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NOTES ON PARTITION CALCULUS

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§0. INTRODUCTION

We deal here with some separate problems which appeared in the problem list of Erdős and Hajnal [1].

In §1 we consider problem 3 of [1] asked by Erdős, Hajnal and Rado, which was the only open case (for infinite cardinals) of $\lambda \to (\mu)^2_2$, and we solve it affirmatively. Thus if $\aleph_\omega < 2^{\aleph_{n(0)}} < \ldots < 2^{\aleph_{n(k)}} < \ldots$, then $\sum\limits_{n<\omega}' 2^{\aleph_n} \to (\aleph_\omega, \aleph_\omega)^2$. We prove a canonization lemma for it.

In §2 we deal with problem 32 of [1] asked by Erdős and Hajnal, which asks whether there is a graph G with $\aleph_1$ vertices and with no $[[\aleph_0, \aleph_1]]$ subgraph for which $\aleph_1 \not\to (G, G)^2$. We provide a wide class of such graphs, assuming CH. If $V = L$ is assumed we show that $\aleph_1 \to (G, G)^2$ iff G has coloring number $\leqslant \aleph_0$.

In §3 we deal with problems 48, 50 of [1] (asked by Erdős and Hajnal): partition relations concerning coloring numbers.

In §4 we deal with problem 42 of [1] asked by Erdős and Hajnal (part A, B) and Gustin (part C). We get compactness and incompactness results on the existence of transversals and on property B. We also find sufficient and necessary conditions for the existence of transversals.

§ 1

Canonization Lemma 1.1. *Suppose* κ, λ_i, $(i < \kappa)$ *are regular,* $i < j \to \lambda_i < \lambda_j$ *and* $\lambda^j \overset{\text{def}}{=} \prod_{i<j} \lambda_i^{\mu(i)} < \lambda_j$. *Suppose* $|A_i| = \lambda_i$ *and* F_i $(i < \chi)$ *is an* n_i*-place function from* $A = \bigcup_{i<\kappa} A_i$ *into* χ, $2^{\chi+\kappa} < \lambda_0$, *so* $2^{\chi+\kappa} < \lambda_j$.

Suppose that for every $B_i \subseteq A_i$ $(i < \alpha < \kappa)$, $|B_i| \leqslant \mu(i)$ *and* $a_i \in A_i$ $(\alpha < i < \kappa)$ *and* $C \subseteq A_\alpha$, $|C| = \lambda_\alpha$ *there is* $B_\alpha \subseteq C$, $|B_\alpha| \leqslant \mu_\alpha$ *such that* $P_\alpha(\langle B_i : i \leqslant \alpha\rangle, \langle a_i : \alpha < i < \kappa\rangle)$ *holds, for specified properties* P_α.

Then there are $a_i^* \in A_i$, $B_i \subseteq A_i$ *such that*

(1) *for any* $a_1, \ldots \in \bigcup_{i<\alpha} B_i$, $b, b' \in B_\alpha$, $c, c' \in B_\beta$ $(\alpha < \beta)$ *and* $i < \chi$,

(1A) $F_i(b, a_1, \ldots) = F_i(b', a_1, \ldots)$;

(1B) $F_i(b, c, a_1, \ldots) = F_i(b', c', a_1, \ldots) = F_i(b', a_\beta^*, a_1, \ldots)$;

(2) *for any* $\alpha < \kappa$, $P_\alpha(\langle B_i : i \leqslant \alpha\rangle, \langle a_i^* : \alpha < i < \kappa\rangle)$.

(3) *If* F_i *is three-place,* $2^{\chi+\kappa} < \mathrm{cf}\,[\mu(i)]$ *for every* i, P_α *hereditary for the* B_i*'s for subsets of the same cardinality then:*

$$a, a' \in B_\alpha,\ b, b' \in B_\beta,\ c, c' \in B_\gamma,\ \alpha < \beta < \gamma < \kappa \ \text{ implies}$$

$$F_i(a, b, c) = F_i(a', b', c').$$

Remark. We could refine the lemma along the lines of [7] §5, but there is no application of it. We can assume that the range of F_i is 2^χ.

Proof. We may assume without loss of generality that the set of functions F_i is closed under permutations and identifications of variables.

Define for any $B \subseteq A$, $\bar{a} \in A$, $\mathrm{tf}(\bar{a}, B)$ as the following set of equations

$$\{F_i(\bar{x}, \bar{b}) = c\colon i < \chi,\ F_i(\bar{a}, \bar{b}) = c,\ \bar{b} \in B,\ c \in \chi\}.$$

Clearly $|\{\mathrm{tf}(\bar{a}, B)\colon \bar{a} \in A\}| \leqslant 2^{|B|+\chi}$. Hence for every $\alpha < \kappa$, any $B_i \subseteq A_i$ $(i < \alpha)$, $|B_i| \leqslant \mu_i$,

$$|\{\mathrm{tf}(a, \bigcup_{i<\alpha} B_i)\colon a \in A_\alpha\}| \leqslant$$

$$\leqslant 2^{\chi + \sum_{i<\alpha} \mu(i)} \leqslant 2^\chi \cdot \prod_{i<\alpha} 2^{\mu(i)} \leqslant \lambda^\alpha.$$

Now the number of such $\langle B_i\colon i < \alpha\rangle$ is $\prod_{i<\alpha} \lambda_i^{\mu(i)} \leqslant \lambda^\alpha$.

So the set C_α of $a \in A_\alpha$ such that for some $B_i \subseteq A_i$ $(i < \alpha)$, $|B_i| \leqslant \mu(i)$, $|\{a' \in A_\alpha\colon \mathrm{tf}(a', \bigcup_{i<\alpha} B_i) = \mathrm{tf}(a, \bigcup_{i<\alpha} B_i)\}| < \lambda_\alpha$ is the union of $\leqslant \lambda^\alpha \cdot \lambda^\alpha < \lambda_\alpha$ sets each of cardinality $< \lambda_\alpha$; hence is of cardinality $< \lambda_\alpha$. Choose $a_i^* \in A_i - C_i$ for each $i < \kappa$. Now define inductively $B_i \subseteq A_i$, $|B_i| \leqslant \mu(i)$. Suppose we have defined B_i for $i < \alpha$. As $a_\alpha^* \notin C_\alpha$, the set

$$B_\alpha^1 = \{a \in A_\alpha\colon \mathrm{tf}(a, \bigcup_{i<\alpha} B_i) = \mathrm{tf}(a_\alpha^*, \bigcup_{i<\alpha} B_i)\}$$

has cardinality λ_α. As $\{\mathrm{tf}(a, \bigcup_{i<\alpha} B_i \cup \{a_i^*\colon i < \kappa\})\colon a \in B_\alpha^1\}$ has cardinality $\leqslant 2^{\chi + \sum_{i<\alpha} \mu(i) + \kappa} < \lambda_\alpha = |B_\alpha^1|$; there are $B_\alpha^2 \subseteq B_\alpha^1$, $|B_\alpha^2| = \lambda_\alpha$ and t_α such that for every $a \in B_\alpha^2$

$$t_\alpha = \mathrm{tf}(a, \bigcup_{i<\alpha} B_i \cup \{a_i^*\colon i < \kappa\})$$

Now by assumption there is $B_\alpha \subseteq B_\alpha^2$ such that $|B_\alpha| \leqslant \mu(\alpha)$ and $P_\alpha(\langle B_i\colon i \leqslant \alpha\rangle, \langle a_i^*\colon \alpha < i < a\rangle)$ holds.

