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THE SPECTRUM OF ULTRAPRODUCTS OF FINITE CARDINALS FOR AN ULTRAFILTER

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Abstract. We complete the characterization of the possible spectrum of regular ultrafilters D on a set I, where the spectrum is the set of ultraproducts of (finite) cardinals modulo D which are infinite.

1. Introduction

1.1. Background, questions and results. Ultraproducts were very central in model theory in the sixties, usually for regular ultrafilters. The question of ultraproducts of infinite cardinals was resolved (see [1]): letting D be a regular ultrafilter on a set I (for transparency we ignore the case of a filter)

$(*)_1$ if $\bar\lambda = \langle \lambda_s : s \in I\rangle$ and $\lambda_s \geq \aleph_0$ for $s \in I$ *then* $\prod_{s\in I} \lambda_s/D = \mu^{|I|}$ when $\mu = \limsup_D(\bar\lambda) := \sup\{\chi$: the cardinal χ satisfies $\{s \in I : \lambda_s \geq \chi\} \in D\}$.

What about the ultraproducts of finite cardinals? Of course, under naive interpretation, if $\{\lambda_s : \lambda_s = 0\} \neq \emptyset$ the result is zero, so for notational simplicity we always assume $s \in I \Rightarrow \lambda_s \geq 1$. Also for every $n \geq 1$, letting $\lambda_s = n$ for $s \in I$ we have $\prod_s \lambda_s/D = n$ so the question was

Question 1.1. *Given an infinite set I*

(a) [*the singleton problem*] *what infinite cardinals* μ *belong to* $\mathscr{C}_I = \mathscr{C}_I^{\rm car}$, *i.e. can be represented as* $\big\{ \prod_{s\in I} \lambda_s/D : D$ *a regular ultrafilter on* I, $1 \leq \lambda_s < \aleph_0\big\} \setminus \{\lambda : 1 \leq \lambda < \aleph_0\}$.

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(b) [*the spectrum problem*] *moreover what are the possible spectra, i.e. which sets of cardinals belong to* $\mathbf{C}_I$ *which is the family of sets* $\mathscr{C}$ *such that for some* D*, a regular ultrafilter on* I *we have* $\mathscr{C} = \mathrm{upf}(D)$ *where* $\mathrm{upf}(D) = \{\prod_{s\in I} \lambda_s/D : 1 \leq \lambda_s < \aleph_0$ *for* $s \in I\}\backslash\{\lambda : 1 \leq \lambda < \aleph_0\}$

Keisler [3] asks and has started on 1.1 (assuming GCH was prevalent at the time as the situation was opaque otherwise):

$(*)_2$ assume GCH, a sufficient condition for $\mathscr{C} \in \mathbf{C}_I$ is
- (a) $\mathscr{C}$ is a set of successor (infinite) cardinals,
- (b) $\max(\mathscr{C}) = |I|^+$,
- (c) if $\mu = \sup\{\chi < \mu : \chi \in \mathscr{C}\}$ then $\mu^+ \in \mathscr{C}$,
- (d) if $\mu^+ \in \mathscr{C}$ then $\mu \cap \mathscr{C}$ has cardinality $< \mu$.

Keisler used products and D-sums of ultrafilters. Concerning the problem for singletons a conjecture of Keisler [3, bottom of p. 49] was resolved in [6]:

$(*)_3$ $\mu = \mu^{\aleph_0}$ *when* $\mu \in \mathscr{C}_I$, i.e. when $\mu = \prod_{s\in I} \lambda_s/D$ is infinite, D an ultrafilter on I, each λ_s finite non-zero.

The proof uses coding enough "set theory" on the n's and using the model theory of the ultra-product. This gives a necessary condition (for the singleton version), but is it sufficient? This problem was settled in [8, Ch. V, §3] = [9, Ch. VI, §3] proving that this is also a sufficient condition (+ the obvious condition $\mu \leq 2^{|I|}$), that is

$(*)_4$ $\mu \in \mathscr{C}_I := \bigcup\{\mathscr{C} : \mathscr{C} \in \mathbf{C}_I\}$ *iff* $\mu = \mu^{\aleph_0} \leq 2^{|I|}$.

The constructions in [8, Ch. VI, §3] = [9, Ch. VI, §3], use a family $\mathscr{F}$ of functions with domain I and a filter D on I such that $\mathscr{F}$ is independent over D (earlier Kunen used such family $\mathscr{F} \subseteq {}^\lambda\lambda$ for constructing a good ultrafilter on λ in ZFC; earlier Engelking–Karlowicz proved the existence). In particular in the construction in [8, Ch. VI, §3] of maximal such filters and the Boolean algebra $\mathbb{B} = \mathscr{P}(\lambda)/D$ are central. We decrease the family and increase D; specifically we construct $\mathscr{F}_\ell$ $(\ell \leq n)$ decreasing with ℓ, D_ℓ a filter on I increasing with ℓ, D_ℓ a maximal filter such that $\mathscr{F}_\ell$ is independent mod D_ℓ; so if $\mathscr{F}_n = \emptyset$ then D_0 is an ultrafilter and we have $\mathbb{B}_\ell = \mathscr{P}(I)/D_\ell$ is essentially $\lessdot$-decreasing and in the ultrapowers $\mathbb{N}^I/D_\ell$ the part which $\mathbb{B}_\ell$ induces for $\ell \leq n$, is a sequence of initial segments of $\mathbb{N}^{\mathbb{B}}/D_0$ decreasing with ℓ.

In [8, Ch. VI, Exercise 3.35] this is formalized:

$(*)_5$ if D_0 is a filter on $I, \mathbb{B}_0 = \mathscr{P}(I)/D_0, D_1 \supseteq D_0$ an ultrafilter, $D = \{A/D_0 : A \in D_1\}$ so $D \in \mathrm{uf}(\mathbb{B}_0)$ then $\mathbb{N}^{\mathbb{B}_0}/D_0^+$ is an initial segment of $\mathbb{N}^I/D$; (also $\mathbb{B}$ satisfies the c.c.c., but this is just to ensure $\mathbb{B}$ is complete, anyhow this holds in all relevant cases here).

It follows that we can replace $\mathscr{P}(I)$ by a Boolean algebra $\mathbb{B}_1$ extending $\mathbb{B}_0$. The Boolean algebra related to $\mathscr{F}$ is the completion of the Boolean algebra generated by $\{x_{f,a} : f \in \mathscr{F},\ a \in \mathrm{Rang}(f)\}$ freely except $x_{f,a} \cap x_{f,b} = 0$ for $a \neq b \in \mathrm{Rang}(f)$ and $f \in \mathscr{F}$. So if $\mathrm{Rang}(f)$ is countable for every $f \in \mathscr{F}$, the Boolean algebra satisfies the $\aleph_1$- c.c. (in fact, is free), this was used there to deal with $\ell\text{cf}(\kappa, D)$ for $\kappa = \aleph_0$ (for $\kappa > \aleph_0$ we need $\text{Rang}(f) = \kappa$) and is continued lately in works of Malliaris–Shelah. But for $\text{upf}(D)$ only the case of f's with countable range is used.

The problem of the spectrum (i.e. 1.1(b)) was not needed in [8, Ch. VI, §3] for the model theoretic problems which were the aim of [8, Ch. VI], still the case of finite spectrum was resolved there (also cofinality, i.e. $\text{lcf}(\kappa, D)$ was addressed).

This was continued by Koppelberg [4] using a possibly infinite $\lessdot$-increasing chains of complete Boolean algebras; also she uses a system of projections instead of maximal filters but this is a reformulation as this is equivalent, see 1.11 below. Koppelberg [4] returns to the full spectrum problem proving:

$(*)_6$ $\mathscr{C} \in \mathbf{C}_I$ *when* $\mathscr{C}$ satisfies:

(a) $\mathscr{C} \subseteq \text{Card}$,

(b) $\max(\mathscr{C}) = 2^{|I|}$,

(c) $\mu = \mu^{\aleph_0}$ if $\mu \in \mathscr{C}$,

(d) if $\mu_n \in \mathscr{C}$ for $n < \omega$ then $\prod_n \mu_n \in \mathscr{C}$.

Central in the proof is $(*)_5$ above ([8, Ch. VI, Ex. 3.35, p. 370]). The result of Koppelberg is very strong, still the full characterization is not obtained; also Kanamori in his math review of her work asked about it.

Here we give a complete answer to the spectrum problem 1.1(b), that is, Theorem 2.20 gives a full ZFC answer to 1.1, that is.

THEOREM 1.2. *For any infinite set* I, $\mathscr{C} \in \mathbf{C}_I$ iff $\mathscr{C}$ *is a set of cardinals such that* $\mu \in \mathscr{C} \Rightarrow \mu = \mu^{\aleph_0} \leq 2^{|I|}$ *and* $2^{|I|} \in \mathscr{C}$.

We now comment on some further questions on ultra-powers.

The problem of cofinalities was central in [8, Ch. VI, §3] in particular $\ell\text{cf}(\aleph_0, \lambda)$ (see 1.6 below). [Why? E.g. if $\text{Th}(M)$, the complete first order theory of the model M is unstable then M^I/D is not $\ell\text{cf}(\aleph_0, \lambda)^+$-saturated.] Another question was raised by the author [7, p. 97] and independently by Eklof [2]:

QUESTION 1.3. *Assume* $f_n \in {}^I\mathbb{N}$, $f_{n+1} <_D f_n$ *and* $\mu \leq \prod_{s \in I} f_n(s)/D$ *for every* n then *is there* $f \in {}^I\mathbb{N}$ *such that* $f <_D f_n$ *for every* n *and* $\mu \leq \prod_{s \in I} f(s)/D$*?*

The point in [7, p. 75] was investigating saturation of ultrapowers (and ultraproducts) and Keisler order on first order theories. The point in [2] was ultraproduct of Abelian groups.

To explain the cofinalities problem, see 1.4. We can consider the following: for D a regular ultrafilter on I we consider $M = \mathbb{N}^\lambda/D$; for $a \in M$ let $\lambda_a = |\{b : b <_M a\}|$ and define $E_M = \{(a, b) : a, b \in M \text{ and } \lambda_a = \lambda_b \geq \aleph_0\}$. So E_M is a convex equivalence relation, and the equivalence classes are linearly ordered and let $A_{D,\lambda} = \{a \in M : \lambda_a = \lambda\}$. So $\mathrm{upf}(D) = \{\lambda_a : A_{D,\lambda} \neq \emptyset\}$ and Question 1.3 asks: can the co-initiality of some $A_{D,\lambda}$ be $\aleph_0$. As M is $\aleph_1$-saturated, in this case the cofinality of $M \restriction \{c : \lambda_c < \lambda_a$ (hence $c <_M a)\}$ is $\ell\mathrm{cf}(\aleph_0, D)$ which is the co-initiality of $A_{D,\min(\mathrm{upf}(D))}$.

So a natural question is

QUESTION 1.4. *What are the possible* $\mathrm{spec}_1(D) = \{(\lambda, \theta, \partial) : \lambda \in \mathrm{upf}(D)$, *$\partial$ the cofinality of $A_{D,\lambda}$ and θ the co-initiality of $A_{D,\lambda}\}$ for D a regular ultrafilter on I?*

A further question is:

QUESTION 1.5. *Assume* $\kappa = \mathrm{cf}(\kappa) < \lambda_1 = \lambda_1^{\aleph_0} < \lambda_2 = \lambda_2^{\aleph_0}, \lambda_1^{<\kappa>_{\mathrm{tr}}} \leq 2^\lambda$. *Is there a regular ultrafilter D on λ such that for $n_i \in \mathbb{N}$ for $i < \lambda$ we have $\prod_i n_i/D = \lambda_1$ and $\prod_i 2^{n_i}/D = \lambda_2$?*

This work was presented in the May 2013 Eilat Conference honoring Mati Rubin's retirement. In a work in preparation [5], we try to build a counterexample to Question 1.3.

1.2. Preliminaries. We define $\ell\mathrm{cf}(\kappa, D)$ and $M^{\mathbb{B}}/D$, when $\mathbb{B}$ is a Boolean algebra and more.

DEFINITION 1.6. For D an ultrafilter on I, κ a regular cardinal let $\mu = \ell\mathrm{cf}(\kappa, D)$ be the co-initiality of the linear order $(\kappa^I/D) \restriction \{f/D : f \in {}^I\kappa$ is not D-bounded by any $\varepsilon < \kappa\}$.

NOTATION 1.7. 1) $\mathbb{B}$ denotes a Boolean algebra, usually complete; let $\mathrm{comp}(\mathbb{B})$ be the completion of $\mathbb{B}$.

2) $\mathrm{uf}(\mathbb{B})$ is the set of ultrafilters on $\mathbb{B}$.

3) Let $\mathbb{B}^+ = \mathbb{B} \backslash \{0_{\mathbb{B}}\}$.

4) Let $\mathrm{cc}(\mathbb{B}) = \min\{\kappa : \mathbb{B}$ satisfies the κ-c.c.$\}$, necessarily a regular cardinal.

