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Incompactness for chromatic numbers of graphs

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0 Introduction

We have proved the singular cardinal compactness theorem ([12, 13]). A special case of it is that if G is a graph of size a singular cardinal λ such that every subgraph of power less than λ has colouring number less than or equal to ω, then G has countable colouring number. We asked in [12] if this held for the chromatic number. Komjáth showed in [10] that it is consistent that there exists a counterexample of size $\aleph_{\omega_1}$. In this model the continuum is $\aleph_{\omega_1+1}$. Answering his question, we show that such a counterexample is consistent even with GCH (Section 1) and show that similar examples exist in $V = L$ (Section 1).

P. Erdős and A. Hajnal showed that under GCH there is a graph G of size $\aleph_2$ with $\mathrm{Chr}(G) = \aleph_1$ such that every subgraph of size $\aleph_1$ is countably chromatic. They asked in [5] if a similar example which is $\aleph_2$-chromatic exists. The consistency and independence of this statement were shown by Baumgartner and Foreman & Laver, respectively ([2], [6]). Whether or not similar examples exist under $V = L$ was an old problem. We show that this is the case and much more (Section 3): for every regular non-weakly compact κ there is a graph G on κ, with $\mathrm{Chr}(G) = \kappa$, such that every smaller subgraph is countably chromatic. We notice that our earlier proof with just $\mathrm{Chr}(G) = \omega_1$ was published in [4].

Galvin [9] observed that it is not obvious whether or not an $\aleph_2$-chromatic graph should contain an $\aleph_1$-chromatic subgraph. Komjáth showed that this is in fact independent ([10]). Here we show that, e.g. under $V = L$, no counterexample of size $\aleph_2$ exists.

Under GCH, P. Erdős & A. Hajnal showed that for $0 < k \leq n < \omega$ there is a graph of size $\aleph_n$ with chromatic number with $\aleph_k$ all subgraphs of size $\aleph_{n-1}$ being $\leq \aleph_{k-1}$-chromatic. In Section 5 we show that it is consistent (relative to a supercompact) that every $\aleph_n$-chromatic graph ($0 < n < \omega$) contains an $\aleph_n$-chromatic subgraph of power less than $\aleph_\omega$.*

Notation

A graph is a pair $G = (V, E)$, where $E \subseteq [V^2] = \{x \subseteq V : |x| = 2\}$. E and G are sometimes confused. Chr(G) is the chromatic number of G. tp(A) is the order type of A.

1 Incompactness in singular cardinals via forcing

Theorem 1 (GCH) *If $\lambda > \mathrm{cf}(\lambda) = \omega_1$, then there exists a cardinality, cofinality and GCH-preserving partial ordering which adds an $\aleph_1$-chromatic graph on λ such that every subgraph of power less than λ is countably chromatic.*

We can, of course, replace ω_1 by any regular cardinal.

Proof The proof is broken into a series of definitions and lemmas. Let $\{\kappa_\alpha : \alpha < \omega_1\}$ be an increasing, continuous sequence of singular cardinals converging to λ and let $\lambda_\alpha = \kappa_\alpha^+$. Fix a sequence $\{D_\alpha : \alpha < \omega_1\}$ of disjoint sets with $|D_\alpha| = \lambda_\alpha$, $D = \bigcup \{D_\alpha : \alpha < \omega_1\}$ and $E_\alpha = \bigcup \{D_\beta : \beta < \alpha\}$. For $A, B \subseteq D$ we use the following convention: $A_\alpha = A \cap D_\alpha$ and $B_\alpha = B \cap D_\alpha$.

Definition $p = (A, X) \in P$ if $A \subseteq D$, $|A_\alpha| < \lambda_\alpha$ for $\alpha < \omega_1$ and X is a graph on D with

(a) $X \cap [D_\alpha]^2 = \emptyset$;
(b) if $\{x, y\} \in X$, $y \in E_\alpha$ and $x \in D_\alpha$, then $x \in A$;
(c) for $x \in A$, the set $\{y \in D - A : \{x, y\} \in X\}$ is finite and is included in E_α;
(d) for $x \in A_\alpha$ and $\beta < \alpha$, the set $\{y \in E_\beta : \{y, x\} \in X\}$ is finite;
(e) $\mathrm{Chr}(X) \leq \omega$.

Next we define extension.

