

# Asymmetric tie-points and almost clopen subsets of $\mathbb{N}^*$

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*To the memory of Bohuslav Balcar*

*Abstract.* A tie-point of compact space is analogous to a cut-point: the complement of the point falls apart into two relatively clopen non-compact subsets. We review some of the many consistency results that have depended on the construction of tie-points of  $\mathbb{N}^*$ . One especially important application, due to Veličković, was to the existence of nontrivial involutions on  $\mathbb{N}^*$ . A tie-point of  $\mathbb{N}^*$  has been called symmetric if it is the unique fixed point of an involution. We define the notion of an almost clopen set to be the closure of one of the proper relatively clopen subsets of the complement of a tie-point. We explore asymmetries of almost clopen subsets of  $\mathbb{N}^*$  in the sense of how may an almost clopen set differ from its natural complementary almost clopen set.

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## 1. Introduction

In this introductory section we review some background to motivate our interest in further study of tie-points and almost clopen sets. The Stone–Čech compactification of the integers  $\mathbb{N}$ , is denoted as  $\beta\mathbb{N}$  and, as a set, is equal to  $\mathbb{N}$  together with all the free ultrafilters on  $\mathbb{N}$ . The remainder  $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$  can be topologized as a subspace of the Stone space of the power set of  $\mathbb{N}$  as a Boolean algebra and, in particular, for a subset  $a$  of  $\mathbb{N}$ , the set  $a^*$  of all free ultrafilters with  $a$  as an element, is a basic clopen subset of  $\mathbb{N}^*$ .

A point  $x$  of a space  $X$  is a butterfly point (or  $b$ -point, see [23]) if there are sets  $D, E \subset X \setminus \{x\}$  such that  $\{x\} = \overline{D} \cap \overline{E}$ . In [4], the authors introduced the tie-point terminology.

**Definition 1.1.** A point  $x$  is a tie-point of a space  $X$  if there are closed sets  $A, B$  of  $X$  such that  $X = A \cup B$ ,  $\{x\} = A \cap B$  and  $x$  is a limit point of each of  $A$  and  $B$ . We picture (and denote) this as  $X = A \underset{x}{\bowtie} B$  where  $A, B$  are the closed sets which have a unique common accumulation point  $x$  and say that  $x$  is a tie-point as witnessed by  $A, B$ .

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In this note the focus is on the local properties of  $x$  with respect to each of the closed sets  $A$  and  $B$  such that  $A \underset{x}{\bowtie} B$  in the case when  $A, B$  witness that  $x$  is a tie-point. For this reason we introduce the notion of an almost clopen subset of  $\mathbb{N}^*$ .

**Definition 1.2.** A set  $A \subset \mathbb{N}^*$  is almost clopen if  $A$  is the closure of an open subset of  $\mathbb{N}^*$  and has a unique boundary point, which we denote  $x_A$ .

**Proposition 1.3.** *If  $A$  is an almost clopen subset of  $\mathbb{N}^*$ , then  $B = \{x_A\} \cup (\mathbb{N}^* \setminus A)$  is almost clopen and  $x_B = x_A$ . In addition  $x_A$  is a tie-point as witnessed by  $A, B$ .*

**Definition 1.4** ([4]). A tie-point  $x$  is a symmetric tie-point of  $\mathbb{N}^*$  if there is a pair  $A, B$  witnessing that  $x$  is a tie-point and if there is a homeomorphism  $h: A \rightarrow B$  satisfying that  $h(x) = x$ .

If  $A$  is almost clopen, then we refer to  $B = \{x_A\} \cup (\mathbb{N}^* \setminus A)$  as the almost clopen complement of  $A$ . A more set-theoretically inclined reader would surely prefer a straightforward translation of almost clopen to properties of ideals of subsets of  $\mathbb{N}$  and the usual mod finite ordering  $\subset^*$ .

**Definition 1.5.** If  $A$  is any subset of  $\mathbb{N}^*$ , then  $\mathcal{I}_A$  is defined as the set  $\{a \subset \mathbb{N} : a^* \subset A\}$ .

