Diamonds, Uniformization
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DIAMONDS, UNIFORMIZATION

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Abstract. Assume G.C.H. We prove that for singular λ, $\square_\lambda$ implies the diamonds hold for many $S \subseteq \lambda^+$ (including $S \subseteq \{\delta : \delta \in \lambda^+, \mathrm{cf}\,\delta = \mathrm{cf}\,\lambda\}$.) We also have complementary consistency results.

§0. Introduction. By Gregory [Gr] and Shelah [Sh3], assuming G.C.H., $\diamondsuit^*_{\{\delta < \lambda^+ : \mathrm{cf}\,\delta \neq \mathrm{cf}\,\lambda\}}$ holds for any λ (but is meaningless for $\lambda = \aleph_0$). So $\diamondsuit_{\lambda^+}$ holds. On the other hand, Jensen had proved (before) the consistency of G.C.H. + SH (with ZFC); thus $\diamondsuit_{\aleph_1}$ may fail (see Devlin and Johnsbraten [DJ]); later the author proved that for λ regular $\diamondsuit_{\{\delta < \lambda^+ : \mathrm{cf}\,\delta = \lambda\}}$ may fail (see Steinhorn and King [SK].) Woodin proved that $\diamondsuit_\kappa$ may fail for the first inaccessible κ, but though κ is strong limit, G.C.H. does not hold below κ in his model. He started with a supercompact cardinal and used Radin forcing.

Assuming G.C.H., for simplicity our results are as follows:

1) For λ singular, if ZFC is consistent then it is consistent (with ZFC + G.C.H.) that $\diamondsuit_S (S \subseteq \lambda^+)$ fails for some stationary $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}\,\delta = \mathrm{cf}\,\lambda\}$. However S is nonlarge in some sense: $F(S) = \{\delta : S \cap \delta$ a stationary subset of $\delta\}$ is not stationary.

2) The "$F(S)$ is not stationary" in 1) is necessary. For if $\square_\lambda$ holds (and it holds if e.g. $0^\# \notin V$ or there is no inner model with a measurable cardinal) and G.C.H., $S \subseteq \lambda^+$, $F(S)$ stationary, then $\diamondsuit_S$ holds; moreover, for some stationary $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}\,\delta = \mathrm{cf}\,\lambda\}$, $F(S) = \varnothing$ but $\diamondsuit_S$ holds. So e.g. there is a λ^+-Souslin tree complete at levels of cofinality $\neq \mathrm{cf}\,\lambda$.

3) If κ is strongly inaccessible and $S \subseteq \kappa$ is such that for every closed unbounded subset C of κ, $C \cap S$ and $C - S$ contain closed subsets of arbitrary order-type $< \kappa$, then in some forcing extension V^P of V, no new sequences of ordinals of length $< \kappa$ are added, S preserves its property but $\diamondsuit_S$ fails.

4) In 1) and 3) really stronger results than failure of diamonds (i.e. uniformization properties) hold. Also we observe a bound on improving 3): if e.g. $0^\# \notin V$ then for every limit δ we can find a closed unbounded C_δ of δ, and $f_\delta : C_\delta \to \{0, 1\}$, such that for every closed unbounded $C \subseteq \kappa$ and $f : C \to \{0, 1\}$ for some δ, $C_\delta \subseteq C$, $f_\delta = f \restriction C$.

The proof of 1) and 3) follows that of [Sh2, §1]. Note that the proof of [Sh2, §1] is obsolete as we can get the theorem easily by proper forcing (see [Sh1, Chapter V]), but not so with generalizations.

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CONVENTIONS. Dealing with $(H(\lambda), \in)$ we assume it has a definable well-ordering $<^*$ (or we can expand it by one). We shall always take λ big enough, so that all the sets we consider belong to $H(\lambda)$.

§1. **(E, h)-completeness.**

1.1. CONVENTION. Here κ is a fixed regular cardinal. $\mathscr{S}_{<\kappa}(D) = \{B: B \subseteq D, |B| < \kappa\}$. E denotes a set of increasing continuous sequences of limit length from some $\mathscr{S}_{<\kappa}(D)$; it satisfies

(1) E is unbounded, i.e. $(\forall A \in \mathscr{S}_{<\kappa}(D))(\exists \bar{B})(\bar{B} \in E \wedge A \subseteq B_0)$;

(2) if $\langle B_i: i < \delta\rangle \in E$, $\langle B'_i: i < \delta\rangle$ is an increasing continuous sequence, $B_i \in \mathscr{S}_{<\kappa}(D)$ and $B_i \subseteq B'_{i+1} \subseteq B_{i+2}$, then $\langle B'_i: i < \delta\rangle \in E$;

(3) E is closed under initial segments, i.e. if $\bar{B} \in E$ and $\delta < l(\bar{B})$ is a limit ordinal, then $(\bar{B} \restriction \delta) \in E$, and under end-segments.

By (1) E determines D, so we write $D = \operatorname{Dom} E$; it is an ordinal $\alpha(E)$ if we do not say otherwise. We sometimes define E forgetting (2); then we mean the closure by this operation. If κ is not clear from the context we write $\kappa = \kappa(E)$. Let h denote a two-place function, $h(\mu, i)$, defined for $\mu < \kappa$ regular and $i < \mu$; also $\aleph_0 < h(\mu, 0)$, $h(\mu, i) \leq \kappa$ is increasing in i, and $\lambda_i < h(\mu, i)$ for $i < \delta$ implies $\sum_{i<\delta}\lambda_i < h(\mu, \delta)$. We omit h when $h(\mu, i) = \kappa$ for every μ and i. Let λ denote a large enough regular cardinal, and $\mathrm{SQS}(\lambda, E, h, \mu, \delta) = \mathrm{SQS}^{\mu}_{\delta}(\lambda, E, h)$ denote the set of sequences $\bar{B} = \langle B_i: i < \delta\rangle \in E, |B_i| < h(\mu, i)$. Let $\mathrm{SQM}(\lambda, E, h, \mu, \delta) = \mathrm{SQM}^{\mu}_{\delta}(\lambda, E, h)$ denote the set of sequences $\bar{N} = \langle N_i: i < \delta\rangle$, $N_i \prec (H(\lambda), \in)$, with $\langle N_i \cap \operatorname{Dom} E: i < \delta\rangle \in \mathrm{SQS}^{\mu}_{i}(\mu, E, h)$, $\langle N_i: i \leq j\rangle \in N_{j+1}$ and $\|N_i\| < h(\mu, i)$. We write μ instead of h when we use $h(\mu, i) = \mu$. We omit δ when $\delta = \mu$. In all that follows "λ large enough" can be replaced by "$\lambda \geq \lambda_0$" for some easily computable λ_0.

1.2. DEFINITION. (1) We call E *h-fat* if for every regular $\mu < \kappa$ and λ large enough, player I has no winning strategy in the following game:

For the αth move player I chooses $A_i \subseteq \operatorname{Dom} E$ with $|A_i| < h(\mu, 2i)$ and $\bigcup_{j<i} B_j \subseteq A_i$, and player II chooses $B_i \subseteq \operatorname{Dom} E$ with $|B_i| < h(\mu, 2i + 1)$ and $A_i \subseteq B_i$.

At the end of the game player II wins if $\langle \bigcup_{j<i} B_j: i < \mu\rangle \in E$.

(2) We call E *strongly fat* if it is h-fat with $h(\mu, i) = \mu + \aleph_1$.