Definition $q = (B, Y) \geq p = (A, X)$, that is, (B, Y) extends (A, X), if
(f) $B \supseteq A$ and $Y \supseteq X$;
(g) if $x \in A_\alpha$, then $\{y \in E_\alpha : \{y, x\} \in Y\} = \{y \in E_\alpha : \{y, x\} \in X\}$;

* We thank Peter Komjáth for rewriting the paper.

(h) if $x \in B_\alpha - A_\alpha$, then $\{y \in A : \{y, x\} \in Y - X\}$ is finite and is included in E_α.

Notice that the second clause of (h) follows from (g). It is a trivial calculation to check that the partial order is transitive.

Definition $(B, Y) \geq_\alpha (A, X)$ if $(B, Y) \geq (A, X)$, $B \cap E_{\alpha+1} = A \cap E_{\alpha+1}$ and $Y \cap [E_{\alpha+1}]^2 = X \cap [E_{\alpha+1}]^2$. Similarly $(B, Y) \geq^\alpha (A, X)$ denotes that $(B, Y) \geq (A, X)$, $B \cap (D - E_{\alpha+1}) = A \cap (D - E_{\alpha+1})$ and

$$Y \cap [D - E_{\alpha+1}]^2 = X \cap [D - E_{\alpha+1}]^2.$$

Obviously $\leq_\alpha$ and $\leq^\alpha$ are transitive suborderings.

Lemma 1.2 *If $\theta \leq \kappa_{\alpha+1}$ and $\{p_\xi : \xi < \theta\}$ form a continuous $\leq_\alpha$-increasing sequence, then they have a common $\leq_\alpha$-extension.*

Proof Put $p_\xi = (A^\xi, X^\xi)$. We take $A = \bigcup \{A^\xi : \xi < \theta\}$ and $X = \bigcup \{X^\xi : \xi < \theta\}$. We show that (A, X) is a condition and that $(A, X) \geq_\alpha (A^\xi, X^\xi)$ $(\xi < \theta)$.

Everything is trivial except that $\mathrm{Chr}(X) \leq \omega$. As every D_β $(\beta \leq \alpha)$ is independent in X (i.e. there is no edge of X joining two vertices of D_β), $E_{\alpha+1}$ is certainly countably chromatic. The vertex set of X on $D - E_{\alpha+1}$ is the union of an independent set, $D - A$, and $A - E_{\alpha+1} = \bigcup \{A^{\xi+1} - A^\xi - E_{\alpha+1} : \xi < \theta\}$. X on $A^{\xi+1}$ is countably chromatic, and from every vertex in $A^{\xi+1} - A^\xi$ only finitely many edges go to A^ξ. This implies that $\mathrm{Chr}(X) \leq \omega$. □

Lemma 1.3 *If $q \geq p$ and $\alpha < \omega_1$, then there are r and s with $p \leq_\alpha r \leq^\alpha q$ and $p \leq^\alpha s \leq_\alpha q$.*

Proof If $q = (B, Y)$ and $p = (A, X)$, put $r = (C, Z)$, where $C_\beta = A_\beta$ for $\beta \leq \alpha$, $C_\beta = B_\beta$ for $\beta > \alpha$ and

$$Z = X \cup \{\{x, y\} \in Y : x \in D_\beta, y \in B_\gamma \ (\beta < \gamma, \alpha < \gamma)\}.$$

Similarly for s. □

Lemma 1.4 *Assume that $\alpha < \omega_1$, $p \in P$ and $p \Vdash$ 'τ is a name for an ordinal'. Then there exists a $q \geq_\alpha p$ and a set A ($|A| \leq \lambda_\alpha$) such that*
(i) *$q \Vdash$ '$\tau \in A$';*
(j) *if $q \leq q^*$, q^* decides a value for τ and $q \leq^\alpha r \leq_\alpha q^*$, then r decides a value for τ.*

Proof We let $\{r_\xi : \xi < \lambda_\alpha\}$ enumerate the possible restrictions $(A \cap E_{\alpha+1}, X \cap [E_{\alpha+1}]^2)$ for $(A, X) \in P$. By transfinite recursion on ξ

we construct an $\leqslant_\alpha$-ascending sequence p_ξ with $p_0 = p$ such that $p_{\xi+1} \cup r_\xi$ decides a value for τ if there exists a $q \geqslant_\alpha p$ with $q \cup r_\xi$ deciding τ. $\square$

Lemma 1.5 *Cardinals and cofinalities remain.*

Proof As usual, it suffices to show that if κ is a regular cardinal in the ground model, then $\theta = \mathrm{cf}(\kappa) < \kappa$ is impossible in the enlarged model. As $|P| = \lambda^+$, no problem arises with $\kappa \geqslant \lambda^{++}$.