Now clearly (2) holds by the choice of B_α. (1A) holds as $B_\alpha \subseteq B_\alpha^1$; as for (1B), $B_\alpha \subseteq B_\alpha^2$ implies that $F_i(b, a_\beta^*, a_1, \ldots) = F_i(b', a_\beta^*, a_1, \ldots)$, and $B_\beta \subseteq B_\beta^1$ implies

$$F_i(b, c, a_1, \ldots) = F_i(b, a^*_\beta, a_1, \ldots) = F_i(b, c', a_1, \ldots),$$

$$F_i(b', c, a_1, \ldots) = F_i(b', a^*_\beta, a_1, \ldots) = F_i(b', c', a_1, \ldots),$$

and combining the equalities we get (1B). In order to get (3) we should replace the B_α by a subset of the same cardinality.

Theorem 1.2. *If* $\kappa \to (\kappa)^2_2$, $\kappa = \operatorname{cf} \lambda$, $\langle 2^\mu : \mu < \lambda \rangle$ *is not eventually constant, but is eventually* $\geqslant \kappa$, *then*

$$\chi = \sum_{\mu < \lambda} 2^\mu \to (\lambda)^2_2$$

(and in fact even $\chi \to (\lambda, \lambda, \omega)^2$*).*

Proof. Let f be a function from $[\chi]^2$ into $2 = \{0, 1\}$. Choose $\mu(i) < \lambda$ $(i < \kappa)$ such that $\sum_{i<\kappa} \mu(i) = \lambda$ and $2^{\mu(i)}$ is strictly increasing and $2^{\mu(i)} \geqslant \lambda$, and let $\lambda_i = (2^{\mu(i)})^+$ and $A_i = \left\{\alpha\colon \bigcup_{j<i} \lambda_j \leqslant \alpha < \lambda_i\right\}$. If for some $i < \kappa$ there is a $B \subseteq A_i$, $|B| \geqslant \lambda$ such that f is constant on $[B]^2$, then we are ready; so assume there is no such B. As (by [4]) $\lambda_i \to (\lambda_i, \mu(i))^2$ and $\lambda_i \to (\mu(i), \lambda_i)^2$ hold for every $A'_i \subseteq A_i$, $|A'_i| = \lambda_i$, there are sets $B_{i,0}, B_{i,1} \subseteq A_i$ of cardinality $\mu(i)$ such that f has the constant value 0 (1) on $B_{i,0}$ $(B_{i,1})$. Define $P_\alpha(\langle B_i : i \leqslant \alpha \rangle, \langle a^*_i : \alpha < i < \kappa \rangle) \stackrel{\text{def}}{=\!=} [B_\alpha = B_{\alpha,0} \cup B_{\alpha,1}$, f has the constant value 0 (1) on $B_{\alpha,0}$ $(B_{\alpha,1})$, and $|B_{\alpha,0}| = |B_{\alpha,1}| = \mu(\alpha)]$.

So by Lemma 1.1 there are $B_\alpha \subseteq A$, $|B_\alpha| = \mu(i)$, and (by (1B) in the lemma) there is a two-place function g from κ to $\{0, 1\}$ such that $f(a, b) = g(i, j)$ for $a \in B_i$, $b \in B_j$, $i < j$. As $\kappa \to (\kappa)^2_2$ there is $I \subseteq \kappa$, $|I| = \kappa$ and $\delta \in 2$ such that for any $i < j \in I$, $g(i, j) = \delta$. Let $B = \bigcup_{\alpha \in I} B_{\alpha,\delta}$. Then clearly $|B| = \sum_{i \in I} \mu(i) = \lambda$, and f has on $[B]^2$ the constant value δ.

Corollary 1.3. *If* $\aleph_\omega < 2^{\aleph_{n(0)}} < 2^{\aleph_{n(1)}} < \ldots$ *then* $\sum_{n<\omega} 2^{\aleph_n} \to (\aleph_\omega, \aleph_\omega)^2$.

Remark. This answers problem 3 of [1], and Theorem 2 completes the answer to the question "when $\lambda \to (\mu)^2_2$" for infinite λ, μ.

Conjecture 1A (Hajnal). *If* $\aleph_\omega < 2^{\aleph_{n(0)}} < 2^{\aleph_{n(1)}} < \ldots$ *then* $\sum_{n<\omega} 2^{\aleph_n} \to (\aleph_\omega, 4)^3$.

Remark. For all previously known proved cases of $\lambda \to (\mu, \mu)^2$ also $\lambda \to (\mu, 4)^3$ holds; on the other hand $\sum_{n<\omega} 2^{\aleph_n} \nrightarrow (\aleph_\omega, 5)^3$.

§2. NOTATION

Notation. For a graph G, let $V(G)$ be its set of vertices, and $E(G)$ its set of edges. We write $a \in G$ instead of $a \in V(G)$, etc.

Definition 2.1. $\lambda \to (G_i)^2_{i<\alpha}$ $(\lambda \nrightarrow [G_i]^2_{i<\alpha})$ (G_i graphs) if for every α-colouring f of λ (i.e. a function $f\colon [\lambda]^2 \to \{i\colon i < \alpha\}$) there is an $i < \alpha$ and a one-to-one function F from $V(G_i)$ into λ such that $a \neq b \in G_i \wedge \{ab\} \in E(G_i) \Rightarrow f(F(a), F(b)) = i$, $(f(F(a), F(b)) \neq i)$.

Definition 2.2. $V_\lambda(A, G) = |\{a \in G\colon |\{b \in A\colon a, b$ are connected$\}| \geqslant \lambda\}|$.

Theorem 2.1.

(A) (CH) $\Rightarrow \aleph_1 \nrightarrow (G)^2_2$ *if* $V_{\aleph_0}(A, G) = \aleph_1$ *for some countable* $A \subseteq G$.

(B) $(2^\lambda = \lambda^+)$ $\lambda^+ \nrightarrow [G_i]^2_{i<\lambda^+}$ *if for every* i *there is* A_i *such that* $V_\lambda(A_i, G) = \lambda^+$ *where* $A_i \subseteq G_i$, $|A_i| = \lambda$.