DEFINITION 1.8. For a Boolean algebra $\mathbb{B}$ a filter D on $\mathbb{B}$ a model or a set M.

1) Let $M^{\mathbb{B}}$ be the set of partial functions f from $\mathbb{B}^+$ into M such that for some maximal antichain $\langle a_i : i < i(*)\rangle$ of $\mathbb{B}, \mathrm{Dom}(f)$ includes $\{a_i : i < i(*)\}$ and is included in[1] $\{a \in \mathbb{B}^+ : (\exists i)(a \leq a_i)\}$ and $f \restriction \{a \in \mathrm{Dom}(f) : a \leq a_i\}$ is constant for each i.

1A) Naturally for $f_1, f_2 \in M^{\mathbb{B}}$ we say f_1, f_2 are D-equivalent, or $f_1 = f_2 \bmod D$ *when* for some $b \in D$ we have $a_1 \in \mathrm{Dom}(f_1) \wedge a_2 \in \mathrm{Dom}(f_2) \wedge a_1 \cap a_2 \cap b > 0_{\mathbb{B}} \Rightarrow f_1(a_1) = f_2(a_2)$.

[1] For the $D_\ell \in \mathrm{uf}(\mathbb{B}_\ell)$ ultra-product, without loss of generality $\mathbb{B}$ is complete, then without loss of generality $f \restriction \{a_i : i < i(*)\}$ is one to one.

1B) Abusing notation, not only $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$ but $M^{\mathbb{B}_1}/D_1 \subseteq M^{\mathbb{B}_2}/D_2$ when $\mathbb{B}_1 \lessdot \mathbb{B}_2$, $D_\ell \in \mathrm{uf}(\mathbb{B}_\ell)$ for $\ell = 1, 2$ and $D_1 \subseteq D_2$, that is, for $f \in M^{\mathbb{B}_1}$ we identify f/D_1 and f/D_2.

2) For D an ultrafilter on the completion of the Boolean algebra $\mathbb{B}$ we define $M^{\mathbb{B}}/D$ naturally, as well as $\mathrm{TV}(\varphi(f_0, \dots, f_{n-1})) \in \mathrm{comp}(\mathbb{B})$ *when* $\varphi(x_0, \dots, x_{n-1}) \in \mathbb{L}(\tau_M)$ and $f_0, \dots, f_{n-1} \in M^{\mathbb{B}}$ where TV stands for truth value and $M^{\mathbb{B}}/D \models \varphi[f_0/D, \dots, f_{n-1}/D]$ iff $\mathrm{TV}_M(\varphi(f_0, \dots, f_{n-1})) \in D$.

3) We say $\langle a_n : n < \omega \rangle$ D-represents $f \in \mathbb{N}^{\mathbb{B}}$ *when* $\langle a_n : n < \omega \rangle$ is a maximal antichain of $\mathbb{B}$ (allowing $a_n = 0_{\mathbb{B}}$) and for some $f' \in \mathbb{N}^{\mathbb{B}}$ which is D-equivalent to f (see 1.8(1A)) we have $f'(a_n) = n$. We may omit D if $D = \{1_{\mathbb{B}}\}$ and say just $\langle a_n : n < \omega \rangle$ represents f.

4) We say $\langle (a_n, k_n) : n < \omega \rangle$ represents $f \in \mathbb{N}^{\mathbb{B}}$ *when*:

(a) the k_n are natural numbers with no repetition,

(b) $\langle a_n : n < \omega \rangle$ is a maximal antichain of $\mathbb{B}$,

(c) $f(a_n) = k_n$.

The proofs in [8, Ch. VI, §] use downward induction on the cardinals.

Observation 1.9. 1) *If $\mathbb{B}$ is a complete Boolean algebra and $f \in \mathbb{N}^{\mathbb{B}}$* then *some sequence $\langle a_n : n < \omega \rangle$ represents f. Some $\langle a_n, b_n \rangle$ represent $f \in \mathbb{N}^{\mathbb{B}}$ when $\mathbb{B}$ is a c.c.c. Boolean algebra.*

2) *For a model M and Boolean algebra $\mathbb{B}_1$ and ultrafilter D on its completion $\mathbb{B}_2$ we have $M^{\mathbb{B}_1}/D = M^{\mathbb{B}_2}/D$.*

Fact 1.10. 1) *If $\mathbb{B}_1 \subseteq \mathbb{B}_2$ are Boolean algebras, $\mathbb{B}$ is a complete Boolean algebra and π_1 is a homomorphism from $\mathbb{B}_1$ into $\mathbb{B}$* then *there is a homomorphism π_2 from $\mathbb{B}_2$ into $\mathbb{B}$ extending π_1.*

2) *There is a homomorphism π_3 from $\mathbb{B}_3$ into $\mathbb{B}$ extending π_ℓ for $\ell =$* $0, 1, 2$ when:

(a) *$\mathbb{B}_0 \subseteq \mathbb{B}_\iota \subseteq \mathbb{B}_3$ are Boolean algebras for $\iota = 1, 2$,*

(b) *$\mathbb{B}_1, \mathbb{B}_2$ are freely amalgamated over $\mathbb{B}_0$ inside $\mathbb{B}_3$,*

(c) *$\mathbb{B}$ is a complete Boolean algebra,*

(d) *π_ℓ is a homomorphism from $\mathbb{B}_\ell$ into $\mathbb{B}$ for $\ell = 0, 1, 2$,*

(e) *$\pi_0 \subseteq \pi_1$ and $\pi_0 \subseteq \pi_2$.*

Proof. 1) Well known. 2) Straightforward. $\square$

Observation 1.11. *Assume $\mathbb{B}_1 \lessdot \mathbb{B}_2$ are Boolean algebras and $\mathbb{B}_1$ is complete.*

1) *The following properties of D are equivalent:*

(a) *D is a maximal filter on $\mathbb{B}_2$ (among those) disjoint to $\mathbb{B}_1 \setminus \{1_{\mathbb{B}_1}\}$,*

(b) *there is a projection π of $\mathbb{B}_2$ onto $\mathbb{B}_1$ such that $D = \{a \in \mathbb{B}_2 : \pi(a) = 1_{\mathbb{B}_1}\}$.*

1A) *Moreover D determines π uniquely and vice versa, in particular $\pi(c)$ is the unique $c' \in \mathbb{B}_1$ such that $c = c'$ mod D.*

2) *If* D *satisfies* (1)(a) *and* D_1 *is an ultrafilter of* $\mathbb{B}_1$, then *there is a one and only one ultrafilter* $D_2 \in \mathrm{uf}(\mathbb{B}_2)$ *extending* $D_1 \cup D$.

PROOF. 1) *Clause* (a) *implies clause* (b): As D is a filter on $\mathbb{B}_2$ clearly for some Boolean algebra $\mathbb{B}_2'$, there is a homomorphism $\mathbf{j}_0 : \mathbb{B}_2 \to \mathbb{B}_2'$ which is onto, such that $a \in \mathbb{B}_2 \Rightarrow (a \in D \leftrightarrow \mathbf{j}_0(a) = 1_{\mathbb{B}_2'})$. As $D \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$ necessarily $\mathbf{j}_0 \restriction \mathbb{B}_1$ is one-to-one. Let $\mathbb{B}_1' = \mathbf{j}_0(\mathbb{B}_1)$ so $\mathbf{j}_1 = (\mathbf{j}_0 \restriction \mathbb{B}_1)^{-1}$ is an isomorphism from $\mathbb{B}_1'$ onto $\mathbb{B}_1$ hence by 1.10(1) there is a homomorphism $\mathbf{j}_2$ from $\mathbb{B}_2'$ onto $\mathbb{B}_1$ extending $\mathbf{j}_1$. Hence $\mathbf{j}_3 = \mathbf{j}_2 \circ \mathbf{j}_0$ is a homomorphism from $\mathbb{B}_2$ onto $\mathbb{B}_1$ extending $\mathrm{id}_{\mathbb{B}_1}$, so it is a projection.

Lastly, $\mathbf{j}_3^{-1}\{1_{\mathbb{B}_1}\}$ is a filter extending D and disjoint to $\mathbb{B}_1 \backslash \{1_{\mathbb{B}_1}\}$. By the maximality of D we have equality.

An alternative proof is: Let $\mathbb{B}_2'$ be the sub-algebra of $\mathbb{B}_2$ generated by $\mathbb{B}_1 \cup D$. Clearly every member of $\mathbb{B}_2'$ can be represented as $(a \cap b) \cup ((1-a) \cap \sigma(\bar{a}, \bar{b}))$ with $a, a_m \in D$ for $m < n = \ell g(\bar{a})$ and $b \in \mathbb{B}_1, b_k \in \mathbb{B}_1$ for $k < \ell g(\bar{b})$, σ a Boolean term such that $\bigwedge_{k<n} a \le a_k$, equivalently $\bigwedge_{k<n} a \cap (1 - a_k) = 0$. We try to define a function π from $\mathbb{B}_2'$ into $\mathbb{B}_1$ by:

$\oplus$ $\pi((a \cap b) \cup ((1-a) \cap \sigma(\bar{a}, \bar{b}))) = b$ for a, $\bar{a}$, b, $\bar{b}$ as above.

We have to prove that π is as promised.

$(*)_1$ π is a well defined (function from $\mathbb{B}_2'$ into $\mathbb{B}_1$).

Why? Obviously for every $c \in \mathbb{B}_2'$ there are $a, \bar{a}, b, \bar{b}, \sigma$ as above, so $\pi(c)$ has at least one definition, still we have to prove that any two such definitions agree. So assume $c = (a_\ell \cap b_\ell) \cup ((1 - a_\ell) \cap \sigma_\ell(\bar{a}_\ell, \bar{b}_\ell))$ for $\ell = 1, 2$ as above so with $a_1, a_2, a_{1,k}, a_{2,m} \in D$ and $b_1, b_2, \bar{b}_1, \bar{b}_2 \in \mathbb{B}_1$ such that $a_1 \le a_{1,k}$, $a_2 \le a_{2,m}$. We should prove that $b_1 = b_2$, if not without loss of generality $b_1 \nleq b_2$ hence $b := b_1 - b_2 > 0$. Clearly $a := a_1 \cap a_2 \in D$ and computing $c \cap b \cap a$ in two ways we get $a \cap b \cap b_1 = a \cap b \cap b_2$ hence $a \cap b = a \cap b \cap b_1 = a \cap b \cap b_2 = a \cap 0 = 0$ recalling $b = b_1 - b_2$, hence $a \le 1 - b$ so as $a \in D$ necessarily $1 - b \in D$. But $b \in \mathbb{B}_1^+$ so $1 - b \in \mathbb{B}_1 \backslash \{1_{\mathbb{B}_1}\}$, contradiction to the assumption on D.

$(*)_2$ π commutes with "$x \cap y$".

Why? Assume that for $\ell = 1, 2$ we have $c_\ell = (a_\ell \cap b_\ell) \cup ((1 - a_\ell) \cap \sigma_\ell(\bar{a}_\ell, \bar{b}_\ell))$ with a_ℓ, b_ℓ, $\bar{a}_\ell$, $\bar{b}_\ell$, σ_ℓ as above.

So $\pi(c_\ell) = b_\ell$ and letting $a = a_1 \cap a_2 \in D$ we have

$$c := c_1 \cap c_2 = (a \cap (b_1 \cap b_2)) \cup ((1-a) \cap \sigma(\bar{a}, \bar{b}))$$

where $\bar{a} = \bar{a}_1{}^\frown\langle a_1\rangle{}^\frown\bar{a}_2{}^\frown\langle a_2\rangle, \bar{b} = \bar{b}_1{}^\frown\langle b_1\rangle{}^\frown\bar{b}_2{}^\frown\langle b_2\rangle$ for some suitable term σ.

As $a \in D$, clearly $\pi(c) = b_1 \cap b_2 = \pi(c_1) \cap \pi(c_2)$, as required.

$(*)_3$ π commutes with "$1 - x$".

Why? Let $c = (a \cap b) \cup ((1-a) \cap \sigma(\bar{a}, \bar{b}))$ hence $1 - c = (a \cap (1-b)) \cup ((1-a) \cap (1 - \sigma(\bar{a}, \bar{b}))$ hence $\pi(1-c) = 1 - b = 1 - \pi(c)$ so we are done.

$(*)_4$ π is a projection onto $\mathbb{B}_1$.

[Why? By $(*)_1$, $(*)_2$, $(*)_3$ clearly π is a homomorphism from $\mathbb{B}'_2$ into $\mathbb{B}_1$. So its range is $\subseteq \mathbb{B}_1$ and if $c \in \mathbb{B}_1$ let $b = c$, $a = 1_{\mathbb{B}_1}$, $\bar{a} = \langle\rangle = \bar{b}$ and $\sigma(\bar{a}, \bar{b}) = 0_{\mathbb{B}_1}$ so $c = (a \cap b) \cup ((1 - a) \cap \sigma(\bar{a}, \bar{b})$ and a, b, $\bar{a}$, $\bar{b}$, σ are as required so $\pi((a \cap b) \cap ((1 - a) \cap \sigma(\bar{a}, \bar{b})) = b$ which means $\pi(c) = b = c$.]