Assume first that $\lambda_\alpha < \kappa < \lambda_{\alpha+1}$, $p \Vdash$ '$S \subseteq \kappa$ is cofinal and $|S| = \theta$' and $\theta < \kappa$. By Lemma 1.4 there is a $q \geqslant p$ and a T with $|T| \leqslant \theta + \lambda_\alpha < \kappa$ such that $q \Vdash$ 'T is cofinal' with T in the ground model: a contradiction.

If $\alpha < \omega_1$, α a limit and $\kappa = \kappa_\alpha^+$, then (as κ_α is singular) $\theta < \kappa_\alpha$, so that $\theta < \kappa_\beta$ for some $\beta < \alpha$. Again, we get that $\mathrm{cf}(\kappa) \leqslant \lambda_\beta < \kappa$ in the ground model. Assume, finally, that $\kappa = \lambda^+$. Then $\theta \leqslant \lambda_\alpha$ for some $\alpha < \omega_1$ and we may proceed as in the previous case. $\square$

Lemma 1.6 *GCH survives.*

Proof As P is ω_1-closed, it suffices (by Silver's theorem) to show that $2^\theta = \theta^+$ holds in an enlarged model for every regular cardinal θ. There is no problem for $\theta > \lambda$; so assume that $\kappa_\alpha^+ \leqslant \theta < \kappa_{\alpha+1}$ and that $p \Vdash$ '$T_\xi \subseteq \theta$ are different ($\xi < \theta^{++}$)'. By Lemma 1.4, there is a q, as there, and a partial function $F(r, \xi, \zeta)$ such that if $r = (A, X)$ with $A \subseteq E_{\alpha+1}$ and $X \subseteq [E_{\alpha+1}]^2$, then $r \cup q$ forces either that $\zeta \in T_\xi$ or that $\zeta \notin T_\xi$ according to whether $F(r, \xi, \zeta)$ is 0 or 1. As the number of different r's is λ_α, the number of $F(r, \cdot, \zeta)$ functions is $\leqslant (\lambda_\alpha^+)^\theta = \theta^+$, and so there are $\xi_1 \neq \xi_2$ with $F(r, \xi_1, \zeta) = F(r, \xi_2, \zeta)$, that is, $q \Vdash$ '$T_{\xi_1} = T_{\xi_2}$': a contradiction. $\square$

Lemma 1.7 *P forces that the generic graph is countably chromatic on every set of size less than λ.*

Proof Assume that $p \Vdash$ '$\tau \subseteq D$ with $|\tau| \leqslant \lambda_\alpha$'. There is a $q \geqslant_\alpha p$ such that $q \Vdash$ '$\tau \subseteq F$', where $|F| \leqslant \lambda_\alpha$, by Lemma 1.4. Extend q to an $r = (X, A)$ with $F \subseteq A$; then we are done by (e). $\square$

Lemma 1.8 *The generic graph is $\aleph_1$-chromatic.*

Assume that $p \Vdash$ '$f: \lambda \to \omega$ is a good colouring'. We let $p_0 = p$ and, by induction, define p_n, x_n and α_n with $p_n = (A^n, X^n)$, $p_n \leqslant p_{n+1}$ and $\alpha_n < \alpha_{n+1}$ for $n < \omega$ and such that either $p_{n+1} \Vdash$ '$f(x_n) = n$ and $x_n \in D_{\alpha_{n+1}} - A^n$', or else $p_{n+1} \Vdash$ '$f(x_n) \neq n$ and $x_n \in D_{\alpha_{n+1}} - A^n - E_{\alpha_n}$ and for every $x \in D - A^n - E_{\alpha_n}$ we have $h(x) \neq n$'. This can easily be done. Put $q = (B, Y)$, where $\alpha = \sup(\alpha_n)$, $y \in D_\alpha - \bigcup\{A^n_{\alpha_n} : n < \omega\}$, $B = \bigcup A^n \cup \{y\}$ and $Y = \bigcup\{X^n : n < \omega\} \cup \{\{x_n, y\} : n < \omega\}$. Obviously, q is a condition and $q \geqslant p_n$ for $n < \omega$. If $r \geqslant q$ forces $f(y) = n$, then r forces a contradiction. $\square$