Proof. Clearly it suffices to prove (B). Let $\{\langle \alpha_i, F_i\rangle\colon \lambda \leqslant i,\ i < \lambda^+\}$ be a list of the pairs $\langle \alpha, F\rangle$, where $\alpha < \lambda^+$ and F is a one-to-one function from A_α into λ^+; we may assume without loss of generality that the range of F_i is $\subseteq \{j\colon j < i\}$.

Let $V(G_{\alpha_i}) = \{a^i_j\colon j < \lambda^+\}$ and $B^i_j = \{F_i(a^i_\beta)\colon a^i_\beta \in A_{\alpha_i},\ \beta < \lambda^+,\ (a^i_\beta, a^i_j) \in E(G_{\alpha_i})\}$, We may choose the enumeration of $V(G_{\alpha_i})$ so that $|B^i_j| = \lambda$ holds for limit j.

Now we define by induction on i the coloring f on $[i]^2$. For $i, j < \lambda$ we define $f(i,j)$ arbitrarily. Suppose f is defined on $[i]^2$.

Now we define $f(i,j)$ for $j<i$. It is well-known that if $\{C_j: j<\lambda\}$ is a family of λ sets, $|C_i|=\lambda$, then we can find pairwise disjoint $C'_i \subseteq C_i$, $|C'_i|=\lambda$, hence we can define $f(i,\alpha)$ such that if $j,\beta \leqslant i$, $|B^j_\beta| = \lambda$, $B^j_\beta \subseteq i$ then

$$\{f(i,\gamma): \gamma \in B^j_\beta\} = i = \{\gamma: \gamma < i\}.$$

Now the coloring f is defined. Suppose F is an embedding of G_α into λ^+ contradicting our claim, then for some $i<\lambda^+$, $\alpha=\alpha_i, F_i = F \upharpoonright A_\alpha$ and for some limit δ we have $i,\lambda<\delta<\lambda^+$; $a^i_\beta \in A_\alpha \Rightarrow \beta<\delta$; and $\beta<\delta \Leftrightarrow F(a^i_\beta)<\delta$. But then $f(F(a^i_\delta), b) \neq \alpha < \delta$ for every $b \in B^i_\delta$, contradicting the definition of f.

Corollary 2.2. *There exists a graph G with $\aleph_1$ vertices which does not have a subgraph of type $[[\aleph_0, \aleph_1]]$ (bipartite graph) but $\aleph_1 \nrightarrow (G,G)$.*

Remark. This answers problem 32 from [1] affirmatively.

Proof. Let $\{A_\alpha: \alpha<\omega_1\}$ be a set of subsets of ω such that $\alpha\neq\beta$ implies $|A_\alpha \cap A_\beta| < \aleph_0$. G will have ω_1 as the set of vertices, and α,β are connected iff $\alpha<\omega\leqslant\beta$, $\alpha\in A_\beta$. Then by 2.1 $\aleph_1 \nrightarrow (G)^2_2$, but G satisfies the other requirements by the construction.

Definition 2.2. The coloring number of a graph G, $\mathrm{cl}(G)$ is the minimal cardinal λ such that we can list its set of vertices $\{a_i: i<l_0\}$ such that each a_i is connected to $<\lambda$ a_j's for $j<i$;

Theorem 2.3. *If G has coloring number $\aleph_0$ and $\leqslant\lambda$ vertices then $\lambda \rightarrow (G)^2_n$ for every $n<\aleph_0$.*

Proof. By [3] we can assume the set of vertices of G is $\{a_i: i<\mu\}$, where $\mu\leqslant\lambda$, such that each a_i is connected to $<\aleph_0$ a_j's with $j<i$.

Let $f: [\lambda]^2 \rightarrow \{0,\ldots,n-1\}$ be an n-coloring of λ.

Let D be a uniform ultrafilter over λ and define $g(\alpha)=$ the $i\in n$ such that $A_i=\{\beta<\lambda: f(\alpha,\beta)=i\}\in D$. ($g$ is well defined because if $\lambda = A_0 \cup \ldots \cup A_{n-1}$, the A_i's are disjoint, and so $A_i\in D$ for exactly one $i<n$.) Let $i_0<n$ be such that $\{\alpha<\lambda: g(\alpha)=i_0\}\in D$. Now define $b_j<\lambda$ by induction on j, such that $g(b_j)=i_0$, $f(b_k,b_j)=i_0$, if a_k, a_j is connected, and $k \neq j \Rightarrow b_k \neq b_j$. If for $k < j$, $b_k < \lambda$ is defined, let $\{k_1, \ldots, k_m\}$ be the set of $k < j$ such that a_k, a_j are connected in G. Then $A_{k_1}, \ldots, A_{k_n} \in D$ hence $A_{k_1} \cap \ldots \cap A_{k_m} \in D$, hence it has cardinality λ so there is $b_j \in A_{k_1} \cap \ldots \cap A_{k_m} - \{b_k : k < j\}$. Clearly $a_i \rightarrow b_i$ is the embedding we seek.

Theorem 2.4 (V = L).

(A) *If G is a graph with $\aleph_1$ vertices which has coloring number $> \aleph_0$ (that is $\aleph_1$) then $\aleph_1 \nrightarrow (G)^2_2$.*

(B) *If each G_i, $(i < \omega_1)$ has $\aleph_1$ vertices and coloring number $> \aleph_0$ then $\aleph_1 \nrightarrow [G_i]^2_{i<\omega_1}$.*

Remarks.

(1) I first claimed the theorem incorrectly without V = L, and A. Hajnal and A. Máté, who tried to reconstruct the proof, also proved this theorem.

(2) We prove only (A). The proof of (B) is similar.

Proof. (A) We may assume without loss of generality that ω_1 is the set of vertices of G. We first show that

(∗) the set $I \subseteq \omega_1$ of $\alpha < \omega_1$ such that, for some $j = j_\alpha \geq \alpha$, j is connected to infinitely many $\beta < \alpha$ is stationary.

In fact, assuming the contrary, let $C \subseteq \omega_1$ be closed and unbounded and disjoint to I, and write $C = \{c_i : i < \omega_1\}$ $(c_0 = 0)$. Now, for each $i < \omega_1$, rearrange $[c_i, c_{i+1})$ in a sequence of type ω; this rearrangement shows that G has coloring number $\aleph_0$, which is a contradiction. So (∗) holds.