1.3. DEFINITION. (1) We call a forcing notion P weakly (E, h)-*complete* if for every large enough λ, and every regular $\mu < \kappa$ and $\delta \leq \mu$, if $\bar{N} \in \mathrm{SQM}^{\mu}_{\delta}(\lambda, E, h)$, $P \in N_0$ and $\bar{p}$ is a generic sequence for $(\bar{N}, \bar{P})$ (see below), then $\{p_i: i < \delta\}$ has an upper bound in P.

(2) We say $\bar{p} = \langle p_i: i < \delta\rangle$ is a *generic sequence* for $(\langle N_i: i < \delta\rangle, P)$ if $P \in N_0$, $\bar{N} \in \mathrm{SQM}(\lambda, E)$, $\bar{p} \restriction i \in N_{i+1}$, and for every i, for every dense open subset $\mathscr{I} \in N_i$ of P for some n, $p_{i+n} \in \mathscr{I}$.

(3) We call P (E, h)-*complete* if it is weakly (E, h)-complete and forcing by P does not add new sequences of ordinals of length $< \kappa$.

REMARK. In 1.3(2) it may be more convenient to interchange the quantification on $\mathscr{I}$ and n. The only change this entails is in 1.5, where we have to assume that P does not add ω-sequences of ordinals.

1.4. REMARK. In 1.3(3) we can demand equivalently that no new sequences of ordinals of length $\mu, \mu < \kappa$ regular, are added.

1.5. LEMMA. *If E is strongly fat and P is weakly (E, h)-complete then P* is *(E, h)-complete.*

PROOF. We prove by induction on $\mu(\mu < \kappa, \mu$ regular) that if $p \in P$, and $\mathscr{I}_\beta(\beta < \mu)$ are dense open subsets of P, then there is $q, p \leq q \in P$, with $q \in \mathscr{I}_\beta$ for each $\beta < \mu$. This clearly suffices.

For $\mu = \aleph_0$, we can by Definition 1.3(1) find $N_n \prec \langle H(\lambda), \in \rangle$, N_n countable, p, $P \in N$, $\mathscr{I}_\beta \in N_0$ for $\beta < \mu$, and $\langle N_n \cap \alpha(E) : n < \omega \rangle \in E$. As N_n is countable there is a sequence $\langle p_n : n < \omega \rangle$, $p_0 = p$, with $p_n \leq p_{n+1}$, $p_n \in P \cap N_{n+1}$, and for every dense $\mathscr{I} \subseteq P$, if $\mathscr{I} \in \bigcup N_n$ then $p_n \in \mathscr{I}$ for some n. So $\langle p_n : n < \omega \rangle$ is a generic sequence for $\langle N_n : n < \omega \rangle$; hence it has an upper bound q in P, as required.

Suppose $\mu > \aleph_0$; then (choosing λ large enough) (by Definition 1.3) we can find $\bar{N} \in \mathrm{SQS}^\mu_\mu(\lambda, E, \mu)$. Remember $<^* \restriction P$ is a well-ordering of the members of P. Now we define p_i by induction on $i \leq \mu$, as follows:

1) $p_0 = p$ and $p_i \in N_{i+1}$;

2) p_i is the $<^*$-first member of P which is above p_j for $j < i$, and is in every open dense subset of P which belongs to $\bigcup_{j<i} N_j$.

Now why is p_i well defined? If i is the first failure, then $\langle p_j : j < i \rangle$ is still defined, and obviously belongs to N_{i+1} (as $\langle N_j : j \leq i \rangle \in N_{i+1}$, and $\langle p_i : j < i \rangle$ is easily defined from $\langle N_j : j \leq i \rangle$, P, p and $<^*$). If i is a limit, $\langle p_j : j < i \rangle$ is a generic sequence for $\langle N_j : j < i \rangle$; and as $\langle \mathrm{Dom}(E) \cap N_j : j < i \rangle \in E$, it has an upper bound, and the $<^*$-first such upper bound belong to N_{i+1}, and satisfies the requirements on p_i (note that it is automatically in every dense open set which belongs to $N_j, j < i$, as it is above p_{j+1}).

So we remain with the case when i is a successor and use the induction hypothesis on μ (and $\|N_i\| < \mu$).

1.6. LEMMA. (1) *If E is h-fat and P is (E, h)-complete, then E is still h-fat in V^P.*

(2) *If $\bar{N} \in \mathrm{SQM}^\mu_\delta(\lambda, E, h)$, $\bar{p}$ is a generic sequence for $\bar{N}$, $p_i \leq q \in P$ for every i, and forcing by P does not add sequences of ordinals of length $< \kappa$, then*

$$q \Vdash_P \text{“}\langle N_i[G] : i < \delta \rangle \in \mathrm{SQS}^\mu_\delta(\lambda, E, h)\text{”}.$$

PROOF. Left to the reader.

1.7. LEMMA. *Suppose $\bar{Q} = \langle P_i, \mathbf{Q}_i; i < \gamma \rangle$ is a $(<\kappa)$-support iteration, and each Q_i is (E, h)-complete, P_γ the limit. If E is h-fat (in V) then P_γ is (E, h)-complete and E is still h-fat in V^P.*

PROOF. The “weak (E, h)-completeness” is preserved trivially. So we need $\Vdash_P$ “$(\forall \alpha)[{}^{\kappa >}\alpha \subseteq V]$”. The proof is by induction on γ. For γ successor the proof is totally straightforward. For γ limit we first prove that, for every regular $\mu < \kappa$, every $p \in P_\gamma$, every $\gamma_i < \delta$ $(i < \mu)$, and every dense open subset $\mathscr{I}_i$ of P_{γ_i} (for $i < \mu$), there is a $q \in P_\gamma$ with $p \leq q$ and $q \restriction \gamma_i \in \mathscr{I}_i$ for $i < \mu$ [if $\mu < \mathrm{cf}\,\gamma$, then $\sup_{i<\delta} \gamma_i < \delta$, and we use the induction hypothesis; if $\mu \geq \mathrm{cf}\,\gamma$, without loss of generality we can take $\gamma = \mathrm{cf}\,\gamma$ and also $\mu = \mathrm{cf}\,\gamma$ (as $\bigcap_{\gamma_i = \beta} \mathscr{I}_i$ is dense in P_β) and use (E, h)-completeness for μ; for suitable $\bar{N}$, by induction on $i < \mu$ we define $\langle q^i_j : j < i \rangle \in P_i \cap N_{i+1}$, increasing in i, belonging to every dense subset of P_{i-1} which belongs to N_i], and then prove the clause about “not adding sequences of length $< \mu$” (Definition 1.3(3)) using (E, h)-completeness for μ.

1.8. DEFINITION. For an iteration $\langle P_i, \mathbf{Q}_i : i < \gamma \rangle$ with $(<\kappa)$-support, assuming for notational simplicity that each Q_i is ordered by inclusion, we make the following definitions:

(1) $\mathrm{Tr}(\gamma) = \{\mathscr{T}:\mathscr{T} = (T, <, f), (T, <)$ a well-founded tree, closed under limits, $f: T \to \gamma, f(\mathrm{rt}_T) = 0$ for the root rt_T, and f is increasing and continuous$\}$.

(2) Let $t \in \mathscr{T}$ mean $t \in T$, and for $t \in \mathscr{T}$ let $\mathrm{lev}(t)$ be its level (i.e. the order-type of $\{s: s < t\}$) and $t \upharpoonright \alpha$ the unique $s \le t$ of level α (for $\alpha \le \mathrm{lev}(t)$). We call the tree *leveled* if $f(t)$ depends on the level of t only. If confusion may arise, we write $<^{\mathscr{T}}$ and $f^{\mathscr{T}}$.