That completes the proof of Theorem 1.1. $\square$

2 Incompactness in singular cardinals under $V = L$.

Theorem 2.1 ($V = L$) *If $\kappa = \mathrm{cf}(\kappa)$ is not weakly compact, $\omega \leqslant \theta < \kappa$ and $\lambda > \mathrm{cf}(\lambda) = \kappa$, then there is a θ^+-chromatic graph of power λ in which every subgraph of power less than λ is $\leqslant \theta$-chromatic.*

Definition If f and g are functions on a common domain, a set of ordinals of limit type, then $f <^* g$ denotes that there is a $\beta \in \mathrm{Dom} f$ such that $f(\beta') < g(\beta')$ holds for every $\beta' > \beta$.

Lemma 2.2 ([16]) ($V = L$) *Assume that λ_i ($i \leqslant \mu$) is an increasing continuous sequence of singular cardinals. Put*

$$\Gamma = \{f : \mathrm{Dom} f = \mu,\ f(i) < \lambda_i^+ \ (i < \mu)\}.$$

Then there is a $<^*$-increasing, $<^*$-cofinal sequence $\{f_\xi : \xi < \lambda_\mu^+\}$ in Γ such that for every $\xi < \lambda_\mu^+$ the system $\{f_\zeta : \zeta < \xi\}$ can be disjointed, that is, there is a function $g: \xi \to \mu$ such that if $\zeta_0 < \zeta_1 < \xi$ and $i > g(\zeta_0), g(\zeta_1)$, then $f_{\zeta_0}(i) < f_{\zeta_1}(i)$ holds.

By the result of in Section 3, there is a graph G on κ with $\mathrm{Chr}(G) = \theta$ and $\mathrm{Chr}(G \restriction \alpha) \leqslant \theta^+$ for $\alpha < \kappa$, and if, for $i < \kappa$, $G(i) := \{j < i : \{j, i\} \in G\}$, then $G(i)$ is either empty or of type θ.

Let $\{\lambda_i : i < \kappa\}$ be a continuous, increasing sequence of singular cardinals, converging to λ, with $\lambda_0 > \kappa$. Put $A_i = \{i\} \times \lambda_i^+ \times \kappa$. We are going to build a graph H on $\bigcup\{A_i : i < \kappa\}$ such that, for every $x \in A_i$, there are g_x and h_x defined on $G(i)$, with $g_x(j) < \lambda_j^+$ and $h_x(j) < \kappa$, and the vertices in $\bigcup\{A_j : j < i\}$ joined to x are $H(x) = \{\langle j, g_x(j), h_x(j)\rangle : j \in G(i)\}$. As there is a natural projection of H onto G, mapping A_i onto i, $\mathrm{Chr}(H) \leqslant \theta^+$ is obvious. We stipulate that $h_x(j) > i$ holds for $x \in A_i$ and $j \in G(i)$.

Definition $X \subseteq A_i$ is *large* if, for every $\xi < \lambda_i^+$ and $\nu < \kappa$, there is an $\langle i, \xi', \nu'\rangle \in X$ with $\xi' > \xi$ and $\nu' > \nu$.

We add the following stipulation on H. Let $\{f^i_\xi : \xi < \lambda_i^+\}$ be a $<^*$-cofinal sequence, as in Lemma 2.2, for $G(i) \neq \emptyset$. So $\mathrm{Dom} f^i_\xi = G(i)$ and $f^i_\xi(j) < \lambda_j^+$.

(a) For $x \in A_i$ there are $\gamma_x < \delta_x < \lambda_i^+$ such that $f^i_{\gamma_x} <^* g_x <^* f^i_{\delta_x}$ and the intervals $[\gamma_x, \delta_x]$ $(x \in A_i)$ are pairwise disjoint;

(b) if, for $j \in G(i)$, $B_j \subseteq A_j$ is large, then

$$\{x \in A_i : H(x) \subseteq \bigcup \{B_j : j \in G(i)\}\}$$

is large.

This selection can be made by an obvious transfinite recursion. The graph H is already constructed: we first show that $\mathrm{Chr}(\theta) = \theta^+$. If $F: \bigcup \{A_i : i < \kappa\} \to \theta$ is a good colouring, by recursion on $i < \kappa$ we can choose a large $X_i \subseteq A_i$ such that

(c) F on X_i is constant;

(d) if $x \in X_i$, $H(x) \subseteq \{X_j : j \in G(i)\}$.