Clearly, we may assume that if $\alpha \in I$ then α is limit and $j_\alpha = \alpha$ holds. By Jensen [5] there are functions $F_\alpha : \alpha \rightarrow \alpha$, $(\alpha \in I)$ such that the set $\{\alpha \in I : F \restriction \alpha = F_\alpha\}$ is stationary for every function $F : \omega_1 \rightarrow \omega_1$. We are about to define the coloring function $f : [\omega_1]^2 \rightarrow \{0, 1\}$. We define $f(i, j)$ by induction on $\max\{i, j\}$. f is defined arbitrarily on $[\omega]^2$.

If f is defined on $[\beta]^2$, $\beta < \omega_1$, then define $f(\beta, i)$ for every $i < \beta$ such that $\{f(\beta, F_\alpha(j))\colon \alpha \leqslant \beta,\ \alpha \in I,\ \text{and}\ j \in A_\alpha\} = \{0, 1\}$, where $A_\alpha = \{\gamma < \alpha\colon \{\gamma, \alpha\} \in G\}$ (note that A_α is infinite).

Assume that $F\colon \omega_1 \to \omega_1$ is an embedding of G into ω_1 such that if i and j are connected then $f(F(i), F(j)) = \delta$ for a fixed δ ($\delta = 0$ or 1). Clearly, $C = \{\alpha < \omega_1\colon F(j) < \alpha \text{ iff } j < \alpha\}$ is closed and unbounded. By the definition of the F_α's, the set $\{\alpha \in I\colon F \restriction \alpha = F_\alpha\}$ is stationary; hence there is an $\alpha \in C \cap I$ such that $F \restriction \alpha = F_\alpha$. Then the definition of f with $\beta = F(\alpha)$ shows that there are $\gamma, \gamma' < \alpha$ such that $\{\gamma, \alpha\}, \{\gamma', \alpha\} \in G$ and $f(F(\alpha), F(\gamma)) = 0$ and $f(F(\alpha), F(\gamma')) = 1$, which contradicts our assumption on F, completing the proof.

Conclusion 2.5. (V = L). *For graphs G with $\aleph_1$ vertices, $\aleph_1 \not\twoheadrightarrow (G)^2_2$ iff G has coloring number $\leqslant \aleph_0$.*

§3.

Definition 3.1. $(\lambda, \mu) \to (\kappa, \chi)$ holds if every graph G with λ vertices all whose subgraphs spanned by a set of $< \mu$ vertices have colouring number $\leqslant \kappa$ has coloring number $\leqslant \chi$.

See [7] for more material on this. Just as Erdős and Hajnal [2] notice that V = L implies a positive anwer to problem 42c of [1], we can notice:

Lemma 3.1 (V = L).

(1) *Assume λ is regular. Then $(\lambda, \lambda) \to (\aleph_0, \aleph_0)$ iff λ is weakly compact.*

(2) *If λ is not weakly compact, then there is a graph of cardinality λ showing $(\lambda, \lambda) \not\twoheadrightarrow (\aleph_0, \aleph_0)$ that has chromatic number $\aleph_1$ and every subgraph of smaller cardinality has chromatic number $\aleph_0$.*

Remark. In fact we can replace $\aleph_0$ by any $\mu < \lambda$; this partially answers 48A [1] (when μ is regular).

Proof.

(1) If λ is weakly compact, it is immediate that $(\lambda, \lambda) \to (\aleph_0, \aleph_0)$ (formulate a suitable set of $L_{\lambda,\lambda}$-sentences such that every subset of power $< \lambda$ has a model by the assumption of $(\lambda, \lambda) \to (\aleph_0, \aleph_0)$, and a model of it gives the conclusion).

Suppose now λ is not weakly compact. By J e n s e n [5] there is a stationary $C \subseteq \lambda$, $\alpha \in C \Rightarrow \mathrm{cf}\,\alpha = \omega$, and we may assume that $\beta < \alpha$, $\alpha \in C \Rightarrow \beta + \omega < \alpha$, but for every limit ordinal $\delta < \lambda$, $C \cap \delta$ is not stationary. Choose $A_\alpha \subseteq \alpha$ for $\alpha \in C$ such that the order type of A_α is ω and $\sup A_\alpha = \alpha$. Define a graph G with set of vertices $\{\alpha: \alpha < \lambda\}$ and set of edges $\{(i, \alpha): i \in A_\alpha,\ \alpha \in C\}$. Now we prove by induction on α that the restriction of G to $\{i: i < \alpha\}$ has coloring number $\aleph_0$. For $\alpha = 0$, or α successor it is immediate; and if α is limit, choose a continuous increasing unbounded sequence $\alpha_i < \alpha$, $i < \mathrm{cf}\,\alpha$, $\alpha_i \notin C$. By the induction hypothesis the restriction of G to $[\alpha_i; \alpha_{i+1})$ has coloring number $\aleph_0$, so let $<_i$ be a suitable order. Define an order $<^*$ on $[0, \alpha)$: $a <^* b$ iff $a < \alpha_i \leqslant b$ for some i or $\alpha_i \leqslant a, b < \alpha_{i+1}$, $a <_i b$ for some i.

For any $a < \alpha$, let $\alpha_i \leqslant a < \alpha_{i+1}$; then

$$\{b <^* a:\ a, b \text{ are connected}\} =$$

$$= \{b < \alpha_i:\ a, b \text{ are connected}\} \cup \{b:\ a, b \text{ are connected},\ b <_i a,\ \alpha_i < b < \alpha_{i+1}\} = \{b < \alpha_i:\ b \in A_a,\ a \in C\} \cup$$

$$\cup \{b:\ b <_i a,\ \alpha_i \leqslant b < \alpha_{i+1},\ a, b \text{ are connected}\}\,.$$

Both sets are finite (the first one since A_a has order type ω and $\alpha_i \notin C$, the second by the definition of $<_i$). Hence the coloring number of G restricted to $\{i: i < \alpha\}$ is $\aleph_0$.

Suppose the coloring number of G is $\leqslant \aleph_0$. By [3] there is an order $<^*$ of $\{\alpha: \alpha < \lambda\}$ of order-type λ such that $|\{\beta <^* \alpha:\ \beta, \alpha$ are connected in $G\}| < \aleph_0$ for every α. It is well known that $S = \{\alpha < \lambda:\ \beta < \alpha \Rightarrow \beta <^* \alpha$ for any $\beta < \lambda\}$ is a closed unbounded subset of λ, hence there is $\alpha \in S \cap C$, and so $A_\alpha \subseteq \{\beta: \beta <^* \alpha\}$; a contradiction.

(2) By $\diamondsuit_\lambda$ of Jensen [5], there are partitions $\langle B_n^\alpha : n < \omega \rangle$ of α such that the set $\{\alpha < \lambda : \operatorname{cf} \alpha = \omega,\ \forall n < \omega [B_n \cap \alpha = B_n^\alpha]\}$ is stationary for any partition $\langle B_n : n < \omega \rangle$ of λ. Choose A_α in the construction above so that if B_n^α is unbounded in α, then $A_\alpha \cap B_n^\alpha \neq \phi$.