(3) $F\mathrm{Tr}(\bar{Q}) = \{\langle p_t: t \in \mathscr{T}\rangle: \mathscr{T} \in \mathrm{Tr}(\gamma)$, and $p_{t\upharpoonright\alpha} = p_t \upharpoonright f(t \upharpoonright \alpha)$; p_t is a function with domain a subset of $f(t)$ of power $<\kappa$, $p_t(i)$ a P_i-name$\}$.

(4) $P'_i = \{p: p$ a function with domain a subset of i of power $<\kappa$, $p(j)$ a P_j-name$\}$. For $j \notin \mathrm{Dom}\, p$ let $p(j) = \varnothing$. For $p, q \in P'_i$, we write $p \le q$ if $q \upharpoonright j \Vdash_{P_j}$ "$p(j) \subseteq q(j)$" for every $j < i$.

(5) $F\mathrm{Tr}_0(\bar{Q}) = \{\langle p_t: t \in \mathscr{T}: \mathscr{T} \in \mathrm{Tr}(\gamma), \langle p_t: t \in \mathscr{T}\rangle \in F\mathrm{Tr}(\bar{Q})$ and $\Vdash_{P_i}$ "$p_t(i) \in \mathbf{Q}_i$" for every $t \in T$ and $i \in \mathrm{Dom}\, p_t\}$.

(6) $F\mathrm{Tr}_1(\bar{Q}) = \{\langle p_\eta: \eta \in \mathscr{T}\rangle \in F\mathrm{Tr}(\bar{Q})$: for every nonmaximal $t \in \mathscr{T}$, and $q \in P_{f(t)}$ if $p_t \le q$ (though maybe $p_t \notin P_{f(t)}$), then for some immediate successor s of t (in $\mathscr{T}$), and $r \in P_{f(s)}$, we have $p_s \le r$ and $q \le r\}$.

1.9. LEMMA. *Suppose Q is as in* 1.7, $\langle p_\eta: \eta \in \mathscr{T}\rangle \in F\mathrm{Tr}_1(\bar{Q})$, *$\mathscr{T}$ has $<\kappa$ levels, and each Q_i is (E, h)-complete. Then, for some maximal $t \in \mathscr{T}$ and $q \in P_\gamma$, $p_\eta \le q$.*

PROOF. Like the proof in [Sh2, 1.7].

1.10. LEMMA. *Suppose P_γ and $\bar{Q}$ are as in* 1.7, $\gamma = l(\bar{Q})$, $\mathscr{T} \in \mathrm{Tr}(\gamma)$, $f(t) = \gamma$ *for every maximal $t \in \mathscr{T}$, and $|\mathscr{T}| \le \mu$, $|\mathscr{T}| < h(\mu, i)$ for some $i < \mu < \kappa, \mu$ regular. If $\langle p_t: t \in \mathscr{T}\rangle \in F\mathrm{Tr}_0(\bar{Q})$, and $\mathscr{I}$ is a dense subset of P_γ, then there is $\langle q_t: t \in \mathscr{T}\rangle \in F\mathrm{Tr}_0(\bar{Q})$ such that $p_t \le q_t$ (for $t \in \mathscr{T}$) and $q_t \in \mathscr{I}$ for t maximal in $\mathscr{T}$.*

PROOF. Again as in the proof of [Sh2, 1.7] (and 1.7 of the present paper).

An inconvenient aspect of Definition 1.3 is that we are interested in sequences of submodels of $H(\lambda)$, whereas E is usually a sequence of sets of ordinals.

1.11. CLAIM. *Suppose E^0* and *E^1 are given, and for some one-to-one function g from $D^0 = \mathrm{Dom}\, E^0$ onto $D^1 = \mathrm{Dom}\, E^1$,*

$$E^0 = \{\langle A_i: i < \delta\rangle: \langle g(A_i): i < \delta\rangle \in E^1\}$$

(*in such case we say that E^0* and *E^1 are isomorphic*). *Then*

a) *E^0 is h-fat iff E^1 is h-fat, and*

b) *any forcing notion P is weakly (E^0, h)-complete iff it is weakly (E^1, h)-complete.*

PROOF. Trivial.

§2. (E, H)-completeness.

2.1. NOTATION. E is as in §1.1, H is a function with domain E, and $H(\langle B_i: i < \delta\rangle) = \langle \alpha_i: i < \delta\rangle$ (usually $\alpha_i \in B_{i+1}$). We let $H(\bar{N}) = H(\langle N_i \cap \alpha(E): i < l(\bar{N})\rangle)$.

2.2. DEFINITION. (1) We call (E, H) *h-fat* if for every regular $\mu < \kappa$, player I has no winning strategy in the following game:

For the ith move, player I chooses $A_i \in S_{<\kappa}(\alpha(E))$ with $|A_i| < h(\mu 2i)$ and $\bigcup_{j<i} B_j \subseteq A_i$, and player II chooses α_i and $B_i \in S_{<\kappa}(\alpha(E))$ with $|B_i| < h(\mu, 2i + 1)$ and $A_i \subseteq B_i$.

At the end of the game, player II wins if $\langle B_j: j \le \mu\rangle \in E$ and $\langle \alpha_i: i < \delta\rangle = H(\langle B_j: j \le \mu\rangle)$.

(2) We call (E, H) *strongly fat* if it is h-fat for $h(\mu, i) = \mu + \aleph_1$.

2.3. DEFINITION. We say that P is (E, H, h)-*complete* if for every regular $\mu < \kappa$ there is a function F_μ such that if $\bar{N} = \langle N_i : i < \mu \rangle \in \mathrm{SQM}(\lambda, E, h, \mu, \mu)$, $p \in N_0 \cap P$ and $\bar{\alpha} = \langle \alpha_i : i < \mu \rangle = H(\bar{N})$, then the following conditions hold:

(A) If $\bar{p} = \langle p_j : j < i \rangle$ is generic for $\bar{N} \restriction i = \langle N_j : j < i \rangle$ then $F_\mu(\bar{p} \restriction i, \bar{N} \restriction i, \bar{\alpha} \restriction (i+1))$ is a sequence of length $< h(\mu, i)$ of bounds of $\bar{p}$.

(B) There is a sequence $\bar{\gamma} = \langle \gamma_i : i < \mu \rangle$, $\gamma_i \in N_{i+1}$, $\bar{\gamma} \restriction i \in N_{i+1}$, such that any sequence $\bar{p} = \langle p_j : j < \delta \rangle (\delta \leq \mu$ limit) satisfying the following has an upper bound:

(α) $\langle p_j : j < \delta \rangle$ is generic for $\bar{N} \restriction \delta$, and

(β) p_i appears in $F_\mu(\bar{p} \restriction i, \bar{N} \restriction i, \bar{\alpha} \restriction (i+1))$; in fact its place is

$$F_\mu(\bar{p} \restriction i, \bar{N} \restriction i, \bar{\alpha} \restriction (i+1), \gamma \restriction (i+1)).$$

REMARKS. (1) The requirement $\bar{\gamma} \restriction i \in N_{i+1}$ will be omitted if

$$(\forall \chi < h(\mu, i))(\chi^{|i|} < h(\mu, i)).$$

(2) We omit h in Definition 2.3 when $h(\mu, i) = \mu$.

2.4. LEMMA. *If (E, H) is h-fat, P is (E, H, h)-complete, and $h(\mu) \leq \kappa\,(h(\mu) \leq \mu)$, then (E, H) is still h-fat in V^P.*

PROOF. Easy.

