algebra by the functions $\pi_{\{f\},\mathcal{F}}$ for $f \in \mathcal{F}$. This follows from the real Stone–Weierstrass theorem, as the coordinates separate points in products. □

Lemma 3.12. *Suppose that $f \in [\mathcal{F}] \subseteq C_I(L)$. Then there is a unique $f(\mathcal{F})$ in $C(\nabla\mathcal{F})$ satisfying*

$$f = f(\mathcal{F}) \circ \Pi\mathcal{F}.$$

$f(\mathcal{F})$ will be called the factorization of f through $\mathcal{F}$. If $\mathcal{F} \subseteq \mathcal{G} \subseteq C_I(L)$, then $f(\mathcal{F}) \circ \pi_{\mathcal{F},\mathcal{G}} = f(\mathcal{G})$.

Proof. Put $f(\mathcal{F})$ to be $T_{\mathcal{F}}^{-1}(f)$ where $T_{\mathcal{F}}$ is the isometry from Proposition 3.11. The second part follows from the fact that $\pi_{\mathcal{F},\mathcal{G}} \circ \Pi\mathcal{G} = \Pi\mathcal{F}$ and the uniqueness of the factorization. □

Lemma 3.13. *Suppose that $\alpha \in \kappa$, then*

(1) if $\mathcal{F}$ depends on $\kappa\setminus\{\alpha\}$, then there is a homeomorphism $\phi : \nabla(\mathcal{F}\cup\{d_\alpha\}) \to (\nabla\mathcal{F})\times I$ such that $\pi \circ \phi = \pi_{\mathcal{F},\mathcal{F}\cup\{d_\alpha\}}$ where π is the natural projection from $(\nabla\mathcal{F}) \times I$ onto $\nabla\mathcal{F}$,
(2) $\nabla\mathcal{D} = I^\kappa$.

Proof. Fix $x \in I$ and $y \in \nabla\mathcal{F}$. Using Lemma 3.6 let $t \in L$ be such that $d_\alpha(t) = x$. Fix $s \in L$ such that $(\Pi\mathcal{F})(s) = y$. Use Lemma 3.3 to find $u \in L$ such that $p_{\{\alpha\}}(u) = p_{\{\alpha\}}(t)$ and $p_{\kappa\setminus\{\alpha\}}(u) = p_{\kappa\setminus\{\alpha\}}(s)$. Since d_α depends on α by Lemma 3.6 and $\mathcal{F}$ depends on $\kappa \setminus \{\alpha\}$ by the hypothesis, we obtain that $(\Pi(\mathcal{F} \cup \{d_\alpha\}))(u) = (y, x)$ which completes the proof of part (1). (2) follows from (1) applied inductively and Lemma 3.6. □

For a Banach space X a density character of X is a cardinality of a minimal dense subset of X and it is denoted by $d(X)$.

Lemma 3.14 *(GCH). Let $\mathcal{D} \subseteq \mathcal{F} \subseteq C_I(L)$. Then the density character of $C(\nabla\mathcal{F})$ equals κ.*

Proof. Using the surjections $\pi_{\mathcal{D},\mathcal{F}}$ and $\Pi\mathcal{F}$ we obtain isometric injection of $C(\nabla\mathcal{D})$ into $C(\nabla\mathcal{F})$ and $C(\nabla\mathcal{F})$ into $C(L)$, so $\kappa \leq d(C(\nabla\mathcal{F})) \leq d(C(L))$ by Lemma 3.13. On the other hand, by the Stone–Weierstrass theorem, the density of the Banach space $C(L)$ is not bigger than the cardinality of the Boolean algebra of clopen subsets of L which is isomorphic to the algebra $\overline{Fr}(\kappa)$, which is c.c.c. (Proposition 3.2) and contains a dense subalgebra $Fr(\kappa)$ of cardinality κ. So each element of $\overline{Fr}(\kappa)$ is the supremum of a countable subset of $Fr(\kappa)$, hence $|\overline{Fr}(\kappa)| \leq \kappa^\omega$. So, we obtain $d(C(L)) \leq \kappa^\omega = \kappa$ (see Lemma 5.1), which completes the proof. □

It turns out to be convenient to talk about open subsets of the Gelfand spaces $\nabla\mathcal{F}$ of the algebras $[\mathcal{F}]$ using a language purely depending on $\mathcal{F}$. The next definition is aiming at this purpose.

Definition 3.15. Suppose that $\mathcal{F} \subseteq \mathcal{G} \subseteq C_I(L)$. Let $\mathcal{J}$ denote the family of all nonempty open subintervals of $[0, 1]$ with rational endpoints. By $\mathcal{B}(\mathcal{F})$ we denote the family of all partial functions

$$U : dom(U) \to \mathcal{J},$$

where the domain $dom(U)$ of U is a finite subset of $\mathcal{F}$. We will consider the evaluation $U(\mathcal{G})$ of U at $\mathcal{G}$ which is defined as

$$U(\mathcal{G}) = \{x \in \nabla\mathcal{G} : x(f) \in U(f) \text{ for all } f \in dom(U)\}.$$

By $U(L)$ we will mean the set $\{x \in L : f(x) \in U(f) \text{ for all } f \in dom(U)\} = \bigcap_{f\in dom(U)} f^{-1}[U(f)]$.

Note that with the above notation $U(f)$ is the same subinterval of I as the one defined as $U(\{f\})$.

Lemma 3.16. *Suppose that $\mathcal{F} \subseteq \mathcal{G} \subseteq C_I(L)$ and $U, V \in \mathcal{B}(\mathcal{F})$. Then*

(1) The family of all sets of the form $W(\mathcal{F})$ for $W \in \mathcal{B}(\mathcal{F})$ forms a basis of open sets for $\nabla\mathcal{F}$,
(2) $(\Pi\mathcal{F})^{-1}[U(\mathcal{F})] = U(L)$,
(3) $\Pi\mathcal{F}[U(L)] = U(\mathcal{F})$,
(4) $\pi^{-1}_{\mathcal{F},\mathcal{G}}[U(\mathcal{F})] = U(\mathcal{G})$,
(5) $\pi_{\mathcal{F},\mathcal{G}}[U(\mathcal{G})] = U(\mathcal{F})$,
(6) $U(\mathcal{F}) \cap V(\mathcal{F}) = \emptyset$, if and only if $U(\mathcal{G}) \cap V(\mathcal{G}) = \emptyset$,
(7) $U(\mathcal{F}) \subseteq V(\mathcal{F})$, if and only if $U(\mathcal{G}) \subseteq V(\mathcal{G})$,
(8) $\overline{U(\mathcal{F})} \subseteq V(\mathcal{F})$, if and only if $\overline{U(\mathcal{G})} \subseteq V(\mathcal{G})$, where the closures are taken in $\nabla\mathcal{F}$ and $\nabla\mathcal{G}$ respectively.

Proof. The first item is clear from the definition of the product topology. Item (2) follows directly from Definition 3.15. Item (3) is the immediate consequence of (2). Item (4) follows from the fact that $\pi^{-1}_{\mathcal{F},\mathcal{G}}[X] = \Pi\mathcal{G}[(\Pi\mathcal{F})^{-1}[X]$ for any $X \subseteq \nabla\mathcal{F}$ and (2)–(3), namely $\pi^{-1}_{\mathcal{F},\mathcal{G}}[U(\mathcal{F})] = \Pi\mathcal{G}[(\Pi\mathcal{F})^{-1}[U(\mathcal{F})] = \Pi\mathcal{G}[U(L)] = U(\mathcal{G})$. Item (5) is the immediate consequence of (4). Items (6)–(7) are the immediate consequences of (4) and the properties of the preimages of functions. For the forward direction of (8), note that always $\pi^{-1}_{\mathcal{F},\mathcal{G}}[U(\mathcal{F})] \subseteq \pi^{-1}_{\mathcal{F},\mathcal{G}}[\overline{U(\mathcal{F})}]$ and apply (4). For the backward direction of (8), note that always $\overline{\pi_{\mathcal{F},\mathcal{G}}[U(\mathcal{G})]} = \pi_{\mathcal{F},\mathcal{G}}[\overline{U(\mathcal{G})}]$ by Lemma 3.9 and apply (5). □

Definition 3.17. Suppose that $\mathcal{F} \subseteq C_I(L)$. A family $\mathcal{U} \subseteq \mathcal{B}(\mathcal{F})$ is called an antichain if and only if $U(\mathcal{F}) \cap V(\mathcal{F}) = \emptyset$ for all $U, V \in \mathcal{U}$.

We see by Lemma 3.16 that the property of being of antichain does not change if we pass from $\mathcal{F}$ to a bigger $\mathcal{G}$. Despite of Lemma 3.16 a nontrivial interplay between properties $U(\mathcal{F})$s and $U(\mathcal{G})$s for $\mathcal{F} \subseteq \mathcal{G} \subseteq C_I(L)$ is possible and will actually be at the heart of the difficulties of the main construction of this paper. For example, as we want $\nabla\mathcal{F}$ to be connected, we would need $\Pi\mathcal{F}[U] \cap \Pi\mathcal{F}[L \setminus U] \neq \emptyset$ for any clopen $U \subseteq L$. In fact, the main properties of $\nabla\mathcal{F}$ for the main construction of $\mathcal{F}$ (Section 5) corresponding

to Definition 1.4 (b) are expressed in terms of the nonemptyness of the intersection $\overline{\bigcup_{n\in M} U_n(\mathcal{F})} \cap \overline{\bigcup_{n\in\mathbb{N}\setminus M} U_n(\mathcal{F})}$ for some antichain $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F})$ and $M \subseteq \mathbb{N}$, while $\overline{\bigcup_{n\in M} U_n(L)} \cap \overline{\bigcup_{n\in\mathbb{N}\setminus M} U_n(L)}$ is always empty, since L is extremally disconnected, which implies that it is empty if we replace $U_n(L)$s by $U_n(\mathcal{G})$s for sufficiently big $\mathcal{G}$. The following lemma is the first of a series of observations aiming at developing techniques of increasing the family $\mathcal{F}$ to a bigger $\mathcal{G}$ with preserving the nonemptyness of the intersections of the closures of unions as above.