Definition 3.2. $\operatorname{Col}(\lambda, \mu, \kappa, \chi)$ holds if every graph G with $|V(G)| = |V(G)| = \lambda$, $\operatorname{cl}(G) > \mu$, contains a $[[\kappa, \chi]]$ subgraph.

Theorem 3.2 (G.C.H.). *The following are equivalent:*

(A) *not* $\operatorname{Col}(\aleph_{\omega+1}, \aleph_1, \aleph_1, \aleph_0)$

(B) *not* $\operatorname{Col}(\aleph_{\omega+1}, \aleph_1, \aleph_2, \aleph_0)$

(C) $(\aleph_{\omega+1}, \aleph_3) \not\rightarrow (\aleph_1, \aleph_1)$

(D) *there are a stationary set*, $C \subseteq \{\alpha < \aleph_{\omega+1} : \operatorname{cf} \alpha = \aleph_1\}$ *and sets* $S_\alpha \subseteq \alpha$, $\operatorname{tp}(S_\alpha) = \omega_1$, $\sup S_\alpha = \alpha$, *such that* $\alpha, \beta \in C$, $\alpha \neq \beta \Rightarrow$ $\Rightarrow |S_\alpha \cap S_\beta| < \aleph_0$.

Remark. This gives a partial answer to problem 5.7 of [3], which is between (A) and (B).

Proof.

(D) $\Rightarrow$ (A). Define G by $V(G) = \aleph_{\omega+1}$, $E(G) = \{(\alpha, \beta) : \alpha \in S_\beta,\ \beta \in C\}$. Suppose $A \times B \subseteq E(G)$, $|A| = \aleph_0$, $|B| = \aleph_1$. As $|A| \neq |B|$, we may assume $A < B$ (i.e. $a \in A, b \in B \Rightarrow a < b$) or $B < A$. If $A < B$ then choose $b_1 \neq b_2 \in B$, so $S_{b_1} \cap S_{b_2} \supseteq A$ is infinite, a contradiction. If $B < A$, the contradiction is similar. So G does not have an $[[\aleph_0, \aleph_1]]$ subgraph.

On the other hand, trivially, $\operatorname{cl} G \leqslant \aleph_2$, as the natural ordering of ordinals shows. Suppose $\operatorname{cl}(G) \leqslant \aleph_1$, and $<^*$ is an order of $\aleph_{\omega+1}$ confirming this; we may assume by [3] that $<^*$ has order-type $\omega_{\omega+1}$. It is well known that $\{\delta < \omega_{\omega+1} : (\forall \alpha)(\alpha < \delta \Rightarrow \alpha <^* \delta)\}$ is a closed and unbounded subset of $\omega_{\omega+1}$, so some $\delta \in C$ belongs to it, hence it is connected to $\aleph_1$ of its predecessors, a contradiction. Hence $\operatorname{cl}(G) = \aleph_2$; but we have proved that G has no $[[\aleph_0, \aleph_1]]$ subgraph, so (A) holds.

(A) $\Rightarrow$ (B). Trivial.

(B) $\Rightarrow$ (C). Suppose G shows (B), that is $|V(G)| = \aleph_{\omega+1}$, $\mathrm{cl}\,(G) > \aleph_1$ but G has no $[[\aleph_2, \aleph_0]]$ subgraph; so by [3] 5.5, every subgraph G' of G with $\leqslant \aleph_\omega$ vertices has coloring number $\leqslant \aleph_1$. Hence this G shows that (C) holds.

(C) $\Rightarrow$ (D). Let G show (C), and assume $V(G) = \aleph_{\omega+1}$. For any $\alpha < \aleph_{\omega+1}$, choose A_n^α, $\bigcup_{n<\omega} A_n^\alpha = \{i\colon i < \alpha\}$, $|A_n^\alpha| \leqslant \aleph_n$, and let $F(\alpha) < \aleph_{\omega+1}$ be the first ordinal such that if $n < \omega$, $B \subseteq A_n$ and

$$c(B, A_n) = \{\gamma < \aleph_{\omega+1}\colon (\forall a \in A_n)[(a, \gamma) \in E(G) \equiv a \in B]\}$$

has cardinality $\leqslant \aleph_\omega$ then $c(B, A_n) < F(\alpha)$, and if $|c(B, A_\alpha)| = \aleph_{\omega+1}$ then $c(B, A_n) \cap \{i\colon i < F(\alpha)\}$ has cardinality $\aleph_\omega$; and also if $a \leqslant \alpha$, $A = \{b < a\colon (b, a) \in E(G)\}$, $|A| < \aleph_\omega$, $B \subseteq A$, $|c(B, A)| < \aleph_{\omega+1}$, then $c(B, A) \leqslant F(\alpha)$. Now let $C_1 = \{\delta < \aleph_{\omega+1}\colon \alpha < \delta \rightarrow F(\alpha) < \delta\}$. We may assume that if $\delta \in C_1$ and $|\{a < \delta\colon (a, c) \in E(G)\}| \geqslant \mathrm{cf}\,\delta$ for some $c > \delta$, *then* $|\{a < \delta\colon (a, \delta) \in E(G)\}| \geqslant \mathrm{cf}\,\delta$.

By our choice, for every $G' \subseteq G$ with $|V(G')| < \aleph_3$ we have $\mathrm{cl}\,(G') \leqslant \aleph_1$, hence it is easy to see that G has no $[[\aleph_1, \aleph_2]]$ subgraph. So if $|\{a < \alpha\colon (a, c) \in E(G)\}| \geqslant \aleph_1$ *then* $c < F(\alpha)$.

Let $C = \{\delta \in C_1\colon |\{\alpha < \delta\colon (\alpha, \delta) \in E(G)\}| \geqslant \aleph_1\}$.

Let us show that C is stationary. If not, let $C_2 \subseteq C_1$ be closed, unbounded and disjoint to C, and let

$$C_2 = \{\delta_i\colon i < \aleph_{\omega+1}\}.$$

Clearly for any i, the subgraph of G spanned by $[\delta_i, \delta_{i+1})$, G_i, has coloring number $\leqslant \aleph_1$. By our construction and definition of C_2, if $a \in [\delta_i, \delta_{i+1})$ then $|\{c\colon c < \delta_i, (c, a) \in E(G)\}| \leqslant \aleph_0$. So it is easy to see that $\mathrm{cl}\,(G) \leqslant \aleph_1$, a contradiction. So C is stationary. For $\delta \in C$ let $S_\delta = \{\alpha < \delta\colon (\alpha, \delta) \in E(G)\}$.