Now we can finish: as $\mathbb{B}_1 \subseteq \mathbb{B}'_2 \subseteq \mathbb{B}_2$ and π is a homomorphism from $\mathbb{B}'_2$ into $\mathbb{B}_1$ which is a complete Boolean algebra, we can extend π to π^+, a homomorphism from $\mathbb{B}_2$ into $\mathbb{B}_1$, see 1.10. But π is a projection hence so is π^+. Clearly $(\pi^+)^{-1}\{1_{\mathbb{B}_1}\}$ includes D and equality holds by the assumption on the maximality of D and we have proved the implication.

Clause (b) *implies clause* (a): First, clearly D is a filter of $\mathbb{B}_2$; also $a \in \mathbb{B}_1 \backslash \{1_{\mathbb{B}_1}\} \Rightarrow \pi(a) = a \neq 1_{\mathbb{B}_1} \Rightarrow a \notin D$.

Toward contradiction assume D_2 is a filter on $\mathbb{B}_2$, $D \subsetneqq D_2$ and $D_2 \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$. Choose $c_2 \in D_2 \backslash D$ and let $c_1 = \pi(c_2)$, consider the symmetric difference, $c_1 \Delta c_2$ it is mapped by π to $c_1 \Delta c_1 = 0_{\mathbb{B}_2}$ hence $\pi(1_{\mathbb{B}_2} - (c_1 \Delta c_2)) = 1_{\mathbb{B}_2} - \pi(c_1 \Delta c_2) = 1_{\mathbb{B}_2} - 0_{\mathbb{B}_2} = 1_{\mathbb{B}_2}$, so $1_{\mathbb{B}_2} - (c_1 \Delta c_2) \in D$ so $c_1 = c_2 \mod D$, hence (recalling $D \subseteq D_2$) we have $c_1 = c_2 \mod D_2$ but $c_2 \in D_2$ hence $c_1 \in D_2$. But

- $c_1 \in \mathbb{B}_1$ being $\pi(c_2)$
- $c_1 \neq 1_{\mathbb{B}_1}$ as $\pi(c_2) = c_1$ and $c_2 \notin D$

and recall

- $c_1 \in D_2$

so c_1 contradicts $D_2 \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$. We comment that for this direction we do not use the completeness of $\mathbb{B}_1$.

1A) Now π determines D in the statement (b). Also D determines π because if π_1, π_2 are projections from $\mathbb{B}_2$ onto $\mathbb{B}_1$ such that $D = \{a \in \mathbb{B}_2 : \pi_\ell(a) = 1_{\mathbb{B}_1}\}$ for $\ell = 1, 2$ and $\pi_1 \neq \pi_2$ let $a \in \mathbb{B}_2$ be such that $\pi_1(a) \neq \pi_2(a)$; then as in (b) $\Rightarrow$ (a) in the proof of part (1), $\pi_\ell(a) = a \mod D$ for $\ell = 1, 2$ hence $\pi_1(a) = \pi_2(a) \mod D$, but $\pi_1(a), \pi_2(a) \in \mathbb{B}_1$ and $D_2 \cap \mathbb{B}_1 = 1_{\mathbb{B}_1}$ and $D \cap \mathbb{B}_1 = 1_{\mathbb{B}_1}$ hence $\pi_1(a) = \pi_2(a)$, contradiction.

2) Straightforward, e.g. by part (1A), or as in (b) $\Rightarrow$ (a) above. □

FACT 1.12. *Assume* $\mathbb{B}_1 \lessdot \mathbb{B}_2$ *are complete Boolean algebras,* $D_\ell \in \mathrm{uf}(\mathbb{B}_\ell)$ *for* $\ell = 1, 2$. *If* D *is a maximal filter on* $\mathbb{B}_2$ *disjoint to* $\mathbb{B}_1 \backslash \{1_{\mathbb{B}_1}\}$ *and* $D \cup D_1 \subseteq D_2$ then $\mathbb{N}^{\mathbb{B}_1}/D_1$ *is an initial segment of* $\mathbb{N}^{\mathbb{B}_2}/D_2$.

REMARK 1.13. 1) This is [8, Ch. VI, Example 3.35].

2) We can prove: if $\mathbf{j} \colon \mathbb{B}_2 \rightarrow_{\text{onto}} \mathbb{B}_1$ maps $D_2 \in \mathrm{uf}(\mathbb{B}_2)$ onto $D_1 \in \mathrm{uf}(\mathbb{B}_1)$ then $\mathbb{N}^{\mathbb{B}_1}/D_1$ is canonically isomorphic to an initial segment of $\mathbb{N}^{\mathbb{B}_2}/D_2$ as in 1.12.

PROOF. The desired conclusion will follow by $(*)_3$ below:

$(*)_1$ If $\mathscr{I}$ is a maximal antichain of $\mathbb{B}_1$ *then* $\{a/D : a \in \mathscr{I}\}$ is a maximal antichain of $\mathbb{B}_2/D$.

[Why? First,

• $a \in \mathscr{I} \Rightarrow a \in \mathbb{B}_1^+ \Rightarrow a/D \in (\mathbb{B}_2/D)^+$

• if $a \neq b \in \mathscr{I}$ then $\mathbb{B}_2 \models$ "$a \cap b = 0_{\mathbb{B}_1}$" hence $\mathbb{B}_2/D \models$ "$(a/D) \cap (b \cap D) = 0_{\mathbb{B}_2/D}$".

Hence, obviously $\mathscr{I}^* := \{a/D : a \in \mathscr{I}\}$ is an antichain of $\mathbb{B}_2/D$. Toward contradiction assume $\mathscr{I}^*$ is not maximal and let c/D witness it. By 1.11 there is $c' \in \mathbb{B}_1$ such that $c = c' \mod D$ and so without loss of generality $c \in \mathbb{B}_1$.

As $c/D \neq 0/D$ necessarily $c \in \mathbb{B}_1^+$ and if $b \in \mathscr{I}$ then $(b/D) \cap (c/D) = 0/D$ hence $b \cap c = 0 \mod D$ but $b, c \in \mathbb{B}_1$ hence $b \cap c = 0$, so c contradicts "$\mathscr{I}$ is a maximal antichain of $\mathbb{B}_1$".]

$(*)_2$ If $f \in \mathbb{N}^{\mathbb{B}_2}$, $c \in \mathbb{B}_1 \backslash D_1$ and $\mathrm{TV}(f > n) \cup c \in D$ for every n *then* $g \in \mathbb{N}^{\mathbb{B}_1} \Rightarrow g/D_2 < f/D_2$.

[Why? If g is a counterexample, then $\mathrm{TV}(f \leq g)$ belongs to D_2 but $1 - c \in D_1 \subseteq D_2$ so $\mathrm{TV}(f \leq g) - c$ belongs to D_2 hence to $D^+ := \{a \in \mathbb{B}_2 : 1 - a \notin D\}$ since $D \subseteq D_2$. Let $\langle b_n : n < \omega \rangle$ represent g as a member of $\mathbb{N}^{\mathbb{B}_1}$, then by $(*)_1$, $\langle b_n/D : n < \omega \rangle$ is a maximal anti-chain of $\mathbb{B}_2/D$ hence for some n, $\mathrm{TV}(f \leq g) \cap b_n - c \in D^+$ but $\mathrm{TV}(f \leq n) - c \geq \mathrm{TV}(f \leq g) \cap b_n - c$ hence $\mathrm{TV}(f \leq n) - c \in D^+$, contradiction to an assumption of $(*)_2$; so $(*)_2$ holds indeed.]

$(*)_3$ If $f \in \mathbb{N}^{\mathbb{B}_2}$, $g \in \mathbb{N}^{\mathbb{B}_1}$ and $f/D_2 \leq g/D_2$ then for some $g' \in \mathbb{N}^{\mathbb{B}_1}/D$ we have $f/D_2 = g'/D_2$.

[Why? Let $\langle a_n : n < \omega \rangle$ represent f and let $a_{\geq n} = \bigcup_{k \geq n} a_k \in \mathbb{B}_2$. If for some $b \in D_2$, we have $n < \omega \Rightarrow a_{\geq n} \cup (1 - b) \in D$ then there is $f' \in \mathbb{N}^{\mathbb{B}_2}$ such that $f'/D_2 = f/D_2$ and $n < \omega \Rightarrow \mathrm{TV}(f' \geq n) \in D$. Now we apply $(*)_2$ with f', $0_{\mathbb{B}_1}$ here standing for f, c there and we get contradiction to "$f/D_2 \leq g/D_2$". So we can assume there is no such b.

Let $a'_n \in \mathbb{B}_1$ be such that $a_n = a'_n \mod D$ so possibly $a'_n = 0_{\mathbb{B}_1}$, such a'_n exists by 1.11(1A). Clearly $\langle a'_n/D : n < \omega \rangle$ is an antichain of $\mathbb{B}_2/D$, so as $D \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$ clearly $\langle a'_n : n < \omega \rangle$ is an antichain of $\mathbb{B}_1$.

Case 1: $c := \bigcup_n a'_n \notin D_1$. Then for every $n \in \omega, \mathrm{TV}(f > n) \cup c \in D$ as otherwise there is some $n \in \omega$ such that $\mathrm{TV}(f \leq n) - c \in D^+$ hence for some $\ell \leq n, a_\ell - c \in D^+$ hence (by the choice of a'_ℓ) we have $a'_\ell - c \in D^+$, contradiction to the choice c. As $c \in \mathbb{B}_1$, by $(*)_2$ we get a contradiction to the assumption "$f/D_2 \leq g/D_2$" of $(*)_3$.

Case 2: $c := \bigcup_n a'_n \in D_1$ *and* $d = \bigcup_n (a_n \Delta a'_n) \notin D_2$. As D_2 is an ultrafilter of $\mathbb{B}_2$, clearly $c' := c - d \in D_2$. We define $g' \in \mathbb{N}^{\mathbb{B}_1}$ as the function represented by $\langle a'_n : n < \omega \rangle$ and $g'' \in \mathbb{N}^{\mathbb{B}_2}$ as the function represented by $\langle a''_n : n < \omega \rangle$, where a''_n is $a'_n \cap c'$ if $n > 0$ and $a'_n \cup (1 - c')$ if $n = 0$. Easily $f/D_2 = g''/D_2$ because f, g'' "agree" on c' which belongs to D_2 and the choice of d; also $g''/D_2 = g'/D_2$ because $c' \in D_2$. Together we are done.

Case 3: $c := \bigcup_n a'_n \in D_1$ *and* $d = \bigcup_n (a_n \Delta a'_n) \in D_2$.

Let $d' \in \mathbb{B}_1$ be such that $d'/D = d/D$. Let $d_1 := \bigcup_n (a_n - a'_n)$ and $d_2 := \bigcup_n (a'_n - a_n)$ hence $d = d_1 \cup d_2$. Let $k < \omega$, now modulo D we have $d' \cap \bigcup_{n \le k} a'_n = d \cap \bigcup_{n \le k} a'_n = \bigcup_{\ell=1}^{2} (d_\ell \cap \bigcup_{n \le k} a'_n)$ and we shall deal separately with each term.

First,

$$d_1 \cap \bigcup_{n \le k} a'_n = \bigcup_{\ell \le k} \Big((a_\ell - a'_\ell) \cap \bigcup_{n \le k} a'_n \Big) \cup \Big(\bigcup_{\ell > k} (a_\ell - a'_\ell) \cap \bigcup_{n \le k} a'_n \Big).$$

Now the first term $\bigcup_{\ell \le k}((a_\ell - a'_\ell) \cap \bigcup_{n \le k} a'_n)$ is equal mod D to $(\bigcup_{n \le k} 0)$ $\cap \bigcup_{n \le k} a'_k = 0_{\mathbb{B}_1}$, by the choice of the a'_ℓ. Next, the second term in the union, $(\bigcup_{\ell > k}(a_\ell - a'_\ell) \cap \bigcup_{n \le k} a'_n)$ is modulo D again by the choice of the a'_ℓ, equal to $(\bigcup_{\ell > k}(a_\ell - a'_\ell)) \cap \bigcup_{n \le k} a_n$ which is zero as $\langle a_n : n < \omega \rangle$ is an antichain; together by the previous sentences $d_1 \cap \bigcup_{n \le k} a'_n = 0_{\mathbb{B}_2}$.