Lemma 3.18. *Let $\mathcal{F} \subseteq C_I(L)$, and let $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F})$ be an antichain. Let $M \subseteq \mathbb{N}$ be such that there exists $x \in \nabla\mathcal{F}$ with*

$$x \in \overline{\bigcup_{n\in M} U_n(\mathcal{F})} \cap \overline{\bigcup_{n\in\mathbb{N}\setminus M} U_n(\mathcal{F})}.$$

Then there exist $s, t \in L$ such that

(1) $s \in \overline{\bigcup_{n\in M} U_n(L)}$ and $t \in \overline{\bigcup_{n\in\mathbb{N}\setminus M} U_n(L)}$, and
(2) $\Pi\mathcal{F}(s) = \Pi\mathcal{F}(t) = x$.

Moreover if $f \in C_I(L)$ is such that

(3) $f(s) = f(t)$ for any s, t satisfying (2),

then

$$\overline{\bigcup_{n\in M} U_n(\mathcal{G})} \cap \overline{\bigcup_{n\in\mathbb{N}\setminus M} U_n(\mathcal{G})} \neq \emptyset,$$

where $\mathcal{G} = \mathcal{F} \cup \{f\}$.

Proof. By Lemma 3.9 and Lemma 3.16 (3) we have

$$\Pi\mathcal{F}[\overline{\bigcup_{n\in M} U_n(L)}] = \overline{\Pi\mathcal{F}[\bigcup_{n\in M} U_n(L)]} = \overline{\bigcup_{n\in M} U_n(\mathcal{F})} \ni x,$$

so the existence of s, t as in (1)–(2) follows. For (3) again use Lemma 3.9 and Lemma 3.16 (3) to note that

$$\Pi(\mathcal{F}\cup\{f\})(s) \in \Pi(\mathcal{F}\cup\{f\})[\overline{\bigcup_{n\in M} U_n(L)}] = \overline{\Pi(\mathcal{F}\cup\{f\})[\bigcup_{n\in M} U_n(L)]} = \overline{\bigcup_{n\in M} U_n(\mathcal{G})},$$

and similarly $\Pi(\mathcal{F}\cup\{f\})(t) \in \overline{\bigcup_{n\in\mathbb{N}\setminus M} U_n(\mathcal{G})}$, which finishes the proof, since $\Pi(\mathcal{F}\cup\{f\})(t) = (x, f(t)) = (x, f(s)) = \Pi(\mathcal{F}\cup\{f\})(s)$ by (2) and by the hypothesis of (3). □

3.3. Adding suprema

If we pass from a smaller algebra of functions to a bigger one, a sequence of functions from the smaller one may acquire its supremum in the bigger one. This is quite natural when we work with subalgebras of an algebra where all bounded sequences have suprema, i.e., in the algebra of all continuous functions on an extremally disconnected compact space. However in general a function which is the supremum in a subalgebra may no longer be the supremum in a bigger algebra. In this section we recall and prove lemmas describing the above phenomena in the context we are interested in. One should note that adding suprema in a careful way while extending algebras is one of the main techniques behind our construction in Section 5 as the main property of the required Banach space is expressed in terms of the suprema in Definition 1.4.

Definition 3.19. Suppose that K is a compact Hausdorff space. For a pairwise disjoint sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_I(K)$ and a function $f \in C_I(K)$ we define the set

$$\Delta\big(f,(f_n)_{n\in\mathbb{N}}\big) = \{x \in K \mid f(x) \neq \sum_{n\in\mathbb{N}} f_n(x)\}.$$

Lemma 3.20 *(4.1.(a) [21]). Let K be a compact Hausdorff space. A function $f \in C_I(K)$ is the supremum in $C(K)$ of a pairwise disjoint sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_I(K)$ if and only if the set $\Delta\big(f,(f_n)_{n\in\mathbb{N}}\big)$ is nowhere dense.*

Lemma 3.21. *Let K be a compact Hausdorff space. For a pairwise disjoint sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_I(K)$ the following set*

$$D((f_n)_{n\in\mathbb{N}}) = \bigcup\{U \mid U \text{ is open in } K \text{ and } \{n \in \mathbb{N}\colon U \cap \operatorname{supp}(f_n) \neq \emptyset\} \text{ is finite}\}$$

is dense open and for $f = \bigvee_{n\in\mathbb{N}} f_n$ in $C(K)$ we have

$$f \restriction D((f_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} f_n \restriction D((f_n)_{n\in\mathbb{N}}).$$

Proof. The first part is the first part of 4.1. (b) of [21]. The second part follows from the second part of Lemma 4.1. (b) which says that $\sum_{n\in\mathbb{N}} f_n$ is continuous on the open set $D((f_n)_{n\in\mathbb{N}})$, from the fact that two distinct continuous function differ on an open set and from Lemma 3.20. □

Lemma 3.22. *Let K be a compact Hausdorff space. Suppose that $(f_n)_{n\in\mathbb{N}} \subseteq C_I(K)$ is an antichain and $f = \bigvee_{n\in\mathbb{N}} f_n$ in $C(K)$. Then*

$$\operatorname{supp}(f) \subseteq \overline{\bigcup_{n\in\mathbb{N}} \operatorname{supp}(f_n)}.$$

Proof. The set

$$\operatorname{supp}(f) \setminus \overline{\bigcup_{n\in\mathbb{N}} \operatorname{supp}(f_n)} \subseteq \Delta\big(f, (f_n)_{n\in\mathbb{N}}\big)$$

is open. If it was nonempty, it would contradict Lemma 3.20. □

In general, when passing from $C(\nabla\mathcal{F})$ to $C(\nabla\mathcal{G})$ for $\mathcal{F} \subseteq \mathcal{G} \subseteq C_I(L)$ the supremum f of a pairwise disjoint sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $C(\nabla\mathcal{F})$ may no longer be its supremum in $C(\nabla\mathcal{G})$, i.e., $f\circ\pi_{\mathcal{F},\mathcal{G}}$ may not be the supremum of $(f_n\circ\pi_{\mathcal{F},\mathcal{G}})_{n\in\mathbb{N}}$. However, if we use the supremum of $(f_n\circ\Pi\mathcal{F})_{n\in\mathbb{N}}$ in $C(L)$, this will not happen as stated in the following:

Lemma 3.23. *Suppose that $\mathcal{F} \subseteq C_I(L)$ and $(f_n)_{n\in\mathbb{N}} \subseteq [\mathcal{F}]$ is a pairwise disjoint sequence of functions, and let $f = \bigvee_{n\in\mathbb{N}} f_n$ in $C(L)$. Then for every $\mathcal{G} \subseteq C_I(L)$ such that $\mathcal{F} \cup \{f\} \subseteq \mathcal{G}$ we have that the factorization $f(\mathcal{G})$ of f is the supremum of the factorizations $(f_n(\mathcal{G}))_{n\in\mathbb{N}}$ in $C_I(\nabla\mathcal{G})$.*

Proof. Use Definition 5.11 and Lemma 5.12 of [24] and the isometric isomorphism between $[\mathcal{F}]$ and $\nabla\mathcal{F}$ from Proposition 3.11 and Lemma 3.12. □

Lemma 3.24. *Let $A \subseteq \kappa$. Suppose that $(f_n)_{n\in\mathbb{N}} \subseteq C_I(L)$ is pairwise disjoint sequence of functions which all depend on A. Then the supremum $\bigvee_{n\in\mathbb{N}} f_n$ in $C(L)$ depends on A.*

Proof. Let f'_n be such a function in $C(L_A)$ that $f'_n\circ p_A = f_n$ for $n \in \mathbb{N}$. Its existence follows from Lemma 3.8. It is clear that f'_ns are pairwise disjoint as well. As L_A is extremally disconnected (Proposition 3.2), we can take the suprema of bounded sequences in $C(L_A)$. So let $g = \bigvee_{n\in\mathbb{N}} f'_n$, where the supremum is taken in $C(L_A)$. It is clear that $g\circ p_A$ depends on A. So it is enough to prove that $g\circ p_A$ is the supremum $\bigvee_{n\in\mathbb{N}} f_n$ in $C(L)$. Recall Definition 3.19 and let $X = \Delta(g, (f'_n)_{n\in\mathbb{N}})$. It is clear that $Y = p_A^{-1}[X] = \Delta(g\circ p_A, (f_n)_{n\in\mathbb{N}})$. So by Lemma 3.20 it is enough to prove that preimages of nowhere dense sets under p_A are nowhere dense, or that images of open sets under p_A have nonempty interior.

As $Fr(\kappa)$ is a dense subalgebra of $\overline{Fr}(\kappa)$, it is enough to prove that $p_A[s_{\overline{Fr}(\kappa)}(a)]$ has a nonempty interior in L_A for any $a \in Fr(\kappa)$ (see Proposition 3.2). But by the independence of the generators of $Fr(\kappa)$ such an a is a finite sum of elements of the form $a' \wedge a''$ where $a' \in Fr(A)$ and $a'' \in Fr(\kappa \setminus A)$. Moreover $p_A[s_{\overline{Fr}(\kappa)}(a' \wedge a'')] = s_{\overline{Fr}(A)}(a')$ by the definition of the Stone functor and the fact that any ultrafilter in $\overline{Fr}(A)$ which contains a' can be extended to one in $\overline{Fr}(\kappa)$ which contains $a' \wedge a''$. It follows that images of open sets under p_A have nonempty interior which completes the proof. □

Lemma 3.25. *Let $\mathcal{F} \subseteq C_I(L)$, $(f_n)_{n\in\mathbb{N}} \subseteq [\mathcal{F}]$ be pairwise disjoint, $(\nu_n)_{n\in\mathbb{N}} \subseteq \{\pm 1\}$, $(\eta_n)_{n\in\mathbb{N}} \subseteq \kappa$ be increasing with $d_{\eta_n} \in \mathcal{F}$ for all $n \in \mathbb{N}$ and let $f = \bigvee_{n\in\mathbb{N}}(f_n d_{\nu_n,\eta_n})$ in $C_I(L)$. Then*

$$f \restriction \bigcup_{n\in\mathbb{N}} \operatorname{supp}(f_n) = \sum_{n\in\mathbb{N}} (f_n d_{\nu_n,\eta_n}) \restriction \bigcup_{n\in\mathbb{N}} \operatorname{supp}(f_n).$$

Proof. Note that $\operatorname{supp}(f_n) \subseteq D((f_n d_{\nu_n,\eta_n})_{n\in\mathbb{N}})$ and so the second part of the Lemma 3.21 may be used. □

3.4. Extensions and preserving the connectedness

For sets X, Y and a function $f\colon X \to Y$ let us denote the graph of f by $\Gamma(f)$.

