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ON A PROBLEM OF KUROSH, JONSSON GROUPS, AND APPLICATIONS

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Abstract. We prove some results in group theory in a model theoretic spirit.

(i) We construct Jonsson groups of cardinality $\aleph_1$ and other cardinalities as well. This answers an old question of Kurosh.

(ii) Our group is simple with no maximal subgroup; so it follows that taking Frattini subgroups does not commute with direct products.

(ii) Assuming the continuum hypothesis, our group is not a topological group, except with the trivial topologies. This answers a quite old question of A.A. Markov.

In the construction we use small cancellation theory. We try to make the paper intelligible to both group theorists and model theorists. Only a knowledge of naive set theory and group theory is needed.

§0. Introduction

We first give the background, state the results, and then explain the proof. Schmidt asked whether infinite groups with no infinite proper subgroups exist; there has been much work in Schmidt groups, see e.g. [16]. Kurosh generalized this question to the following: Does there exist a group of cardinality $\aleph_1$ which has no proper subgroup of the same cardinality? Later Jonsson asked the same question for any algebra; so now an algebra with no proper subalgebra of the same cardinality is called a Jonsson algebra. Chang and Keisler [5] in their list of open problems, repeat Kurosh's question in this terminology: is there a Jonsson group of cardinality $\aleph_1$? McKenzie [11] proved that, for almost any cardinal λ, every Jonsson semi-group of cardinality λ is a group. Much work was done on the following question: for which λ

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is there a Jonsson algebra of cardinality λ (with countably many operations, of course). See e.g. [5]. Magidor and Malitz showed, from a general theorem, that if there is a Jonsson group in some regular cardinal, and $V = L$ (or even $\diamondsuit_{\aleph_1}$) holds, then there is a Jonsson group of cardinality $\aleph_1$.

The main body of the paper is a proof of the following result:

Theorem A. *There is a Jonsson group of cardinality $\aleph_1$ and also of cardinality λ^+ when $\lambda^+ = 2^\lambda$.*

The groups we construct can serve as counterexamples for some problems; more properties of these groups are stated in Theorem 1, and more hold by the construction.

It has been asked whether the operation of taking the Frattini subgroup commutes with direct products. Now $\tau(G)$, the Frattini subgroup of G, is the intersection of all proper maximal subgroups. It was well known that any simple group G with no maximal subgroup will provide a counterexample (because $\tau(G) = G$, but $\tau(G \times G) =$ the diagonal subgroup $= \{\langle a, a\rangle : a \in G\}$). Our groups serve as examples.

Theorem B. (1) *There is a simple group (of power $\aleph_1$) with no maximal subgroup.*

(2) *Taking Frattini subgroups does not commute with direct products.*

Problem. Does Theorem B hold for countable groups?

A.A. Markov [18] asked about the existence of an untopologizable group; i.e., a group which admits only the discrete topology. A.A. Markov [19] and Podewski [12] reported this question and proved that for any Hausdorff indiscrete topologizable group, any finite set of inequations which has at least one solution has at least two; and that for countable groups this condition on the group is necessary and sufficient for the existence of a Hausdorff indiscrete topology for the group. (The demand of "Hausdorff" is quite natural.) It is still unknown whether countable groups not satisfying this condition exist. Podowski [12] also gives a sufficient condition on a not necessarily countable group G (any system of less than $|G|$ inequations which has at least one solution has at least two) for the existence of such a topology. He also deals with other algebras. Previously Hanson [7] gave an untopologizable groupoid and Arnautov [2], [3] proved results on rings similar to those of Podewski. Kertesz and Szele [8] showed that for abelian groups a nontrivial topology always exists.

Theorem C. *Assume CH (i.e. $2^{\aleph_0} = \aleph_1$). Then there is an untopologizable group of cardinality $\aleph_1$.*

In fact, every countable subgroup of this group is topologized, thus answering negatively a question of Makowski. Of course we can replace $\aleph_1$ by any $\lambda^+ = 2^\lambda$.

Bokut [4] asked whether every algebra M over a field K (M not necessarily associative) of infinite dimension can be represented as $\bigcup_{n<\omega} M_n$, M_n a subalgebra over K, $\dim(M_n) = \dim(M_{n+1}, M_n) = \lambda$ for every n.

Theorem D. *There is a group G of cardinality $\aleph_1$ such that for every field K, the group-ring $K[G]$ is not the union of a strictly increasing chain of length ω.*

Note that Sabbagh [13] had proved the result for modules and groups; but they were of cardinality $\lambda = \lambda^{\aleph_0}$; but his group can serve as well, and we can easily change it to cardinality $\aleph_1$.

Koppelberg and Tits [9] have shown that no complete Boolean algebra is the union of a strictly ascending chain of subalgebra of length ω.

I would like to thank G. Hesse wholeheartedly for pointing out the incorrectness of 2.11 as it was stated (it was needed for Theorem 2.9) and suggesting a proof of 2.9 avoiding 2.11. The error was that if $K \setminus H$ contains torsion elements we cannot get $L <_{md} L^{**}$ so we added in 2.11 the hypothesis "all groups are torsion-free" and added a proof to 2.11 (which was left to the reader in the first version). Hesse's proof is added too. (Note that by his method in 2.9 for any uncountable set of elements I of the group, and element a, a belongs to the subgroup generated by three elements instead of two, but the length of the word is shorter.)

We shall first present small cancellation theory. We prove everything except the main theorem. We also slightly improve an application from [14]. We shall prove (in Theorem 2.1) that for $\lambda^+ = 2^\lambda$, $\lambda > \aleph_0$, there is a Jonsson group of cardinality λ^+ satisfying some other conditions. Then we show that with small changes our proof works for $\lambda = \aleph_0$. Next with less details (Theorem 2.9) we prove there is a Jonsson group of cardinality $\aleph_1$ (without assuming CH). Now Theorems A, B, follow immediately from 2.1, 2.9; we then give the (short) proof of theorems C, D, and we finish the paper by making a few remarks.

I would like to thank Paul Schupp wholeheartedly for explaining small cancellation theory to me, and for checking the proof.

Let me try to explain the proof of Theorem 1. We construct the group M as the union of an ascending chain of length λ^+, of groups of cardinality λ, say $M_\alpha\,(\alpha < \lambda^+)$. For definiteness we assume that the set of elements of M_α is $\lambda(1+\alpha)$ ($=$ the set of ordinals $< \lambda(1+\alpha)$). We try to prevent the existence of large proper subgroups G. In stage α M_α is already defined, and we are defining $M_{\alpha+1}$: we have a list of proper subgroups of M_α of cardinality λ $\{S_\beta : \beta \leq \alpha\}$, we look at them as approximations in some $\alpha(0) \leq \alpha$ to a G (i.e. as $G \cap M_{\alpha(0)}$) and try not to let them "grow". More specifically, we want that for every $a \in M_{\alpha+1} - M_\alpha$, and $\beta \leq \alpha$, the subgroup of $M_{\alpha+1}$ generated by S_β and a includes M_α. This means that if G is a proper subgroup of $M, M_{\alpha(0)} \not\subseteq G$, $S_\beta = G \cap M_{\alpha(0)}$ then G is disjoint to $M_{\alpha+1} - M_\alpha$. If we choose S_β $(\beta < \lambda^+)$ so that every subgroup of M of cardinality λ appears (and this can be done by $2^\lambda = \lambda^+$) this scheme works, provided that we can define $M_{\alpha+1}$ from M_α. We construct $M_{\alpha+1}$ by a series of approximations of smaller cardinality $L_\beta\,(\beta < \lambda)$ (L_β ascending and continuous, of course) and let $H_\beta = L_\beta \cap M_\beta$ (of course $\bigcup_\beta H_\beta = M_\alpha$, $\bigcup_\beta L_\beta = M_{\alpha+1}$). So we have to amalgamate L_β, $H_{\beta+1}$ over H_β, faithfully. The free product of L_β, $H_{\beta+1}$ with amalgamation over H_β satisfies this; but we have tasks to fulfil. To ensure $\bigcup_{\beta<\lambda} H_\beta = M_\alpha$ is easy; but we also have to ensure that each $b \in M_\alpha$ belongs to $\langle a, S_\gamma\rangle$ for $a \in L_\beta - H_\beta$, $\gamma \leq \alpha$. There are λ such tasks, so in each amalgamation we can deal with one such task only. We can choose $H_{\beta+1}$ such that $b \in H_{\beta+1}$, and $S_\gamma \cap H_{\beta+1}$ is "quite" large. But how to do this? We want something like free amalgamation, with an extra relation, saying that a word in a and some $x \in S_\gamma$ is equal to b.