Similarly $d_2 \cap \bigcup_{n \le k} a'_n = 0_{\mathbb{B}_2} \bmod D$ noting that $\langle a'_n : n < \omega \rangle$ is necessarily an antichain of $\mathbb{B}_1$. Hence

$$d' \cap \bigcup_{n \le k} a'_n = d \cap \bigcup_{n \le k} a'_n = \bigcup_{\ell=1}^{2} (d_\ell \cap \bigcup_{n \le k} a'_n)) = 0_{\mathbb{B}_2} \cup 0_{\mathbb{B}_2} = 0_{\mathbb{B}_2} \bmod D.$$

But $d' \in \mathbb{B}_1$ and $a'_n \in \mathbb{B}_1$ for every n (and $D \cap \mathbb{B}_1 = \{1_{\mathbb{B}_1}\}$, of course), hence $d' \cap \bigcup_{n \le k} a'_n = 0_{\mathbb{B}_1}$. However, as this holds for every k it follows that $d' \cap c = 0$, but by the case first assumption $c \in D_1 \subseteq D_2$ so $d' \notin D_2$, but by the case assumption $d'/D = d/D$ and $d \in D_2$ contradiction. $\square$

2. Spectrum of the ultraproducts of finite cardinals

DEFINITION 2.1. Assume D is an ultra-filter on I.

1) Let $\mathrm{upf}(D)$ be the spectrum of ultra-products mod D of finite cardinals, that is; $\{\prod_{i \in I} n_i/D : n_i \in \mathbb{N}$ for $i \in I$ and $\prod_{i \in I} n_i/D$ is infinite$\}$.

2) For $\lambda \in \mathrm{upf}(D)$ let $A_{D,\lambda} = \{a : a \in \mathbb{N}^I/D$ and $\{b \in \mathbb{N}^I/D : \mathbb{N}^I/D \models$ "$b < a$"$\}$ has cardinality $\lambda\}$; we consider it as a linearly ordered set by the order inherited from $\mathbb{N}^I/D$.

3) Let $\mathrm{spec}_1(D) = \{(\lambda, \theta, \partial) : \lambda \in \mathrm{upf}(D)$ and $A_{D,\lambda}$ has cofinality ∂ and co-initiality $\theta\}$.

4) Let $\mathrm{spec}_2(D)$ be the set of triples $(\lambda, \theta, \partial)$ such that:

(a) $\lambda \in \mathrm{upf}(D)$,

(b) (α) if $\lambda < \sup(\mathrm{upf}(D))$ then $A_{D,\lambda}$ has cofinality ∂,

(β) if $\lambda = \max(\mathrm{upf}(D))$ then $\partial = 0$ (or $*$),

(c) θ is the co-initiality of $A_{D,\lambda}$.

5) For D an ultrafilter on a complete Boolean algebra $\mathbb{B}$ we define the above similarly considering $\mathbb{N}^{\mathbb{B}}/D$ instead of $\mathbb{N}^J/D$ but in clause (b), ∂ is the cofinality of $A_{D,\lambda}$ in all cases.

Definition 2.2. Let K_α be the class of objects $\mathbf{k}$ consisting of:
(a) $\mathbb{B}_\beta$ is a Boolean algebra for $\beta \leq \alpha$,
(b) $\langle \mathbb{B}_\beta : \beta \leq \alpha \rangle$ is increasing,
(c) $\mathbb{B}_\beta$ is complete for $\beta < \alpha, \mathbb{B}_0$ is trivial,
(d) $\mathbb{B}_\beta \lessdot \mathbb{B}_\gamma$ if $\beta < \gamma \leq \alpha$ and $\bigcup\{\mathbb{B}_{\beta_1} : \beta_1 < \gamma\} \lessdot \mathbb{B}_\gamma$ for limit $\gamma \leq \alpha$,
(e) D_β is a filter on $\mathbb{B}_\alpha$ such that $\mathbb{B}_\beta \cap D_\beta = \{1_{\mathbb{B}_\beta}\}$,
(f) D_β is maximal under clause (e), so D_0 is an ultrafilter and $D_\alpha = \{1_{\mathbb{B}_\alpha}\}$,
(g) $\langle D_\beta : \beta \leq \alpha \rangle$ is $\subseteq$-decreasing.

Definition 2.3. 1) Above let $\mathbb{B}[\mathbf{k}] = \mathbb{B}_{\mathbf{k}} = \mathbb{B}_\alpha$, $\mathbb{B}[\mathbf{k},\beta] = \mathbb{B}_{\mathbf{k},\beta} = \mathbb{B}_\beta$, $\bar{\mathbb{B}}_{\mathbf{k}} = \langle \mathbb{B}_{\mathbf{k},\beta} : \beta \leq \alpha \rangle$, $D_{\mathbf{k},\beta} = D_\beta$, $D_{\mathbf{k}} = D_{\mathbf{k},0}$, $\ell g(\mathbf{k}) = \alpha_{\mathbf{k}} = \alpha(\mathbf{k}) = \alpha$.

1A) Let $K^{\rm com}_\alpha$ be the class of $\mathbf{k} \in K_\alpha$ such that $\mathbb{B}_{\mathbf{k}}$ is a complete Boolean algebra.

2) Assume $\kappa > \aleph_0$ is regular. Let $K^{{\rm cc}(\kappa),1}_\alpha$ be the class of $\mathbf{k} \in K_\alpha$ such that $\mathbb{B}_\alpha$ satisfies the κ-c.c.

3) Let $K^{{\rm cc}(\kappa),2}_\alpha$ be the class of $\mathbf{k} \in K^{{\rm cc}(\kappa),1}_\alpha$ such that:
- $\mathbb{B}_{\mathbf{k}}$ is complete; recall that for every $\beta < \alpha$, $\mathbb{B}_\beta$ is complete,
- if $\delta \leq \alpha$ has cofinality $\geq \kappa$ *then* $\mathbb{B}_{\mathbf{k},\delta} = \bigcup_{\beta<\delta} \mathbb{B}_{\mathbf{k},\beta}$,
- if $\delta \leq \alpha$ is limit of cofinality $< \kappa$, *then* $\mathbb{B}_{\mathbf{k},\delta}$ is the completion of $\bigcup_{\beta<\delta} \mathbb{B}_{\mathbf{k},\beta}$.

3A) We may omit κ when $\kappa = \aleph_1$ so $K^{{\rm cc},\iota}_\alpha = K^{{\rm cc}(\aleph_1),\iota}_\alpha$; if we omit ι we mean 1.

4) Let $K = \bigcup_\alpha K_\alpha$ and $K^{{\rm cc}(\kappa),\iota} = \bigcup\{K^{{\rm cc}(\kappa),\iota}_\alpha : \alpha \text{ an ordinal}\}$ so $K^{\rm cc} = \bigcup_\alpha K^{\rm cc}_\alpha$.

5) We say $\mathbb{B}$ is above $\bar{\mathbb{B}}_{\mathbf{k}}$ when $\mathbb{B}_{\mathbf{k}} \subseteq \mathbb{B}$ and $\mathbb{B}_{\mathbf{k},\beta} \lessdot \mathbb{B}$ for $\beta < \alpha_{\mathbf{k}}$.

6) $K^{{\rm fr}(\kappa)}_\alpha$ is the class of $\mathbf{f}$ consisting of:
(a) $\mathbf{k}_{\mathbf{f}} = (\bar{\mathbb{B}}, \bar{D})$ as in 2.2,
(b) $\bar{\xi} = \langle \xi_\gamma : \gamma \leq \alpha \rangle$ and $\bar{x} = \langle x_{\beta,i} : i < \kappa,\ \beta < \xi_\alpha \rangle$, $x_{\beta,i} \in \mathbb{B}_{\mathbf{k}}$ are such that $\bar{x}$ is free except that $\beta < \xi_\alpha \wedge i < j < \kappa \Rightarrow x_{\beta,i} \cap x_{\beta,j} = 0$,
(c) the sub-algebra which $\langle x_{\beta,i} : \beta < \xi_\gamma,\ i < \kappa \rangle$ generates is dense in $\mathbb{B}_{\mathbf{k},\gamma}$,
(d) so $\bar{\xi}_{\mathbf{f}} = \bar{\xi}$, $\bar{x}_{\mathbf{f}} = \bar{x}$, $\bar{\mathbb{B}}_{\mathbf{f}} = \mathbb{B}_{\mathbf{k}}$, etc.

7) Let ${}_*K_\alpha$ be defined like K_α in 2.2 omitting clause (d), and define ${}_*K$, as above; not really needed here but we may comment.

Definition 2.4. 1) If $\beta \leq \gamma$ and $\mathbf{m} \in K_\gamma$ then $\mathbf{k} = \mathbf{m} \restriction \beta$ is the unique $\mathbf{k} \in K_\beta$ such that $\mathbb{B}_{\mathbf{k},\alpha} = \mathbb{B}_{\mathbf{m},\alpha}$, $D_{\mathbf{k},\alpha} = D_{\mathbf{m},\alpha} \cap \mathbb{B}_{\mathbf{k}}$ for $\alpha \leq \beta$.

1A) If $\mathbf{k} \in K_\alpha$ and $\beta < \alpha$ *then* $\pi_{\mathbf{k},\beta}$ is the unique projection from $\mathbb{B}_{\mathbf{k}}$ onto $\mathbb{B}_{\mathbf{k},\beta}$ such that $\pi^{-1}_{\mathbf{k},\beta}\{1_{\mathbb{B}_{\mathbf{k},\beta}}\} = D_{\mathbf{k},\beta}$ recalling 1.11; let $\pi_{\mathbf{k},\alpha} = \mathrm{id}_{\mathbb{B}_{\mathbf{k},\alpha}}$ and if $\gamma \le \beta \le \alpha$ then $\pi_{\mathbf{k},\beta,\gamma} = \pi_{\mathbf{k},\gamma} \restriction \mathbb{B}_{\mathbf{k},\beta}$.

2) We define the following two-place relations on K:

(A) $\mathbf{k} \le^{\mathrm{at}}_K \mathbf{m}$ where at stands for atomic *iff*:

(a) $\alpha_{\mathbf{k}} = \alpha_{\mathbf{m}}$,

(b) $\mathbb{B}_{\mathbf{k},\beta} = \mathbb{B}_{\mathbf{m},\beta}$ for $\beta < \alpha_{\mathbf{k}}$,

(c) $\mathbb{B}_{\mathbf{k},\alpha(\mathbf{k})} \lessdot \mathbb{B}_{\mathbf{m},\alpha(\mathbf{m})}$,

(d) $D_{\mathbf{k},\beta} \subseteq D_{\mathbf{m},\beta}$ for $\beta \le \alpha_{\mathbf{k}}$.

(B) $\mathbf{k} \le^{\mathrm{ver}}_K \mathbf{m}$, where ver stands for vertical *iff*

(a) $\alpha_{\mathbf{k}} \le \alpha_{\mathbf{m}}$

(b) $\mathbf{k} \le^{\mathrm{at}}_K (\mathbf{m} \restriction \alpha_{\mathbf{k}})$

(C) $\mathbf{k} \le^{\mathrm{hor}}_K \mathbf{m}$, where hor stands for horizontal *iff*

(a) $\alpha_{\mathbf{k}} = \alpha_{\mathbf{m}}$,

(b) $\mathbb{B}_{\mathbf{k},\beta} \lessdot \mathbb{B}_{\mathbf{m},\beta}$ for $\beta, \le \alpha$

(c) $D_{\mathbf{k},\beta} \subseteq D_{\mathbf{m},\beta}$ for $\beta \le \alpha$.

(D) (a) $\mathbf{f}_1 \le^{\mathrm{fr}(\kappa)}_K \mathbf{f}_2$ iff $\mathbf{f}_\ell \in K^{\mathrm{fr}(\kappa)}_{\alpha_\ell}$ and $\mathbf{k}_{\mathbf{f}_1} \le^{\mathrm{ver}}_K \mathbf{k}_{\mathbf{f}_2}$ and $\bar{x}_{\mathbf{f}_1} \trianglelefteq \bar{x}_{\mathbf{f}_2}$ which means, i.e., $\beta < \alpha_1 \Rightarrow \bar{x}_{\mathbf{f}_1,\beta} = \bar{x}_{\mathbf{f}_2,\beta}$ and $\beta = \alpha_1 \Rightarrow \xi_{\mathbf{f}_1,\beta} \le \xi_{\mathbf{f}_2,\beta} \wedge \bar{x}_{\mathbf{f}_1,\beta} = \bar{x}_{\mathbf{f}_2,\beta} \restriction \xi_{\mathbf{f}_2,\beta}$.

(b) $\le^{\mathrm{at-fr}(\kappa)}_K$ is defined similarly.

(E) $\mathbf{k} \le^{\mathrm{wa}}_K \mathbf{m}$ where wa stands for weakly atomic *iff*

(a), (b), (d) as in Clause (A)

(c) $\mathbb{B}_{\mathbf{k},\alpha(\mathbf{k})} \subseteq \mathbb{B}_{\mathbf{m},\alpha(\mathbf{m})}$.

Remark 2.5. Note that for the present work it is not a loss to use exclusively c.c.c. Boolean algebras; moreover ones which have a dense subalgebra which is free. So using only free Boolean algebras or their completion, i.e. $(K^{\mathrm{fr}(\aleph_1)}_\alpha, \le^{\mathrm{fr}(\kappa)}_K)$; so we are giving for $\mathbb{B}$ a set of generators (and the orders respects this).

Observation 2.6. *The relations* $\le^{\mathrm{at}}_K$, $\le^{\mathrm{wa}}_K$ *and* $\le^{\mathrm{ver}}_K$ *and* $\le^{\mathrm{hor}}_K$ (*the last one is not used*) *are partial orders on* K.