Definition 3.26 *(4.2. [21]).* Let K be a compact Hausdorff space and let $(f_n)_{n\in\mathbb{N}} \subseteq C_I(K)$ be pairwise disjoint. Then the closure of $\Gamma\big(\sum_{n\in\mathbb{N}} f_n \restriction D((f_n)_{n\in\mathbb{N}})\big)$ in $K \times I$ is called an extension of K by $(f_n)_{n\in\mathbb{N}}$. We denote it by $\operatorname{ext}(K, (f_n)_{n\in\mathbb{N}})$. The extension is called strong if $\Gamma\big(\sum_{n\in\mathbb{N}} f_n\big) \subseteq \operatorname{ext}(K, (f_n)_{n\in\mathbb{N}})$. If $K = \nabla\mathcal{F}$ for some $\mathcal{F} \subseteq C_I(K)$ and $(f_n)_{n\in\mathbb{N}} \subseteq [\mathcal{F}]$, then an extension of $\nabla\mathcal{F}$ by $(f_n)_{n\in\mathbb{N}}$ means the extension of $\nabla\mathcal{F}$ by $(f_n(\mathcal{F}))_{n\in\mathbb{N}}$ and is denoted $\operatorname{ext}(\nabla\mathcal{F}, (f_n)_{n\in\mathbb{N}})$.

Indiscriminate adding of suprema leads to a complete lattice $C(K)$ and implies that K is extremally disconnected, so in general extensions of compact spaces do not need to preserve the connectedness (for explicit analysis of this phenomenon in the case of pairwise disjoint sequences of functions see [4]), however we have the following:

Lemma 3.27 *([21] Lemma 4.4.). Let K be a compact and connected Hausdorff space and let $(f_n)_{n\in\mathbb{N}} \subseteq C_I(K)$ be pairwise disjoint. The strong extension of K by $(f_n)_{n\in\mathbb{N}}$ is a compact and connected space.*

Lemma 3.28. *Let $\mathcal{F} \subseteq C_I(L)$ and $A \subseteq \kappa$ be such that the family $\mathcal{F}$ depends on A and $\{d_\alpha \mid \alpha \in A\} \subseteq \mathcal{F}$. Let $(f_n)_{n\in\mathbb{N}} \subseteq [\mathcal{F}]$ be pairwise disjoint and let $f \in C_I(L)$ be the supremum of $(f_n)_{n\in\mathbb{N}}$. Then $\operatorname{ext}(\nabla\mathcal{F}, (f_n)_{n\in\mathbb{N}}) = \nabla(\mathcal{F} \cup \{f\})$.*

Proof. Use Lemma 5.13. of [24] and Lemma 3.12. □

Lemma 3.29 *([24] Lemma 2.7.). Let $\mathcal{F} \subseteq C(L)$. Then $\nabla\mathcal{F}$ is connected if and only if ∇F is connected for all finite $F \subseteq \mathcal{F}$.*

Lemma 3.30. *Let*

(1) $\mathcal{F} \subseteq C_I(L)$,
(2) $(f_n)_{n\in\mathbb{N}} \subseteq [\mathcal{F}]_I$ be an pairwise disjoint,
(3) $\mathcal{F}$ depends on $A \subseteq \kappa$,
(4) $(\eta_n)_{n\in\mathbb{N}} \subseteq \kappa$ be such that the set $\{n \in \mathbb{N} \mid \eta_n \in A\}$ is finite,
(5) $(\nu_n)_{n\in\mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$.

Then for all infinite $M \subseteq \mathbb{N}$ and for $\mathcal{G} = \mathcal{F} \cup \{d_{\eta_n} \mid n \in \mathbb{N}\}$ the extension of $\nabla\mathcal{G}$ by $(f_n d_{\nu_n,\eta_n})_{n\in M}$ is strong. Moreover, if $\nabla\mathcal{G}$ is compact and connected, then $\operatorname{ext}(\nabla\mathcal{G}, (f_n d_{\nu_n,\eta_n})_{n\in M})$ is compact and connected as well.

Proof. Fix some infinite $M \subseteq \mathbb{N}$ and $(x,s) \in \Gamma(\sum_{n\in M}(f_n d_{\nu_n,\eta_n})(\mathcal{G}))$. We need to check that

$$(x,s) \in \overline{\Gamma\big(\sum_{n\in M}(f_n d_{\nu_n,\eta_n})(\mathcal{G}) \restriction D(((f_n d_{\nu_n,\eta_n})(\mathcal{G}))_{n\in M})\big)}.$$

If $s > 0$, then $x \in D(((f_n d_{\nu_n,\eta_n})(\mathcal{G}))_{n\in M})$, so we can assume that $s = 0$. Fix a neighbourhood of $(x,0)$ of the form $U(\mathcal{G}) \times (-\varepsilon,\varepsilon)$ where $U \in \mathcal{B}(\mathcal{G})$ and $\varepsilon > 0$. It will be sufficient to show that

$$(U(\mathcal{G}) \times (-\varepsilon,\varepsilon)) \cap \Gamma\big(\sum_{n\in M}(f_n d_{\nu_n,\eta_n})(\mathcal{G}) \restriction D(((f_n d_{\nu_n,\eta_n})(\mathcal{G}))_{n\in M})\big) \neq \emptyset.$$

If $U(\mathcal{G})$ intersects only finitely many sets $\mathrm{supp}((f_n d_{\nu_n,\eta_n})(\mathcal{G}))$, then we have the inclusion $U(\mathcal{G}) \subseteq D(((f_n d_{\nu_n,\eta_n})(\mathcal{G}))_{n\in M})$ and so the point $(x,0)$ belongs to the graph $\Gamma\big(\sum_{n\in M}(f_n d_{\nu_n,\eta_n})(\mathcal{F}) \restriction D(((f_n d_{\nu_n,\eta_n})(\mathcal{G}))_{n\in M})\big)$.

If $U(\mathcal{G})$ intersects $\mathrm{supp}((f_n d_{\nu_n,\eta_n})(\mathcal{G}))$ for infinitely many $n \in M$ then, by the hypothesis of the lemma, we can pick a number $n_0 \in M$ such that $\eta_{n_0} \notin A \cup B$ where $B = \{\eta_n \mid d_{\eta_n} \in dom(U)\}$, and there is an $x \in U(\mathcal{G}) \cap \mathrm{supp}((f_{n_0} d_{\nu_{n_0},\eta_{n_0}})(\mathcal{G}))$. In particular $f_{n_0}(\mathcal{G})(x) \neq 0$. Let $u \in L$ be such that $\Pi\mathcal{G}(u) = x$. Let $v \in L$ be such that

$$0 < d_{\nu_{n_0},\eta_{n_0}}(v) < \varepsilon / f_{n_0}(\mathcal{G})(x), \tag{1}$$

which exists by Lemma 3.6 (2). By Lemma 3.3 there is $t \in L$ such that $p_{A\cup B}(t) = p_{A\cup B}(u)$ and $p_{\{\eta_{n_0}\}}(t) = p_{\{\eta_{n_0}\}}(v)$. Put $y = \Pi\mathcal{G}(t)$. It follows that $f(t) = f(u)$ for every $f \in dom(U)$ and so

$$y \in U(\mathcal{G}), \tag{2}$$

since $\Pi\mathcal{G}(u) = x \in U(\mathcal{G})$. Also by the dependence of $\mathcal{F}$ by A, Lemma 3.12 and Lemma 3.6 we have

$$f_{n_0}(\mathcal{G})(y) = f_{n_0}(t) = f_{n_0}(u) = f_{n_0}(\mathcal{G})(x) \neq 0, \tag{3}$$

and $d_{\nu_{n_0},\eta_{n_0}}(\mathcal{G})(y) = d_{\nu_{n_0},\eta_{n_0}}(t) = d_{\nu_{n_0},\eta_{n_0}}(v)$ and so by (1) and (3) we have

$$0 < (f_{n_0} d_{\nu_{n_0},\eta_{n_0}})(\mathcal{G})(y) < \varepsilon. \tag{4}$$

The first inequality of (4) implies that $y \in D(((f_n d_{\nu_n,\eta_n})(\mathcal{G}))_{n\in M})$ and so that $(y, f_{n_0} d_{\nu_{n_0},\eta_{n_0}})(\mathcal{G})(y))$ is in $\Gamma\big(\sum_{n\in M} f_n d_{\nu_n,\eta_n} \restriction D((f_n d_{\nu_n,\eta_n})_{n\in M})$, while (2) and the second inequality of (4) imply that it is in $U(\mathcal{G}) \times (-\varepsilon,\varepsilon)$ which completes the proof of the first part of the lemma. The moreover part follows from Lemma 3.27. □

3.5. Separations

Definition 1.4 requires us in (a) to have many suprema while maintaining the disjointness of certain closures in (b). In this sections we prove lemmas concerning the disjointness of these closures.

Definition 3.31. Let $\mathcal{F} \subseteq C_I(L)$. We say that an antichain $(U_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{B}(\mathcal{F})$ is separated along M in $[\mathcal{F}]$ iff

$$\overline{\bigcup_{n\in M} U_n(\mathcal{F})} \cap \overline{\bigcup_{n\notin M} U_n(\mathcal{F})} = \emptyset,$$

where the closures are taken in $\nabla\mathcal{F}$.