Small cancellation theory is just the right theory. However one needs some hypothesis on a and x. For this we make the induction assumption that H_β is a malnormal subgroup of L_β (see "Notation", just below, for a definition), and we use the fact that $|H_\beta| < \lambda \doteq |S'_\gamma|$ to find suitable $x, y \in S_\gamma \cap H_{\beta+1}$. Now a, x, y satisfy a variant of the blocking pair condition, so small cancellation theory works.

For $\lambda = \aleph_0$ (see 2.7) we should replace usually "of cardinaltiy $< \lambda$" by "finitely generated". The main point is that a set $S \subseteq M_\alpha$ which is not included in a finitely generated subgroup of M_α may be included in a finitely generated subgroup of some $M_\beta, \beta \geq \alpha$. However our construction prevents this possibility, as shown in fact by 2.8, which is useful also in replacing $\diamondsuit_{\aleph_1}$ by CH in Macintyre [10] for algebraically closed groups (see [17]).

Theorem 2.9, which says that in power $\aleph_1$ Jonsson groups always exist, demands more changes, but we give fewer details and the ideas are essentially the same.

Notation. G, H, K, L, M will be groups. $G \leqslant H$ means G is a subgroup of H. If $G \leqslant H$, $x \in H$, then we call x *malnormal* over G (relative to H, of course) if G, G^x are disjoint, except for the unit element which we ignore where $G^x = \{xgx^{-1} : g \in G\}$. If $G \leqslant H$ and every x in $H - G$ is malnormal over G, then H is a *malnormal extension* of G, and G a *malnormal subgroup* H and we write $G \leqslant_m H$. If $G \leqslant H$, $x \in H$ then the *right*, *left*, *double* coset of x over G in H is xG, Gx, GxG resp; and belonging to the same right (or left, or double) coset, is an equivalence relation; and inequivalent elements have disjoint right (left or double) cosets.

We do not distinguish strictly between a group and its set of elements. $|A|$ is the number of elements of A, i.e., its cardinality.

For $X \subseteq G$, $\langle X \rangle$ is the subgroup of G generated by X. We call $x, y \in H$ *good fellows* over G if $x, y \in G - H$ with $x^{\pm 1} \notin Gy^{\pm 1}G$. Note that among any three elements of $G - H$ with distinct double cosets at least two are good fellows over G. Note also that for good fellows x, y $(Gx^{\pm 1}G) \cap (Gy^{\pm 1}G) = \emptyset$; and that G itself is a double coset. The relation of not being good fellows is an equivalence relation.

§1. Free products with amalgamation

Definition 1.1. Suppose H, K, L are groups, $K \cap L = H$ (for notational simplicity only). We now define the free product with amalgamation of K, L over H, denoted by $L^* = K *_H L$ as follows:

i) Let F_1 be the free group generated by the elements $K \cup L$.

ii) Let N_1 be the normal subgroup of F_1 generated by $\{g_1g_2g_3 : g_1, g_2, g_3 \in K,\ g_1g_2 = g_3^{-1} \text{ or } g_1, g_2, g_3 \in L,\ g_1g_2 = g_3^{-1}\}$.

iii) Let $L^* = F_1/N_1$. Now K, L, H have natural homomorphisms into $L^* : g \mapsto g/N_1$.

Fact 1.2. These homomorphisms are embeddings, and the intersection of the images of K and L is the image of their intersection, H (see e.g. [4]).

We shall not from now on distinguish between g and g/N_1, and we shall call an element of F_1 a word, and of $K \cup L$ a letter. It follows that every element of L^* which is not in H is equal to a product of the form $g_1 \cdots g_n$ such that for each l, $g_l \in L \cup K - H$, and $g_l \in L$ iff

$g_{l+1} \in K$. Such a product will be called a *canonical representation.* For $g \in H$, g itself is the canonical representation. A subword of (a canonical representation) $g_1 \cdots g_n$ is $g_l \cdots g_m$, $1 \leq l \leq m \leq n$. In general canonical representation is not unique but

Fact 1.3. If $g \in L^*$ has the canonical representations $g_1 \cdots g_n$, $g_1^1 \cdots g_m^1$ then:

(i) $n = m$.

(ii) $g_l \in K$ iff $g_l^1 \in K$.

(iii) There are $h_1, \ldots, h_{n-1} \in H$ such that letting $h_0 = h_n = e$, for every l, $g_l^1 = h_l g_l h_{l+1}^{-1}$.

Definition 1.4. If $g \in L^*$ has a canonical representation $g_1 \cdots g_n$, n will be called the length of g and denoted by $|g|$.

Definition 1.5. The canonical representation $g_1 \cdots g_n$ is called *weakly cyclically reduced* if n is even, or $n = 1$ or $g_n g_1 \notin H$ (equivalently, $g_1 \cdots g_n$ has no conjugate of length $< n - 1$). Notice that this is a property of the element.

Small cancellation theory

The aim of this theory in this context, is as follows: We have in $L^* = L *_H K$, a set R of words which are "long and complicated", which we want to make equal to the identity without "hurting" L, K, and short words in general. More accurately, we want to divide L^* by the normal subgroup N of L^* generated by R, and want that N will be disjoint to K, L and moreover will not have "short" elements.

For simplicity R will always be a set of weakly cyclically reduced elements.

Definition 1.6. (1) The symmetrized closure of R is obtained from R by the following operations:

(i) Add the inverses of the elements in R.

(ii) Add the conjugates (in L^*) which are weakly cyclically reduced.

(2) R is symmetrized if it is equal to its symmetrized closure.

(3) A part of a cyclically reduced word is a subword of a weakly cyclically reduced conjugate of it. A part of R is a part of one of its elements.

Claim 1.7. If $R = \{g_1 \cdots g_n\}$, n even, $g_1 \cdots g_n$ a canonical representation, the symmetrized closure of R consists of the following elements: for each $1 \leq l \leq n$, $g_l = g_l^1 g_l^2$, $g_l^i \in K \Leftrightarrow g_l \in K$, the words:

$g_l^2 g_{l+1}\cdots g_n g_1\cdots g_{l-1} g_l^1$ and $(g_l^1)^{-1} g_{l-1}^{-1}\cdots g_1^{-1} g_n^{-1}\cdots g_{l+1}^{-1}(g_l^2)^{-1}$. (Note that we write the elements in canonical form except that maybe $g_l^\delta \in H$ and then we multiply it with its neighbor.)

Definition 1.8. A symmetrized R satisfies the condition $C'(\theta)$ (θ a real positive number < 1) if whenever $g^* = g^m \cdots g^1$, $g_* = g_1 \cdots g_n$ are canonical representations of elements of R, $g^* \neq g_*^{-1}$ and $g^{l(0)} \cdots g^1 g_1 \cdots g_{l(0)} \in H$ (hence this hold for any $l \leqslant l(0)$) *then* $l(0) < \theta n$, $l(0) < \theta m$ and $n, m > (1/\theta)$ (this means $g^*, (g_*)^{-1}$ have no common segment of this length. Clearly $l(0)$ depends on g^* and g_* and not on the representation).

Main Theorem 1.9. *If $L^* = K *_H L$, $R \subseteq L^*$ symmetrized and satisfies $C'(1/k)$, $k \geqslant 6$, N the normal subgroup of L^* that R generates, $w = g_1 \cdots g_n \in N$ in a weakly cyclically reduced canonical form, then w has a part w_0 which is part of some $w_1 \in R$, and $|w_0| \geqslant [(k-3)/k]\,|w_1|$.*

Now we have to find suitable R. Schupp [14], [15] suggests, as a nice sufficient condition for existence, that L (or K) contains a blocking pair $\{x, y\}$ over H, which means a pair of malnormal elements over H which are good fellows except that possibly $x = y^{-1}$. Then for a $a \in L \setminus H$ the symmetrized closure of $\{axayax(ay)^2ax(ay)^3 \cdots ax(ay)^{80}\}$ satisfies $C'(1/10)$. We notice two weaker conditions:

(1) there is a malnormal $a \in L$ over H, and two good fellows $x, y \in K - H$;

(2) there is a malnormal $a \in L$, and $x \neq y \in K - H$ (this is weaker than (1) but the word contains inverses). So we can somewhat improve Theorem 10, p. 582 of Schupp [14] (see [15]).