2.5. THEOREM. *Suppose*

(a) *κ is strongly inaccessible,*

(b) *E_0 is fat, i.e. h_0-fat where $h_0(\mu, i) = \mu + \aleph_1$,*

(c) *(E_1, H) is fat,*

(d) *$\bar{Q} = \langle P_i, \mathbf{Q}_i : i < \gamma \rangle$ is a $(<\kappa)$-support iteration with limit P_γ, and*

(e) *each Q_i is E_0-complete and (E_1, H)-complete.*

Then P_γ is E_0-complete (and so does not add new sequences of ordinals of lengths $< \kappa$) and (E_1, H) is still fat in V^{P_γ}.

PROOF. The E_0-completeness follows by 1.7. Now (E_1, H) is still fat by 1.9 and 1.10, imitating [Sh2, §1].

2.6. DEFINITION. Let h^* be a function from ordinals to ordinals [or from sequences of ordinals to ordinals] and $\eta_\delta\,(\delta \in S)$ a sequence of ordinals. We say that $\langle \eta_\delta : \delta \in S \rangle$ has the *h^*-uniformization property* if *for every* $\langle g_\delta : \delta \in S \rangle$, g_δ a function with domain $\mathrm{Rang}(\eta_\delta)$, $g_\delta(\alpha) < h^*(\alpha)$ [or $g_\delta(\alpha) < h^*(\eta_\delta \restriction (\alpha+1))$], *there is* a function g with domain $\bigcup_{\delta \in S} \mathrm{Rang}(\eta_\delta)$, such that for every $\delta \in S$,

$$(\exists i < l(\eta_\delta))(\forall j)[i < j < l(\eta_\delta) \Rightarrow g(\eta_\delta(j)) = g_\delta(\eta_\delta(j))].$$

REMARK. On this property see [DS], [Sh1], [Sh2], [Sh4] and [SK].

2.7. DEFINITION. We say $\langle \eta_\delta : \delta \in S \rangle$ is *free* if there is a function f, $\mathrm{Dom}\, f = S$, $f(\delta) < l(\eta_\delta)$, such that the sets $\{\eta_\delta(\alpha) : f(\delta) < \alpha < l(\eta_\delta)\}$ are pairwise disjoint (for $\delta \in S$) (clearly, free implies the h^*-uniformization property).

2.8. CONCLUSION. *Suppose κ is strongly inaccessible, $h^* : \kappa \to \kappa$, $S \subseteq \kappa$, and for every closed unbounded $C \subseteq \kappa$ there are, in $S \cap C$ and in $C - S$, closed subsets of any order-type $< \kappa$.*

For some forcing notion P:

(a) *V^P and V have the same sequences of ordinals of length $< \kappa$.*

(b) *P satisfies the κ^+-chain condition, and e.g. $|P| = 2^\kappa$.*

(c) *S satisfies in V^P the assumption we have on it (in V).*

(d) *There is* $\langle \eta_\delta : \delta \in S \rangle, \eta_\delta$ *an increasing sequence converging to* δ, *which has the* h^*-*uniformization property.*

(e) P *is* E_0-*complete, where* $E_0 = \{\langle B_i : i < \delta \rangle : B_i$ *and* $\bigcup_{i<\delta} B_i$ *are ordinals in* $\kappa - S$, B_i *increasing continuous*$\}$.

PROOF. For given $\langle \eta_\delta : \delta \in S)$ let

$$E_1 = \{\langle B_i : i < \delta \rangle : B_i \text{ is an ordinal in } S, B_i \text{ increasing continuous}\}$$

(or replace S by κ), and put $H(\langle B_i : i < \delta \rangle) = \langle \alpha_i : i < \delta \rangle$ if the α_i "code" the set $(\bigcup_{i<\delta} \mathrm{Rang}(\eta_{B_i}) \cap B_{i+1})$.

Can we define $\langle \eta_\delta : \delta \in S \rangle$ so that (E_1, H) is h_1-fat and $\{\eta_\delta : \delta \in S, \delta < \alpha\}$ is free for every $\alpha < \kappa$? The easiest way to do it is by forcing such $\langle \eta_\delta : \delta \in S \rangle$, a condition being an initial segment (alternatively use squares). Now we can define a $(<\kappa)$-support iteration $\bar{Q} = \langle P_i, \mathbf{Q}_i : i < 2^k \rangle$ such that

(A) each $\mathbf{Q}_i$ has the form $Q\langle g^i_\delta : \delta \in S \rangle$, where g^i_δ is a function with domain $\mathrm{Rang}(\eta_\delta)$, $g^i_\delta(i) < h^*(i)$ ($\langle g^i_\delta : \delta \in S \rangle \in V^{P_i}$ of course), and $Q\langle g^i_\delta : \delta \in S \rangle = \{g : g$ a function with domain $j < \kappa$ and for every $i \in S \cap (j+1)$, for some $i^* < i$, $(\forall \xi)\,[\xi \in \mathrm{Rang}(\eta_\delta) \wedge i^* < \xi < i \rightarrow g(\xi) = g^i_\delta(\xi)]\}$; and

(B) if $\langle g_\delta : \delta \in S \rangle \in V^P$, $\delta < 2^\kappa$, then for some i,

$$\langle g^i_\delta : \delta \in S \rangle = \langle g_\delta : \delta \in S \rangle.$$

This is not hard to do. Easily each Q_i is E_0-complete and (E_1, H)-complete; hence by 2.5 P_{2^κ} is. Now P_{2^κ} satisfies the κ^+-chain condition (see [Sh1, Chapter VIII, §2]).

2.9. THEOREM. *Suppose*

(a) $\kappa = \chi^+$, *where* χ *is a singular strong limit,*

(b) E_0 *is fat,*

(c) (E_1, H) *is* χ-*fat* (*i.e.* h_1-*fat*, $h_1(\mu, i) = \chi$), $\mathrm{Dom}\, E_1 = \mathrm{Dom}\, E_0$, *and* $(\exists \bar{B} \in E_1)$ $[l(\bar{B}_1) \leq \mathrm{cf}\, \kappa]$, *and* $\bar{B} \in E_1$, $l(\bar{B}) < \mathrm{cf}\, \kappa$ *implies* $\bar{B} \in E_0$,

(d) $Q = \langle P_i, \mathbf{Q}_i : i < \gamma \rangle$ *is a* $(<\kappa)$-*support iteration with limit* P_γ, *and*

(e) *each* Q_i *is* E_0-*complete and* (E_1, H, h_1)-*complete.*

Then P_γ *is* E_0-*complete and, in* V^{P_γ}, (E_1, H_1) *is still* h_1-*fat.*

PROOF. As in 2.5, only simpler: we use trees of power $<\chi$ to get an inverse limit of power $\chi^{\mathrm{cf}\,\chi}$, and then use 1.9.

2.10. CONCLUSION. *Suppose* $\kappa = \chi^+ = 2^\chi$, χ *a singular strong limit, and* $S \subseteq \{\delta < \kappa : \mathrm{cf}\,\delta = \mathrm{cf}\,\chi\}$ *is stationary, but no initial segment of it is stationary. Then for some forcing motion* P:

(a) V^P *and* V *have the same sequences of ordinals of length* $<\kappa$,

(b) P *satisfies the* κ^+-*chain condition,*

(c) S *is stationary in* V^P, *and*

(d) *there is* $\langle \eta_\delta : \delta \in S \rangle$, η_δ *an increasing sequence converging to* δ *of order-type* $\mathrm{cf}\,\chi$ *and* $h^* : {}^{\mathrm{cf}\,\chi >}\kappa \rightarrow \kappa$ *such that* $\langle \eta_\delta : \delta \in S \rangle$ *has the* h^*-*uniformization property.*

PROOF. Like 2.8, using 2.9 instead 2.5.