We need various claims on extending members of K, existence of upper bounds to an increasing sequence and amalgamation.

Claim 2.7. *Let* δ *be a limit ordinal.*

1) *If* $\langle \mathbf{k}_i : i < \delta \rangle$ *is* $\le^{\mathrm{at}}_K$*-increasing* then *it has a* $\le^{\mathrm{at}}_K$*-lub* $\mathbf{k}_\delta$, *the union naturally defined so* $|\mathbb{B}_{\mathbf{k}_\delta}| \le \Sigma\{|\mathbb{B}_{\mathbf{k}_i}| : i < \delta\}$.

1A) *Like part* (1) *for* $\le^{\mathrm{wa}}_K$.

2) *If* $\langle \mathbf{k}_i : i < \delta \rangle$ *is a* $\le^{\mathrm{ver}}_K$*-increasing sequence*, then *it has a* $\le^{\mathrm{ver}}_K$*-upper bound* $\mathbf{k} = \mathbf{k}_\delta$ *which is the union which means*:

(a) $\ell g(\mathbf{k}) = \bigcup\{\ell g(\mathbf{k}_i) : i < \delta\}$ *call it* α,

(b) *if* $\beta < \alpha$ *then* $\mathbb{B}_{\mathbf{k},\beta} = \mathbb{B}_{\mathbf{k}_i,\beta}$ *for every large enough* i,

(c) (α) *if* $\langle \ell g(\mathbf{k}_i) : i < \delta \rangle$ *is eventually constant* (*so* $\ell g(\mathbf{k}_i) = \alpha$ *for every* $i < \delta$ *large enough*) then

- $\mathbb{B}_{\mathbf{k},\alpha} = \bigcup \{\mathbb{B}_{\mathbf{k}_i,\alpha} : i < \delta$ *is such that* $\ell g(\mathbf{k}_i) = \alpha\}$,
- $D_{\mathbf{k},\alpha} = \{1_{\mathbb{B}_{\mathbf{k},\alpha}}\}$, *redundant*;

(β) *if* $\langle \ell g(\mathbf{k}_i) : i < \delta \rangle$ *is not eventually constant* then

- $\mathbb{B}_{\mathbf{k},\alpha} = \bigcup \{\mathbb{B}_{\mathbf{k}_i,\ell g(\mathbf{k}_i)} : i < \delta\}$,
- $D_{\mathbf{k},\alpha} = \{1_{\mathbb{B}_{\mathbf{k},\alpha}}\}$, *redundant*;

(d) *if* $\beta < \alpha$ *then* $D_{\mathbf{k},\beta} = \bigcup \{D_{\mathbf{k}_i,\beta} : i < \delta$ *is such that* $\beta \leq \ell g(\mathbf{k}_i)\}$.

3) *In part* (2) *if* $\mathbf{k}_i \in K^{\mathrm{cc}(\kappa),2}, \mathrm{cf}(\delta) \geq \kappa$ *and the sequence* $\langle \ell g(\mathbf{k}_i) : i < \delta \rangle$ *is not eventually constant* then $\mathbb{B}_{\mathbf{k}}$ *is complete and* $\mathrm{upf}(D_{\mathbf{k}}) = \bigcup \{\mathrm{upf}(D_{\mathbf{k}_i}) : i < \delta\}$.

4) *Similarly for the* ${}_*K$ *version.*

PROOF. Straightforward (concerning part (3) note that recalling $\mathrm{cf}(\delta) \geq \kappa$ we have $\mathbb{N}^{\mathbb{B}(\mathbf{k})} = \bigcup \{\mathbb{N}^{\mathbb{B}(\mathbf{k}_i)} : i < \delta\}$. $\square$

CLAIM 2.8. 1) *If* $\mathbf{k} \in K_\alpha$ *and* $\mathbb{B}$ *is above* $\bar{\mathbb{B}}_{\mathbf{k}}$ (*i.e.* $\mathbb{B}_{\mathbf{k},\alpha} \subseteq \mathbb{B}$ *and* $\beta < \alpha \Rightarrow \mathbb{B}_{\mathbf{k},\beta} \lessdot \mathbb{B}$) then *there is* $\mathbf{m} \in K_\alpha$ *such that* $\mathbf{k} \leq^{\mathrm{wa}}_K \mathbf{m}$ *and* $\mathbb{B}_{\mathbf{m},\alpha} = \mathbb{B}$.

2) *If* $\mathbf{k} \in K_\alpha$ *and* $\mathbb{B} \subseteq \mathbb{B}_{\mathbf{k}}$ *and* $\beta < \alpha \Rightarrow \mathbb{B}_{\mathbf{k},\beta} \subseteq \mathbb{B}$ (*hence* $\mathbb{B}_{\mathbf{k},\beta} \lessdot \mathbb{B}$) then *there is* $\mathbf{m} \in K_\alpha$ *such that* $\mathbf{m} \leq^{\mathrm{wa}}_K \mathbf{k}$ *and* $\mathbb{B}_{\mathbf{m}} = \mathbb{B}$.

3) *If* $\mathbf{k} \in K_\alpha, \mathbb{B}_{\mathbf{k}} \lessdot \mathbb{B}$ then *for some* $\mathbf{m} \in K_\alpha, \mathbf{k} \leq^{\mathrm{at}}_K \mathbf{m}, \mathbb{B}_{\mathbf{m}} = \mathbb{B}$.

4) *In part* (2) *if* $\mathbb{B} \lessdot \mathbb{B}_{\mathbf{k}}$ then *we can add* $\mathbf{m} \leq^{\mathrm{at}}_K \mathbf{k}$.

5) *If* $\mathbf{k} \leq^{\mathrm{wa}}_K \mathbf{m}$ *and* $\mathbb{B}_{\mathbf{k}} \lessdot \mathbb{B}_{\mathbf{m}}$ then $\mathbf{k} \leq^{\mathrm{at}}_K \mathbf{m}$.

PROOF. 1) For transparency we assume $\mathbb{B}$ satisfies the c.c.c. By 2.7(1A) without loss of generality $\mathbb{B}$ is generated by $\mathbb{B}_{\mathbf{k},\alpha} \cup \{a\}$ where $a \notin \mathbb{B}_{\mathbf{k},\alpha}$. So $\mathbb{B}_{\mathbf{m}}$ is uniquely defined and as required in Definition 2.2, but we have to define the $D_{\mathbf{m},\beta}$'s and, of course, let $D_{\mathbf{m},\alpha} = \{1_{\mathbb{B}_{\mathbf{m}}}\}$.

Case 1: $\alpha = 0$. This is trivial.

Case 2: $\alpha = \beta + 1$. As $\mathbb{B}_{\mathbf{k},\beta} \lessdot \mathbb{B}$ and $\pi_{\mathbf{k},\beta}$ is a projection from $\mathbb{B}_{\mathbf{k},\alpha}$ onto $\mathbb{B}_{\mathbf{k},\beta}$ and [$B_{\mathbf{k},\alpha} \subseteq \mathbb{B}$ and $\mathbb{B}_{\mathbf{k},\beta}$ is complete by a projection π from $\mathbb{B}$ onto $\mathbb{B}_{\mathbf{k},\beta}$ extending $\pi_{\mathbf{k},\beta}$. Now for $\gamma < \alpha$ let $D_{\mathbf{m},\gamma}$ be the filter on $\mathbb{B}$ generated by $D_{\mathbf{k},\gamma} \cup \{(a \Delta \pi(a)\}$.

Case 3: *α is a limit ordinal, $\mathbb{B}$ is c.c.c.* (*the main*) *and α is of cofinality* $> \aleph_0$. In this case, $\mathbb{B}' = \bigcup_{\gamma<\alpha} \mathbb{B}_{\mathbf{k},\gamma}$ is complete and $\lessdot \mathbb{B}$ so we can continue as in Case 2 using $\mathbb{B}'$ instead $\mathbb{B}_{\mathbf{k},\beta}$.

Case 4: $\mathrm{cf}(\alpha) = \aleph_0$. Let $\alpha = \bigcup_n \alpha_n, \alpha_n < \alpha_{n+1}$. For $\beta < \alpha$ let π_β be the projection of $\mathbb{B}_{\mathbf{k}}$ onto $\mathbb{B}_{\mathbf{k},\beta}$ which maps $D_{\mathbf{k},\beta}$ onto $1_{\mathbb{B}_{\mathbf{k},\beta}}$. Let Π_β be the set of homomorphisms from $\mathbb{B}$ into $\mathbb{B}_{\mathbf{k},\beta}$ extending π_β, so not empty hence (recalling $\mathbb{B}_{\mathbf{k},\beta}$ is complete) there are $b_\beta \leq c_\beta$ from $\mathbb{B}_{\mathbf{k},\beta}$ such that $\{\pi(a) : \pi \in \Pi_\beta\}$ is $\{a' \in \mathbb{B}_{\mathbf{k},\beta} : b_\beta \leq a' \leq c_\beta\}$. Clearly $\gamma < \beta < \alpha \Rightarrow b_\gamma \leq b_\beta \leq c_\beta \leq c_\gamma$ in $\mathbb{B}_{\mathbf{k},\beta}$.

Now by induction on $\zeta < (\|\mathbb{B}\|^{(\alpha)})^+$ we defined $\langle (b_{\beta,\zeta}, c_{\beta,\zeta}) : \beta < \alpha \rangle$ such that:

$(*)_\zeta$ (a) $\mathbb{B}_{\mathbf{k},\beta} \models$ "$b_{\beta,\zeta} \leq c_{\beta,\zeta}$" for $\beta < \alpha$,

(b) if $\gamma < \beta < \alpha$ then $\mathbb{B}_{\mathbf{k},\beta} \models$ "$b_{\gamma,\zeta} \le b_{\beta,\zeta} \le c_{\beta,\zeta} \le c_{\gamma,\zeta}$",

(c) if $\varepsilon < \zeta$ and $\beta < \alpha$ then $\mathbb{B}_{\mathbf{k},\beta} \models$ "$b_{\beta,\varepsilon} \le b_{\beta,\zeta} \le c_{\beta,\zeta} \le c_{\beta,\varepsilon}$".

Case 1: For $\zeta = 0$ let $(b_{\beta,\zeta}, c_{\beta,\zeta}) = (b_\beta, c_\beta)$, so clauses (a), (b) hold as said above and clause (c) is empty.

Case 2: *ζ is a limit ordinal.* Let for $\beta < \alpha$.

- $b_{\beta,\zeta} = \bigcup\{b_{\gamma,\varepsilon} : \varepsilon < \zeta\}$ in $\mathbb{B}_{\mathbf{k},\beta}$
- $c_{\beta,\zeta} = \bigcap\{b_{\gamma,\varepsilon} : \varepsilon < \zeta\}$.

They are well defined because $\mathbb{B}_{\mathbf{k},\beta}$ is a complete Boolean algebra and it is easy to check that (a),(b),(c) holds.

Case 3: $\zeta = \varepsilon + 1$. Let

$$b_{\beta,\zeta} = \bigcup\{\pi_{\mathbf{k},\gamma,\beta}(b_{\gamma,\varepsilon}) : \gamma \in (\beta,\alpha)\}, \quad c_{\beta,\zeta} = \bigcap\{\pi_{\mathbf{k},\gamma,\beta}(c_{\gamma,\varepsilon}) : \gamma \in (\beta,\alpha)\}.$$

Now check. Having carried the induction, by $(*)_\zeta$ for $\zeta < (\|\mathbb{B}\|^{|\alpha|})^+$ for some $\zeta_* < (\|\mathbb{B}\|^{|\alpha|})^+, \langle (b_{\zeta,\zeta}, c_{\beta,\zeta}) : \beta < \alpha\rangle$ is the same for all $\zeta \ge \zeta_*$ and let $a_\beta = b_{\beta,\zeta_*}$ for $\beta < \alpha$.

Easily $\gamma < \beta < \alpha \Rightarrow \pi_\beta(a_\beta) = a_\gamma$ and let $D_{\mathbf{m},\beta}$ be the filter of $\mathbb{B}$ generated by $D_{\mathbf{k},\beta} \cup \{(a\Delta a_\beta)\}$.

2) Easy.

3) By (1).

4), 5) Should be easy. □

Claim 2.9. 1) *If* $\mathbf{k} \in K_\alpha, \langle \beta_i : i \le i(*)\rangle$ *is increasing with* $\beta_{i(*)} = \alpha$ then *there is one and only one* $\mathbf{m} \in K_{i(*)}$ *such that* $(\mathbb{B}_{\mathbf{m},i}, D_{\mathbf{m},i}) = (\mathbb{B}_{\mathbf{k},\beta_i}, D_{\mathbf{m},\beta_i})$ *for* $i \le i(*)$.

2) *Above if* $\mathbf{m} \le^{\mathrm{at}}_K \mathbf{m}_1$ then *there is* $\mathbf{k}_1$ *such that* $\mathbf{k} \le^{\mathrm{at}}_K \mathbf{k}_1$ *and* $\mathbb{B}_{\mathbf{k}_1} = \mathbb{B}_{\mathbf{m}_1}$ *and* $D_{\mathbf{k}_1,\beta_i} = D_{\mathbf{m}_1,i}$ *for* $i \le i(*)$.