Lemma 3.32. *Let $\mathcal{F} \subseteq C_I(L)$ and let $(U_n)_{n\in\mathbb{N}}$ be an antichain of elements of $\mathcal{B}(\mathcal{F})$. Then $(U_n)_{n\in\mathbb{N}}$ is separated along M in $[\mathcal{F}]$ if and only if there are elements $\{V_j \mid j \in J\}, \{V'_j \mid j \in J'\} \subseteq \mathcal{B}(\mathcal{F})$, with J and J' finite, such that for $V(\mathcal{F}) = \bigcup_{j\in J} V_j(\mathcal{F})$ and $V'(\mathcal{F}) = \bigcup_{j\in J'} V'_j(\mathcal{F})$ and we have*

$$\bigcup_{n\in M} U_n(\mathcal{F}) \subseteq V(\mathcal{F}) \subseteq \overline{V(\mathcal{F})} \subseteq V'(\mathcal{F}) \subseteq \nabla\mathcal{F} \setminus \big(\bigcup_{n\notin M} U_n(\mathcal{F})\big).$$

Proof. It follows directly from the normality of the compact space ∇F and from Lemma 3.16 (1). □

Note that by Lemma 3.16 the above condition from Lemma 3.32 is preserved if we pass from $\mathcal{F}$ to a bigger $\mathcal{G}$. However if an antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along $M \subseteq \mathbb{N}$ in $[\mathcal{F}]$, it may become separated along $M \subseteq \mathbb{N}$ in $[\mathcal{G}]$.

Lemma 3.33. *Let $\mathcal{F} \subseteq \mathcal{G} \subseteq C_I(L)$ and let $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F})$ be an antichain, $M \subseteq \mathbb{N}$. Then the antichain $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{G})$ is separated along M in $[\mathcal{G}]$ if and only if it is separated in $[\mathcal{F} \cup \mathcal{H}]$ for some finite $\mathcal{H} \subseteq \mathcal{G}$.*

Proof. Suppose that $(U_n)_{n\in\mathbb{N}}$ is separated along M in $[\mathcal{G}]$ and let $V, V', \{V_j \mid j \in J\}, \{V'_j \mid j \in J'\}$ be as in Lemma 3.32. Let $\mathcal{H} \subseteq \mathcal{G}$ be a finite set including domains of all V_j for $j \in J$ and all V'_j for all $j \in J'$. Now use Lemma 3.16 (7) and (8). □

Lemma 3.34. *Let $\mathcal{F} \subseteq C_I(L)$ be countable and let $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F})$ be an antichain. Then there is a set $M \subseteq \mathbb{N}$ such that the antichain $(U_n(\mathcal{F}))_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}]$.*

Proof. The separation of $(U_n)_{n\in\mathbb{N}}$ along M would yield finite sets $\{V_j^M : j \in J\}, \{(V'_j)^M : j \in J'\} \subseteq \mathcal{B}(\mathcal{F})$ as in Lemma 3.32. For distinct $M_1, M_2 \subseteq N$, these finite families must

be distinct. However there are continuum many subsets of $\mathbb{N}$ while $\mathcal{B}(\mathcal{F})$ is countable for countable $\mathcal{F}$. □

Definition 3.35. Let $\mathcal{F} \subseteq \mathcal{G} \subseteq C_I(L)$ and let $(U_n)_{n\in\mathbb{N}}$ be an antichain in $\mathcal{B}(\mathcal{F})$. We say that $\mathcal{F}$ is separating for $(U_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$, if whenever $(U_n(\mathcal{G}))_{n\in\mathbb{N}}$ is separated along some $M \subseteq \mathbb{N}$ in $[\mathcal{G}]$, then already $(U_n(\mathcal{F}))_{n\in\mathbb{N}}$ is separated along M in $[\mathcal{F}]$.

Lemma 3.36. *Let $\mathcal{F} \subseteq \mathcal{G} \subseteq C_I(L)$ and let $(U_n)_{n\in\mathbb{N}}$ be an antichain in $\mathcal{B}(\mathcal{F})$ such that $\mathcal{F}$ is separating for $(U_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$. Suppose that $(f_n)_{n\in\mathbb{N}} \subseteq C_I(L)$ is an antichain such that $U_n(L) \cap \operatorname{supp}(f_m) = \emptyset$ for every $n, m \in \mathbb{N}$ and let $f \in C_I(L)$ be the supremum of $(f_n)_{n\in\mathbb{N}}$ in $C(L)$. Then $\mathcal{F}$ is separating for $(U_n)_{n\in\mathbb{N}}$ in $\mathcal{G} \cup \{f\}$.*

Proof. Fix $M \subseteq \mathbb{N}$ and suppose that the antichain $(U_n(\mathcal{F}))_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}]$. Then the hypothesis of the lemma guarantees that $(U_n(\mathcal{G}))_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{G}]$ and so we can pick $x \in \overline{\bigcup_{n\in M} U_n(\mathcal{G})} \cap \overline{\bigcup_{n\in\mathbb{N}\setminus M} U_n(\mathcal{G})} \subseteq \nabla\mathcal{G}$. Now it is enough to find appropriate $s, t \in L$ and use Lemma 3.18.

By the hypothesis we have for every $P \subseteq \mathbb{N}$:

$$\bigcup_{n\in P} U_n(L) \cap \bigcup_{n\in\mathbb{N}} \operatorname{supp}(f_n) = \emptyset,$$

but the space L is extremally disconnected, hence for every $P \subseteq \mathbb{N}$ we have

$$\overline{\bigcup_{n\in P} U_n(L)} \cap \overline{\bigcup_{n\in\mathbb{N}} \operatorname{supp}(f_n)} = \emptyset. \tag{$*$}$$

By Lemma 3.18 (1)–(2) there are $s \in \overline{\bigcup_{n\in M} U_n(L)}$ such that $(\Pi\mathcal{G})(s) = x$ and $t \in \overline{\bigcup_{n\in\mathbb{N}\setminus M} U_n(L)}$ such that $(\Pi\mathcal{G})(t) = x$. By $(*)$ we have that $s, t \notin \overline{\bigcup_{n\in\mathbb{N}} \operatorname{supp}(f_n)}$, so by Lemma 3.22 we have $s, t \notin \operatorname{supp}(f)$, which means exactly that $f(s) = f(t) = 0$ so an application of Lemma 3.18 (3) completes the proof. □

Lemma 3.37. *Let $A \subseteq \kappa$, $\alpha \in \kappa \setminus A$ and suppose that $\mathcal{F} \subseteq C_I(L)$ depends on A and $(U_n)_{n\in\mathbb{N}}$ is an antichain in $\mathcal{B}(\mathcal{F})$. Then $\mathcal{F}$ is separating for $(U_n)_{n\in\mathbb{N}}$ in $\mathcal{F} \cup \{d_\alpha\}$.*

Proof. By Lemma 3.13 (1) the sets $U_n(\mathcal{F} \cup \{d_\alpha\})$ correspond to $U_n(\mathcal{F}) \times I$, so the lemma follows. □

Lemma 3.38. *Suppose that we are given*

(1) $A \subseteq \kappa$,
(2) $\mathcal{F} \subseteq C_I(L)$ which depends on A with $\{d_\alpha \mid \alpha \in A\} \subseteq \mathcal{F}$,
(3) an antichain $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F})$,
(4) a pairwise disjoint $(f_n)_{n\in\mathbb{N}} \subseteq [\mathcal{F}]_I$,

(5) $(\nu_n)_{n\in\mathbb{N}} \subseteq \{\pm 1\}$,
(6) a strictly increasing $(\eta_n)_{n\in\mathbb{N}} \subseteq \kappa$ *with the set* $\{n \in \mathbb{N} \mid \eta_n \in A\}$ *finite.*

Let

$$f = \bigvee_{n\in\mathbb{N}} f_n d_{\nu_n,\eta_n}$$

in $C_I(L)$. *Then* $\mathcal{G}$ *is separating for* $(U_n)_{n\in\mathbb{N}}$ *in* $\mathcal{G}\cup\{f\}$ *where* $\mathcal{G} = \mathcal{F}\cup\{d_{\eta_n} \mid n\in\mathbb{N}\}$.

Proof. Fix $M \subseteq \mathbb{N}$ such that the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{G}]$, we will show that $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{G}\cup\{f\}]$. Let $F \subseteq \mathbb{N}$ be the finite set of all n's such that $\eta_n \in A$ and let $\mathcal{H} = \mathcal{F}\cup\{\eta_n : n\in F\} \subseteq \mathcal{G}$. Lemma 3.33 implies that $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{H}]$. Hence there is $x \in \overline{\bigcup_{n\in M} U_n(\mathcal{H})} \cap \overline{\bigcup_{n\in\mathbb{N}\setminus M} U_n(\mathcal{H})} \subseteq \nabla\mathcal{H}$. Using Lemma 3.18 (1)–(2) fix $t \in \overline{\bigcup_{n\in M} U_n(L)}$ and $s \in \overline{\bigcup_{n\in\mathbb{N}\setminus M} U_n(L)}$ such that $\Pi\mathcal{H}(t) = \Pi\mathcal{H}(s) = x$.