2.11. THEOREM. *Suppose*

(a) $\kappa_1 = \kappa_0^+$, κ_0 *strongly inaccessible,*

(b) E_0 *is fat,* $\alpha(E_0) = \kappa_0$,

(c) $\kappa(E_1) = \kappa_1$ *and* (E_1, H) *is* κ-*complete,* (*i.e.* h_1-*complete* $h_1(\mu, i) = \kappa_0$ *for*

$i < \mu < \kappa_1$, and

$$(\forall \bar{B} \in E_1)(l(\bar{B}) \leq \kappa_0), \qquad (\forall \bar{B} \in E_1)(l(\bar{B}) < \kappa_0 \Rightarrow \bar{B} \in E_0),$$

(d) *$\bar{a} = \langle P_i, Q_i : i < \gamma \rangle$ is a $(<\kappa)$-support iteration with limit P_γ, and*

(e) *each Q_i is E_0-complete and (E_1, H, h_1)-complete.*

Then P_γ is E_0-complete and, in V^{P_γ}, (E_1, H) is still h_1-fat.

REMARK. We can let E_0 be essentially the set of all sequences of the right power and length.

PROOF. As in [Sh1, §1].

2.12. THEOREM. *Suppose*

(a) *$\kappa_1 = \kappa_0^+$, $2^{\kappa_0} = \kappa_1$, and $\diamondsuit_{\kappa_0}$ holds.*

(b) *E_0 is fat, with $\alpha(E_0) = \kappa_1$.*

(c) *$\kappa(E_1) = \kappa_1$, (E_1, H) is κ_0-complete and*

$$(\forall \bar{B} \in E_1)(l(\bar{B}) \leq \kappa_0), \qquad (\forall \bar{B} \in E_1)(l(\bar{B}) < \kappa_0 \Rightarrow \bar{B} \in E_0).$$

(d) *We make a change in Definition* 2.3(b) *for $\mu = \kappa_0$: there is a stationary subset $S = F_\mu(\langle N_i \cap \mathrm{Dom}\, E_1 : i < l(\bar{N}) \rangle)$ of κ_0, satisfying $\diamondsuit_S$, and we restrict (β) to $i \notin S$ (or to $i \notin S \cap C$, C a closed unbounded subset of κ_0; the truth value of $\alpha \in C$ depends on $\beta \restriction \alpha$* and *$N$).*

(e) *$\bar{Q} = \langle P_i, Q_i : i < \gamma \rangle$ is a $(<\kappa)$-support iteration with limit P_γ.*

(f) *Each Q_i is E_0-complete and (E_1, H, κ_0)-complete.*

Then P_γ is E_0-complete and in $V^{P_\gamma}(E_1, H)$ is still h_1-fat (so $(\kappa_1 > \alpha)^{V^P} = (\kappa_1 > \alpha)^V$).

PROOF. As in [SK] (we use the diamond to compensate for 1.10 which is not applicable).

§3. Diamonds and Souslin trees on successors of singular λ.

3.1. THEOREM. *Suppose λ is singular, $\chi \leq \lambda$, $\lambda^+ = 2^\lambda$, $(\forall \kappa < \chi)(\forall \mu < \lambda)\mu^\kappa < \lambda$ and $\square_\lambda$ holds. Then we can define for every $\alpha < \lambda^+$ a family $\mathscr{P}_\alpha$ of $\leq \lambda$ subsets of α, such that for every $A \subseteq \lambda^+$, for some closed unbounded $C \subseteq \lambda^+$, for no $\delta \in C$ do we have that $\aleph_0 < \mathrm{cf}(\delta) < \chi$ and $Gu(A) \cap \delta$ is a stationary subset of δ, where $Gu(A) = \{\alpha : A \cap \alpha \notin \mathscr{P}_\alpha\}$.*

REMARK. If λ is a strong limit (which is the important case), then $\chi = \lambda$ is okay.

PROOF. We imitate part of the proof of the strong covering lemma [SH1, XIII, 2.3].

We have assumed $\square_\lambda$, so there is $\langle C_\delta : \lambda < \delta < \lambda^+, \delta \text{ limit} \rangle$ such that C_δ is a closed unbounded subset of λ, $|C_\delta| < \lambda$ and if $\gamma \in C'_\delta$ (the set of limit points of C_δ) then $C_\gamma = C_\delta \cap \gamma$.

Let $\kappa = \mathrm{cf}\, \lambda$, $R = \{\theta : \theta$ a regular cardinal, $\kappa < \theta < \lambda\}$. As $2^\lambda = \lambda^+$ we can find $f_i^* \, (i < \lambda^+)$ such that

1) $\mathrm{Dom}\, f_i^* = R, f_i^*(\theta) < \theta$,

2) $f_i^* <^* f_j^*$ for $i < j$ (which means that, for every large enough $\theta \in R$, $f_i^*(\theta) < f_j^*(\theta)$),

3) if $i \in C_j, \theta \in R$ and $\theta > |C_j|$, then $f_i^*(\theta) < f_j^*(\theta)$,

4) if $\mathrm{Dom}\, f = R$ and $(\forall \theta)[f(\theta) < \theta]$, then $f <^* f_i^*$ for some i, and

5) if the length of C_j is divisible by ω^2 and $\theta > |C_j|$, then $f_j^*(\theta) = \sup_{i \in C_j} f_i^*(\theta)$.

Also, as $2^\lambda = \lambda^+$ there is a list $\{A_\alpha : \alpha < \lambda^+\}$ of all bounded subsets of λ^+.

Now let the model $M^2 = M^2_{\lambda^+}$ be defined as follows: its universe is λ^+, and it has the following functions: $F^0(\beta, -)$ is a one-to-one mapping from β onto $|\beta|$; G^0 is essentially an inverse of F^0, i.e. $G^0(\beta, F^0(\beta,\gamma)) = \gamma$ for $\gamma < \beta$;

S: the successor function, $S(\alpha) = \alpha + 1$; $\mathrm{CF}(\alpha)$ is $\mathrm{cf}(\delta)$ if δ is limit, and $\alpha - 1$ if α is a successor ordinal;

H^0: for β limit, $\langle H^0(\beta,i): i < \mathrm{CF}(\beta)\rangle$ is an increasing continuous sequence converging to β, while for β successor $H^0(\beta,0) = |\beta|, H^0(\beta,1) = |\beta|^+$ ($\mathrm{cf}\,\beta < \lambda$);

0 and λ are individual constants;

$<$ is the order relation;

$$F^1(i,\theta) = f_i^*(\theta) \quad \text{for } \theta \in R \text{ and } i < \lambda^+;$$

G^2: for limit δ, $\lambda < \delta < \lambda^+$, $\langle G^2(\delta,i): i < G^2(\delta,\delta)\rangle$ is an increasing continuous sequence, whose set of elements is C_δ.