Theorem 1.10. *Let $L^* = L *_H K$ be a free product with amalgamation where $L \neq H$, $K \neq H$, and in at least one of them there is a malnormal element over H. Then L^* is SQ-universal except when L, K, H are cyclic groups of order 2, 2, 1 resp. (SQ-universal means every countable group can be embedded in a quotient group of K).*

Now we present this more accurately. We denote by $w(\bar{x}) = w(x_1, \ldots, x_r)$ a sequence composed of the letters $x_1, \ldots, x_r$. For every group (or semigroup) G and $a_1, \ldots, a_r \in G$ the meaning of $w(a_1, \ldots, a_r)$ is clear.

Definition 1.11. $w(\bar{x}) = w(x_1, \ldots, x_r) = x^1 \cdot \cdots \cdot x^k$, where $x^l \in \{x_1, \ldots, x_r\}$ and we stipulate $x^j = x^i$ when $i = j \bmod k$ (for latter use).

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(1) We call $w(\bar{x})$ n-random if

(i) whenever $1 \leq p,\ q \leq k$, and $(\forall i)\ (1 \leq i \leq k/n \rightarrow x^{p+i} = x^{q+i})$ then $p = q$,

(ii) for every p, $1 \leq p,\ q \leq k$, for some i, $1 \leq i \leq k/n$ and $x^{p+i} = x^{q-i}$.

(2) We call $w(\bar{x})$ strongly *n-random* if

(iii) if $1 \leq p,\ q \leq k$, $v_1 \cdots v_n$, $u_1 \cdots u_n$ sequences of x_i's, $\varepsilon, \delta \in \{1, -1\}$ *then* for some l, $1 \leq l < k/n - n$, $x^{p+\delta(l+i)} = u_i$, $x^{q+\varepsilon(l+i)} = v_i$ for every i, $1 \leq i \leq n$, except when an apparent contradiction arises (that is $\delta = \varepsilon$, and for some i, j, $1 \leq j \leq n$, $p + \delta i = q + \varepsilon j$, $u_i \neq v'_j$ (so $|p - q| < n$).

Remark. It is easy to check that a strongly n-random sequence is an n-random sequence.

Claim 1.12. For every n, r for every big enough k, there is an n-random word $w^r_n(\bar{x}) = w^r_n(x_1, \ldots, x_r)$ which is even strongly n-random.

Proof. For every large enough k, compute the number of sequences, and number of non-strongly n-random sequences.

Claim 1.13. (1) Suppose $L^* = L *_H K$, $z \in L \cup K$, $a \in L$ is malnormal over H, and $x, y \in K$ are good fellows over H. If $w(x_1, x_2) = x^1 \cdots x^k$ is $4/\theta$-random *then* the symmetrized closure R of $\{zw(ax, ay)\}$ satisfies $C'(\theta)$.

(2) Suppose $L^* = L *_H K$, $z \in L \cup K$, $a \in L$ is malnormal over H, and $x, y \in K - H$, $x \neq y$. If $w(x_1, x_2, x_3, x_4)$ is strongly $(4/\theta)$-random *then* the symmetrized closure of $\{zw(ax, ay, a^{-1}x, a^{-1}y)\}$ satisfies $C'(\theta)$.

Proof. (1) The length of every word in R is $2k$ or $2k + 1$ (note that $w(ax, ay)$ is already in canonical form, and of even length $2k$). Also if $g = g_1 g_2 \cdots g_m \in R$ (in canonical form) then we can assume for some $\varepsilon \in \{0, 1\}$, $\delta \in \{1, -1\}$ and p the following hold:

$$g_{\varepsilon+2i} = a^{\delta},\ g_{\varepsilon+2i+1} = (x)^{\delta} \text{ when } x^{p+\delta i} = x_1;$$

$$g_{\varepsilon+2i} = a^{\delta},\ g_{\varepsilon+2i+1} = (y)^{\delta} \text{ when } x^{p+\delta i} = x_2$$

when the index of g is $< m$ and > 2, except possibly once (because of z). Of course ε, δ and p depends on g, so we shall write $\varepsilon(g)$, $\delta(g)$, and $p(g)$.

Suppose now $g^*, g_*, l(0)$ are as in 1.8, and let $h_l = g^l \cdots g^1$, $g_1 \cdots g_l \in H$ for $l \leq l(0)$; so $h_{l+1} = g^{l+1} h_l g_{l+1}$, $h_0 = e$; and note $g^l \in K \Leftrightarrow g_l \in K$ (otherwise $l(0) = 0$). Except for at most three l's (1, and one exception for g_*, and one for g^*) $g^l \in L \Rightarrow [g^l, g_l = a^{\pm 1}]$ and $g^l \in K \Rightarrow [g_l, g^l \in \{x^{\pm 1}, y^{\pm 1}\}]$. There are $1 < l(1) < l(2) \leqslant l(0)$, $l(2) - l(1) \geqslant l(0)/3$, such that no l, $l(1) < l \leqslant l(2)$ is exceptional. So for every such l, if $g^l \in K$, (as x, y are good fellows and $g^l h_{l-1} g_l \in H$), then $g_l = x^{\pm 1}$ iff $g^l = x^{\pm 1}$. This means for $l(1)/2 < i < l(2)/2$, $x^{l(0)-[p(g^*)+\delta(g^*)i]} = x^{[p(g_*)+\delta(g_*)i-(\varepsilon(g^*)-\varepsilon(g_*))/2]}$ (note g^* was written in a reverse order).

By the $(4/\theta)$-randomness of $w(x_1, x_2)$ this implies $\delta(g_*) = -\delta(g^*)$ and $g^l = (g_l)^{-1}$ for $1 < l < \min\{m, n\}$. If $g^1 g_1 \neq e$, then for every $l < l(0)$, $h_l \neq e$ (as a conjugate of an element $\neq e$ is $\neq e$) and for some l, $g^l = a^\delta$ so $g_l = a^{-\delta}$ and $h_{l+1} = a^\delta h_l a^{-\delta}$, so by the malnormality of a, $h_{l+1} \notin H$, contradiction. So necessarily $g^1 = g_1^{-1}$, hence $n = m$, $g^n = g_n^{-1}$, so $g^* = g_*^{-1}$. So we finish.

(2) We proved as before and let $v \in \{x, y\}$, $v \neq x^{-1}$. Suppose $\delta(g^*) = \delta(g_*) = 1$ for simplicity. But now by strong $(4/\theta)$-randomness we can find $l, l(1) < l$, $l + 2 < l(2)$ such that $g^l = a$, $g_l = a^{-1}$, $g^{l+1} = x$, $g_{l+1} = v$, $g^{l+2} = a$, $g_{l+2} = a^{-1}$. As $h_l = g^l h_{l-1} g_l \in H$, a malnormal over H, clearly $h_{l-1}, h_l \in H$ so $h_l = h_{l-1} = e$ (otherwise $h_l \notin H$ by a being malnormal over H) so $h_{l+1} = xv \neq e$ so $h_{l+2} = a(xv)a^{-1} \notin H$, contradiction. In the other cases we get similar contradictions except in the desired case.

§2. The theorems

Theorem 2.1. *Suppose λ is an infinite uncountable cardinal and $\lambda^+ = 2^\lambda$.*

(1) *There is a Jonsson group of cardinality λ^+.*

(2) *Moreover this group is a Jonsson semigroup (i.e. has no proper semigroup of the same cardinality), is simple, and there is a natural number n_0 such that for any subset S of the group of cardinality λ^+, any element of the group is equal to the product of n_0 elements of S.*

Proof. We will note some facts on groups, then we describe the construction, and at last prove it works.

Fact 2.2. Suppose H is a subgroup of K and A a subset of K such that either

(i) $3|H|^2 < |A|$, or

(ii) H is included in a finitely generated subgroup of K and A is not included in a finitely generated subgroup of K.

Then there are $x, y \in A$ which are good fellows over H.

Proof. Say $x \approx y$ if x or $x^{-1} \in HyH$. Now $\approx$ is an equivalence relation on $A - H$. Then x, y are *good fellows* over H iff they are not

$\approx$-equivalent. Thus if no $x, y \in K$ are good fellows over H, $A - H$ is contained in the union of at most 2 double cosets: HaH, $Ha^{-1}H$ (a any element of $A - H$); which contradicts (i), and contradicts (ii) as if H is contained in a finitely generated subgroup of K, so is $(HaH) \cup (Ha^{-1}H) \cup H$.