One only needs to notice that the union of θ non-large sets is not large, either. By (c), we have a θ-colouring on G and so we are finished by $\mathrm{Chr}(G) = \theta^+$.

We finally show that every $B \subseteq \bigcup \{A_i : i < \kappa\}$ with $|B| < \lambda$ spans a subgraph which is θ-chromatic. Let $|B| \leqslant \lambda_i$. The graph on $B \cap \bigcup \{A_j : j \leqslant i\} \subseteq \bigcup \{A_j : j \leqslant i\}$ is θ-chromatic by our assumptions on G (using the projection). Assume now that $B \subseteq \bigcup \{A_j : j > i\}$ (and that $|B| \leqslant \lambda_i$). For every $j > i$, there is, by Lemma 2.2, a disjointing function ξ_x $(x \in B \cap A_j)$ for g_x. Decompose the edges of $H \restriction B$ into two classes: $\{y, x\} \in H_1$ if $y = \langle j, g_x(j), h_x(j)\rangle$ if $j \leqslant \xi_x$ and $\{y, x\} \in H_2$ otherwise. Now H_1 has the property that there is a well-ordering (the ordered sum of $A_j \cap B$ $(j > i)$) such that every vertex is joined to less than θ smaller vertices. As is well known, this implies that $\mathrm{Chr}(H_1) \leqslant \theta$. It suffices to show that $\mathrm{Chr}(H_2) \leqslant \theta$. If $\{y, x\} \in H_2$, $y \in A_j$, $x \in A_i$, $j < i$ and if $y = \langle j, g_x(j), h_x(j)\rangle$, then, given y and i, there is at most one x, and $i < h_x(j)$. Therefore, every vertex has not more than θ edges 'going down' and less than κ edges 'going up'. So every connected component is of size less than κ. By the properties of G, every component is $\leqslant \theta$-chromatic. Thus so is H_2. □

Note Even $\bigcap_i |B \cap A_i| \leqslant \lambda_i$ implies that $\mathrm{Chr}(G \restriction B) \leqslant \theta$.

3 Large gaps in regular cardinals under $V = L$

Theorem 3.1 $(V = L)$ *If κ is a cardinal then there is a graph G on κ^+ such that $\mathrm{Chr}(G) = \kappa^+$ but, for every $\alpha < \kappa$, $\mathrm{Chr}(G \restriction \alpha) \leqslant \omega$ holds.*

Proof We use the following principle deduced from $V = L$ in [1].

(⊠) There is a sequence $\langle C_\delta, M_\delta : \delta < \kappa^+, \text{limit}\rangle$ such that

(a) $C_\delta \subseteq \delta$ is a club;

(b) if $\alpha \subseteq C'_\delta$ then $C_\alpha = C_\delta \cap \alpha$;

(c) M_δ is a model on δ;

(d) if $\alpha \in C'_\delta$, then $M_\alpha \prec M_\delta$;

(e) if M is a model on κ^+ with vocabulary $\leqslant \kappa$, then

$$\{\delta < \kappa^+ : \mathrm{tp}(C_\delta) = \kappa,\ M_\delta \prec M\}$$

is stationary.

We assume that for every limit δ, $M_\delta = \langle \delta, f_\delta\rangle$, where f_δ is a function from δ into κ. We define for every $\delta < \kappa^+$ (δ a limit) $g_\delta : C'_\delta \to \kappa^+$ as follows. Let $B = \{\delta < \kappa^+ : \mathrm{tp}(C_\delta) = \kappa\}$ and, for $\delta \in B$, let $h^*(\delta) = \min C'_\delta$. Then

(f) if $\alpha \in C'_\delta$, then $g_\alpha \subseteq g_\delta$;

(g) if $\mathrm{tp}(C'_\delta) = \xi + 1$ and $\epsilon = \max(C'_\delta)$, then

$$g_\delta(\epsilon) = \min\{\tau : \tau \in B,\ h^*(\tau) \geqslant \epsilon,\ f_\delta(\tau) = \xi\}$$

if such a τ exists and is undefined otherwise.