Now we can define the $\mathscr{P}_\alpha$'s. So for every limit δ and $\mu < \lambda$ we define a model $M_{\delta,\mu}$: it is the closure of $\{i: i < \mu\} \cup C_\delta$ under the functions of M^2 (we do not strictly distinguish between a submodel and its set of elements). When $\mathrm{cf}\,\delta \geq \chi$ let $\mathscr{P}_\delta = \varnothing$; otherwise let

$$\mathscr{P}_\delta = \{\textstyle\bigcup_{\alpha \in I} A_\alpha: \text{for some } \mu < \lambda, I \text{ is a subset of } M_{\delta,\mu} \text{ of power } \mathrm{cf}\,\delta\}.$$

So we have to prove only that $\langle \mathscr{P}_\delta: \delta < \lambda^+\rangle$ is as required. So let $A \subseteq \lambda^+$ and $h: \lambda^+ \to \lambda^+$ be such that $A \cap \alpha = A_{h(\alpha)}$. Now we define, by induction on δ, $\lambda < \delta < \lambda^+$, an elementary submodel N_δ of M^2 such that:

a) $\delta \in N_\delta$, $C_\delta \subseteq N_\delta$, N_δ is closed under h, and $\|N_\delta\| \leq |C_\delta|$;

b) the closure (in the order topology) of $\bigcup\{N_i: i \in C'_\delta\}$ is contained in N_δ; and

c) there is $i = i_\delta \in N_\delta$ such that, for every large enough $\theta \in R$,

$$\sup(\textstyle\bigcup\{N_i: i \in C_\delta\} \cap \theta) < f_i^*(\theta).$$

If $\delta = \sup C'_\delta$, let $N_\delta^* = \bigcup\{N_\alpha: \alpha \in C'_\delta\}$. There is no problem in doing so (for (c) use (4) in the conditions on the f_i^*). Let

$$C^* = \{\alpha < \lambda^+: \alpha \text{ is limit}, \alpha > \lambda, \text{ and for every } \delta < \alpha, \sup(N_\delta) < \alpha\}.$$

Clearly C^* is a closed unbounded subset of λ^+. We shall prove:

FACT A. *If $\delta \in C^*$ and $\mathrm{cf}\,\lambda < \mathrm{cf}\,\delta \leq \chi$, then for a closed unbounded set of $\gamma < \delta$, $(\exists\mu)[N_\gamma^* \subseteq M_{\gamma,\mu}]$.*

This is enough, because the case $\mathrm{cf}\,\delta \leq \mathrm{cf}\,\lambda$ holds by [Sh3], and then we can find an unbounded subset D of $\delta \cap N_\delta^*$ of power $\mathrm{cf}\,\delta$; hence $\{h(\alpha): \alpha \in D\} \subseteq N_\delta^* \subseteq M_{\delta,\mu}$, wherefore $\bigcup_{\alpha \in D} A_{h(\alpha)} \in \mathscr{P}_\delta$, and as $A_{h(\alpha)} = A \cap \alpha$ for $\alpha \in D$ clearly $A \cap \delta = \bigcup_{\alpha \in D} A_{h(\alpha)} \in \mathscr{P}_\delta$.

PROOF OF FACT A. Let $(C_\delta)' = \{\beta(\zeta): \zeta < \zeta_0\}, \beta_\zeta = \beta(\zeta)$ increasing continuous, so $C_\delta \cap \beta(\zeta)$ has order-type divisible by ω^2. Let Ch_ζ be the function with domain R, $\mathrm{Ch}_\zeta(\theta) = \mathrm{Sup}(\theta \cap \bigcup\{N_\beta: \beta \in C'_{\beta(\zeta)}\})$.

By the choice of $i_{\beta(\zeta)}$, $\mathrm{Ch}_\zeta <^* f^*_{i_{\beta(\zeta)}}$. On the other hand, as $i_{\beta(\zeta)} \in N_{\beta(\zeta)}$, for every $\theta \in \mathrm{Dom}(\mathrm{Ch}_\zeta)$, $\theta > |C_\delta|$, we have $f^*_{i_{\beta(\zeta)}}(\theta) < \mathrm{Ch}_{\zeta+1}(\theta)$ for every $\xi, \zeta < \xi < \zeta_0$. So for some $\mu_\zeta < \lambda$:

$$(\alpha) \quad (\forall\theta \in R)[\theta \geq \mu_\zeta \wedge \theta \in \mathrm{Dom}(\mathrm{Ch}_\zeta) \Rightarrow \mathrm{Ch}_\zeta(\theta) < f^*_{i_{\beta(\zeta)}}(\theta) < \mathrm{Ch}_{\zeta+1}(\theta)],$$

$$(\alpha_1) \qquad \beta(\xi) \leq i_{\beta(\xi)} < \beta(\xi + 1).$$

As $\operatorname{cf}\zeta_0 = \operatorname{cf}\delta > \operatorname{cf}\lambda$, there is μ^* such that $\mu^* > |C_\delta|$ and $\{\zeta < \zeta_0 : \mu_\zeta < \mu^* < \lambda\}$ is an unbounded subset of ζ_0 and by their definition (see (3) and (α_1)):

(β) $(\forall \zeta < \zeta < \zeta_0)(\forall \theta \in R)[\theta \geq \mu^* \wedge \beta(\xi) \in C^* \rightarrow f^*_{\beta(\zeta)}(\theta) < f^*_{i_{\beta(\zeta)}}(\theta) < f^*_{\beta(\xi)}(\theta)]$

and, even more trivially,

(γ) $(\forall \zeta < \xi < \zeta_0)(\forall \theta \in R)[\theta \geq \mu^* \wedge \theta \in \operatorname{Dom}\operatorname{Ch}_\zeta \Rightarrow \operatorname{Ch}_\zeta(\theta) < \operatorname{Ch}_\xi(\theta)]$.

Also, by (5),

(δ) For every limit $\zeta < \zeta_0$

$$f^*_{\beta(\zeta)}(\theta) = \sup_{\xi < \zeta} f^*_{\beta(\xi)}(\theta) \quad \text{for } \theta \geq \mu^*.$$

Note also

(ε) for every limit $\zeta < \zeta_0$ and $\theta \in \operatorname{Dom}\operatorname{Ch}_\zeta$,

$$\operatorname{Ch}_\zeta(\theta) = \sup_{\xi < \zeta} \operatorname{Ch}_\xi(\theta).$$

Now choose a closed unbounded $E \subseteq \zeta_0$ such that $(\forall \zeta \in E)(\beta(\xi) \in C^*)$ and for every $\zeta_1 < \zeta_2$ in E for some ζ, $\zeta_1 < \zeta < \zeta_2 \wedge \mu_\zeta < \mu^*$. By (α)–(ε) it is easy to see that

$(*)$ for every $\zeta \in E$ and $\theta \geq \mu^* \wedge \theta \in \operatorname{Dom}\operatorname{Ch}_\zeta$,

$$\operatorname{Ch}_\zeta(\theta) = f^*_{\beta(\zeta)}(\theta).$$

As $\{\beta(\zeta) : \zeta \in E\}$ is a closed unbounded subset of δ, for proving Fact A (and thus the theorem), it suffices to prove:

$(**)$ for $\zeta \in E'$, $N^*_{\beta(\zeta)}$ is the closure of $(N^*_{\beta(\zeta)} \cap \mu^*) \cup C_{\beta(\zeta)}$ (hence is included in $M_{\beta(\zeta),\mu^*}$).

To prove $(**)$ let B be the closure of $(|N^*_{\beta(\zeta)}| \cap \mu^*) \cup C_{\beta(\zeta)}$ (closure in M^2). So clearly $B \subseteq N^*_{\beta(\zeta)}$ (it is easy to check that $C_{\beta(\zeta)} \subseteq N_{\beta(\zeta)}$). Suppose $B \neq N^*_{\beta(\zeta)}$; then there is a minimal ordinal i in $N^*_{\beta(\zeta)} - B$. As $C_{\beta(\zeta)}$ is unbounded in $\beta(\zeta)$ and $\operatorname{Sup} N^*_{\beta(\zeta)} = \beta(\zeta)$ (as $\beta(\xi) \in C^*$), clearly B has a member $> i$. Let j be the first ordinal in $B - i$. So B is necessarily disjoint to $[i, j)$, and $j > i$.