Fact 2.3. Suppose $H = K \cap L$ (so H is a subgroup of K and of L), and $H \leqslant_m L$, and $L^* = K *_H L$ (i.e. L^* is the free product of K and L with amalgamation over H). *Then* $K \leqslant_m L^*$

Proof. Let $g \in K$, $g \neq e$, $p = w_1 w_2 \cdots w_n \in L^* - K$ (a canonical form, the w's are letters; so each w_i belongs to $L - H$ or $K - H$, and successive letters do not belong to the same one). Note that $n = 1 \Rightarrow w_1 \in L - H$ as $p \notin K$. We should prove that $q = pgp^{-1} = w_1 \cdots w_n g w_n^{-1} \cdots w_1^{-1}$ does not belong to K.

(i) If $w_n \in L - H$, $g \in K - H$, q is already written in canonical form (of length $2n + 1 > 1$) hence $q \notin K$.

(ii) If $w_n \in L - H$, $g \in H$, then $w_n g w_n^{-1} \in L - H$ by the malnormality of H in L, hence q has the canonical form $w_1 w_2 \cdots w_{n-1}(w_n^{-1} g w_n) w_{n-1}^{-1} \cdots w_2^{-1} w_1^{-1}$ and $w_n^{-1} g w_n \notin K$. So $q \notin K$. So we can assume $w_n \in K - H$, so as $p \notin K$, $n > 1$.

(iii) If $w_n g w_n^{-1} \notin H$, then $w_1 \cdots w_{n-1}(w_n g w_n^{-1}) w_{n-1}^{-1} \cdots w_1^{-1}$ is a canonical form of q hence $q \notin K$.

(iv) If $g_1 = w_n g w_n^{-1} \in H$, then clearly $g_1 \neq 1$, and as $n > 1$, we are reduced to case (ii).

We exhaust all possibilities, thus finish.

We shall need the following fact about small cancellation theory over free products with amalgamation.

Fact 2.4. $L^* = K *_H L$ where $H \leqslant_m L$. Suppose that $x, y \in K - H$ are good fellows over H and let $a \in L - H$, $z \in K$.

Let $r = r(a, x, y, z) = z^{-1} xzyaxa(ya)^2 xa(ya)^3 \cdots xa(ya)^{80}$, and let R be the symmetrized closure of r. Let N be the normal closure of R in L^*, and let $L^{**} = L^*/N$. Then

(i) the natural map $r : L^* \to L^{**}$ embeds H and K, so that their intersection does not increase,

(ii) $K \leqslant_m L^{**}$.

Remarks. In fact r has to be a long enough random word consisting of instances of xa, ya; then multiplied by z^{-1}; but group theorists like this particular r. "Random" means that no two distinct "large" (e.g. of length at least 1/100 of that of r) segments of r are the same, ***even if we allow to inverting the order*** (but a segment and its inversion are considered distinct).

We could have replaced "good fellows" by "with distinct double coset representations" but then the random word *should* consist also of instances of ax^{-1}, ay^{-1}, so later we should not get a Jonsson semigroup.

Proof. For (i) see 1.13 (1) and then 1.9.

To see that (ii) holds, let $e \neq k \in K$. Suppose $u \notin K$. We can assume that u does not contain more that half of an element of R. Write $u = u_1 u_2 \cdots u_n$ in reduced form in L^*. If $u^{-1}ku = k' \in K$, we have the equation $u^{-1}ku(k')^{-1} = e$ in L^{**} and thus the left-hand side contains a part of more than 7/10 of an element of R by length. This cannot occur in u^{-1} nor in u (as they do not contain more than half of a generator), so this occurs around k (or $(k')^{-1}$), and at least 2/10 of it lies on each side of k. This means that for some v (initial segment of u) $v^{-1}kv$ is a part of more than 1/10 of a member of R. But no element of R contains a long subword of the form $c^{-1}gc$ where g has length 1. Thus the above equation cannot hold. If the part is around $(k')^{-1}$, the proof is similar.

The construction 2.5 (*for Theorem* 2.1). As we have assumed $2^\lambda = \lambda^+$ the number of subsets of λ^+ (as a set of ordinals) of cardinality exactly λ is $2^\lambda = \lambda^+$, so let $\{S_\alpha : \alpha < \lambda^+\}$ be a list of them.

We now define an increasing sequence of groups $M_\alpha (\alpha < \lambda^+)$ such that the set of elements of M_α is $\lambda(1+\alpha)$. We define them such that

(i) M_α is a malnormal subgroup of $M_{\alpha+1}$.

(ii) For every $\gamma \leq \alpha$ and $a \in M_{\alpha+1} - M_\alpha$, if $S_\gamma \subseteq M_\alpha$ then the subgroup $\langle S_\gamma \cup \{a\}\rangle$ includes M_α. In fact, every element of M_α is a product of length n_0 elements from $S_\gamma \cup \{a\}$ (where n_0 is the length of the r in Fact 2.4).

(iii) For limit δ, $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$.

We do the induction step later, and now let us prove that $M = \bigcup_{\alpha<\lambda^+} M_\alpha$ is as desired. Suppose by contradiction that M has a proper sub-semigroup, M^*, of cardinality λ^+. Clearly, for some $\alpha(0) < \lambda^+$, $S = M^* \cap M_{\alpha(0)}$ has cardinality λ, and let $a \in M - M^*$, for some $\alpha(1)$, $a \in M_{\alpha(1)}$. Now S is a subset of λ^+ of cardinality λ, hence for some $\alpha(3)$, $S = S_{\alpha(3)}$. Let $\alpha = \max\{\alpha(0), \alpha(1), \alpha(3)\}$. Since M^* has cardinality λ^+, it is not included in M_α and hence for some $b \in M^* - M_\alpha$. So necessarily for some $\beta \geq \alpha$, $b \in M_{\beta+1} - M_\beta$. Thus by (ii) ($\beta, \alpha(3)$ here stands for α, γ there) the subgroup generated by b and $S_{\alpha(3)}$ includes

M_α. Therefore it includes $M_{\alpha(1)}$ and hence a belongs to it. But $b \in M^*$, $S_{\alpha(3)} \subseteq M^*$, hence $a \in M^*$, contradicting its choice.

This finishes the proof of 2.1(1). As for part 2 of Theorem 2.1, M is a Jonsson semi-group as just proved; moreover checking how the group generated by b and $S_{\alpha(3)}$ includes M_β (by (ii)) we see that we have proved that if S is any subset of M of cardinality λ^+, any element of M is equal to the product of n_0 elements of S, here n_0 is the length of the word r from Fact 2.4 and is a fixed natural number. From 2.1(2) only the simplicity remains, but if $a \in M$, a not the unit, for some $\alpha, a \in M_\alpha$, so for every $\beta \geqslant \alpha$ a has a conjugate in $M_{\beta+1} - M_\beta$ (as $M_{\beta+1}$ is a malnormal extension of M_β, for each $b \in M_{\beta+1} - M_\beta$ we have $bab^{-1} \in M_{\beta+1} - M_\beta$). Hence the set of conjugates of a has power λ^+, so the subgroup they generate is the whole group M. In other words the normal subgroup a generates is M, so M is simple.

The induction step 2.6. So M_α is given and we should define $M_{\alpha+1}$. Let $\{c_i : i < \lambda\}$ be a list of the elements of M_α. Let $\{\langle a_\beta, \gamma_\beta, b_\beta\rangle : \beta < \lambda\}$ be a list of all triples $\langle a, \gamma, b\rangle$, $b \in M_\alpha$, $\gamma \leqslant \alpha$, $a \in M_{\alpha+1} - M_\alpha$ ($M_{\alpha+1}$ is not yet defined, but $M_{\alpha+1} - M_\alpha$ is $A_\alpha = \{\xi : \lambda(1+\alpha) \leqslant \xi < \lambda(1+\alpha+1)\}$) and each triple appearing λ times. (This can be done by cardinality considerations.) Now we shall define groups L_β, $H_\beta (\beta < \lambda)$ (more accurately L^α_β, H^α_β) and later let $M_{\alpha+1} = \bigcup_{\beta<\alpha} L_\beta$.