To define G we join every $\delta < \kappa^+$ with $\mathrm{tp}(C_\delta) = \kappa$ into the vertex set $\{g_\delta(\xi) : \xi \in C'_\delta\}$.

We show that $\mathrm{Chr}(G) = \kappa^+$. Assume that $f: \kappa^+ \to \kappa$ is a good colouring. Select a δ as in (e). Then, for every $\xi < \kappa$ with $E_\xi = \{h^*(\delta) : \delta \in B, f(\delta) = \xi\}$ unbounded (in κ^+), $g_\delta(\xi)$ is defined and so $f(\delta) = \xi$ is ruled out by construction. If E_ξ is bounded, then this bound is less than $h^*(\delta)$, and so $f(\delta) = \xi$ is impossible again.

We now turn to the proof of the other property.

Definition $F: \alpha \to \omega$ is *suitable* if it is a good colouring and, for every limit $\beta \leqslant \alpha$, $|\omega - \{F(g_\beta(\xi)) : \xi \in C'_\beta\}| = \omega$.

The following claim clearly suffices for the proof.

Claim *If $\beta < \alpha$, $\mathrm{tp}(C_\beta) \neq \kappa$, F is a suitable colouring of β and F' is a colouring of a finite subset of $[\beta, \kappa^+)$ such that $F \cup F'$ is a good colouring, then there is a good colouring on α, compatible with $F \cup F'$.*

Proof of the claim (by transfinite induction on α) If $\alpha = \alpha' + 1$, add α' to the domain of F' and apply the claim.

Assume that α is a limit. Enumerate C_α as $\{\gamma_\xi : \xi < \mathrm{tp}(C_\alpha)\}$ and suppose that $\gamma_\zeta \leqslant \beta < \gamma_{\zeta+1}$ ($\gamma_0 = 0$ is assumed). As F is suitable on β, $A = \omega - F(g_\alpha(\gamma_{\omega\xi})) : \omega\xi < \zeta\}$ is infinite. Select $k^* \in A$. Applying the claim we can extend F from β to $\gamma_{\zeta+1}$, from $\gamma_{\zeta+1}$ to γ_{ζ_2}, and so on, but colouring vertices $g_\alpha(\gamma_{\omega\epsilon})$ $(\gamma_\zeta \leqslant \epsilon \in C'_\alpha)$ only with the colour k^*. For a limit ordinal $\xi \leqslant \mathrm{tp}(C_\alpha)$, $\{F(g_\alpha(\gamma_{\omega\tau})) : \omega\tau < \xi\}$ contains only one element of A. The inductive step is possible as $g_\alpha(\gamma_{\omega\xi})$ is connected by an edge to no ordinal less than or equal to $h^*(g_\alpha(\gamma_{\omega\xi}))$ which is greater or equal to $\gamma_{\omega\xi}$. □

Theorem 3.2 $(V = L)$ *If κ is an inaccessible, not weakly compact cardinal, then there is a graph G on κ with* $\mathrm{Chr}(G) = \kappa$, *but for $\alpha < \kappa$,* $\mathrm{Chr}(G \restriction \alpha) \leqslant \omega$.

Proof Similar to the proof of the previous theorem, only we use the appropriate principle with

(e*) if M is a model on κ with vocabulary $\leqslant \kappa$ and $\mu < \kappa$, then

$$\{\delta < \kappa : \mathrm{tp}(C_\delta) = \mu,\ M_\delta \prec M\}$$

is stationary. □

Remark It is easy to modify the construction to get graphs as in Theorems 3.1 and 3.2 with arbitrary chromatic number less than $|G|$.

4 Non-spanned subgraphs

Theorem 4.1 $(V = L)$ *If G is a graph on $\lambda = \mathrm{cf}(\lambda) > \omega$ with $\mathrm{Chr}(G) \geqslant \theta \geqslant \omega$ and, for every $\alpha < \lambda$ we have $\mathrm{Chr}(G \restriction \alpha) < \theta$, then there exists a subgraph G' of G with $\mathrm{Chr}(G') = \theta$.*

Proof We are going to use the following consequence of $V = L$, proved like the proof of ◇ by R. L. Jensen. Let $L_{\mathrm{a}} \subseteq L_{\mathrm{b}} \subseteq L_{\mathrm{c}}$ be extensions of ZF vocabulary by finitely many new symbols. $M^{\mathrm{a}}(\delta)$ denotes a model of L_{a} and similarly for $M^{\mathrm{b}}(\delta)$, etc.