By definition of C, $|S_\delta| \geqslant \aleph_1$. If for some $\alpha < \delta$, $S_1 = S_\delta \cap \{i\colon i < \alpha\}$ has cardinality $\aleph_1$, then for some n, $|A_n^\alpha \cap S| \geqslant \aleph_1$, hence

$\delta < F(\alpha)$, contradicting $C \subseteq C_1$. So $|S_\delta| = \aleph_1$, and $\mathrm{tp}(S_\delta) = \omega_1$, and $\sup S_\delta = \delta$, so $\mathrm{cf}\,\delta = \omega_1$. Suppose $\delta_1 < \delta_2 \in C$, $S_{\delta_1} \cap S_{\delta_2}$ is infinite. (Note that $\mathrm{cf}\,\delta_1 = \mathrm{cf}\,\delta_2 = \omega_1$.)

Let $a = \delta_1$, $A = S_{\delta_1}$, $B = S_{\delta_1} \cap S_{\delta_2}$; then by the definition of F, $F(\delta_1) > \delta_2$, a contradiction. So C and the S_α's show that (D) holds.

Lemma 3.3. *If the coloring number of G is $< \mu$, then there are no sets A, B of vertices such that:*

$$|B| > |A| \geqslant \mu, \text{ and for every } b \in B$$

$$|\{a \in A\colon (a, b) \in E(G)\}| \geqslant \mu .$$

Proof. Easy.

This enables us to eliminate G.C.H. from some results in this section and from similar results after suitable changes.

§4.

Definition 4.1. Let $\lambda^+ \geqslant \kappa$.

(A) PT (λ, κ) holds if there is an indexed family S of λ sets, each of cardinality $< \kappa$, such that S has no transversal, but every $S' \subseteq S$, $|S'| < \lambda$ has a transversal.

PT*(λ, κ) is defined in the same way except that $A \in S \Rightarrow |A| = \kappa$; $\lambda \geqslant \kappa$. A transversal of S is a one-to-one function f such that for any $A \in S$, $f(A) \in A$.

(B) PB (λ, κ), PB* (λ, κ) are defined similarly replacing "has a transversal" by having property B. A family S has property B if there is a set C such that $A \in S \Rightarrow A \cap C \neq \phi$, $A - C \neq \phi$.

(C) PD (λ, μ) hold if there is a graph G with λ vertices such that G does not have the property $D(\mu)$, but every subgraph with $< \lambda$ vertices has. G has property $D(\mu)$ if we can direct its edges so that the number of directed edges emanating from any vertex is $< \mu$.

These properties arise from problem 42 [1] ($\lambda = \aleph_2$ there); PT from a question of Gustin; PB, PD from questions of Erdős and Hajnal. We give partial answers.

Lemma 4.1.

(A) $\mathrm{PT}^*(\lambda, \kappa)$ *implies* $\mathrm{PT}(\lambda, \kappa^+)$; $\mathrm{PB}^*(\lambda, \kappa)$ *implies* $\mathrm{PB}(\lambda, \kappa^+)$.

(B) *If* $\lambda > \kappa$ *then* $\mathrm{PT}^*(\lambda, \kappa)$ *iff* $\mathrm{PT}(\lambda, \kappa^+)$.

(C) $\mathrm{PT}(\mu, \mu^+)$, $\mathrm{PT}^*(\kappa^+, \kappa)$, $\mathrm{PB}^*(2^\kappa, \kappa)$, *and* $\mathrm{PD}(\kappa^+, \kappa)$ *hold, but* $\mathrm{PB}(\lambda, \aleph_0)$, $\mathrm{PT}(\lambda, \aleph_0)$, $\mathrm{PT}^*(\kappa, \kappa)$, $\mathrm{PB}^*(\kappa, \kappa)$, $\mathrm{PD}(\kappa, \kappa)$ *do not hold.*

(D) *If* $\kappa_2 \geqslant \kappa_1$ *then* $\mathrm{PT}(\lambda, \kappa_1)$ *implies* $\mathrm{PT}(\lambda, \kappa_2)$; $\mathrm{PB}(\lambda, \kappa_1)$ *implies* $\mathrm{PB}(\lambda, \kappa_2)$.

(E) $\mathrm{PT}(\lambda, \kappa) \Rightarrow \mathrm{PT}(\lambda^+, \kappa)$ *for regular* λ, *hence* $\mathrm{PT}(\aleph_{\alpha+n}, \aleph_\alpha)$ *holds.*

(F) *If* κ *is singular, then* $\mathrm{PT}(\kappa, \kappa)$.

(G) *If* $\mathrm{cf}\,\lambda < \mathrm{cf}\,\mu$ *then* $\mathrm{PD}(\lambda, \mu)$ *fails.*

Proof.

(A) Immediate by the definitions: use the same family.

(B) Let the family S exemplify $\mathrm{PT}(\lambda, \kappa^+)$; we may assume that the elements of each $A \in S$ are ordinals. Let

$$S' = \{\kappa \cup (A \times \{\alpha\}) \colon A \in S,\ \alpha < \kappa^+\}.$$

It is easy to check that S' exemplifies $\mathrm{PT}^*(\lambda, \kappa)$

(C) For $\mathrm{PT}(\mu, \mu^+)$ take

$$S = \{\mu\} \cup \{\{\alpha\} \colon \alpha < \mu\},$$

for $\mathrm{PT}^*(\kappa^+, \kappa)$ take $S = \{\alpha \colon \kappa \leqslant \alpha < \kappa^+\}$ (these examples are well known). For $\mathrm{PB}^*(2^\kappa, \kappa)$ take a maximal family S of subsets of κ of power 2^κ such that $A \neq B \in S \Rightarrow A \not\subset B$, and S is closed under complements (there is such family of power 2^κ, and we can extend it to a maximal one). For PD (κ^+, κ) take the complete graph with κ^+ vertices.

It is easy to show that PT* (κ, κ), PB* (κ, κ), and PD (κ, κ) fail. PT $(\lambda, \aleph_0)$ fails by Hall's theorem, and PB $(\lambda, \aleph_0)$ fails by compactness arguments.

(D) Immediate by the definitions (use the same family).

(E) See [9].

(F) Let $\kappa = \sum_{i<\mu} \kappa_i$, $\mu < \kappa_0 < \kappa_1 < \ldots$, $\kappa_\delta = \bigcup_{i<\delta} \kappa_i$ for limit δ. Let

$$S = \{\{\alpha\}: \alpha < \kappa,\ \alpha \neq \kappa_i \text{ for any } i < \mu\} \cup$$

$$\cup \{\{\alpha: \kappa_i \leqslant \alpha < \kappa_{i+1}\}: i < \mu\} \cup \{\{\kappa_i: i < \mu\}\}.$$

(G) Immediate.