Case A. j is a successor ordinal: then $\operatorname{CF}(j) = j - 1$, so $j \in B \Rightarrow j - 1 \in B$; but $(j-1) \in [i, j)$, contradiction.

Case B. j is a limit ordinal but not a regular cardinal. Then $\operatorname{CF}(j) \in B$, and $\operatorname{CF}(j) = \operatorname{cf}(j) < j$. Hence $\operatorname{CF}(j) < i$ and there is $\varepsilon < \operatorname{CF}(j)$ such that $i \leq F(j, \varepsilon) < j$ (as $\langle \operatorname{CF}(j, \varepsilon) : \varepsilon < \operatorname{CF}(j) \rangle$ converge to j); but $j, \varepsilon \in B \Rightarrow \operatorname{CF}(j, \varepsilon) \in B$, contradiction.

Case C. j is a regular cardinal. Necessarily $j < \lambda$, and as $j > i$, $j \geq \mu^*$ so by $(*)$

$$\begin{aligned} i \leq \operatorname{Sup}(N^*_{\beta(\zeta)} \cap j) \leq f^*_{\beta(\zeta)}(j) &= \operatorname{Sup}\{f^*_\varepsilon(j) : \varepsilon \in C_{\beta(\zeta)}\} \\ &= \operatorname{Sup}\{F^1(\varepsilon, j) : \varepsilon \in C_{\beta(\zeta)}\} \leq \operatorname{Sup}(B \cap j) < i, \end{aligned}$$

contradiction.

3.2. Conclusion. *Suppose* $\square_\lambda$, $2^\lambda = \lambda^+$, *and* $(\forall \mu < \lambda)[\mu^{\operatorname{cf}\lambda} < \lambda]$.

1) *If* $S \subseteq S^* = \{\delta < \lambda^+ : \operatorname{cf}\delta = \operatorname{cf}\lambda\}$, *and* $F(S) = \{\delta < \lambda^+ : \delta \cap S$ *is a stationary subset of* $\delta\}$ *is stationary, then* $\diamondsuit_S$ *holds.*

2) *There are a stationary* $S \subseteq S^*$, $F(S) = \varnothing$, $\diamondsuit_S$, *and a square sequence* $\langle C_\delta : \lambda < \delta < \lambda^+ \rangle$ (*i.e.* C_δ *is a closed unbounded subset of* δ, $\alpha \in C'_\delta \Rightarrow C_\alpha = C_\delta \cap \alpha$, $|C_\delta| < \lambda$) *such that* $C_\delta \cap S = \varnothing$.

3) *There is a λ^+-Souslin tree complete at levels of cofinality $\neq$ cf λ.*

4) *Suppose T is a complete first order theory, T has a model M in which $(P^M, <)$ is a Souslin tree, $(Q^M, <) \cong (\omega_1, <)$, and $F^M: P^M \to Q^M$ gives the level. Then T has a model N, $(Q^N, <)$ is a λ^+-like ordering, and $(P^N, <)$ is a κ^+-Souslin tree (except that its set of levels is not well-ordered).*

PROOF. (1) By the previous theorem there are $\mathscr{P}_\alpha \subseteq \mathscr{P}(\alpha)$ $(\alpha \in S)$, $|\mathscr{P}_\alpha| \leq \lambda$, such that, for every $A \subseteq \lambda$, $\{\alpha \in S: A \cap \alpha \in \mathscr{P}_\alpha\}$ is stationary (as its complement in S is not so large). By a theorem of Kunen it follows that $\diamondsuit_S$ holds.

(2) It is known that $I = \{S \subseteq \lambda^+: \diamondsuit_S \text{ does not hold}\}$ is a normal ideal (see Devlin and Shelah [DS]). Let $\langle C^0_\delta: \lambda < \delta < \lambda^+\rangle$ be a square sequence. For $\alpha < \lambda$ let $S^*_\alpha = \{\delta \in S^*: C^0_\alpha \text{ has order-type } \alpha\}$. So $\bigcup_{\alpha<\lambda} S^*_\alpha \notin I$ (by part (1)); hence $S^*_\alpha \notin I$ for some α. Let $S = S^*_\alpha$; $F(S) = \varnothing$ because C^0_δ is a close unbounded subset of δ, $|C^0_\delta \cap S| \leq 1$. Now define C^1_δ: if $C^0_\delta \cap S = \varnothing$, then $C^1_\delta = C^0_\delta$, and if $C^0_\alpha \cap S = \{\gamma_\alpha\}$, then $C^1_\alpha = C^0_\alpha - (\gamma_\alpha + 1)$. It is easy to check that S and $\langle C^1_\delta: \delta < \lambda^+\rangle$ are as required.

(3) Part (2) of the conclusion provides the necessary assumptions for the theorem of Jensen [J] on the existence of such a λ^+-Souslin tree.

(4) Keisler and Kunen (see Keisler [K]) prove such a theorem for successor of regular. We just have to combine this with the proof of $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$ (the theorem is due to Jensen; for a proof by Silver, see [J]).

Notice that if e.g. $0^\# \notin V$ and κ is strongly inaccessible, the hypothesis of 3.3 will hold (e.g. for μ a successor of a strong limit cardinal).

3.3. LEMMA. *Suppose κ is strongly inaccessible and there is a square sequence $\langle C^0_\delta: \delta < \kappa, \text{cf } \delta < \mu^+\rangle$, C_δ having order-type $<\delta$. Let μ be regular. Suppose $S \subseteq \mu$ and $\diamondsuit_S$ holds.*

Then we can choose for every $\delta < \kappa$, $\text{cf}\,\delta < \mu$, a closed unbounded subset B_δ and $f_\delta: B_\delta \to \{0, 1\}$ such that for every closed unbounded $C \subseteq \kappa$ and $f: C \to \{0, 1\}$, for stationarily many $\delta < \kappa$ we have $B_\delta \subseteq C$ and $f_\delta \subseteq f$.

PROOF. For some γ the set $S_1 = \{\delta < \kappa: \text{cf}\,\delta = \mu \text{ and } C_\delta \text{ has order-type } \gamma\}$ is stationary (by Fodor's lemma). Let g be an increasing continuous function from μ into γ, $\text{Sup}(\text{Rang}\, g) = \gamma$.

Let $\{(C^1_i, f^1_i): i \in S\}$ be such that C_i is a closed unbounded subset of i, f_i a function from i to $\{0, 1\}$, and, for every closed unbounded $C \subset \mu$ and $f: \mu \to \{0, 1\}$ for stationarily many i's, $C \cap i = C^1_i$ and $f \restriction i = f^1_i$. Now for some $\delta < \kappa$ we shall define B_δ and f_δ. If C^0_δ has order-type γ_δ, and $\gamma_\delta < \gamma$, let h_δ be a one-to-one monotonic function from γ_δ onto C^0_δ. If γ_δ is in the range of g, let $\beta_\delta < \mu$ be such that $g(\beta_\delta) = \gamma_\delta$. Now

$$B_\delta = \{h_\delta(g(\varepsilon)): \varepsilon \in C^1_{\beta_\delta}\}, \qquad f_\delta[h_\delta(g(\varepsilon))] = f^1_{\beta_\delta}(\varepsilon).$$

The rest should be clear.