We define them such that:

(a) $|L_\beta| < \lambda$, moreover $|L_\beta| \leqslant \aleph_0 + |\beta|$.

(b) L_β is a subset of $M_\alpha \cup A_\alpha$; $L_\beta \cap M_\alpha = H_\beta \leqslant H_{\beta+1} \leqslant M_\alpha$, and for limit $\delta < \lambda$, $L_\delta = \bigcup_{\beta<\delta} L_\beta$ so $H_\delta = \bigcup_{\alpha<\delta} H_\beta$.

(c) $c_\beta \in H_{\beta+1}$ (so $M_\alpha = \bigcup_{\beta<\lambda} H_\beta$) and $\lambda(1+\alpha)+\beta \in L_{\beta+1}$ (so $M_{\alpha+1}$ will have the appropriate set of elements). (c_β is from the list of elements of M_α.)

(d) If $a_\beta \in L_\beta$, $S_{\gamma_\beta} \subseteq M_\alpha$ then $b_\beta \in \langle S_{\gamma_\beta} \cap H_{\beta+1}, a_\beta\rangle$, in fact it is the product of length n_0 of elements of $(S_{\gamma_\beta} \cap H_{\beta+1}) \cup \{a_\beta\}$.

(e) H_β is a malnormal subgroup of L_β.

We define L_β, H_β by induction on $\beta : H_0$ is the trivial group, L_0 the free group with one generator. For limit β define by (b). Suppose we have defined for β and define for $\beta+1$. Now we can by Fact 2.1(i) find $x, y \in S_{\gamma_\beta}$ which are good fellows over H_β, and let $H_{\beta+1} = \langle H_\beta, x, y, b_\beta, c_\beta\rangle$. Clearly $H_{\beta+1}$ satisfies (c), (b), and we can define $L_{\beta+1}$ by Fact 2.4, so that (d), (e) holds.

It is easy to check that $M_{\alpha+1} = \bigcup_{\beta<\lambda} L_\beta$ is as required: part (i) as $H_\beta \leqslant_m L_\beta$ (and easy checking) part (ii) as for every $a \in M_{\alpha+1} - M_\alpha$ for some β_0, $a \in L_{\beta_0}$, and for every $\gamma \leqslant \alpha$, $b \in M_\alpha$, the triple $\langle a, \gamma, b\rangle$ is $\langle a_\beta, \gamma_\beta, b_\beta\rangle$ for many $\beta \geqslant \beta_0$, (as each triple appears λ times) so we can

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apply (d). Part (iii) and the condition on the set of elements of $M_{\alpha+1}$ holds by (b) and (c).

Statement 2.7. The proof works for $\lambda = \aleph_0$, too, the only changes being:

(A) In the construction 2.5 in (ii) we have to say "if $S_\gamma \subseteq M_\alpha$ is not included in a finitely generated subgroup of M_α".

(B) In the induction step 2.6 at the end we have to use Fact 2.2(ii) instead of 2.2(i); and in (a) we should assert that L_β, H_β are finitely generated and $\{\xi < \omega(1+\alpha+1): \xi \not\in M_\alpha,\ \xi \not\in L_\beta\}$ is infinite.

(C) In the construction 2.5 when we prove the construction works, we have to define $S = M^* \cap M_{\alpha(0)}$ somewhat more carefully. First of all we have to choose it so that it is not included in a finitely generated subgroup of $M_{\alpha(0)}$ which can be done by a Lowenheim–Skolem argument. However, a priori, maybe in M_β this is no longer true; but the following fact closes the gap.

Fact 2.8. Suppose $M_\alpha\,(\alpha < \delta)$ is an increasing continuous chain of groups, and $M_{\alpha+1}$ is defined from M_α as in 2.6 for $\lambda = \aleph_0$, i.e. $M_{\alpha+1} = \bigcup_{\beta<\omega} L^\alpha_\beta$, $L^\alpha_\beta \cap M_\alpha = H^\alpha_\beta$, H^α_β is a finitely generated subgroup of M_α (or even is included in such a subgroup) $M_\alpha = \bigcup_{\beta<\omega} H^\alpha_\beta$. *Then* if $\alpha < \delta$, and G a subgroup of M_α not included in a finitely generated subgroup of M_α, *then* G is not included in any finitely generated subgroup of M_δ (of course, we can replace group by algebra of some fixed kind).

Proof. We prove there is no finitely generated subgroup of $M_\beta\,(\beta \leq \delta)$ which includes G, by induction on β. For $\beta \leq \alpha$ this is a hypothesis, for β limit it is easy. For $\beta+1$, suppose $a_1, \dots, a_k$ generates some such subgroup. Then for some n $a_1, \dots, a_k$ are in L^β_n, hence $G \subseteq \langle a_1, \dots, a_k\rangle \cap M_\alpha \subseteq L^\beta_n \cap M_\alpha = H_n$, so G is included in a finitely generated subgroup of M_β, contradiction.

Theorem 2.9. (1) *There is a Jonsson group of cardinality* $\aleph_1$.

(2) *This group is simple, and is a Jonsson semigroup.*

Proof. We first we state a definition and a fact.

Definition 2.10. (1) When $H \subseteq L$, $x \in L$ is *made over* H if for n, m natural numbers, $h_1, h_2 \in H$, $h_1 x^n h_2 = x^m$ implies $n = m$, $h_1 = h_2 = e$.

(2) H is a mad subgroup of L, and L is a mad extension of H, if every $x \in L - H$ is mad over H; and we write $H \subseteq_{md} L$.

Fact 2.11. Let $L^* = L *_H K$, and $H \leqslant_{md} L$, all of them countable torsion-free. Let $\{\langle b_n, c_n, z_n\rangle : n < \omega\}$ be a list of all triples $\langle b, c, z\rangle$, $b \in L - H$, $c \in K - H$, and $z \in K$. We can find natural numbers $k(n)$ $(n < \omega)$ such that the following holds.

Let R be the symmetrized closure of $\{r(c_n b_n^{k(2n)}, c_n b_n^{k(2n+1)}) z_n^{-1} : n < \omega\}$ where r is strongly 4000-random and N the normal closure of R, and $L^{**} = L^*/N$. Then

(i) R satisfies the small cancellation condition C′(1/1000),

(ii) the natural maps $L^* \to L^{**}$ embed K and L, and the intersection of their images is the image of H,

(iii) $K \leqslant_{md} L^{**}$,

(iv) L^{**} is torsion-free.

Proof of 2.11. For each n, the classes $Hb_n^k H$ $(0 < k < \omega)$ are distinct (hence disjoint), because H is a mad subgroup of L. A similar assertion holds for $Hb_n^{-k}H$ $(0 < k < \omega)$. Hence we can define by induction on n, $k(n) > 0$ such that $2n + p \neq 2m + q$, n, m natural numbers, $p = 0, 1$, $q = 0, 1$, $h_1, h_2 \in H$ implies $h_1 b_n^{\pm k(2n+p)} h_2 \neq b_m^{\pm k(2m+q)}$ or, equivalently, $b_n^{\pm k(2n+p)} h_2 b_m^{\pm k(2m+q)} \neq h_1$.

Let us check each part:

(i) We use the notation of Definition 1.8, and let $g^m \cdots g^1 g_1 \cdots g_m = h_m \in H$, so $h_0 = e$, $g^{m+1} h_m g_{m+1} = h_{m+1}$, so by the choice of the $k(n)$, for some $r_n = r(c_n b_n^{k(2n)}, c_n h_n^{k(2n+1)}) z_n^{-1}$, g^*, g_* are cyclically reduced conjugations of r_n, r_n^{-1} respectively or of r_n^{-1}, r_n respectively. By the strong 4000-randomness we get the desired contradiction (like 1.13).

(ii) Follows from (i).

(iii) Suppose $x \in L^{**} - K$, $k_1 \in K$, $k_2 \in K$, and n, m are distinct natural numbers > 0, and $k_1 x^m k_2 = x^n$ and we shall get a contradiction.