2A) *Similarly for* $\le^{\mathrm{wa}}_K$.

3) *Above if* $\mathbf{m} \le^{\mathrm{ver}}_K \mathbf{m}_1 \in K_{i(*)+j(*)}$ then *there is* $\mathbf{k}_1 \in K_{\alpha+j(*)}$ *such that* $\mathbf{k} \le^{\mathrm{ver}}_K \mathbf{k}_1$, $\mathbb{B}_{\mathbf{k}_1} = \mathbb{B}_{\mathbf{m}_1}$, $\mathbb{B}_{\mathbf{k}_1,\alpha+j} = \mathbb{B}_{\mathbf{m}_1,i(*)+j}$, $D_{\mathbf{k}_1,\alpha+j} = D_{\mathbf{m}_1,i(*)+j}$, $D_{\mathbf{k}_1,\beta_i} = D_{\mathbf{m}_1,i}$ *for* $j < j(*)$, $i \le i(*)$.

4) *If* $\mathbf{k} \in K_\alpha$ *and* $\beta_0 \le \beta_1 \le \beta_2 \le \alpha$ then $\pi_{\mathbf{k},\beta_2,\beta_0} = \pi_{\mathbf{k},\beta_1,\beta_0} \circ \pi_{\mathbf{k},\beta_2,\beta_1}$.

Proof. Straightforward. □

Conclusion 2.10. 1) *If* $\mathbf{k} \in K_\alpha$ then *there is* $\mathbf{m}$ *such that* $\mathbf{k} \le^{\mathrm{at}}_K \mathbf{m}$ *and* $\mathbb{B}_{\mathbf{m},\alpha}$ *is the completion of* $\mathbb{B}_{\mathbf{k},\alpha}$ *so* $\mathbf{m} \in K^{\mathrm{com}}_\alpha$ *and* $\mathbf{k} \le^{\mathrm{ver}}_K \mathbf{m}$; *so if* $\mathbf{k} \in K^{\mathrm{cc}(\kappa)}_\alpha$ *then* $\mathbf{m} \in K^{\mathrm{cc}(\kappa)}_\alpha \cap K^{\mathrm{com}}$.

2) *If* $\alpha < \beta$, $\mathbf{k} \in K_\alpha$, $\mathbf{n} \in K_\beta$ *and* $\mathbf{k} \le^{\mathrm{ver}}_K \mathbf{n}$, then *for some* $\mathbf{m}$ *we have* $\mathbf{k} \le^{\mathrm{ver}}_K \mathbf{m} \le^{\mathrm{ver}}_K \mathbf{n}$ *and* $\mathbb{B}_{\mathbf{m}}$ *is the completion of* $\mathbb{B}_{\mathbf{k}}$ *inside* $\mathbb{B}_{\mathbf{n}}$; *so if* $\mathbf{n} \in K^{\mathrm{cc}(\kappa)}$ *then* $\mathbf{m} \in K^{\mathrm{cc}(\kappa)}$.

Proof. 1) By 2.8. 2) Check the definitions. □

CLAIM 2.11. *There is* $\mathbf{m}$ *such that* $\mathbf{k} \leq_K^{\mathrm{wa}} \mathbf{m}$, $\mathbb{B}_{\mathbf{m}} = \mathbb{B}$ *and* $Y \subseteq D_{\mathbf{m}}$ *and* $\mathbb{B}_{\mathbf{k}} \lessdot \mathbb{B} \Rightarrow \mathbf{k} \leq_K^{\mathrm{at}} \mathbf{m}$ when:

(a) $\mathbf{k} \in K_\alpha$,

(b) $\mathbb{B}$ *is a Boolean algebra*, $Y \subseteq \mathbb{B}$,

(c) (α) $\mathbb{B}_{\mathbf{k}} \subseteq \mathbb{B}$,

(β) $\mathbb{B}_{\mathbf{k},\beta} \lessdot \mathbb{B}$ *for* $\beta < \alpha_{\mathbf{k}}$,

(d) *if* $\beta < \alpha_{\mathbf{k}}$ then *for some* X_β *we have*:

(α) $X_\beta \subseteq Y$,

(β) X_β *is a downward directed subset of* $\mathbb{B}$,

(γ) *if* $x \in X_\beta$ *and* $b \in D_{\mathbf{k},\beta}$ then $x \cap b$ *is not disjoint to any* $a \in \mathbb{B}^+_{\mathbf{k},\beta}$,

(δ) *if* $y \in Y$ then *for some* $b \in D_{\mathbf{k},\beta}$ *and* $x \in X_\beta$ *we*[2] *have* $b \cap x \leq y$.

PROOF. By 2.8 without loss of generality $\mathbb{B}$ is generated by $\mathbb{B}_{\mathbf{k}} \cup Y$ and let $\alpha = \alpha_{\mathbf{k}}$ and let X_β be as in clause (d) in the claim for $\beta < \alpha$ and define $\mathbf{m} \in K_\alpha$ as follows:

- $D_{\mathbf{m},\alpha} = \{1_{\mathbb{B}}\}$, $\mathbb{B}_{\mathbf{m},\alpha} = \mathbb{B}$
- for $\beta < \alpha$ let $\mathbb{B}_{\mathbf{m},\beta} = \mathbb{B}_{\mathbf{k},\beta}$ and $D_{\mathbf{m},\beta}$ be the filter on $\mathbb{B}_{\mathbf{m},\alpha}$ generated by $D_{\mathbf{k},\beta} \cup X_\beta$.

The point is to check $\mathbf{m} \in K_\alpha$ as then $\mathbf{k} \leq_K^{\mathrm{at}} \mathbf{m}$ and $Y \subseteq D_{\mathbf{m}}$ are obvious, also $\bar{\mathbb{B}}_{\mathbf{m}}$ is as required and $D_{\mathbf{m},\beta}$ a filter on $\mathbb{B}_{\mathbf{m},\alpha}$ including $D_{\mathbf{k},\beta}$ are obvious.

So proving $(*)_1$, $(*)_2$, $(*)_3$ below will suffice

$(*)_1$ if $\beta < \gamma < \alpha$ then $D_{\mathbf{m},\gamma} \subseteq D_{\mathbf{m},\beta}$.

[Why? If $a \in D_{\mathbf{m},\gamma}$ then by the choice of $D_{\mathbf{m},\gamma}$ (recalling $D_{\mathbf{k},\gamma}$ is downward directed being a filter and X_γ is downward directed by its choice (i.e. Clause $(d)(\beta)$ of the claim) for some $b \in D_{\mathbf{k},\gamma}$ and $x \in X_\gamma$ we have $b \cap x \leq a$. So by $(d)(\alpha)$ applied to γ we have $x \in Y$ hence by $(d)(\delta)$ applied to β for some $b_1 \in D_{\mathbf{k},\beta}$ and $x_1 \in X_\beta$ we have $b_1 \cap x_1 \leq x$ hence $(b \cap b_1) \cap x_1 = b \cap (b_1 \cap x_1)$ $\leq b \cap x \leq a$ but $b \in D_{\mathbf{k},\gamma} \subseteq D_{\mathbf{k},\beta}, b_1 \in D_{\mathbf{k},\beta}$ hence $b \cap b_1 \in D_{\mathbf{k},\beta}$ and $x_1 \in X_\beta$ hence $a \in D_{\mathbf{m},\beta}$ by the choice of $D_{\mathbf{m},\beta}$.]

$(*)_2$ $D_{\mathbf{m},\beta}$ is a filter on $\mathbb{B}_{\mathbf{m},\alpha} = \mathbb{B}$ disjoint to $\mathbb{B}_{\mathbf{m},\beta} \backslash \{1_{\mathbb{B}_{\mathbf{k}}}\} = \mathbb{B}_{\mathbf{k},\beta} \backslash \{1_{\mathbb{B}_{\mathbf{k}}}\}$.

[Why? By the definition of $D_{\mathbf{m},\beta}$ and clause $(d)(\gamma)$.]

$(*)_3$ if $\beta < \alpha$ then $D_{\mathbf{m},\beta}$ is a maximal filter of $\mathbb{B}_{\mathbf{m}}$ disjoint to $\mathbb{B}_{\mathbf{k},\beta} \backslash \{1_{\mathbf{m},\beta}\} = \mathbb{B}_{\mathbf{m},\beta} \backslash \{1_{\mathbb{B}_{\mathbf{m},\beta}}\}$.

Why? If $b \in \mathbb{B} = \mathbb{B}_{\mathbf{m},\alpha}$ then for some Boolean terms $\sigma(y_0, y_1, \ldots, z_0, \ldots)$ and $a_0, a_1, \ldots \in \mathbb{B}_{\mathbf{k}}$ and $x_0, x_1, \ldots \in Y$ we have $b = \sigma(a_0, a_1, \ldots, x_0, x_1, \ldots)$ hence modulo the filter $D_{\mathbf{m},\beta}, b$ is equal to $\sigma(a_0, a_1, \ldots, 1_{\mathbb{B}_{\mathbf{k},\beta}}, 1_{\mathbb{B}_{\mathbf{k},\beta}}, \ldots)$. But for each a_ℓ there is $c_\ell \in \mathbb{B}_{\mathbf{k},\beta}$ such that $a_\ell = c_\ell \bmod D_{\mathbf{k},\beta}$ hence b is equal to $\sigma(c_0, c_1, \ldots, 1_{\mathbb{B}_{\mathbf{k},\beta}}, 1_{\mathbb{B}_{\mathbf{k},\beta}}, \ldots)$ which belongs to $\mathbb{B}_{\mathbf{k},\beta}$.

As this holds for any $b \in \mathbb{B}$ we are easily done. $\square$

[2] Does this contradict (d)(γ)? No, as $\mathbf{D}_{\mathbf{k},\beta}$ is disjoint to $\mathbb{B}_{\mathbf{k},k} \backslash \{1_{\mathbb{B}_{\mathbf{k}}}\}$.

DEFINITION 2.12. 1) We say $\mathbf{k}$ is reasonable in α *when* $\alpha+1 \le \alpha_{\mathbf{k}}$ (so $\mathbb{B}_{\mathbf{k},\alpha}$ is complete) and there is a maximal antichain of $\mathbb{B}_{\alpha+1}$ included in $\{a \in \mathbb{B}_{\mathbf{k},\alpha+1} : \pi_{\mathbf{k},\alpha+1,\alpha}(a) = 0\}$.

2) We say $\mathbf{k}$ is reasonable *when* it is reasonable in α whenever $\alpha+1 \le \alpha_{\mathbf{k}}$.

3) Let

- $A^1_{\mathbf{k},\alpha} = \{f \in \mathbb{N}^{\mathbb{B}[\mathbf{k}]} : f \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\alpha]}$ and if $\beta < \alpha$ then $f/D_{\mathbf{k}} \notin \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}/D_{\mathbf{k}}\}$
- $A^2_{\mathbf{k},\alpha} = \{f/D_{\mathbf{k}} : f \in A^1_{\mathbf{k},\alpha}\}$
- $A^1_{\mathbf{k},<\alpha} = \bigcup_{\beta<\alpha} A^1_{\mathbf{k},\beta}$ and $A^2_{\mathbf{k},<\alpha} = \bigcup_{\beta<\alpha} A^2_{\mathbf{k},\beta}$, etc.

4) We say f is reasonable in $(\mathbf{k},\alpha,\beta)$ when $\alpha < \beta < \alpha_{\mathbf{k}}$ and $f \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$ and for some $f' \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$, we have $f'/D_{\mathbf{k}} = f/D_{\mathbf{k}}$ and f' is represented by $\langle a_n : n<\omega\rangle$ and $\pi_{\mathbf{k},\alpha}(a_n)=0$ for every n large enough and $a_n \notin D_k$ for every k. If $\beta = \alpha+1$ we may omit it.

5) We say f is reasonable in $(\mathbf{k},<\alpha)$ when it is reasonable in $(\mathbf{k},\beta,\gamma)$ for some $\beta+1=\gamma<\alpha$.

OBSERVATION 2.13. *If* $\beta < \alpha_{\mathbf{k}}, \mathbf{k} \in K^{\mathrm{com}}$ *and* $f \in \mathbb{N}^{\mathbb{B}[\mathbf{k}]}$ *is represented by* $\langle a_n : n < \omega\rangle$, then $f \in A^1_{\mathbf{k},\le\beta}$ iff $\bigcup_n (a_n \Delta \pi_{\mathbf{k},\beta}(a_n)) \notin D_{\mathbf{k}}$.

PROOF. By the proof of 1.12. □

CLAIM 2.14. 1) *If* $\mathbf{k} \in K^{\mathrm{cc}}_{\alpha+1}$ then *there is* $\mathbf{m} \in K^{\mathrm{cc}}_{\alpha+1}$ *such that* $\mathbf{k} \le^{\mathrm{at}}_K \mathbf{m}$, $\mathbb{B}_{\mathbf{m}}$ *is complete and* $\mathbf{m}$ *is reasonable in* α *and* $\|\mathbb{B}_{\mathbf{m}}\| = \|\mathbb{B}_{\mathbf{k}}\|^{\aleph_0}$.