As $\mathcal{H}$ depends on A we have $\Pi\mathcal{H}(t') = \Pi\mathcal{H}(s') = x$ for any $t', s' \in L$ such that $p_A(t') = p_A(t)$ and $p_A(s') = p_A(s)$. Using this observation and inductively Lemma 3.6 and Lemma 3.3 we may assume that $d_{\nu_n,\eta_n}(s) = d_{\nu_n,\eta_n}(t) = 0$ for all $n \in \mathbb{N}\setminus F$. Inductive application of Lemma 3.18 and later Lemma 3.33 implies that $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{G}\cup\{g\}]$ where

$$g = \bigvee_{n\in\mathbb{N}\setminus F} f_n d_{\nu_n,\eta_n}$$

in $C_I(L)$. So it is enough to show that there is a continuous surjection $\phi : \nabla(\mathcal{G}\cup\{g\}) \to \nabla(\mathcal{G}\cup\{f\})$ such that $\phi[U(\mathcal{G}\cup\{g\})] = U(\mathcal{G}\cup\{f\})$ for every $U \in \mathcal{B}(\mathcal{G})$. For this it is enough to have a continuous surjection $\psi : \nabla(\mathcal{G}\cup\{g\}) \to \nabla(\mathcal{G}\cup\{f,g\})$ such that $\phi[U(\mathcal{G}\cup\{g\})] = U(\mathcal{G}\cup\{f,g\})$ for every $U \in \mathcal{B}(\mathcal{F})$ since then we can consider $\phi = \pi_{\mathcal{G}\cup\{f\},\mathcal{G}\cup\{f,g\}} \circ \psi$ and Lemma 3.16 (5). To get ψ note that f is the composition of $\Pi(\mathcal{G}\cup\{g\})$ with the sum h of two continuous functions on $\nabla(\mathcal{G}\cup\{g\})$ namely $g(\mathcal{G}\cup\{g\})$ and $\Sigma_{n\in F}((f_n d_{\nu_n,\eta_n})(\mathcal{G}\cup\{g\}))$, so $\psi(x) = (x, h(x))$ works. □

3.6. Butterfly points

Recall Definition 1.1 (4) of a butterfly point and Theorem 1.2. Thus, if we want to construct a space $C(K)$ with few operators, we would like to obtain K with no butterfly points. The purpose of this subsection is to prove a lemma which provides a sufficient condition for the nonexistence of butterfly points.

Lemma 3.39. *Let* $\mathcal{F} \subseteq C_I(L)$ *and let* $U \subseteq \nabla\mathcal{F}$ *be open. Then there is a countable subset* $\mathcal{F}_0 \subseteq \mathcal{F}$ *and an antichain* $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F}_0)$ *such that*

$$\overline{U} = \overline{\bigcup_{n\in\mathbb{N}} U_n(\mathcal{F})} \text{ in } \nabla\mathcal{F}.$$

Proof. Let $\mathcal{U}$ be a maximal with respect to inclusion subfamily of $\mathcal{B}(\mathcal{F})$ such that

- $V(\mathcal{F}) \subseteq U$ for all $V \in \mathcal{U}$,
- $V(\mathcal{F}) \cap W(\mathcal{F}) = \emptyset$ for any two distinct $V, W \in \mathcal{U}$.

Note that $\mathcal{U}$ is countable since ∇F satisfies the c.c.c. as a continuous image of L and by Proposition 3.2. The maximality together with Lemma 3.16 (1) gives $\overline{U} = \overline{\bigcup_{n\in\mathbb{N}} U_n(\mathcal{F})}$. □

Recall the notion of a butterfly point from Definition 1.1.

Lemma 3.40. *Let $\mathcal{D} = \{d_\alpha : \alpha < \kappa\} \subseteq \mathcal{F} \subseteq C_I(L)$ be of the form $\mathcal{F} = \bigcup_{\alpha<\kappa} \mathcal{F}_\alpha$ for $\mathcal{F}_\alpha$s satisfying $\mathcal{F}_{\alpha'} \subseteq \mathcal{F}_\alpha$, $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{d_\alpha\}$, $\mathcal{F}_\alpha$ depends on α and $\mathcal{F}_\alpha \cup \{d_\alpha\}$ is separating in $\mathcal{F}$ for every antichain in $\mathcal{B}(\mathcal{F}_{\alpha'})$ for each $\alpha' < \alpha < \kappa$. Then $\nabla\mathcal{F}$ has no butterfly points.*

Proof. Fix two disjoint open set $U, V \subset \nabla\mathcal{F}$ such that there exists $x \in \overline{U} \cap \overline{V}$. We will show that $\overline{U} \cap \overline{V}$ contains at least two distinct points. By Lemmas 3.39 and 3.7 there exist countable sets $A \subseteq \kappa$ and $\mathcal{G} \subseteq \mathcal{F}$ and antichains $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{G})$ and $(V_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{G})$ such that $\mathcal{G}$ depends on A and

$$\overline{U} = \overline{\bigcup_{n\in\mathbb{N}} U_n(\mathcal{F})} \text{ and } \overline{V} = \overline{\bigcup_{n\in\mathbb{N}} V_n(\mathcal{F})} \text{ in } \nabla\mathcal{F}.$$

Using the regularity of κ we see that there exists $\alpha < \kappa$ such that $A \subseteq \alpha$ and $\mathcal{G} \subseteq \mathcal{F}_\alpha$

By Lemma 3.16 (6) we have that $\pi_{\mathcal{F}_\alpha,\mathcal{F}}[U_n(\mathcal{F})] = U_n(\mathcal{F}_\alpha)$ and $\pi_{\mathcal{F}_\alpha,\mathcal{F}}[V_n(\mathcal{F})] = V_n(\mathcal{F}_\alpha)$, so $x' \in \overline{\bigcup_{n\in\mathbb{N}} U_n(\mathcal{F}_\alpha)} \cap \overline{\bigcup_{n\in\mathbb{N}} V_n(\mathcal{F}_\alpha)}$ where $x' = \pi_{\mathcal{F}_\alpha,\mathcal{F}}(x)$. It is clear that in $\nabla\mathcal{F}_\alpha \times [0,1]$ we have $(x', \frac{u+v}{2}) \in \overline{\bigcup_{n\in\mathbb{N}} U_n(\mathcal{F}_\alpha) \times (u,v)} \cap \overline{\bigcup_{n\in\mathbb{N}} V_n(\mathcal{F}_\alpha) \times (u,v)}$ for any $0 < u < v < 1$.

Now define $W^{u,v}_{2n} = U_n \cup \{\langle d_\alpha, (u,v)\rangle\}$ in $\mathcal{B}(\mathcal{F}_{\alpha+1})$ and $W^{u,v}_{2n+1} = V_n \cup \{\langle d_\alpha, (u,v)\rangle\}$ in $\mathcal{B}(\mathcal{F}_{\alpha+1})$ for all $n \in \mathbb{N}$. By the above observation and Lemma 3.13 (1), for every $0 < u < v < 1$ the sequence $(W^{u,v}_n)_{n\in\mathbb{N}}$ is an antichain in $\mathcal{B}(\mathcal{F}_{\alpha+1})$ which is not separated along $2\mathbb{N}$ in $\nabla(\mathcal{F}_{\alpha+1})$. By Lemma 3.37 and the hypothesis that $\mathcal{F}_{\alpha+1}$ depends on $\alpha+1$ the sequence $(W^{u,v}_n)_{n\in\mathbb{N}}$ is an antichain in $\mathcal{B}(\mathcal{F}_{\alpha+1})$ which is not separated along $2\mathbb{N}$ in $\nabla(\mathcal{F}_{\alpha+2})$. Hence by the hypothesis that $\mathcal{F}_{\alpha+2}$ is separating in $\mathcal{F}$ for every antichain in $\mathcal{F}_{\alpha+1}$, the antichain $(W^{u,v}_n)_{n\in\mathbb{N}}$ is not separated along $2\mathbb{N}$ in $\nabla\mathcal{F}$.

Now let $X = \langle d_\alpha, (0, 1/3)\rangle$ and $Y = \langle d_\alpha, (2/3, 1)\rangle$ be elements of $\mathcal{B}(\mathcal{F})$. We see that $W^{0,1/3}_n(\mathcal{F}) \subseteq X(\mathcal{F})$ as well as $W^{2/3,1}_n(\mathcal{F}) \subseteq Y(\mathcal{F})$ while $\overline{X(\mathcal{F})} \cap \overline{Y(\mathcal{F})} = \emptyset$ which shows that the points witnessing the nonseparation of $(W^{0,1/3}_n(\mathcal{F}))_{n\in\mathbb{N}}$ and $(W^{2/3,1}_n(\mathcal{F}))_{n\in\mathbb{N}}$ along $2\mathbb{N}$ in $\nabla\mathcal{F}$ must be distinct.

On the other hand we have $W^{u,v}_{2n}(\mathcal{F}) \subseteq U_n(\mathcal{F})$ and $W^{u,v}_{2n+1}(\mathcal{F}) \subseteq V_n(\mathcal{F})$ for any $0 < u < v < 1$ and any $n \in \mathbb{N}$ which shows that these two distinct points must belong to $\overline{\bigcup_{n\in\mathbb{N}} U_n(\mathcal{F})} \cap \overline{\bigcup_{n\in\mathbb{N}} V_n(\mathcal{F})}$ and consequently to $\overline{U} \cap \overline{V}$ which completes the proof. □

4. Ladder families

This section is devoted to a particular type of construction of a family of functions $\mathcal{F} \subseteq C_I(L_\lambda)$ which we call a ladder family. The construction depends, among others, on a choice of "ladders" i.e., sequences $(\eta^\alpha_n)_{n\in\mathbb{N}} \subseteq \alpha$ increasingly convergent to α for each $\alpha \in S \subseteq E^\lambda_\omega$ where E^λ_ω denotes the set of ordinals less then λ of countable cofinality. In Section 5 the choice of the ladders will be provided by a combinatorial principle which follows from GCH.

Definition 4.1. Let $\lambda < \kappa$ and $S \subseteq E^\lambda_\omega$. We say that a family $\mathcal{F} \subseteq C_I(L_\lambda)$ is a ladder family of length λ given by the following parameters defined for all $\alpha \in S$:

(1) $(\nu^\alpha_n)_{n\in\mathbb{N}} \subseteq \{-1, 1\}$,
(2) sequences $(\eta^\alpha_n)_{n\in\mathbb{N}} \subseteq \alpha$ increasingly convergent to α,
(3) pairwise disjoint $(f^\alpha_n)_{n\in\mathbb{N}} \subseteq [\mathcal{F}]$, which depends on some $\beta_\alpha < \alpha$,
(4) infinite coinfinite set of integers $M_\alpha \subseteq \mathbb{N}$,

if $\mathcal{F} = \{d_\alpha \mid \alpha < \lambda\} \cup \{g_\alpha \mid \alpha \in S\}$ where

$$g_\alpha = \bigvee_{n\in M_\alpha} f^\alpha_n d_{\nu^\alpha_n, \eta^\alpha_n} \text{ in } C(L)$$

and each f^α_n belongs to

$$[\{d_\beta \mid \beta < \beta_\alpha\} \cup \{g_\beta \mid \beta \in S \cap \beta_\alpha\}].$$

Given $B \subseteq \lambda$ we denote the family $\{d_\alpha \mid \alpha \in B\}$ by $\mathcal{D}[B]$ and the family $\mathcal{D}[B] \cup \{g_\alpha \mid \alpha \in B \cap S\}$ by $\mathcal{F}[B]$.

