PACIFIC JOURNAL OF MATHEMATICS
Vol. 121, No. 1, 1986

THE SOLUTION TO CRAWLEY'S PROBLEM

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In this supplement to our paper, ω-elongations and Crawley's problem, we show that if $(V = L)$ every Crawley group is a direct sum of cyclic groups.

A simple argument completes the work begun in [**MS**] and allows us to show if that if $(V = L)$ then every Crawley group (no matter what its cardinality) is a direct sum of cyclic groups. (See [**MS**] for definitions and conventions.) We first note that in Theorem 2.2 of [**MS**] we prove something stronger. Namely

THEOREM 1. *Assume $(V = L)$ and let G be a group of cardinality at most $\aleph_1$ such that $p^\omega G \simeq Z(p)$. A separable group A is a direct sum of cyclic groups iff for every ω-elongation H of $Z(p)$ by A there is a homomorphism f from H to G such that $f(p^\omega H) \neq 0$.*

LEMMA 2. *Suppose A is a separable group of length ω. There is an ω-elongation H of $Z(p)$ by A and a group G such that $p^\omega G \simeq Z(p)$, $|G| \leq 2^{\aleph_0}$, and there is a homomorphism f from H to G with $f(p^\omega H) \neq 0$.*

Proof. Choose $B \subseteq A$ a basic subgroup and write $B = B_0 \oplus B_1$ where B_0 is countable with elements of arbitrarily large order. Let A^* be the closure of B_1 (i.e. A^* is the maximal subgroup of A so that A^*/B_1 is divisible). The subgroup $A^* + B_0 = A^* \oplus B_0$. Choose H_0 an ω-elongation of $Z(p)$ by B_0. Let $H_1 = A^* \oplus H_0$. Finally choose H an ω-elongation of $Z(p)$ by A containing H_1. Since $H \supseteq A^*$, we can let $G = H/A^*$. Let t generate $p^\omega H$. We have the sequence $\langle t + A^* \rangle \rightarrowtail G \twoheadrightarrow A/A^*$. Since $t \notin A^*$, to complete the proof we only need to note that A/A^* is separable and $|A/A^*| \leq 2^{\aleph_0}$. Both of these claims are easy to establish by first choosing an independent set of generators of B_0 and then identifying A/A^* with a group of formal sums of multiples of these generators.

THEOREM 3. *Assume $(V = L)$. Every Crawley group is a direct sum of cyclic groups.*

Proof. Suppose A is a separable group which is not the direct sum of cyclic groups. By Lemma 2 we can choose an ω-elongation H of $Z(p)$ by A and a group G such that $p^\omega G \simeq Z(p)$; $|G| \le 2^{\aleph_0}(= \aleph_1)$; and there is a homomorphism $f\colon H \to G$ with $f(p^\omega H) \ne 0$. But by Theorem 2.2 there is an ω-elongation H' of $Z(p)$ by A such that there is no homomorphism $g\colon H' \to G$ with $g(p^\omega H') \ne 0$. Hence A is not a Crawley group.

References

[MS] A. Mekler, and S. Shelah, *ω-elongations and Crawley's problem*, Pacific J. Math., **121** (1986), 121–132.

Received June 19, 1985. Research by the first author was supported by NSERC Grant A8948. Research by the second author was supported by NSERC Grant A3040.

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