2) *If* $\mathbf{k} \in K^{\mathrm{cc}}_{\alpha+1}$ *is reasonable in* α *and* $\mathbf{k} \le^{\mathrm{at}}_K \mathbf{m}$ *or* $\mathbf{k} \le^{\mathrm{ver}}_K \mathbf{m}$ then $\mathbf{m}$ *is reasonable in* α.

3) *If* $\langle \mathbf{k}_i : i < \delta\rangle$ *is* $\le^{\mathrm{ver}}_K$*-increasing in* K^{cc} *and each* $\mathbf{k}_i$ *is reasonable* then *there is a* $\le^{\mathrm{ver}}_K$*-upper bound* $\mathbf{k}$ *of cardinality* $(\sum_i \|\mathbb{B}_{\mathbf{k}_i}\|)^{\aleph_0}$ *which is reasonable.*

4) *If* f *is reasonable in* $(\mathbf{k},\alpha)$ then *it is reasonable in* $(\mathbf{k},<\alpha+1)$.

5) *If* $f \in A^1_{\mathbf{k},\alpha}$ then f *is reasonable in* $(\mathbf{k},\alpha)$.

6) *In* 2.7,(2),(3) *if* $\mathbf{k}_i$ *is reasonable for every* $i < \delta$ then *so is* $\mathbf{k}$.

PROOF. Straightforward, e.g.:

2) Because $\mathbb{B}_{\mathbf{k}} \lessdot \mathbb{B}_{\mathbf{m},\alpha(\mathbf{k})}$, see Definition 2.4(2) and read Definition 2.12(1).

5) Let $\langle a_n : n < \omega\rangle$ represent f. Let $a'_n = \pi_{\mathbf{k},\alpha+1}(a_n)$, so $\pi_{\mathbf{k},\alpha+1,\alpha}(a_n - a'_n)$ $= \pi_{\mathbf{k},\alpha+1,\alpha}(a_n) - \pi_{\mathbf{k},\alpha+1,\alpha}(a'_n) = a'_n - a'_n = 0$ and $b = \bigcup_n (a_n - a'_n) \in D_{\mathbf{k}}$ by 2.12 so changing $\bar a$ but not b, without loss of generality $\bigwedge_{n\ge1} a'_n = 0$, so f is reasonable in $(\mathbf{k},\alpha)$. □

CLAIM 2.15. *If* (A) *then* (B) *where*:

(A) (a) $\mathbf{k} \in K^{\mathrm{cc}}_\alpha$,
  (b) $\beta_n < \beta_{n+1} < \beta = \bigcup_k \beta_k \le \alpha$,
  (c) $\mathbf{k}$ *is reasonable in* β_n *for every* $n \in \omega$;

(B) *if* $f_1 \in A^1_{\mathbf{k},\beta}$, *i.e.* $f_1 \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}, f_1/D_{\mathbf{k}} \notin \bigcup\{\mathbb{N}^{\mathbb{B}[\mathbf{k},\gamma]}/D_{\mathbf{k}} : \gamma < \beta\}$ then *there is* f_2 *such that*:

(a) $f_2 \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$,
(b) $f_2/D_{\mathbf{k}} \notin \bigcup\{\mathbb{N}^{\mathbb{B}[\mathbf{k},\gamma]}/D_{\mathbf{k}} : \gamma < \beta\}$,
(c) $f_2/D_{\mathbf{k}} < f_1/D_{\mathbf{k}}$,
(d) *there is* $\langle (a_i, k_i) : i < \omega\rangle$ *representing* f_2 *such that:*
(α) *for each* i, *letting* $k(i) = k_i$ *we have*

$$a_i \in \mathbb{B}_{\mathbf{k},\beta_{k(i)+1}} \quad \textit{and} \quad \pi_{\mathbf{k},\beta_{k(i)+1},\beta_{k(i)}}(a_i) = 0,$$

(β) *for each* ℓ *the set* $\{i : k_i < \ell\}$ *is finite.*

Proof. For each n let $\langle a_{n,\ell} : \ell < \omega\rangle$ be a maximal antichain of $\mathbb{B}_{\beta_n+1}$ such that $\pi_{\mathbf{k},\beta_n+1,\beta_n}(a_{n,\ell}) = 0$ for $\ell < \omega$, exists as $\mathbf{k}$ is reasonable in β_n for every $n \in \omega$, see Definition 2.12(2).

Let

$(*)_0$ (a) $\mathscr{T}_n = \left\{\eta : \eta \in {}^n\omega \text{ and } \bigcap_{k<n} a_{k,\eta(k)} > 0\right\}$,
(b) $\mathscr{T} = \bigcup_n \mathscr{T}_n$.

Hence

$(*)_1$ (a) $\langle a_\eta : \eta \in \mathscr{T}_n\rangle$ is a maximal antichain of $\mathbb{B}_{\beta_{n+1}}$ on which $\pi_{\mathbf{k},\beta_n}$ is zero,
(b) $\mathscr{T}$ is a subtree of ${}^{\omega>}\omega$.

Now choose a sequence of natural numbers $\bar{k}$ such that:

$(*)_2$ (a) $\bar{k} = \langle k_s : s \in \mathscr{T}\rangle$,
(b) if $\nu \triangleleft \eta$ then $k_\nu < k_\eta$,
(c) if $k_\eta = k_\nu$ then $\eta = \nu$.

$(*)_3$ let $g_n \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta(n)+1]}$ be represented by $\langle (a_\eta, k_\eta) : \eta \in \mathscr{T}_n\rangle$, see Definition 1.8(4).

[Why? By $(*)_1$(a).]

$(*)_4$ Let $\mathbf{M}_{\mathbf{k}} = \{\bar{c} : \bar{c} = \langle c_\ell : \ell < \omega\rangle$ is a maximal antichain of $\mathbb{B}_{\beta_0+1}$ disjoint to $D_{\mathbf{k}}\}$.

What is the point of $\mathbf{M}_{\mathbf{k}}$? $g_n \in A^1_{\mathbf{k},\beta(n)}$ hence $\langle g_n/D_{\mathbf{k}} : n < \omega\rangle$ is increasing and cofinal in $\bigcup\{\mathbb{N}^{\mathbb{B}[\mathbf{k},\beta(n)]}/D : n < \omega\}$ hence if in $\mathbb{N}^{\mathbb{B}[\mathbf{k}]}/D_{\mathbf{k}}$ we have a definable sequence, the n-th try being g_n/D, in "non-standard places" we have the $g_{\bar{c}}$'s defined below members of $A^2_{\mathbf{k},\alpha}$ and those are co-initial in it.

$(*)_5$ For each $\bar{c} \in \mathbf{M}_{\mathbf{k}}$ let
(a) $S_{\bar{c}} = \{(\ell,\eta) : \ell < \omega, \eta \in \mathscr{T}_\ell \text{ and } c_\ell \cap a_\eta > 0\}$,
(b) for $(\ell,\eta) \in S_{\bar{c}}$ let $a_{(\ell,\eta)} = c_\ell \cap a_\eta$,
(c) $g_{\bar{c}} \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$ be represented by $\langle (a_{(\ell,\eta)}, k_\eta) : (\ell,\eta) \in S_{\bar{c}}\rangle$.

$(*)_6$ g_n is $(\mathbf{k},\beta_n)$-reasonable.

[Why? By the Definition 2.12.]

$(*)_7$ $g_n \in A^1_{\mathbf{k},\beta_n+1}$.

[Why? Follows from the definition of g_n in $(*)_3$ and the choice of the a_η-s and b_ν-s in $(*)_1$ and $(*)_2$.]

$(*)_8$ $g_{\bar{c}} \in A^1_{\mathbf{k},\beta}$ for $\bar{c} \in \mathbf{M_k}$.

[Why? As $\bar{k}$ is with no repetition and the definition.]

$(*)_9$ there is $\bar{c} \in \mathbf{M_k}$ such that $g_{\bar{c}}/D_\mathbf{k} < f_1/D_\mathbf{k}$.

[Why? See explanation after $(*)_4$.] □

CLAIM 2.16. *If* (A) *then* (B) *where*

(A) (*a*) $\mathbf{k} \in K^{\rm cc}_\alpha$ *is* $\mathbb{B}_{\mathbf{k},\alpha}$ *infinite,*

(*b*) $\beta_n = \beta(n) < \alpha$ *is increasing with limit* α,

(*c*) $\mathbf{k}$ *is reasonable in* β_n,

(*d*) $f \in A^1_{\mathbf{k},\alpha}$;

(B) *there are* $\mathbf{m}$, g *such that*

(*a*) $\mathbf{k} \le^{\rm at}_K \mathbf{m}$ *and* $\|\mathbb{B}_\mathbf{m}\| = (\|\mathbb{B}_\mathbf{k}\|^{\aleph_0})^+$,

(*b*) $g \in A^1_{\mathbf{m},\alpha}$,

(*c*) $g/D_\mathbf{m} < f/D_\mathbf{m}$,

(*d*) $g/D_\mathbf{m} \notin \mathbb{N}^{\mathbb{B}[\mathbf{k}]}/D_\mathbf{m}$.

PROOF. Without loss of generality $\mathbb{B}_\mathbf{k}$ is complete of cardinality λ.

Let f_2, $\langle (a_n, k_n) : n < \omega\rangle$ be as in 2.15 for $f_1 = f$ and let $u_n = \{\ell : a_\ell \in \mathbb{B}_{\mathbf{k},\beta_n}\}$, by 2.15 clearly u_n is finite.

Let $\mathbb{B}^0$ be the Boolean algebra extending $\mathbb{B}_\mathbf{k}$ generated by $\mathbb{B}_\mathbf{k} \cup \{x_{\varepsilon,n,\ell} : \ell \le n$ and $\varepsilon < \lambda^+\}$ freely except the equation $x_{\varepsilon,n,\ell} \le a_n$, $x_{\varepsilon,n,\ell_1} \cap x_{\varepsilon,n,\ell_2} = 0$, $\bigcup_{\ell \le n} x_{\varepsilon,n,\ell} = a_n$ for $\varepsilon < \lambda^+$, $\ell \le n$, $\ell_1 < \ell_2 \le n$ and let $\mathbb{B}$ be the completion on $\mathbb{B}^0$. Let $g_\varepsilon \in \mathbb{N}^\mathbb{B}$ be represented by $\langle x_{g_\varepsilon,\ell} := \bigcup\{x_{\varepsilon,n,\ell} : n \text{ satisfies } \ell \le n\} : \ell < \omega\rangle$, clearly

$(*)_1$ $g_\varepsilon/D \le f_2/D$ for any $D \in {\rm uf}(\mathbb{B})$.

For $\varepsilon \ne \zeta < \lambda^+$ let $c_{\varepsilon,\zeta} = \bigcup_\ell (x_{g_\varepsilon,\ell} \Delta x_{g_\zeta,\ell})$.

Now

$(*)_2$ $c_{\varepsilon,\zeta} = \bigcup_n \bigcup_{\ell \le n} (x_{\varepsilon,n,\ell} \Delta x_{\zeta,n,\ell})$.

[Why? As $x_{\varepsilon,n,\ell} \le a_n$ and $\langle a_n : n < \omega\rangle$ is an antichain of $\mathbb{B}$.]

Let $\mathbb{B}'$ be the sub-algebra of $\mathbb{B}$ generated by $\mathbb{B}_\mathbf{k} \cup Y$, $Y := \{c_{\varepsilon,\zeta} : \varepsilon < \zeta < \lambda^+\}$.

$(*)_3$ We define $\pi^1_n : \mathbb{B}_\mathbf{k} \cup Y \to \mathbb{B}_{\beta_n}$ by:

- $\pi^1_n \restriction \mathbb{B}_\mathbf{k} = \pi_{\mathbf{k},\alpha(\mathbf{k}),\beta(n)}$,
- $\pi^1_n(c_{\varepsilon,\zeta}) = 1_{\mathbb{B}_{\beta(n)}}$ for $\varepsilon \ne \zeta < \lambda^+$.

$(*)_4$ π^1_n has an extension $\pi^2_n \in {\rm Hom}(\mathbb{B}', \mathbb{B}_{\beta(n)})$, necessarily unique.