Lemma 4.2. *Assume* V = L *and let* λ, μ *be regular,* $\lambda > \mu$. *Then the following are equivalent:*

(A) λ *is not weakly compact.*

(B) PT* (λ, μ)

(C) PB* (λ, μ)

(D) PD (λ, μ)

(E) *If* λ *is inaccessible there is a family* S *exemplifying* PB (λ, λ) *(and also* PT (λ, λ)*) such that no two members of* S *have the same cardinality.*

Remark. Erdős and Hajnal [2] already noticed (A) ⇒ (B) and the proof of 3.1 is similar to it. Therefore, we only give it here concisely. Parts (B)-(D) give answers to problem 42 [1]. (E) is a privately communicated problem of Erdős.

Proof. Clearly if λ is weakly compact then (B)-(E) fail. So suppose λ is not weakly compact. Then by Jensen [5] if $\mu = \aleph_0$, and by a slight improvement of A. Beler otherwise, there is a stationary $C \subseteq \subseteq \{\alpha < \lambda\colon \operatorname{cf} \alpha = \operatorname{cf} \mu,\ \beta + \mu \leqslant \alpha \text{ for } \beta < \alpha\}$ such that for every $\delta < \lambda$, $C \cap \delta$ is not stationary. For each $\alpha \in C$, choose $A_\alpha \subseteq \alpha$, $\sup A_\alpha = \alpha$, $\operatorname{tp} A_\alpha = \mu$; then $S = \{A_\alpha\colon \alpha \in C\}$ proves (B). Let G be a graph whose set of vertices is λ, and its set of edges $\{(\alpha, \beta)\colon \alpha \in A_\beta\}$; this proves (D). Now by Jensen [5], there are sets $T^\alpha \subseteq \alpha$ such that for any $A \subseteq \lambda$ of cardinality λ, $\{\alpha \in C\colon A \cap \alpha = T^\alpha\}$ is stationary, and $|T^\alpha| = |\alpha|$. Choose $A^\alpha \subseteq T^\alpha$, $|A^\alpha| = \mu$ and then $\{A^\alpha\colon \alpha \in C\}$ proves (C). Now (E) is proved by $S = \{T^\alpha\colon \alpha \in C,\ \alpha \text{ is a limit cardinal}\}$ (any $S' \subseteq S$, $|S'| < |S|$ has property B because $S' \cup S$ (which is $\neq S'$, as it is an indexed family) has a transversal by [9]).

Notation. For a family S, and sets A, B

$$S(A) = \{C\colon C \in S,\ C \subseteq A\}$$

$$S(A, B) = S^A_B = \{C - B\colon C \in S,\ C \subseteq A,\ C \nsubseteq B\}.$$

Definition 4.2. Define $m(S, \kappa)$ recursively, where S is a family of sets of cardinality $< \kappa$, as follows:

Case I. $|S| < \kappa$, then $m(S, \kappa)$ is 0 if S has a transversal, and -1 otherwise.

Case II. $\lambda = |S| \geqslant \kappa$ is regular. Then $m(S, \kappa)$ is $-\lambda$ if there is a continuous increasing sequence of sets A_α, $\alpha < \lambda$, $|A_\alpha| < \lambda$, and $A \in S \Rightarrow A \subseteq \bigcup_{\alpha < \lambda} A_\alpha$, and C_1 or C_2 is stationary where

$$C_1 = \Big\{\alpha\colon S(A_\alpha) - \bigcup_{\beta < \alpha} S(A_\beta) \neq \phi\Big\}$$

$$C_2 = \{\alpha\colon m[S(A_{\alpha+1}, A_\alpha), \kappa] < 0\}$$

or $|\bigcup_{\alpha < \lambda} A_\alpha| < \lambda$; and $m(S, \kappa) = 0$ otherwise.

Case III. $\lambda = |S| \geqslant \kappa$ has cofinality $\aleph_0$, $\lambda = \aleph_{\alpha+\beta}$, where $\kappa = \aleph_\alpha$ and $0 < \beta < \omega_1$. Then $m(S, \kappa) = 0$.

Remark. The definition is interesting only if κ is regular $\kappa = \aleph_\beta$, $|S| < \aleph_{\beta + \omega_1}$.

Lemma 4.3.

(A) *If for some* A, $m(S(A),\kappa) < 0$, *then* S *has no transversal.*

(B) *If* F *is a transversal of* S, *and* $m(S^A_B,\kappa) = -\mu$ *then* $\{b \in B$: *for some* $C \in S^A_B$, $F(C) = b)\}$ *has cardinality* μ.

Proof. *Immediate.*

Theorem 4.4. *Suppose* $\kappa = \aleph_\alpha$ *is regular,* $|S| < \aleph_{\alpha+\omega_1}$. *Then* S *has a transversal iff* $m[S(A),\kappa] \geq 0$ *for every* A.

Proof. The only if part is 4.3 (A). So we now prove by induction on $|S|$ that

(*) if for every A, $m[S(A),\kappa] \geq 0$, then S has a transversal. Note that if $|S(A)| > |A|$, $S(A)$ has no transversal.

Case I. $|S| < \kappa$, there is nothing to prove, by definition.

Case II. $\lambda = |S|$ is regular.

We may assume that S is a family of subsets of λ and $|S(\beta)| < \lambda$ for $\beta < \lambda$. Let

$$C_1 = \{\alpha < \lambda\text{: there is } A \in S,\ A \subseteq \alpha,\ \sup A = \alpha\}$$

$$C_2 = \{\alpha < \lambda\text{: there is } A \subseteq \lambda,\ \alpha \subseteq A,\ |A| < \lambda \text{ such that } m(S^A_\alpha,\kappa) < 0\}\,.$$

If C_1 is stationary, put $A_\alpha = \alpha$; then $m[S(\lambda),\kappa] = -\lambda$, a contradiction.

If C_2 is stationary, define A_α by induction. If $A_\alpha = \alpha \in C_2$, $A_{\alpha+1}$ is the A mentioned in C_2, otherwise $A_{\alpha+1} = \beta$ where β is the first ordinal bigger than any $\gamma \in A_\alpha$ and than α. $A_0 = \phi$, $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$ for limit δ. This shows $m[S(\lambda),\kappa] = -\lambda$, a contradiction. Hence there is a closed unbounded $C \subseteq \lambda$ disjoint to $C_1 \cup C_2$. Let $C = \{\alpha(i)\colon i < \lambda\}$, $S_i = S^{\alpha(i+1)}_{\alpha(i)}$. As $C \cap C_1 = \phi$, $S = \bigcup_{i<\lambda} \{A\colon A - \alpha_i \in S_i\}$ and as $C \cap C_2 = \phi$ for every A, $m[S_i(A),\kappa] \geq 0$. So by the induction hypothesis each S_i has a transversal F_i, and F defined by $F(A) = F_i(A - \alpha_i)$ if $A \subseteq \alpha_{i+1}$, $A \nsubseteq \alpha_i$ is a transversal of S.