Thus a ladder family is a family determined by S and the parameters as in (1)–(4) and constructed in a recursive manner following the values of these parameters.

Lemma 4.2. *Suppose that $\mathcal{F}$ is a ladder family of length λ. Then $\mathcal{F}[\alpha]$ depends on α for every $\alpha < \lambda$.*

Proof. Use the recursive definition of $\mathcal{F}[\alpha]$, Lemma 3.6 (1) and Lemma 3.24. □

Lemma 4.3. *Let $\lambda < \kappa$ and let $\mathcal{F}$ be a ladder family of length λ and let $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F}[\lambda_0])$ be an antichain for some $\lambda_0 < \lambda$. Then the family $\mathcal{F}[\lambda_0 \cup \{\lambda_0\}]$ is separating for $(U_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$.*

Proof. Let $S \subseteq E^\lambda_\omega$ be as in the definition of a ladder family and let $M \subseteq \mathbb{N}$ be such that the antichain $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F}[\lambda_0])$ is not separated along M in $[\mathcal{F}[\lambda_0 \cup \{\lambda_0\}]]$. By

Lemma 3.33 it is enough to show that the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[\alpha+1]]$ for all $\alpha \in [\lambda_0, \lambda)$. We proceed by induction. The base step $\alpha = \lambda_0$ follows from the choice of M. For the inductive step we fix $\alpha \in (\lambda_0, \lambda)$ and assume the hypothesis for all ordinals smaller than α. Then by Lemma 3.33 the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $\mathcal{F}[\alpha]$ and we have the following two cases:

Case 1. $\alpha \notin S$. Then $\mathcal{F}[\alpha+1] = \mathcal{F}[\alpha] \cup \{d_\alpha\}$ and the family $\mathcal{F}[\alpha]$ depends on α so we can use Lemma 3.37.

Case 2. $\alpha \in S$. Then $\mathcal{F}[\alpha+1] = \mathcal{F}[\alpha] \cup \{g_\alpha, d_\alpha\}$. First, we show that the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[\alpha] \cup \{g_\alpha\}]$. By the definition of the ladder family we have that $g_\alpha = \bigvee_{n\in M_\alpha}(f^\alpha_n d_{\nu^\alpha_n, \eta^\alpha_n})$ and all f^α_ns are from $[\mathcal{F}[\beta_\alpha]]$. Now, as $\beta_\alpha < \alpha$, we observe that by Lemma 3.33, it is enough to show that the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[\beta] \cup \{g_\alpha\}]$ for all $\beta \in [\beta_\alpha, \alpha)$. But this follows from Lemma 3.38 for $\mathcal{F} = \mathcal{F}[\beta+1]$, $A = \beta$, $f = g_\alpha$ and the inductive hypothesis that for all $\beta \in [\beta_\alpha, \alpha)$ the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[\beta+1]]$.

Finally, to conclude that the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[\alpha] \cup \{g_\alpha, d_\alpha\}]$ we use Lemma 3.37 as in the Case 1 and Lemma 3.24. The proof of the Lemma is finished. □

Lemma 4.4. *Suppose that κ is an uncountable regular cardinal, $S \subseteq E^\kappa_\omega$ and a strictly increasing sequence $(\eta^\alpha_n)_{n\in\mathbb{N}} \subseteq \kappa$ is convergent to α for every $\alpha \in S$. Then for every countable $A \subseteq \kappa$ the set S_A of all $\alpha \in S$ such that $\{\eta^\alpha_n \mid n \in \mathbb{N}\} \cap A$ is infinite is at most countable.*

Proof. Define $f : S_A \to A \cup \{\sup A\}$ by putting for $\alpha \in S_A$ the value $f(\alpha)$ to be the least upper bound of the set $\{\eta^\alpha_n \mid n \in \mathbb{N}\} \cap A$ among the elements of the set $A \cup \{\sup A\}$. Since $A \cup \{\sup A\}$ is countable it is enough to check the injectivity of f. Fix $\alpha, \alpha' \in S_A$ such that $\alpha < \alpha'$. $A \cap \{\eta^{\alpha'}_n \mid n \in \mathbb{N}\}$ is cofinal in α', so we can pick n_0 such that $\alpha < \eta^{\alpha'}_{n_0}$ and $\eta^{\alpha'}_{n_0} \in A$. Then we see that $f(\alpha) \leq \eta^{\alpha'}_{n_0} < f(\alpha')$. □

Lemma 4.5. *Let $\lambda < \kappa$, let $\mathcal{F}$ be a ladder family of length λ and let $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F})$ be an antichain. Then there is a countable $A \subseteq \lambda$ such that $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F}[A])$ and $\mathcal{F}[A]$ is separating for $(U_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$.*

Proof. Fix the set S from the definition of a ladder family.

Claim 1. *There exists an increasing sequence $(A_n)_{n\in\mathbb{N}}$ of countable subsets of λ such that*

(i) $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F}[A_0])$,
(ii) *for all $n \in \mathbb{N}$ the family $\mathcal{F}[A_n]$ depends on A_n,*
(iii) *for all $\alpha \in S$ and all $n \in \mathbb{N}$ if $\eta^\alpha_k \in A_n$ for infinitely many k then $\alpha \in A_{n+1}$.*

Proof. By the assumption the domain of every U_n is some finite set $F_n \subseteq \mathcal{F}$ of coordinates. Fix an arbitrary countable set $A_{0,0} \subseteq \lambda$ such that $\bigcup_{n\in\mathbb{N}} F_n \subseteq \mathcal{F}[A_{0,0}]$. This choice guarantees that $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F}[A_{0,0}])$. Then for every $n \in \mathbb{N}$ define $A_{0,n+1} \subseteq \lambda$ as the union of $A_{0,n}$ and the countable set $\bigcup\{Y_\alpha \mid \alpha \in A_{n,0} \cap S\}$ where $Y_\alpha \subseteq \alpha$ is some countable set such that g_α depends on Y_α for $\alpha \in S$. We see that $A_{0,n}$ is countable for every $n \in \mathbb{N}$ and so is the set $A_0 = \bigcup_{n\in\mathbb{N}} A_{0,n}$. Now we have (i) and (ii) for $n = 0$.

Fix $n \in \mathbb{N}$ and assume we have defined A_n such that (ii) and (iii) hold. We define A_{n+1} in two steps. First, we use Lemma 4.4 with $S = S$ and $A = A_n$ to obtain the countable set S_{A_n} so that we know the set $A_{n+1,0} = A_n \cup S_{A_n}$ is countable and that any superset A_{n+1} of $A_{n+1,0}$ satisfies (iii). Then we apply the procedure outlined above for constructing A_0 to obtain countable A_{n+1} such that (i) and (ii) hold. This completes the proof of the Claim. □

Fix a sequence $(A_n)_{n\in\mathbb{N}}$ from the above claim and set $A = \bigcup_{n\in\mathbb{N}} A_n$. We will show that $\mathcal{F}[A]$ is separating for $(U_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$. So suppose that $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[A]]$. By Lemma 3.33 it is enough to show that $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[A \cup \alpha + 1]]$ for every $\alpha < \lambda$. We prove it by induction on $\alpha < \lambda$. The base step for $\alpha = 0$ that $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[A]]$ follows from the choice of M. Now assume that $\alpha < \lambda$ and that the hypothesis is true for all ordinals smaller than α that is the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[A \cup \alpha]]$. We have the following three cases:

Case 1. $\alpha \in A$. Then $\mathcal{F}[A \cup (\alpha + 1)] = \mathcal{F}[A \cup \alpha \cup \{\alpha\}] = \mathcal{F}[A \cup \alpha]$ and we are done by inductive hypothesis.

Case 2. $\alpha \notin A$ and $\alpha \notin S$. Then

$$\mathcal{F}[A \cup (\alpha + 1)] = \mathcal{F}[A \cup \alpha] \cup \{d_\alpha\}.$$

The family $\mathcal{F}[A \cup \alpha]$ depends on $A \cup \alpha$ by (ii) of the above claim, by Lemma 4.2 and by definition of ladder family. Therefore, we can use Lemma 3.37 to conclude that the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[A \cup (\alpha + 1)]]$.

Case 3. $\alpha \notin A$ and $\alpha \in S$. Then

$$\mathcal{F}[A \cup (\alpha + 1)] = \mathcal{F}[A \cup \alpha] \cup \{g_\alpha, d_\alpha\}.$$

We prove the inductive step in this case in two steps. In the first step we show that the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[A_k \cup \alpha] \cup \{g_\alpha\}]$ for every $k \in \mathbb{N}$, which, by Lemma 3.33, implies that the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[A \cup \alpha] \cup \{g_\alpha\}]$. Fix $k \in \mathbb{N}$ and let $\beta_\alpha < \alpha$ be as in the definition of a ladder family. Using Lemma 3.33 once more we see that it is sufficient to show that the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[A_k \cup \beta] \cup \{g_\alpha\}]$ for all $\beta \in (\beta_\alpha, \alpha)$. Fix $\beta \in (\beta_\alpha, \alpha)$ and

apply Lemma 3.38 with $\mathcal{F} = \mathcal{F}[A_k \cup \beta]$, $A = A_k \cup \beta$ and $f = g_\alpha$. Let us check the assumptions of Lemma 3.38:

- the family $\mathcal{F}[A_k \cup \beta]$ depends on the set $A_k \cup \beta$ by (ii) of the above claim, Lemma 4.2 and the definition of ladder family,
- we have $\mathcal{D}[A_k \cup \beta] \subseteq \mathcal{F}[A_k \cup \beta]$ by the definition of operation $\mathcal{F}[(\cdot)]$,
- the elements f^α_n all depend on β because we have $\beta_\alpha < \beta$,
- the set $\{n \in \mathbb{N} \mid \eta^\alpha_n \in A_k \cup \beta\}$ is finite because by the assumption of this case $\alpha \notin A_{k+1}$ and $(\eta^\alpha_n)_{n\in\mathbb{N}}$ increasingly converges to $\alpha > \beta$.