Lemma 4.2 $(V = L)$ *If $\lambda = \mathrm{cf}(\lambda) > \omega$, M^{a} is a model on λ and φ is a first-order sentence in L_{c}, then there exist models*

$$\langle M^{\mathrm{c}}_\xi(\delta) : \xi < \epsilon_\delta,\ \delta < \lambda \text{ limit}\rangle$$

such that

(a) $M^{\mathrm{c}}_\xi(\delta)$ expands $M^{\mathrm{a}} \restriction \delta$;

(b) for $\xi \neq \zeta$, $M^{\mathrm{c}}_\xi(\delta) \restriction L_{\mathrm{b}} \neq M^{\mathrm{c}}_\zeta(\delta) \restriction L_{\mathrm{b}}$;

(c) if M^{c} expand M^{a} satisfies φ, then there is an $N^{\mathrm{c}} \supseteq M^{\mathrm{a}}$ satisfying φ, such that for a closed unbounded set of δ there is a $\xi < \epsilon_\delta$ with $M^{\mathrm{c}}_\xi(\delta) = N^{\mathrm{c}} \restriction \delta$.

If ◇* holds, we can take $\epsilon_\delta = \delta$.

Definition Let C and D be closed, unbounded sets in $\lambda = \mathrm{cf}(\lambda)$ and $\tau(\alpha) = \min(C - (\alpha+1))$ for $\alpha < \lambda$. Then

$$\Delta(C,D) = \tau(0) \cup \bigcup \{[\alpha, \tau(\alpha)) : \alpha \in D\}.$$

Lemma 4.3 *If I has the property that for every club C there is a club D such that $\Delta(C,D) \in I$, then λ is the union of countably many elements of I.*

Proof Let $C_0 = \lambda$ and let C_{n+1} satisfy $\Delta(C_n, C_{n+1}) \in I$. If $\alpha \notin \bigcup \Delta(C_n, C_{n+1})$ and $\alpha_n = \max(\alpha \cap C_n)$, then $\alpha = \alpha_0 > \alpha_1 > \cdots$: a contradiction. □

In order to prove the result, we try to formulate the fact that no subgroup G' of G has $\mathrm{Chr}(G') = \theta$. $\mathrm{Chr}(G') \leqslant \theta$ means that there is an $F\colon \lambda \to \theta$ good colouring of G'. On the other hand, given F, we may assume that G' consists of the edges $\{\alpha,\beta\}$ with $\{\alpha,\beta\} \in G$ *and* $F(\alpha) \neq F(\beta)$. So the property can be translated as follows. For every $F\colon \lambda \to \theta$ there is a $\sigma < \theta$ and an $H\colon \lambda \to \sigma$ with

(d) if $\{\alpha,\beta\}$ is in G and $F(\alpha) \neq F(\beta)$, then $H(\alpha) \neq H(\beta)$.

We now let M^{a} be G, $L_{\mathrm{b}} = L_{\mathrm{a}} \cup \{F, \theta\}$, $L_{\mathrm{c}} = L_{\mathrm{b}} \cup \{H\}$ and φ the sentence in (d).

If I is the collection of subsets of λ spanning subgraphs with chromatic number less than θ, then by Lemma 4.3 and the fact that $\mathrm{Chr}(G) > \theta$ (otherwise we are done), there is a club C such that, for no club D, $\Delta(C,D) \in I$ holds. Enumerate $C \cup \{0\}$ as $\{\gamma_\alpha : \alpha < \lambda\}$. We construct an $F\colon \lambda \to \theta$ by recursively defining $F \restriction [\gamma_\alpha, \gamma_{\alpha+1})$.

If there exists a ξ such that $M^{\mathrm{c}}_\xi(\gamma_\alpha) \restriction L_{\mathrm{b}} = (M^{\mathrm{a}} \restriction \delta, F \restriction \gamma_\alpha, \theta)$, then the range of H in $M^{\mathrm{c}}_\xi(\gamma_\alpha)$ is bounded (in θ) and ξ is unique by (b). For $\gamma_\alpha \leqslant \tau < \gamma_{\alpha+1}$ we then put

$$B(\tau) = \{\beta < \gamma_\alpha : \{\beta,\tau\} \text{ is in } G, \text{ and no } \beta' < \beta \text{ has } \{\beta',\tau\} \in G$$
$$\text{and } M^{\mathrm{c}}_\xi(\delta_\alpha) \models H(\beta') = H(\beta).\}$$

$|B(\tau)| < \theta$ as $\mathrm{Rang}\, H$ is bounded (in θ). Now define

$$F(\tau) = \min\{\theta - F''B(\tau)\}.$$

If no such ξ exists, any extension works.