[Why? It is enough to show that if $d_0, \ldots, d_{m-1} \in \mathbb{B}_\mathbf{k}$ and $\varepsilon_\ell < \zeta_\ell < \lambda^+$ for $\ell < k$ and $\sigma(y_0, \ldots, y_{m-1}, x_0, \ldots, x_{k-1})$ is a Boolean term and $\mathbb{B} \models \sigma(d_0, \ldots, d_{m-1}, c_{\varepsilon_0,\zeta_0}, \ldots, c_{\varepsilon_{k-1},\zeta_{k-1}}) = 0$ then

$$\mathbb{B}_{\mathbf{k},\beta(n)} \models \sigma\big(\pi^1_n(d_0), \ldots, \pi^1_n(d_{m-1}), \pi^1_n(c_{\varepsilon_0,\zeta_0}, \ldots)\big) = 0.$$

As $d_0, \ldots, d_{m-1} \in \mathbb{B}_\mathbf{k}$ and $\pi^1_n(c_{\varepsilon_\ell,\zeta_\ell}) = 1_{\mathbb{B}_{\mathbf{k},\beta(n)}}$ it is sufficient to prove: if $d \in \mathbb{B}_\mathbf{k}$, $\eta \in {}^k 2$ and $\mathbb{B} \models$ "$d \cap \bigcap_{\ell < k} c^{[\eta(\ell)]}_{\varepsilon_\ell,\zeta_\ell} = 0$" then $\mathbb{B}_{\mathbf{k},\beta(n)} \models$ "$(\pi_{\mathbf{k},\alpha(\mathbf{k}),\beta(k)}(d)) = 0$".

Now if for some $\ell \geq 1$, $\ell \notin u_n$, $d \cap a_\ell > 0$ the assumption does not hold and otherwise, necessarily $d \leq \bigcup_{\ell \in u_n} a_\ell \cup a_0$ hence the conclusion holds. So indeed π^2_n is well defined.]

$(*)_5$ $\pi^2_n = \pi_{\mathbf{k},\beta(n+1),\beta(n)} \circ \pi^2_{n+1}$.

[Why? See the definition of π^1_m recalling 2.9(4).]

$(*)_6$ there is $\mathbf{n}$ such that $\mathbf{k} \leq^{\rm at}_K \mathbf{n}$ and $\mathbb{B}_{\mathbf{n}} = \mathbb{B}', \pi_{\mathbf{n},\alpha(\mathbf{k}),\beta(n)} = \pi^2_n$ hence $D_{\mathbf{n}} \supseteq Y$.

[Why? Check the definitions.]

$(*)_7$ There is $\mathbf{m}$ such that $\mathbf{n} \leq^{\rm wa}_K \mathbf{m}, \mathbb{B}_{\mathbf{m}} = \mathbb{B}$ hence $\mathbf{k} \leq^{\rm at}_K \mathbf{m}$.

[Why? By claim 2.8(1),(3), the "hence" by 2.8(5).]

$(*)_8$ There is $\varepsilon < \lambda^+$ such that $g_\varepsilon \in A^1_{\mathbf{n},\alpha(\mathbf{k})}$.

[Why? As $A^1_{\mathbf{n},<\alpha(\mathbf{k})}$ has cardinality $\leq \lambda$.]

$(*)_9$ $\mathbf{m}$ is as required.

[By $(*)_7$ and 2.8(2).] $\square$

Observation 2.17. *In Claim* 2.16 *we can demand* $\|\mathbb{B}_{\mathbf{m}}\| = \|\mathbb{B}_{\mathbf{k}}\|^{\aleph_0}$.

Proof. By the Löwenheim–Skolem–Tarski argument. $\square$

Claim 2.18. *Assume* $\mathbf{k} \in K^{\rm cc}_\alpha$ *and* $\mathrm{cf}(\alpha) > \aleph_0$.

If $f \in \mathbb{N}^{\mathbb{B}[\mathbf{k}]}$ *and* $f/D \notin \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}/D_{\mathbf{k}}$ *for* $\beta < \alpha$, then *for some* $\mathbf{m}$ *and* g *we have*:

$(*)$ (a) $\mathbf{k} \leq^{\rm at}_K \mathbf{m}$,

(b) $g \in \mathbb{N}^{\mathbb{B}[\mathbf{m}]}$,

(c) $g/D_{\mathbf{m}} < f/D_{\mathbf{m}}$,

(d) $h/D_{\mathbf{m}} < g/D_{\mathbf{m}}$ when $h \in \mathbb{N}^{\mathbb{B}[\mathbf{k},\beta]}$ *for some* $\beta < \alpha$,

(e) $|\mathbb{B}_{\mathbf{m}}| \leq |\mathbb{B}_{\mathbf{k}}|$.

Proof. Like the proof of 2.16 only simpler and shorter. Let $\bar{a} = \langle a_n : n < \omega \rangle$ represent f, $\lambda = \|\mathbb{B}_{\mathbf{k}}\|$. By 2.14(4), without lose of generality f is reasonable in $(\mathbf{k}, \alpha)$; let $\{x_{\varepsilon,n,\ell} : \varepsilon < \lambda^+, \ell \leq n\}$, $\mathbb{B}^0$, $\mathbb{B}$, Y, $\mathbb{B}'$ be as there and define $\pi_1 \colon \mathbb{B}_{\mathbf{k}} \cup Y \to \mathbb{B}_{\mathbf{k},\alpha}$ as there. $\pi_1 \restriction \mathbb{B}_{\mathbf{k},\alpha} = \pi_{\mathbf{k},\alpha+1,\alpha}$, $\pi_1(c_{\varepsilon,\zeta}) = 1_{\mathbb{B}_{\mathbf{k},\alpha}}$ for $\varepsilon < \zeta < \lambda^+$.

The rest is as there. $\square$

Claim 2.19. *If* $\mathbf{k} \in K^{\rm com}_\alpha$, $\lambda \geq \|\mathbb{B}_{\mathbf{k}}\| + 2^{\aleph_0}$ *and* $p(x)$ *is a type in the model* $\mathbb{N}^{\mathbb{B}[\mathbf{k}]}/D_{\mathbf{k}}$ then *for some* $\mathbf{m} \in K_{\alpha+1}$ *we have* $\mathbf{k} \leq^{\rm ver}_K \mathbf{m}$ *and* $p(x)$ *is realized in* $\mathbb{N}^{\mathbb{B}[\mathbf{m}]}/D_{\mathbf{m}}$.

Proof. Easy. $\square$

Having established all these statements, we can prove now the main result of this paper:

THEOREM 2.20. *For any infinite cardinal* λ, *for some regular ultrafilter* D *on* λ *we have* $\mathrm{upf}(D) = \mathscr{C}$ iff:

(∗) (*a*) $\mathscr{C}$ *is a set of cardinals* $\leq 2^\lambda$,
(*b*) $\mu = \mu^{\aleph_0}$ *whenever* $\mu \in \mathscr{C}$,
(*c*) 2^λ *is the maximal member of* $\mathscr{C}$.

PROOF. The implication ⇒, we already know, so we shall deal with the ⇐ implication; the proof relies on earlier definitions and claims so the reader can return to a second reading in the end.

Let $\langle \lambda_\alpha : \alpha \leq \alpha(*) \rangle$ list $\mathscr{C}$ in increasing order. Let $S = \{\alpha : \alpha \leq \alpha(*) + 1$ and $\mathrm{cf}(\alpha) \neq \aleph_0\}$. We choose $\mathbf{k}_\alpha$ by induction on $\alpha \in S \cap (\alpha(*) + 2)$ such that:

⊞ (a) $\mathbf{k}_\alpha \in K^{\mathrm{com}}_\alpha \cap K^{\mathrm{cc}}_\alpha$, see Definition 2.2, 2.3(1A),
(b) $\mathbf{k}_\beta \leq^{\mathrm{ver}}_K \mathbf{k}_\alpha$ for $\beta < \alpha$, i.e. for $\beta \in \alpha \cap S$, see 2.4(2)(B),
(c) if $f \in A_{\mathbf{k},\beta}$, $\beta < \alpha$ then $\lambda_\beta = |\{g/D_{\mathbf{k}} : g \in \mathbb{N}^{\mathbb{B}},\ g/D_{\mathbf{k}} < f/D_{\mathbf{k}}\}|$,
(d) if $\mathrm{cf}(\alpha) > \aleph_0$ then $\mathbb{B}_{\mathbf{k}_\alpha} = \bigcup\{\mathbb{B}_{\mathbf{k}_\beta} : \beta < \alpha\}$,
(e) $\mathbf{k}_\alpha$ is reasonable (see Definition 2.12).

Case 1 $\alpha = 0$. $\mathbb{B}_{\mathbf{k}_0}$ is the trivial Boolean algebra , so really there is nothing to prove.

Case 2: $\mathrm{cf}(\alpha) > \aleph_0$. Use 2.7(2),(3) to find $\mathbf{k}_\alpha$ satisfying clauses (a), (b), (c), (d). Now $\mathbf{k}_\alpha$ satisfies clause (e) by 2.14(6).

Case 3: $\alpha = \beta + 1$. We choose $\mathbf{k}_{\beta,i}$ by induction for $i \leq \lambda_\beta$ such that

(∗) (a)(α) if $\beta \in S$ then $\mathbf{k}_\beta \leq^{\mathrm{ver}}_K \mathbf{k}_{\beta,i} \in K^{\mathrm{com}}_\alpha \cap K^{\mathrm{cc}}_\alpha$,
(β) if $\beta \notin S$ then $\gamma \in \beta \cap S \Rightarrow \mathbf{k}_\gamma \leq^{\mathrm{ver}}_K \mathbf{k}_{\beta,i} \in K^{\mathrm{com}}_\alpha \cap K^{\mathrm{cc}}_\alpha$,
(γ) if $i = 0$ then there is $g \in \mathbb{N}^{\mathbb{B}[\mathbf{k}_{\beta,i}]}$ such that $g/D_{\mathbf{k}_{\beta,i}} \notin \{f/D_{\mathbf{k}_{\beta,i}} : f \in \mathbb{N}^{\mathbb{B}[\mathbf{k}_\gamma]}$ for some $\gamma \in \alpha \cap S\}$,
(δ) $\mathbb{B}_{\mathbf{k}_{\beta,i}}$ is infinite;
(b) $\langle \mathbf{k}_{\beta,j} : j \leq i \rangle$ is $\leq^{\mathrm{at}}_K$-increasing continuous,
(c) $\mathbb{B}_{\mathbf{k}_{\beta,i}}$ has cardinality $\leq \lambda_\beta$,
(d) if $i = j + 1$:
(α) bookkeeping gives us $g_{\beta,j} \in \mathbb{N}^{\mathbb{B}[\mathbf{k}_{\beta,i}]}$ such that $g_{\beta,j}/D_{\mathbf{k}_{\beta,j}} \notin \bigcup\{\mathbb{N}^{\mathbb{B}[\gamma,\mathbf{k}_\beta]} : \gamma \in \alpha \cap S\}$,
(β) there is $f_{\beta,j} \in \mathbb{N}^{\mathbb{B}[\mathbf{k}_{\beta,i}]}$ such that $f_{\beta,j}/D_{\mathbf{k}_{\beta,i}} < g_{\beta,j}/D_{\mathbf{k}_{\beta,i}}$ and $f_{\beta,j}/D_{\mathbf{k}_{\beta,i}} \notin \bigcup\{\mathbb{N}^{\mathbb{B}[\gamma,\mathbf{k}_\beta]} : \gamma \in \alpha \cap S\}$,
(e) if $i < \lambda_\beta$ and g satisfies $(d)(\alpha)$ then for some $i_1 \in [i, \lambda_\beta], g_{\beta,i_1} = g$,
(f) if $i = j + 1$ then $\mathbb{B}_{\mathbf{k}_{\beta,i}}$ is complete and reasonable.

Note that by 2.14(1) we can take care of clause (f), so we shall ignore it.
For $i = 0$ uses 2.14(1) if $\beta \in S$ and we use 2.7 if $\beta \notin S$.
For i limit use 2.7(1).
For $i = j + 1$, $\mathrm{cf}(j) > \aleph_0$ use the claim 2.18.
If $i = j + 1$, $\mathrm{cf}(j) = \aleph_0$ use the claim 2.16.
For $i = j + 1$, $\mathrm{cf}(j) = 1$ we use 2.14(1).

Having carried the induction on $i \leq \lambda_\beta$ let $\mathbf{k}_\alpha = \mathbf{k}_{\beta,\lambda_\beta}$. In particular $\mathbb{B}_{\mathbf{k}_\beta,\lambda_\beta}$ is complete as $\lambda_\beta = \sup\{i < \lambda_\beta : \mathbb{B}_{\mathbf{k},i}$ is complete$\}$ by clause (f) and $\mathrm{cf}(\lambda_\beta) > \aleph_0$ as $\lambda_\beta = \lambda_\beta^{\aleph_0}$.

Having carried the induction on $\alpha \leq \alpha(*)+1$ clearly the pair $(\mathbb{B}_{\mathbf{k}_{\alpha(*)+1}}, D_{\mathbf{k}_{\alpha(*)+1}})$ is almost as required. That is, (see [9, Ch. VI, §]) we know that for some regular filter D_* on $\mathscr{P}(I)$, there is a homomorphism $\mathbf{j}$ from the Boolean algebra $\mathscr{P}(I)$ onto $\mathbb{B}_{\mathbf{k}_{\alpha(*)+1}}$ and let $D = \{A \subseteq \lambda : \mathbf{j}(A) \in D_{\mathbf{k}_{\alpha(*)+1}}\}$. $\square$

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