As the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[A_k \cup \beta]]$, then Lemma 3.38 guarantees that the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[A \cup \beta] \cup \{g_\alpha\}]$.

The second step of the proof of Case 3 consists of showing that the antichain $(U_n)_{n\in\mathbb{N}}$ is not separated along M in $[\mathcal{F}[A \cup \alpha] \cup \{g_\alpha, d_\alpha\}]$. This is done by Lemma 3.37 since the family $\mathcal{F}[A \cup \alpha] \cup \{g_\alpha\}$ depends on $A \cup \alpha$. This completes the inductive step and hence the proof of the lemma. □

Lemma 4.6. *Let $\mathcal{F}$ be a ladder family of length κ. Then $\nabla\mathcal{F}$ is connected.*

Proof. By Lemma 3.29 it is enough to show that $\nabla\mathcal{F}[\alpha + 1]$ is connected for all $\alpha < \kappa$. We use the transfinite induction so let us fix $\alpha < \kappa$ and let us assume that we are done below α. The inductive hypothesis implies that $\nabla\mathcal{F}[\alpha]$ is connected. If $\alpha \notin S$ then

$$\nabla(\mathcal{F}[\alpha + 1]) = \nabla(\mathcal{F}[\alpha] \cup \{d_\alpha\}) = \nabla(\mathcal{F}[\alpha]) \times I$$

by Lemma 3.13, since $\mathcal{F}[\alpha]$ depends on α by Lemma 4.2, so we are done. If $\alpha \in S$ then $\mathcal{F}[\alpha + 1] = \mathcal{F}[\alpha] \cup \{g_\alpha, d_\alpha\}$. By the definition of the ladder family we have

$$g_\alpha = \bigvee_{n\in M_\alpha} f^\alpha_n d_{\nu^\alpha_n, \eta^\alpha_n}$$

and all f^α_ns are from $[\mathcal{F}[\beta_\alpha]]$ where $\beta_\alpha < \alpha$. Using Lemma 3.29 once more we see that it is enough to prove that $\nabla(\mathcal{F}[\beta] \cup \{g_\alpha\})$ is connected for all $\beta \in (\beta_\alpha, \alpha)$. By Lemma 3.28 we have that

$$\nabla(\mathcal{F}[\beta] \cup \{g_\alpha\}) = \operatorname{ext}\left(\nabla(\mathcal{F}[\beta]), (f^\alpha_n d_{\nu^\alpha_n, \eta^\alpha_n})_{n\in\mathbb{N}}\right)$$

and we use Lemma 3.30 to conclude that $\nabla(\mathcal{F}[\beta] \cup \{g_\alpha\})$ is connected for all $\beta \in (\beta_\alpha, \alpha)$ since $\nabla(\mathcal{F}[\beta])$ is connected by the inductive hypothesis. With adding d_α we proceed as in the first case. □

Lemma 4.7. *Let $\mathcal{F}$ be a ladder family of length κ. Then $\nabla\mathcal{F}$ has no butterfly points.*

Proof. Put $\mathcal{F}_\alpha = \mathcal{F}[\alpha+1]$ and note that the hypothesis of Lemma 3.40 is satisfied by Lemmas 4.2 and 4.3, so the proof is completed by applying Lemma 3.40. □

5. The construction

In this section we construct ladder families $\mathcal{F} \subseteq C_I(L_\kappa)$ as described in Section 4 such that the spaces $C(K)$ for $K = \nabla\mathcal{F}$ satisfy Theorem 1.3. To do so we use the previously developed techniques and a combinatorial principle $\diamondsuit(E^\kappa_\omega)$ (Theorem 5.2) which follows from GCH.

Lemma 5.1. *Assume the GCH. Let κ be a regular cardinal which is of the form λ^+ for λ which is a cardinal of uncountable cofinality. Then $\kappa^\omega = \kappa$ and for every $\alpha < \kappa$ we have*

$$\alpha^\omega < \kappa.$$

Proof. We prove the lemma by induction on a cardinal $\alpha < \kappa$. If $cf(\alpha) = \omega$, then $\alpha^\omega \leq 2^\alpha = \alpha^+ \leq \kappa$ by the GCH and $\alpha^+ < \kappa$ by the hypothesis on κ. If $cf(\alpha) > \omega$, then $\alpha^\omega = \sum\{\beta^\omega \mid \beta < \alpha\}$ which is less than κ by the inductive assumption and the regularity of κ. It also follows that $\kappa^\omega = \kappa$. □

When talking about topological concepts like convergence in the context of ordinals we always refer to the order topology on the ordinals. Recall that a subset $C \subseteq \kappa$ is called club if and only if it is unbounded in κ and closed in the order topology. $S \subseteq \kappa$ is called stationary if it intersects all club sets. It is well known that E^κ_ω is stationary for any uncountable regular κ (see [17], [25]). Assuming GCH we have the following theorem due to Gregory:

Theorem 5.2 *(Theorem 23.2 [17]). Assume GCH. There is a sequence $(S_\alpha)_{\alpha\in E^\kappa_\omega}$ such that:*

(1) $S_\alpha \subseteq \alpha$ for every $\alpha \in E^\kappa_\omega$,
(2) for every $X \subseteq \kappa$ the set

$$\{\alpha \in E^\kappa_\omega \mid X \cap \alpha = S_\alpha\}$$

is stationary in κ.

The above statement is called $\diamondsuit(E^\kappa_\omega)$.

Fix a bijection Ψ from κ onto $(\{-2,-1\} \cup \kappa) \times \{-1,1\}^{\mathbb{N}} \times \big(C_I(L)^{\mathbb{N}} \cup \mathcal{B}(C_I(L))^{\mathbb{N}}\big)$, which exists by the fact that the cardinalities of the sets $C_I(L)$ and $\mathcal{B}(C_I(L))$ are κ and $\kappa^\omega = \kappa$ by Lemma 5.1 and Lemma 3.14. By the standard closure argument and the fact that $\alpha^\omega < \kappa$ for all $\alpha < \kappa$ (Lemma 5.1) the set

$$C_\Psi = \{\alpha \in \kappa : \Psi[\alpha] = (\{-2,-1\} \cup \alpha) \times \{-1,1\}^{\mathbb{N}} \times \big(C_I(L_\alpha)^{\mathbb{N}} \cup \mathcal{B}(C_I(L_\alpha))^{\mathbb{N}}\big)\}$$

is a club set in κ. While using Ψ we will only be working with such subsets $T \subseteq \alpha$ for $\alpha \in C_\Psi \cup \{\kappa\}$ that $\Psi[T]$ is a graph of a function with two coordinate functions, first from $\{-2,-1\} \cup \alpha$ into $\{-1,1\}^{\mathbb{N}}$ and the second from $\{-2,-1\} \cup \alpha$ into $C_I(L_\alpha)^{\mathbb{N}} \cup \mathcal{B}(C_I(L_\alpha))^{\mathbb{N}}$ considered as a subset of $(\{-2,-1\} \cup \alpha) \times \{-1,1\}^{\mathbb{N}} \times \big(C_I(L_\alpha)^{\mathbb{N}} \cup \mathcal{B}(C_I(L_\alpha))^{\mathbb{N}}\big)$. That is, Ψ will serve as a coding of such pairs of functions by subsets of ordinals in κ. $\diamondsuit(E^\kappa_\omega)$ from Theorem 5.2 will be our prediction principle which for $\alpha \in C_\Psi$ may provide such a code for the above pair of functions in the form of $T = S_\alpha \subseteq \alpha$.

Theorem 5.3. *Assume GCH. Let κ be the successor of a cardinal of uncountable cofinality. There is a compact Hausdorff connected c.c.c. space K of weight κ without a butterfly point such that $C(K)$ has asymmetric distribution of separations in the direction of some $\mathcal{D} \subseteq C_I(K)$.*

Proof. We will construct a ladder family $\mathcal{F} \subseteq C_I(L_\kappa)$ such that $K = \nabla\mathcal{F}$ satisfies the theorem. Let $(S_\alpha)_{\alpha<\kappa}$ be a $\diamondsuit_\kappa(E^\kappa_\omega)$-sequence as in Theorem 5.2. Let Ψ and C_Ψ be as above. For each ordinal $\alpha \in E^\kappa_\omega$ choose a ladder $(\eta^\alpha_n)_{n\in\mathbb{N}}$, that is an increasing, cofinal in α sequence of type ω. The family $\mathcal{F}$ will depend of $S \subseteq \kappa$ and will be a ladder family with the following parameters for $\alpha \in S$:

- $(\rho^\alpha_n)_{n\in\mathbb{N}} \subseteq \{-1,1\}$,
- $(\eta^\alpha_n)_{n\in\mathbb{N}}$,
- $(f^\alpha_n)_{n\in\mathbb{N}} \subseteq [\mathcal{F}]$, which depends on some $\beta_\alpha < \alpha$,
- $M_\alpha \subseteq \mathbb{N}$.