Having constructed $F\colon \lambda \to \theta$, by our indirect assumption there is $H\colon \lambda \to \sigma < \theta$, a 'better colouring of G'', determined by F. So, by Lemma 4.2, there is (a possibly different) H such that, for a closed unbounded D, if $\delta \in D$ there is a ξ with $M^{\mathrm{c}}_\xi(\delta) = M_{\mathrm{c}} \restriction \delta$. We assume that $D \subseteq C$.

Claim $\mathrm{Chr}(G \restriction \Delta(C,D)) < \theta$.

Clearly this claim gives the desired contradiction.

Proof of the claim As $\mathrm{Chr}(G \restriction \alpha) < \theta$ for every $\alpha < \lambda$ and $\mathrm{cf}(\lambda) > \theta$, there exists a $\sigma < \theta$ such that $\mathrm{Chr}(G \restriction \alpha) \leqslant \sigma$ $(\alpha < \lambda)$. From this, those edges joining vertices in the *same* interval of $\Delta(C,D)$ can get a good colouring by not more than σ colours. It suffices to show that H is a good colouring for the edges between different intervals. Otherwise there is $\{\tau', \tau\} \in G$ with $\tau' < \gamma_\alpha \leqslant \tau < \gamma_{\alpha+1}$ $(\gamma_\alpha \in D)$ and $H(\tau') = H(\tau)$. Fix τ and take τ' *minimal*. Then $F(\tau') \neq F(\tau)$ and so $H(\tau') \neq H(\tau)$: a contradiction. □

5 Compactness is consistent

We mention that Foreman & Laver showed, from an almost huge cardinal, the consistency of GCH and the statement that every graph with power and chromatic number $\aleph_2$ contains a subgraph of power and chromatic number $\aleph_1$. See also [7, 8, 15]. We use the following result.

Lemma 5.1 (Ben-David & Magidor [3]) *If the existence of a supercompact cardinal is consistent, then so is ZFC+GCH and that for every regular $\lambda > \omega_\omega$ there is an ultrafilter D on λ such that $\aleph_n^\lambda/D = \aleph_n$ for $0 < n < \omega$, and there are $A_\xi \in D$ $(\xi < \lambda)$ such that, for $\alpha < \lambda$, the set $w_\alpha = \{\xi : \alpha \in A_\xi\}$ has power less than $\aleph_\omega$.*

Theorem 5.2 *In the model of Lemma 5.1, if $0 < n < \omega$, G is a graph such that every subgraph of G of power less than $\aleph_\omega$ has chromatic number not exceeding $\aleph_n$, then* $\mathrm{Chr}(G) \leqslant \aleph_n$.

Proof We may assume that the vertex set of G is λ, a regular cardinal (otherwise we may use λ^+). By assumption, there is a good colouring, $f_\alpha : w_\alpha \to \omega_n$. For $\xi < \lambda$, let $g_\xi(\alpha) = f_\alpha(\xi)$ and, finally, define $h: \lambda \to \omega_n^\lambda/D$ by $h(\xi) = g_\xi/D$. This h is a good colouring of G by ω_n colours. [It is a good colouring because, if we assume that $\xi, \zeta < \lambda$, $\{\xi, \zeta\} \in G$ and $h(\xi) = h(\zeta)$, then

$$\begin{aligned}\{\alpha : g_\xi(\alpha) = g_\zeta(\alpha)\} &= \{\alpha : f_\alpha(\xi) = f_\alpha(\zeta)\} \\ &\subseteq \{\alpha : \{\xi, \zeta\} \not\subseteq W_\alpha\} \\ &= \{\alpha : \alpha \notin A_\xi \text{ or } \alpha \notin A_\zeta\} \\ &\subseteq \{\alpha : \alpha \notin A_\xi \cap \in A_\zeta\} = \varnothing \pmod D.\end{aligned}$$

Hence $g_\xi/D \neq g_\zeta/D$. The number of colours is $\aleph_n$ because, as $|\omega_n^\lambda/D| = \aleph_n$, the function h has the right number of colours.]

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