For any family  $\mathcal{A}$  of subsets of  $\mathbb{N}$  (or  $\omega$ ), we define  $\mathcal{A}^\perp$  to be the orthogonal ideal  $\{b \subset \mathbb{N} : \forall a \in \mathcal{A} \ b \cap a =^* \emptyset\}$ . We state this next result for easy reference.

**Proposition 1.6.** *If  $\mathcal{I}$  is an ideal that has no  $\subset^*$ -maximal element, then the ideal generated by  $\mathcal{I} \cup \mathcal{I}^\perp$  is a proper ideal.*

**Lemma 1.7.** *If  $A$  is an almost clopen subset of  $\mathbb{N}^*$  with almost clopen complement  $B$ , then  $\mathcal{I}_A \cap \mathcal{I}_B$  is the Fréchet ideal  $[\mathbb{N}]^{<\aleph_0}$ ,  $\mathcal{I}_B = \mathcal{I}_A^\perp$ , and  $x_A$  is the unique ultrafilter that is disjoint from  $\mathcal{I}_A \cup \mathcal{I}_B$ .*

Almost clopen sets (and tie-points) first arose implicitly in the work [9] of N. J. Fine and L. Gillman in the investigation of extending continuous functions on dense subsets of  $\mathbb{N}^*$ . A subset  $Y$  of a space  $X$  is  $C^*$ -embedded if every bounded continuous real-valued function on  $Y$  can be continuously extended to all of  $X$ . The character of a point  $x \in \mathbb{N}^*$  is the minimal cardinality of a filter base for  $x$  as an ultrafilter on  $\mathbb{N}$ .

**Proposition 1.8** ([9]). *If  $x$  is a tie-point of  $\mathbb{N}^*$ , then  $\mathbb{N}^* \setminus \{x\}$  is not  $C^*$ -embedded in  $\mathbb{N}^*$ . Every point of character  $\aleph_1$  is a tie-point of  $\mathbb{N}^*$ .*

It was shown in [3] to be consistent with Zermelo–Fraenkel set theory (ZFC) that  $\mathbb{N}^* \setminus \{x\}$  is  $C^*$ -embedded for all  $x \in \mathbb{N}^*$ . It was also shown by J. E. Baumgartner in [1] that this result holds in models of the Proper forcing axiom (PFA).

**Proposition 1.9** ([3], [1]). *PFA implies that  $\mathbb{N}^* \setminus \{x\}$  is  $C^*$ -embedded in  $\mathbb{N}^*$  for all  $x \in \mathbb{N}^*$*

**Corollary 1.10.** *PFA implies that there are no almost clopen sets and no tie-points in  $\mathbb{N}^*$ .*

Almost clopen sets arise in the study of minimal extensions of Boolean algebras (see [15]) and in the application of this method of construction for building a variety of counterexamples (e.g. [16], [12], [22], [6]). The next application of almost clopen subsets of  $\mathbb{N}^*$  was to the study of nontrivial automorphisms of  $\mathcal{P}(\mathbb{N})/\text{fin}$ , or nontrivial autohomeomorphisms of  $\mathbb{N}^*$ . M. Katětov in [14] proved that the set of fixed points of an autohomeomorphism of  $\beta\mathbb{N}$  will be a clopen set. It is immediate from N. J. Fine and L. Gillman's work [9] that every  $P$ -point of character of  $\aleph_1$  is a fixed point of a nontrivial autohomeomorphism of  $\mathbb{N}^*$ .

**Definition 1.11.** A point  $x$  of  $\mathbb{N}^*$  is a  $P$ -point if the ultrafilter  $x$  is countably complete mod finite. For a cardinal  $\kappa$ , an ultrafilter  $x$  on  $\mathbb{N}$  is a simple  $P_\kappa$ -point if  $x$  has a base well-ordered by mod finite inclusion of order type  $\kappa$ .

**Proposition 1.12** ([9]). *If  $A$  is an almost clopen subset of  $\mathbb{N}^*$  and  $x_A$  is a simple  $P_{\aleph_1}$ -point of  $\mathbb{N}^*$ , then*

- (1)  $A$  is homeomorphic to  $\mathbb{N}^*$ ;
- (2)  $x_A$  is a symmetric tie-point;
- (3) there is an autohomeomorphism  $f$  on  $\mathbb{N}^*$  such that  $\{x\}$  is the only fixed point of  $f$ .