Case III. $|S| = \aleph_{\alpha+\delta}$, $\aleph_\alpha = \kappa$, cf $\delta = \omega$, $\delta < \omega_1$.

We may assume that S is a family of subsets of $\aleph_{\alpha+\delta}$, and let $\lambda_n < \aleph_{\alpha+\delta}$, $\sum_{n<\omega} \lambda_n = \aleph_{\alpha+\delta}$. It suffices to prove

(**) for any $A_0 \subseteq \aleph_{\alpha+\delta}$ with $|A_0| < \aleph_{\alpha+\delta}$, there is an A_1 with $A_0 \subseteq A_1 \subseteq \aleph_{\alpha+\delta}$, $|A_1| < \aleph_{\alpha+\delta}$ such that for any $A_2 \subseteq \aleph_{\alpha+\beta}$ we have $m[S^{A_2}_{A_1}, \kappa] \geqslant 0$.

Indeed, assuming this, define $A(n)$ $(n < \omega)$ recursively such that $|A(n)| < \aleph_{\alpha+\delta}$, $\lambda_n \subseteq A(n)$, and for every A, $m(S^A_{A(n)}, \kappa) \geqslant 0$. Then by the induction hypothesis $S[A(0)]$, $S[A(n+1), A(n)]$ have transversals, and combining them we get a transversal of S.

Suppose that A_0 contradicts (**), and let $\mu = |A_0| + \kappa$. Define A_α, $\alpha \leqslant \mu^+$ by recursion so that $|A_\alpha| \leqslant \mu$. A_0 is already defined, and $A_\delta = \bigcup_{i<\delta} A_i$ for limit δ. If A_α is defined, then, by the definition of A_0, there is an A with $|A| < \aleph_{\alpha+\delta}$, $m[S^A_{A_\alpha}, \kappa] < 0$, and, as is easily seen, with $A \supseteq A_\alpha$, and choose such A with minimal cardinality. If $|A| > \mu$, then we get by definition 4.2 that $m(S^A_{A_\alpha}, \kappa) = -|S^A_{A_\alpha}| \leqslant -|A| < -\mu$, and so, by Lemma 4.3 (B), $S(A)$ has no transversal, but $|S(A)| < \aleph_{\alpha+\delta}$, so by the induction hypothesis we get a contradiction. Thus $|A| \leqslant \mu$; let $A_{\alpha+1} = A$. Now clearly $m[S(A_{\alpha+1}, A_\alpha), \kappa) < 0$, hence $m[S(A_{\mu^+}), \kappa] = -\mu^+$, a contradiction. So (**) holds, completing the proof.

Corollary 4.5. *If* β *is limit,* $0 < \beta < \omega_1$, $\kappa < \aleph_{\alpha+\beta}$ *then* PT $(\aleph_{\alpha+\beta}, \kappa)$ *does not hold.*

A similar theorem is

Theorem 4.6. *If* λ *is a strong limit cardinal of cofinality* $\aleph_0$, $\kappa < \lambda$ *then* PT (λ, κ) *does not hold.*

Proof. We may assume that S is a family of subsets of λ of cardinality $< \kappa$ such that each $S' \subseteq S$, $|S'| < \lambda$ has a transversal. We must have $|S(A)| < \lambda$ for any $A \subseteq \lambda$, $|A| < \lambda$ (as λ is strong limit), hence $|S(A)| \leqslant |A|$. Similarly to the proof of 4.5, it suffices to prove

(***) for any $A \subseteq \lambda$ with $|A| < \lambda$ there is a $B, A \subseteq B \subseteq \lambda$, and a transversal F of $S(B)$ such that if C is the range of F then the family $S' = \{D - C\colon D \in S,\ D \notin S(B)\}$ satisfies: for every $S'' \subseteq S'$, $|S''| < \lambda$, S'' has a transversal.

Suppose $A \subseteq \lambda$, $|A| < \lambda$ is given. If taking $B = A$, there is a transversal F of $S(A)$ satisfying (***), then we are ready. Otherwise, for each transversal F of $S(A)$ there is a subfamily S_F of $S - S(A)$, such that $|S_F| < \lambda$, and $S_F^1 = \{C - \text{Range}\, F\colon C \in S_F\}$ has no transversal. Let $\lambda = \sum_{n<\omega} \lambda_n$, $\lambda_n < \lambda$, and $S^n = \bigcup\{S_F\colon F$ is a transversal of $S(A)$, $|S_F| \leqslant \lambda_n\}$. Clearly $|S^n| \leqslant 2^{|A|} + \lambda_n < \lambda$, so let F^n be a transversal of $S^n \cup S(A)$. Let B be the smallest set with $A \subseteq B$ such that we have $F^n(C) \in B \Rightarrow C \subseteq B$ for any $C \in S$ and $n < \omega$. Clearly B exists and $|B| \leqslant |A| + \kappa + \aleph_0 < \lambda$, hence $S(B)$ hence a transversal F_0, and let F_1 be its restriction to $S(A)$, and let $\lambda_n \geqslant |S_{F_1}|$. Now we shall show that $S_{F_1}^1$ has a transversal F, and so we get a contradiction: if $C \in S_{F_1}$ and $C \nsubseteq B$, then let $F(C - \text{Range}\, F_1) = F^n(C)$ (which $\notin B$) and if $C \in S_{F_1}$ and $C \subseteq B$, then let $F(C - \text{Range}\, F_1) = F_0(C)$ (note that $C \nsubseteq \text{Range}\, F_1$). It is easy to check that the F is a transversal, a contradiction. So (***) holds, and the proof is complete.

Open Problems.

(A) In 4.4, what can we say about singular cardinals λ of cofinality $> \omega$?

(B) In 4.6, can the strong limitness of λ be dropped?

(C) Can we generalize the definition of $m(S, \kappa)$ to all possible $|S|$ so that 4.4 holds?

We can deal with PB as with PT and prove, e.g.

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Theorem 4.7. *If* $\mathrm{cf}\,\lambda = \aleph_0 < \lambda$, λ *is strong limit or* $\lambda = \aleph_{\alpha+\beta}$, $0 < \beta < \omega_1$ *then* $\mathrm{PB}(\lambda, \mu)$ *does not hold for any* $\mu < \lambda$.

Added in proof. The answer to all questions is yes; the proofs will appear in [10]; there we defined $m(S, \kappa) = 0$ whenever $|S|$ was singular.

We noticed long ago that there was a trivial solution of problem B: in case $\lambda = |S|$ is singular, $m(S, \kappa) = -\lambda$ if no $S' \subseteq S$ with $|S - S'| < < \lambda$ has a transversal, and $m(S, \kappa) = 0$ otherwise.

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