Among the representations of x we choose one $y^{-1} z y$, $y = y_1 \cdots y_p$, $z = z_1 \cdots z_q$ (where $y_1 \cdots z_1 \cdots \in K \cup L$) with smallest $p + 2q$ $(p, q$ natural numbers). So $y_p^{-1} \cdots y_1^{-1} z_1 \cdots z_q y_1 \cdots y_p$ is in canonical form except that we may put together $y_1^{-1} z_1$, $z_q y_1$, and $z_1 \cdots z_q$ is weakly cyclically reduced. [Otherwise $q > 1$ is odd, $z_q z_1 \in H$, so $x = (y_p^{-1} \cdots y_1^{-1} z_q^{-1})((z_q z_1 z_2) z_3 \cdots z_{q-1}(z_q y_1 \cdots y_p)$ so we can let $y' = z_q y_1 \cdots y_p$, $z' = (z_q z_1 z_2) z_3 \cdots z_{q-1})$. (Note $z_q z_1 z_2 \in L \cup K - H$) so $p' = p + 1$, $q' = q - 2$, contradicting the minimality of $p + 2q$).] Similarly $q > 1$ implies q is even; $y_1, \ldots, z_1 \cdots \notin H$.

As $x \in L^{**} - K$ clearly $q \geqslant 1$.

Note:

A) y, y^{-1}, z cannot contain more than half of a word from R. (As then we can decrease p without changing q or vice versa.)

B) $y^{-1}z$, zy cannot contain more than 9/10 of a word from R.

By symmetry suppose $z_{i+1}\cdots z_q y_1\cdots y_j$ is a subword of a word w from R, of length $>9|w|/10$, so w.l.o.g. $w = z_{i+1}\cdots z_q y_1 \cdots y_j t$ (t a word $|t|\leqslant|w|/10$). Clearly

$$\begin{aligned} x &= y_p^{-1}\cdots y_1^{-1}z_1\cdots z_q y_1\cdots y_p \\ &= (y_p^{-1}\cdots y_{j+1}^{-1})(y_j^{-1}\cdots y_1^{-1})(z_1\cdots z_i)(z_{i+1}\cdots z_q)(y_1\cdots y_j)(y_{j+1}\cdots y_p) \\ &= (y_p^{-1}\cdots y_{j+1}^{-1})[(y_j^{-1}\cdots y_1^{-1})(z_1\cdots z_i)t^{-1}](y_{j+1}\cdots y_p). \end{aligned}$$

So we get a representation with $y' = y_{j+1}\cdots y_p$, $z' = y_j^{-1}\cdots y_1^{-1}z_1\cdots z_i t^{-1}$. So $p' = p - j$, $q' = j + i + |t^{-1}|$, hence

$$\begin{aligned} (p+2q)-(p'+2q') &\geqslant (p+2q)-(p-j+2j+2i+2|t^{-1}|) \\ &= 2(q-i)-j-2|t^{-1}| \\ &\geqslant 2(q-i)-j-2|w|/10. \end{aligned}$$

But $p+2q$ was minimal, hence $\leqslant p'+2q'$, hence $j \geqslant 2(q-i)-2|w|/10$. But also $j+(q-i)\geqslant 9|w|/10$, hence

$$3j/2 = j + j/2 \geqslant j + (q-i) - |w|/10 \geqslant 9|w|/10 - |w|/10 = 4|w|/5.$$

So $j > |w|(4\cdot 2/5\cdot 3) = |w|(8/15) > |w|/2$, contradiction.

C) zz cannot contain more than 9/10 of a word from R.

So suppose $z_{i+1}\cdots z_q z_1\cdots z_j$ is a subword of a word w from R of length $>9|w|/10$. So w.l.o.g. $w = z_{i+1}\cdots z_q z_1\cdots z_j\, t$ (t a word so $|t|<|w|/10$). Assume first $j<i$.

So

$$\begin{aligned} x &= y_p^{-1}\cdots y_1^{-1}z_1\cdots z_q y_1\cdots y_p \\ &= (y_p^{-1}\cdots y_1^{-1})(z_q^{-1}\cdots z_{i+1}^{-1})(z_{i+1}\cdots z_q)(z_1\cdots z_j)(z_{j+1}\cdots z_q)(y_1\cdots y_p) \\ &= (y_p^{-1}\cdots y_1^{-1}z_q^{-1}\cdots z_{i+1}^{-1})(t^{-1}z_{j+1}\cdots z_i)(z_{i+1}\cdots z_q y_1\cdots y_p). \end{aligned}$$

So let $y' = z_{i+1}\cdots z_q y_1\cdots y_p$, $z' = t^{-1}z_j\cdots z_i$. Then $x = (y')^{-1}z'y'$, and $p'\leqslant p+(q-i)$, $q'\leqslant|t^{-1}|+i-j\leqslant|w|/10+i-j$. Again

$$\begin{aligned} 0 &\geqslant (p+2q)-(p'+2q') \\ &\geqslant (p+2q)-(p+(q-i)+2|w|/10+2(i-j)) \\ &= q-i+2j-|w|/5. \end{aligned}$$

So $q-i\leqslant|w|/5-2j$, but by the choice of w, i, j, t clearly $(q-i)+j\geqslant 9|w|/10$, hence $q-i\geqslant 9|w|/10-j$. Those two inequalities imply $|w|/5-2j \geqslant 9|w|/10-j$, which is a contradiction (as $|w| \geqslant 1$, $j \geqslant 0$).

Now we are left with the case $j \geqslant i$, but then by the strong randomness of r, $j-i \leqslant |w|/1000$, so replace j by $i-8$, and repeat the same argument.

D) In B) and C) we can replace $9|w|/10$ by $|w|(9/10-1/1500)$.

We have assumed $n \neq m > 0$, $k_1 x^m k_2 = x^n$. We shall get weakly cyclically reduced forms of word which is e, and using 1.9 and A)–D) get a contradiction.

Case I. $q = 1$.

Notice $z_1 \in K - H \Rightarrow y_1 \in L - H$ by $p+2q$ minimality. Then $x^n = y_p^{-1} \cdots y_1^{-1} z_1^n y_1 \cdots y_p$.

If $k_1 = k_2 = e$ we get $z_1^{n-m} = e$ and as K, L are torsion free, $n = m$.

If $k_1 = e \neq k_2$, $m > n$ we get $y_p^{-1} \cdots y_1^{-1} z_1^{m-n} y_1 \cdots y_p k_2 = e = y_p^{-1} \cdots y_1^{-1} z^{m-n} y_1 \cdots y_{p-1}(y_p k_2)$.

As before, $z_1^{m-n} \neq e$. As $y_1 \cdots z_1 \cdots$ are in $K \cup L - H$, if $k_2 \in H$, $z^{m-n} \notin H$ then the word is in canonical form.

If $k_2 \in H$, $z^{m-n} \in H$, $y_1 \in L - H$, by the madness condition $w = y_p^{-1} \cdots y_2^{-1}(y_1^{-1} z^{m-n} y_1) y_2 \cdots y_{p-1}(y_p k_2)$ is in canonical form ($z^{m-n} \neq e$ as K is torsion free) when $p > 1$, and $(y_1^{-1} z^{m-n} y_1 h_2) \in L - H$ is in canonical form when $p = 1$. When $k_2 \notin H$ we have a similar situation. So in all cases in the word w is not e in L^*. As it is e it should be long, so we can make it weakly cyclically reduced by small changes and we get easy contradiction by the strong 4000-randomness.

The case $k_1, k_2 \neq e$ is similar.

Case II. $q > 1$ hence q is even.

We get that

$$e = k_1 x^m k_2 x^{-n} = k_1 y_p^{-1} \cdots y_1^{-1} z_1 \cdots z_q \cdots z_1 \cdots z_q y_1 \cdots y_p k_2$$
$$y_p^{-1} \cdots y_1^{-1} z_q^{-1} \cdots z_1^{-1} \cdots z_q^{-1} \cdots z_1^{-1} y_1 \cdots y_p =^{\mathrm{df}} w^*.$$

The word is not weakly cyclically reduced only, possibly in $k_1, k_2, y_1^{-1} z_1, z_q y_1, y_1^{-1} z_q^{-1}, z_1^{-1} y_1$. But if $k_1, k_2 \neq e$, the needed changes involve few letters (much less than a hundred) so we ignore them. Let w^* be the weakly cyclically reduced form. If $k_1 = k_2 = e$ we get $e = x^{m-n} = z_1 \cdots z_q \cdots z_1 \cdots z_q$. If $k_1 = e \neq k_2$

$$e = x^{m-n} k_2 = y_p^{-1} \cdots y_1^{-1} z_1 \cdots z_q \cdots z_1 \cdots z_q y_1 \cdots y_p k_2.$$

Both cases are easier, and we leave them to the reader.