CONCLUDING REMARKS. (1) We can use a weaker variant of the square, e.g. (as Jensen [J] suggested):

$\square'_\lambda$: For every $\alpha < \lambda^+$ we have a family $\mathscr{P}^c_\alpha$ of closed unbounded subsets of α of order-type $< \lambda$, $|\mathscr{P}^c_\alpha| \leq \lambda$, such that $C \in \mathscr{P}^c_\alpha$, $\beta \in C' \Rightarrow C \cap \beta \in \mathscr{P}^c_\beta$.

We can weaken this further (where $S \subseteq \lambda^+$ is stationary):

$\square'_\lambda(S)$: For every $\alpha < \lambda^+$ we have a family $\mathscr{P}^c_\alpha$, $|\mathscr{P}^c_\alpha| \leq \lambda$, of closed unbounded subsets of α of order-type $< \lambda$, such that $C \in \mathscr{P}^c_\alpha$, $\beta \in C'$, $\beta \in S \Rightarrow \beta \cap C \in \mathscr{P}^c_\beta$.

$\square''_\lambda(S)$: For every $\alpha < \lambda^+$ we have a family $\mathscr{P}^c_\alpha$ of closed unbounded subsets of α of order-type $< \lambda$, $|\mathscr{P}^c_\delta| \le \lambda$, such that $C \in \mathscr{P}^c_\alpha$, $\beta \in C' \Rightarrow \beta \cap C \in \mathscr{P}^c_\beta$.

See also [Sh3] on this.

(2) We can rephrase our results in terms of clubs instead of diamonds, or even in the following manner: there are $\mathscr{P}_\alpha \subseteq \{A \subseteq \alpha : |A| < \lambda\}$, $|\mathscr{P}_\alpha| \le \lambda$, such that for every unbounded $A \subseteq \lambda^+$ for "many" α's,

$$(\exists B \in \mathscr{P}_\alpha)(A \cap B \text{ is an unbounded subset of } \alpha).$$

3.4. THEOREM. *Suppose λ is strong limit of cofinality $\kappa > \aleph_0$, with $2^\lambda = \lambda^+$. Then we can find $\langle \mathscr{P}_\alpha : \alpha < \lambda^+ \rangle$, $\mathscr{P}_\alpha$ a family of $\le \lambda$ subsets of α, such that for every $X \subseteq \lambda^+$ there are $S_i \subseteq \lambda^+$ $(i < \kappa)$, $\bigcup_{i<\kappa} S_i = \{\alpha < \lambda^+ : \operatorname{cf} \alpha < \kappa\}$, such that if $\delta \notin S^*_X = \{\delta < \lambda^+ : \operatorname{cf} \delta = \kappa, X \cap \delta \in \mathscr{P}_\delta\}$, then S_i is not stationary below δ (for every $i < \kappa$).*

PROOF. Let $\{A_i : i < \lambda^+\}$ be a list of all bounded subsets of λ^+ such that $A_i \subseteq i$. For each α let $\alpha = \bigcup_{\xi<\kappa} B^\alpha_\xi$, the B^α_ξ increasing with ξ and $|B^\alpha_\xi| \le \lambda_\xi$, where $\lambda = \sum_{\xi<\kappa} \lambda_\xi$, the $\lambda_\xi < \lambda$ increasing continuously. For each $\delta < \lambda^+$, choose a closed unbounded subset C^*_δ of δ of order type $\operatorname{cf} \delta$. Let $\mathscr{P}_{\delta,\xi}$ be the family of sets which are a union of a subfamily of $\{A_i : i \in \bigcup\{B^\alpha_\xi : \alpha \in C^*_\delta\}\}$ and $\mathscr{P}_\alpha = \bigcup_{\xi<\kappa} \mathscr{P}_{\alpha,\xi}$. Clearly $\mathscr{P}_{\alpha,\xi}$ is a family of $\le 2^{\lambda_\xi}$ subsets of α (as $A_i \subseteq i$), and so $\mathscr{P}_\alpha$ is a family of $\le \lambda$ subsets of α.

Let $X \subseteq \lambda^+$, and $S_X = \{\delta < \lambda^+ : X \cap \delta \in \mathscr{P}_\delta, \operatorname{cf} \delta = \kappa\}$, and $C = \{\delta < \lambda^+ :$ for every $\alpha < \delta$, $X \cap \alpha \in \{A_i : i < \delta\}\}$ (so clearly C is closed unbounded), and define a two-place function f on λ^+:

$$f(\alpha, \beta) = \operatorname{Min}\{\xi < \kappa : X \cap \alpha \in \{A_i : i \in B^\beta_\xi\}\}.$$

By the definition of C, $f(\alpha,\beta)$ is well defined for $\alpha < \beta$, $\beta \in C$ (remember $\beta = \bigcup_{\xi<\kappa} B^\beta_\xi$). Moreover, just $(\alpha, \beta] \cap C \ne \varnothing$ is enough.

For $\alpha \in C$, $\operatorname{cf} \alpha < \kappa$, we define

$$\xi(\alpha) = \operatorname{Min}\{\xi < \kappa : \text{for every } \gamma < \alpha, \text{ there is a } \beta \text{ with } \gamma \le \beta < \alpha \text{ and } f(\beta, \alpha) < \xi\}.$$

As $\operatorname{cf} \alpha < \kappa$, clearly $\xi(\alpha)$ is well defined.

FACT. *If $\delta \in C$, $\operatorname{cf} \delta = \kappa$, $\xi < \kappa$, and $\{\gamma \in C^*_\delta \cap C : \xi(\gamma) \le \xi\}$ is unbounded below δ, then $\delta \in S_X$.*

This is because for every $\gamma < \delta$, for some $\beta < \alpha < \delta, \gamma < \beta, \alpha \in C^*_\delta \cap C$, we have $f(\beta, \alpha) \le \xi$, so

$$X \cap \beta \in \{A_i : i \in B^\alpha_\xi\} \subseteq \{A_i : i \in \bigcup\{B^\varepsilon_\xi : \varepsilon \in C^*_\delta\}.$$

As we can find arbitrarily large such $\beta < \delta$, clearly $X \cap \delta \in \mathscr{P}_{\delta,\xi} \subseteq \mathscr{P}_\delta$. So the fact is proved.

We can conclude that for every $X \subseteq \lambda^+$, there are a closed unbounded set $C \subseteq \lambda^+$ and a function ξ from $\{\delta \in C : \operatorname{cf} \delta < \kappa\}$ into κ, such that

$$\delta < \lambda^+, \operatorname{cf} \delta = \kappa, \delta \notin S_X \textit{ implies } \text{for every } \xi_0 < \kappa,$$
$$\{\alpha \in C^*_\delta \cap C : \xi(\alpha) \le \xi_0\} \text{ is bounded below } \delta.$$

REMARK. This shows that, assuming G.C.H., $\diamondsuit_{\{\delta<\lambda^+ : \operatorname{cf} \delta = \operatorname{cf} \lambda\}}$ may follow from properties of cardinals $< \lambda$.

There is one missing point: we prove the conclusion restricted to C. What about the $\delta \notin C$? First we can assume that the points of C which are not limit points of C have cofinality ω, and that $0 \in C$. Now if $\beta < \gamma$ are successive members of C, we define $S_i \cap (\beta, \gamma) (i < \kappa)$ such that for no $\delta \in (\beta, \gamma)$, $\operatorname{cf} \delta = \kappa$, is $S_i \cap \delta$ stationary in δ, and $\bigcup_{i<\kappa} S_i \cap (\beta, \gamma) = \{i \in (\beta, \gamma) : \operatorname{cf} i < \kappa\}$. Why is this possible? Because there is a continuous increasing function from (β, γ) into C.

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