So we will use for it the terminology and notation as in Definition 4.1. In fact the above parameters are build by recursion together with some additional objects which will witness the fact that $\mathcal{F}$ has asymmetric distribution of separations. Namely, the recursive construction involves:

(1) $S = \{\alpha_\xi : \xi < \kappa\} \subseteq E^\kappa_\omega \cap C_\Psi$,
(2) $\{\beta_{\alpha_\xi} : \xi < \kappa\} \subseteq \kappa$, $\beta_{\alpha_\xi} < \alpha_\xi$,
(3) $(\rho^{\alpha_\xi}_n)_{n\in\mathbb{N}} \subseteq \{-1,1\}$,
(4) $\{f^{\alpha_\xi}_n : n \in \mathbb{N}\}$ which is a pairwise disjoint sequence in the algebra $[\mathcal{F}[\beta_\xi]]$,
(5) $\{V^\xi_n : n \in \mathbb{N}\}$ which is an antichain in $\mathcal{B}(\mathcal{F}[\alpha_\xi])$ such that

$$V^\xi_n(L) \cap \operatorname{supp}(f^{\alpha_\xi}_m) = \emptyset, \text{ for all } n,m \in \mathbb{N},$$

(6) A_ξ a countable subset of α_ξ such that $\mathcal{F}[A_\xi]$ is separating in $\mathcal{F}[\alpha_\xi]$ for $\{V^\xi_n : n \in \mathbb{N}\}$ (see Definition 3.35),
(7) $M_{\alpha_\xi} \subseteq \mathbb{N}$ such that $\{V^\xi_n : n \in \mathbb{N}\}$ is not separated in $[\mathcal{F}[A_\xi]]$ along M_{α_ξ},
(8) $g_{\alpha_\xi} = \bigvee_{n\in M_{\alpha_\xi}} (f^{\alpha_\xi}_n d_{\eta^{\alpha_\xi}_n, \rho^{\alpha_\xi}_n})$.

Suppose that we have constructed all these objects for all $\xi < \gamma$ for some $\gamma < \kappa$. This gives ladder families $\mathcal{F}[\alpha]$ for any $\sup\{\alpha_\xi : \xi < \gamma\} \leq \alpha \leq \kappa$, just consisting of the elements $\{g_{\alpha_\xi} : \xi < \gamma\}$ and $\{d_\beta : \beta < \alpha\}$ for α as above. Let α_γ be the first ordinal in $E^\kappa_\omega \cap C_\Psi$ not smaller than $\sup\{\alpha_\xi : \xi < \gamma\}$ such that

(i) $\Psi[S_{\alpha_\gamma}] = (\phi, \psi)$ is a function (identified with its graph) from $(\{-2,-1\} \cup \alpha_\gamma)$ into $\{-1,1\}^{\mathbb{N}} \times \big(C_I(L)^{\mathbb{N}} \cup \mathcal{B}(C_I(L))^{\mathbb{N}}\big)$,
(ii) $\psi(-2)$ is a pairwise disjoint sequence $(f^{\alpha_\gamma}_n)_{n\in\mathbb{N}}$ in $\mathcal{F}[\beta_\gamma]$ for some $\beta_\gamma < \alpha_\gamma$,
(iii) $\psi(-1)$ is an antichain $(V^\gamma_n)_{n\in\mathbb{N}}$ in $\mathcal{B}[\mathcal{F}[\alpha_\gamma]]$ such that for all $n, m \in \mathbb{N}$ we have $V^\gamma_n(L) \cap \operatorname{supp}(f^{\alpha_\gamma}_m) = \emptyset$,
(iv) for $\alpha \in \alpha_\gamma$ the value $\psi(\alpha)$ is an antichain $\{V^\gamma_{\alpha,n} : n \in \mathbb{N}\}$ in $\mathcal{B}[\mathcal{F}[\alpha_\gamma]]$ whose n-th element $V^\gamma_{\alpha,n}$ is below V^γ_n.

In this case we define

(a) β_γ, $f^{\alpha_\gamma}_n$, V^γ_n as above,
(b) $\rho^{\alpha_\gamma}_n = \phi(\eta^{\alpha_\gamma}_n)(n)$,

The existence of α_γ follows from $\diamondsuit(E^\kappa_\omega)$. To define A_γ and M_{α_γ} we need to make some argument: The family $\mathcal{F}[\alpha_\gamma]$ is a ladder family and $\{V^\gamma_{\eta^{\alpha_\gamma}_n,n} : n \in \mathbb{N}\}$ is an antichain in $\mathcal{B}[\mathcal{F}[\alpha_\gamma]]$, hence by Lemma 4.5, there is a countable $A_\gamma \subseteq \alpha_\gamma$ such that $\mathcal{F}[A_\gamma]$ is separating for $\{V^\gamma_{\eta^{\alpha_\gamma}_n,n} : n \in \mathbb{N}\}$ in $\mathcal{F}[\alpha_\gamma]$. Since $\mathcal{F}[A_\gamma]$ is a countable, by Lemma 3.34 we can find an infinite $M_{\alpha_\gamma} \subseteq \mathbb{N}$ such that

(c) $\{V^\gamma_{\eta^{\alpha_\gamma}_n,n} : n \in \mathbb{N}\}$ is not separated in $[\mathcal{F}[A_\gamma]]$ along M_{α_γ}.

Finally put

(d) $g_{\alpha_\gamma} = \bigvee_{n\in M_{\alpha_\gamma}} (f^{\alpha_\gamma}_n d_{\eta^{\alpha_\gamma}_n, \rho^{\alpha_\gamma}_n})$.

This completes the inductive step in the construction of the ladder family $\mathcal{F}$. Now let us prove that $C(\nabla\mathcal{F})$ has asymmetric distribution of separations in the direction of $\mathcal{D} = \{d_\alpha : \alpha < \kappa\}$. So, fix

- Pairwise disjoint $(f_n)_{n\in\mathbb{N}} \subseteq [\mathcal{F}]$ and antichain $(U_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{F})$ such that

$$\operatorname{supp}(f_n) \cap U_n(L) = \emptyset \quad \text{for all } n, m \in \mathbb{N},$$

- $(\nu^\xi_n)_{n\in\mathbb{N}} \subseteq \{-1,1\}$ for all $\xi \in \kappa$,
- $\{\, (U^\xi_n)_{n\in\mathbb{N}} \mid \xi \in \kappa \,\} \subseteq \mathcal{B}(\mathcal{F})$ such that $U^\xi_n(\mathcal{F}) \subseteq U_n(\mathcal{F})$ for every $n \in \mathbb{N}$ and $\xi \in \kappa$.

Let $X \subseteq \kappa$ be such that $\Psi[X]$ is a function (ϕ, ψ) from $\{-2,-1\} \cup \alpha_\gamma$ into $\{-1,1\}^{\mathbb{N}} \times \big(C_I(L)^{\mathbb{N}} \cup \mathcal{B}(C_I(L))^{\mathbb{N}}\big)$ (identified with its graph) such that:

- $\phi(\alpha)(n) = \nu_n^\alpha$ for each $n \in \mathbb{N}$ and each $\alpha \in \kappa$,
- $\psi(-2)$ is $(f_n)_{n\in\mathbb{N}}$,
- $\psi(-1)$ is $(U_n)_{n\in\mathbb{N}}$,
- $\psi(\alpha)$ is $(U_n^\alpha)_{n\in\mathbb{N}}$ for each $\alpha \in \kappa$.

By the properties of the $\diamondsuit_\kappa(E^\kappa_\omega)$-sequence (Theorem 5.2), the facts that E^κ_ω is stationary and $C_\psi \setminus \beta$ is a club set, there is $\alpha \in E^\kappa_\omega \cap [C_\psi \setminus (\beta+1)]$ such that $S_\alpha = X \cap \alpha$ where $\beta < \kappa$ is such that $f_n \in \mathcal{F}[\beta]$ and $U_n \in \mathcal{B}[\mathcal{F}[\beta]]$ for each $n \in \mathbb{N}$ which exists by the uncountable cofinality of κ. By the definition of C_Ψ and the choice of α we have

$$\Psi[S_\alpha] = \Psi[X \cap \alpha] = \Psi[X] \cap \Psi[\alpha] = (\phi,\psi) \cap \Psi[\alpha] = (\phi,\psi) \restriction (\{-2,-1\} \cup \alpha).$$

So, (i)–(iv) are satisfied, moreover, then $\alpha = \alpha_\gamma \in S$ for some $\gamma < \kappa$. In particular, by the construction (1)–(8) we have

- $\rho_n^{\alpha_\gamma} = \nu_n^{\eta_n}$, where $\eta_n = \eta_n^{\alpha_\gamma}$,
- $f_n^{\alpha_\gamma} = f_n$ for each $n \in \mathbb{N}$,
- $\beta_\gamma = \beta$,
- $V_n^\gamma = U_n$ for each $n \in \mathbb{N}$,
- $V^\gamma_{\alpha,n} = U_n^\alpha$ for all $\alpha < \alpha_\gamma$ and each $n \in \mathbb{N}$.

So

$$g_{\alpha_\gamma} = \bigvee_{n\in M_{\alpha_\gamma}} (f_n^{\alpha_\gamma} d_{\eta_n^{\alpha_\gamma},\rho_n^{\alpha_\gamma}}) = \bigvee_{n\in M_{\alpha_\gamma}} (f_n d_{\eta_n^{\alpha_\gamma},\nu_n^{\eta_n}})$$

as required in Definition 1.4. So it remains to prove that the antichain $(U_n^{\eta_n})_{n\in\mathbb{N}}$ is not separated along the set M_γ in $[\mathcal{F}]$. First note that $(U_n^{\eta_n})_{n\in\mathbb{N}}$ is $(V^\gamma_{\eta_n,n})_{n\in\mathbb{N}}$ so is not separated along M_γ in $[\mathcal{F}[\alpha_\gamma]]$ by (c). Now, since $U_n \cap \mathrm{supp}(f_m) = \emptyset$ for all $n,m \in \mathbb{N}$, and $U_n^\alpha \subseteq U_n$ by Lemma 3.36 and Lemma 4.3 we conclude that $(U_n^{\eta_n})_{n\in\mathbb{N}}$ is not separated along M_γ in $[\mathcal{F}]$. So, $C(\nabla\mathcal{F})$ has asymmetric distribution of separations in the direction of $\{d_\alpha : \alpha < \kappa\}$. $\nabla\Delta$ is connected and has no butterfly points by Lemma 4.7 and Lemma 4.6. It is c.c.c. as a continuous image of a c.c.c. space L_κ by Proposition 3.2, and has weight κ by Lemma 3.14, so the proof is completed. □

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