So by 1.9 there is $t = t_1 \cdots t_l$ a part of w^* and of a word w from R $|t| \geqslant 997|w|/1000$.

If for some natural number $j > |w|/1000$, $1 < i < i < i + j < q$, t contains $z_i \cdots z_{i+j}$ from two copies of z, we get a contradiction to the strong 4000-randomness of r. Similarly if t contains $z_1 \cdots z_{i+j}$, $z_{i+j}^{-1} \cdots z_i^{-1}$. Also if for some $j > |w|/4000$, $1 < i < i + j < p$, t contains $y_{i+j}^{-1} \cdots y_i^{-1}$, and $y_i \cdots y_{i+j}$ (or two copies from one of them) we get a contradiction.

But t is a part of w^*, so looking at it we can see that if it intersects two among the four copies of y, y^{-1} in w^* with length $> |w|/4000$, we get a contradiction to the above. With the one left its intersection has length $\leq |w|/2$ (by A).

Necessarily the sum of its intersection with z^n, z^{-n} is $\geq$ $|w|(99/100 - 3/4000 - 1/2) > 496|w|/1000$ so with one of them, e.g. z^m, it is $\geq 2|w|/10$.

If the length of z is $< |w|/100$, then for some i, $1 < i < l$, for every j, $i \leq j < i + |w|/10$ t_j is equal to t_{j+q} (or t_{j+q-1}) contradicting the strong 4000-randomness of r. So the length of z is $> |w|/100$, hence t cannot contain two copies of z (see above).

So summing our observations we have the following possibilities only:

a) t is contained in $y^{-1}zz$, and is not disjoint to y^{-1}.

So its intersection with the second z has length $< |w|/1000$, so $y^{-1}z$ contains a subword of w of length $> (91/100 - 1/100)|w| = 9|w|/10$, contradiction to (B).

b) t is contained in zzy and is not disjoint to y.

The same contradiction.

c) t is contained in zz, so it is

$$z'_i z_{i+1} \cdots z_q z_1 \cdots z_{j-1} z'_j.$$

We get contradiction to (C).

d) t is contained in $zzyk_2y^{-1}zz$ (or similarly with k_1). We can get similar contradictions.

(iv) Left to the reader.

Proof of Theorem 2.9. The construction is as in 2.5, but the list $\{S_\gamma : \gamma < \lambda^+\}$ is no longer necessary; and we make the changes mentioned in 2.7 and assume all groups are torsion-free. The main point is that in the induction step 2.6 we use Fact 2.11 rather than Fact 2.4. So in 2.5, condition (ii) is replaced by

(ii)′ For every $a \in M_{\alpha+1} - M_\alpha$, for an $S \subseteq M_\alpha$ which is not included in a finitely generated subgroup of M_α, $M_\alpha \leq \langle a, S \rangle$. Let us prove that the definition of $M_{\alpha+1}$ (see 2.6) satisfies this.

So let $b \in M_\alpha$, and we should prove $b \in \langle a, S \rangle$. Clearly for big enough n, $a \in L_n$, $b \in H_n$; and we can find $m \geq n$ such that

$S \cap (H_{m+1} - H_m) \neq \emptyset$ (as $M_\alpha = \bigcup_{m<\omega} H_m$, but for no m is $S \subseteq H_m$). Now choose $c \in S \cap (H_{m+1} - H_m)$. As we have used Fact 2.11 in L_{n+1}, $b \in \langle a, c\rangle$; so we finish.

Proof of Theorem C. We assume $2^{\aleph_0} = \aleph_1$, and let M be the group from 2.7. Suppose $\underset{\sim}{U}$ is a non-trivial topology which makes M a topological group. Choose a neighborhood $U_0 \neq M$ of the unit, and define inductively neighbourhoods U_n of the unit such that $U_{n+1} \subseteq U_n$, and $x, y \in U_{n+1} \Rightarrow xy \in U_n$.

If U_{n_0} is uncountable, every element of M is the product of n_0 elements of it and hence belongs to U_0, so $U_0 = M$. Contradiction. If U_{n_0} is countable, it is a subset of M_α for some $\alpha < \omega_1$. Choose $x \in M - M_\alpha$; $xU_{n_0}x^{-1}$ is necessarily open, so $U_{n_0} \cap xU_{n_0}x^{-1}$ is a neighbourhood of the unit; but by the malnormality condition ($M_\alpha \leqslant_m M$, see 2.5(i)) this intersection contains the unit alone. So any singleton is an open set, so any subset of M is open, a contradiction.

We can conclude that $\underset{\sim}{U}$ is either $\{M, \emptyset\}$ or the discrete topology.

Proof of Theorem D. Let M be the group of Theorem 2.9. Clearly we cannot find a strictly increasing sequence of sub-semi-groups M_n, $M = \bigcup_{n<\omega} M_n$ (as some M_n is necessarily uncountable, so $M_n = M$). Hence if K is a field, $K(M)$ the group-ring, $K(M) = \bigcup_{n<\omega} A_n$, then $M = \bigcup_{n<\omega} (A_n \cap M)$. Now $A_n \cap M$ is a sub-semi-group of M. Clearly $M \not\subseteq A_n$, so $A_n \cap M$ is strictly increasing and get a contradiction to the previous observation.

Additional information

2.15. Simplicity of Jonsson groups

Macintyre has shown that in fact "almost" any Jonsson group is simple. More exactly, there is no Jonsson abelian group, hence M has centre $Z(M)$ of power $<|M|$. For any $a \in M - Z(M)$, its centralizer again (is by its choice $\neq M$, hence) has cardinality $<|M|$; so the number of conjugates of a is $|M|$, so the normal subgroup it generates is M. So $M/Z(M)$ is a Jonsson group, and is simple.

2.16. Jonsson groups with centre

This naturally raises the question whether there are Jonsson groups with a non-trivial centre. We can repeat the proofs and constructions of 2.1, 7, 9 by starting with an abelian group Z, $|Z| \leqslant \lambda$, and change the definitions and requirements accordingly. So all groups will extend

Z, and Z will be in their centre; $H \leqslant_m L$ means $a \in L - H$, $b \in H - Z \Rightarrow aba^{-1} \notin H$, etc. So the generalization will be easy.

2.17. On torsion Jonsson groups

The groups we constructed are torsion free, and moreover satisfy $x^n = y^n \Rightarrow x = y$ for $n \neq 0$. We may like to build torsion Jonsson groups. Now free products with "good" amalgamation for some torsion groups (i.e. the given groups and the result are torsion) exists (by Adian [1]) but not with small cancellation, so we can only hope.

2.18. Jonsson group in $\aleph_2$

The proof of Theorem 2.9 works also for $\lambda = \aleph_2$ without any CH but for any $\aleph_n$, we need more complicated amalgamations, and the situation is not clear.

Appendix

Another proof of Theorem 2.9 avoiding 2.11. This proof is due to G. Hesse.

Lemma. *Let H, K, L be groups such that $K \cap L = H$ and $H \leqslant_m L$, and let Q be a subset of $H \times (K \setminus H) \times (L \setminus H)^2$ satisfying:*

(i) *If $(h, a, b, b') \in Q$, then b, b' are good fellows over H.*

(ii) *If (h_1, a_1, b_1, b_1') and (h_2, a_2, b_2, b_2') are different elements of Q, then at least one of the pairs b_1, b_2 and b_1', b_2' is a pair of good fellows. Suppose $L^* = K *_H L$, $w(x_1, x_2)$ is the word $x_1x_2x_1x_2^2 \cdots x_1x_2^{80}$, $R \subseteq L^*$ is the symmetrized closure of $\{h^{-1}w(ba, b'a) \mid (h, a, b, b') \in Q\}$, N is the normal closure of R in L^* and $L^{**} = L^*/N$. Then:*

(1) *R satisfies $C'(1/10)$, and the natural map $\pi : L^* \to L^{**}$ embeds K and L, so that their intersection does not increase.*

(2) *$K \leqslant_m L^{**}$.*

(3) *If $a \in K \setminus H$, $b \in L \setminus H$ and $0 \leqslant m < n \in \omega$, then $(ab)^m$, $(ab)^n$ are good fellows over K in L^*.*

(4) *If $b_1, b_2 \in L \setminus H$ are good fellows over H, then b_1, b_2 are good fellows over K in L^{**}.*

Proof. (1) Let $g^* = g^m \cdots g^1$, $g_* = g_1 \cdots g_n$ be canonical representations of elements of R such that $g^*g_* \neq e$, and let $g^l \cdots g^1 g_1 \cdots g_l \in H$ for some $l \leqslant \min\{m, n\} =: \mu$. An easy computation shows $m, n \in \{k, k+1\}$, where $k = 6640$. We may assume that there are elements (h_1, a_1, b_1, b'_1) and (h_2, a_2, b_2, b'_2) of Q, canonical representations $h_1^{-1}w(b_1a_1, b'_1a_1) = u_1 \cdots u_k$ and $h_2^{-1}w(b_2a_2, b'_2a_2) = V_1 \cdots V_K$, $p, q \in \{1, \ldots, k\}$ and $\delta, \varepsilon \in \{-1, 1\}$ such that

(a) $g' = u^{\delta}_{p-\delta i}$, $g_i = V^{\varepsilon}_{q+\varepsilon i}$ $(1 < i < \mu)$,

(b) $g^1 g^m \cdots g^\mu = u^{\delta}_{p-\delta} u^{\delta}_{p-\delta k} \cdots u^{\delta}_{p-\delta\mu}$, $g_\mu \cdots g_n g_1 = V^{\varepsilon}_{q+\varepsilon\mu} \cdots V^{\varepsilon}_{q+\varepsilon k} V^{\varepsilon}_{q+\varepsilon}$,

where $u_i = u_j$ and $V_i = V_j$ when $i \equiv j \bmod K$.

Case 1. $p \neq q$ or $\varepsilon = \delta$.

Then by (i) and the choice of w we can find $i \in \{2, \ldots, 664\}$ such that $g^i = u^{\delta}_{p-\delta i}$ and $g_i = V^{\varepsilon}_{q+\varepsilon i}$ are good fellows over H. Hence $g^i h g_i \notin H$ for every $h \in H$, and we have $l < i \leqslant 664 \leqslant \mu/10$.

Case 2. $p = q$ and $\varepsilon = -\delta$.

We find $i, j \in \{2, \ldots, 664\}$ such that w.l.o.g. $g^i = u^{-\varepsilon}_{p+\varepsilon i} = b_1^{-\varepsilon}$, $g_i = V^{\varepsilon}_{p+\varepsilon i} = b_2^{\varepsilon}$, $g^j = u^{-\varepsilon}_{p+\varepsilon j} = b_1^{1-\varepsilon}$ and $g_j = V^{\varepsilon}_{p+j} = b_2^{1-\varepsilon}$. If we assume $l \geqslant 664$, then none of the pairs b_1, b_2 and b'_1, b'_2 is a pair of good fellows. By (ii), $(h_1, a_1, b_1, b'_1) = (h_2, a_2, b_2, b'_2)$ and w.l.o.g. $u_i = V_i$ $(i = 1, \ldots, k)$. Choose $v \in \{2, 3\}$ and $b \in L \setminus H$ with $g^\nu = b^{-\varepsilon}$, $g_\nu = b^{\varepsilon}$. Since by assumption $b^{-\varepsilon}(g^{\nu-1} \cdots g^1 g_1 \cdots g_{\nu-1})b^{\varepsilon} \in H$, by the malnormality of b over H we have $g^{\nu-1} \cdots g^1 g_1 \cdots g_{\nu-1} = e$. Therefore by (a) and (b):

$$\begin{aligned} g^* g_* &= g^m \cdots g^\mu g^{\mu-1} \cdots g^\nu g_\nu \cdots g_{\mu-1} g_\mu \cdots g_n \\ &= g^m \cdots g^\mu g_\mu \cdots g_n \\ &= (g^{\nu-1} \cdots g^1)^{-1} g^{\nu-1} \cdots g^1 g^m \cdots g^\mu g_\mu \cdots g_n g_1 \cdots g_{\nu-1} (g_1 \cdots g_{\nu-1})^{-1} \\ &= (g^{\nu-1} \cdots g^1)^{-1} (g_1 \cdots g_{\nu-1})^{-1} = e, \end{aligned}$$

a contradiction.

The embedding property follows now from 1.9.

The proof of (2) is exactly the same as the proof of Fact 2.4(ii).

(3) Suppose $n, n \in \omega$, $m \neq n$, $\varepsilon \in \{-1, 1\}$, $a_1, a_2 \in K$ and $(ab)^m a_1 (ab)^{\varepsilon \cdot n} a_2 = e$ in L^{**}. Let $w = g_1 \cdots g_l$ be a canonical representation of the element $(ab)^m a_1 (ab)^{\varepsilon \cdot n} a_2$ of L^*. Obviously there is at most one $i \in \{1, \ldots, l\}$ such that g_i, b are good fellows over H in L. Since $m \neq n$, we have $l \geqslant 1$. Thus by (1) and the Main Theorem 1.9, $g_1 \cdots g_l$ contains a long subword w_0 which is also a piece of some $w_1 \in R$. It follows from (i) that there are at least two $i \in \{1, \ldots, l\}$ such that g_i, b are good fellows over H in L, a contradiction.

The proof of (4) is left to the reader.

Proof of Theorem 2.9. As before M is constructed as the union of an increasing continuous chain $(M_\alpha \mid \alpha < \aleph_1)$ of countable groups, where no M_α is finitely generated If M_α has already been constructed, we choose a strictly increasing chain $(H_n \mid n \in \omega)$ of finitely generated subgroups such that $H_0 = \{e\}$ and $M_\alpha = \bigcup\{H_n \mid n \in \omega\}$. By induction, we now define a strictly increasing chain $(L_n \mid n < \omega)$ of groups with the following properties:

(a) $|L_n| = \aleph_0$.

(b) $L_n \cap M_\alpha = H <_{n \neq m} L_n$.

(c) If $a \in H_{n+1} \setminus H_n$, $b \in L_n \setminus H_n$, $0 \leq m < n < \omega$, then $(ab)^m$, $(ab)^n$ are good fellows over H_{n+1} in L_{n+1}.

(d) If $b_1, b_2 \in L_n \setminus H_n$ are good fellows over H_n, then b_1, b_2 are good fellows over H_{n+1} in L_{n+1}.

In order to define L_{n+1}, let $((h_i, a_i, a'_i, b'_i) \mid i \in \omega)$ be an enumeration of all (h, a, a', b') such that $h \in H_n$, $a \in H_{n+1} \setminus H_n$, $a' \in H_n \setminus H_m$ and $b' \in L_m \setminus H_m$ for some $m < n$. Using the induction hypothesis and the Lemma above, one can define a sequence $(b_i \mid i \in \omega)$ of elements of $L_n \setminus H_n$ such that:

(1) $b_i \in L_n \setminus H_n$.

(2) b_i is a product of elements of $\{a'_i, b'_i\}$.

(3) b_i, b'_i are good fellows over H.

(4) If $j < i$ then b_i and b_j are good fellows over H.

By induction hypothesis "$H_n \leqslant_m L_n$" and by the lemma, the symmetrized closure R of $\{h_i^{-1} \cdot w(b_i a_i, b'_i a_i) \mid i \in \omega\}$ satisfies C′(1/10) in $L^*_{n+1} := H_{n+1} *_{H_n} L_n$, and we may define $L_{n+1} := L^*_{n+1}/N$, where N is the normal closure of R in L^*_{n+1} and w.l.o.g. $L_{n+1} \cap M_\alpha = H_{n+1}$. The desired properties of L_{n+1} follows from the lemma. This concludes the main step of our construction, and we put $M_{\alpha+1} = \bigcup\{L_n \mid n \in \omega\}$.

Suppose now that S is an uncountable subset of $M = \bigcup\{M_\alpha \mid \alpha < \aleph_1\}$ and h is an element of M. Then there is an $\alpha < \aleph_1$ such that $S \cap (M_{\alpha+1} \setminus M_\alpha) \neq q$ and $S \cap M_\alpha$ contained in no finitely generated subgroup of M_α.

Choose $m < n < \omega$ and $a, a', b' \in S$ such that $h \in H_n$, $a \in H_{n+1} \setminus H_n$, $a' \in H_n \setminus H_m$ and $b' \in L_m \setminus H_m$, where the groups H_i, L_i are as in the construction of $M_{\alpha+1}$. By construction, h is a product of elements of $\{a, a', b'\}$ in L_{n+1} and hence in M.

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