As we have seen above, PFA implies that there are no almost clopen subsets of  $\mathbb{N}^*$ , and of course, PFA also implies that all autohomeomorphisms of  $\mathbb{N}^*$  are trivial, see [24]. However Veličković utilized a similar simple  $P$ -point trick (motivating our definition of symmetric tie-point) in order to prove that this is not a consequence of Martin's axiom (MA).

**Proposition 1.13** ([27]). *It is consistent with MA and  $\mathfrak{c} = \aleph_2$  that there is an almost clopen set  $A$  of  $\mathbb{N}^*$  such that  $x_A$  is a simple  $P_{\aleph_2}$ -point and,*

- (1)  $x_A$  is a symmetric tie-point,
- (2) there is an autohomeomorphism  $f$  on  $\mathbb{N}^*$  such that  $\{x\}$  is the only fixed point of  $f$ .

Veličković's result and approach was further generalized in [25] and [26]. It is very interesting to know if an almost clopen subset of  $\mathbb{N}^*$  is itself homeomorphic to  $\mathbb{N}^*$ , see [8], [13]. This question also arose in the authors' work on two-to-one images of  $\mathbb{N}^*$ , see [5]. Veličković's method was slightly modified in [5] to produce a complementary pair of almost clopen sets so that neither is homeomorphic to  $\mathbb{N}^*$ , but it is not known if there is a symmetric tie-point  $A \underset{x}{\bowtie} B$  where  $A$  is not a copy of  $\mathbb{N}^*$ .

Our final mention of recent interest in almost clopen subsets of  $\mathbb{N}^*$  is in connection to the question, see [19], [7], of whether the Banach space  $l_\infty/c_0$  is necessarily primary. It was noted by P. Koszmider in [20, page 577] that a special almost clopen subset of  $\mathbb{N}^*$  could possibly resolve the problem. For a compact space  $K$ , we let  $C(K)$  denote the Banach space of continuous real-valued functions on  $K$

with the supremum norm. It is well-known that  $C(\mathbb{N}^*)$  is isomorphic (as a Banach space) to  $l_\infty/c_0$ . Naturally if a space  $A$  is homeomorphic to  $\mathbb{N}^*$ , then  $C(A)$  is isomorphic to  $C(\mathbb{N}^*)$ .

**Proposition 1.14** ([20, page 577]). *Suppose that  $A$  is an almost clopen subset of  $\mathbb{N}^*$  and that  $B$  is its almost clopen complement. If  $C(\mathbb{N}^*)$  is not homeomorphic to either of  $C(A)$  or  $C(B)$ , then  $l_\infty/c_0$  is not primary.*

## 2. Asymmetric tie-points

In many of the applications mentioned in the introductory section, the tie-points utilized were symmetric tie-points. In other applications for example the primariness of  $l_\infty/c_0$ , it may be useful to find examples where the witnessing sets  $A, B$  for a tie-point are quite different. There are any number of local topological properties that  $x$  may enjoy as a point in  $A$  that it may not share as a point in  $B$ . We make the following definition in connection with simple  $P_\kappa$ -points.

**Definition 2.1.** Let  $\kappa$  be a regular cardinal. An almost clopen set  $A$  is simple of type  $\kappa$  if  $\mathcal{I}_A$  has a  $\subset^*$ -cofinal  $\subset^*$ -increasing chain  $\{a_\alpha : \alpha \in \kappa\}$  of type  $\kappa$ .

If  $\{a_\alpha : \alpha \in \kappa\}$  is strictly  $\subset^*$ -increasing and  $\subset^*$ -cofinal in  $\mathcal{I}_A$  for an almost clopen set  $A$ , then the family  $\mathcal{A}_A = \{a_{\alpha+1} \setminus a_\alpha : \alpha \in \kappa\}$  cannot be reaped. A family  $\mathcal{A} \subset [\mathbb{N}]^{\mathbb{N}_0}$  is reaped by a set  $c \subset \mathbb{N}$  if  $|a \setminus c| = |a \cap c|$  for all  $a \in \mathcal{A}$ . We verify that  $\mathcal{A}_A$  cannot be reaped. Let  $c \subset \mathbb{N}$  and, by symmetry, assume that  $c$  is not an element of the ultrafilter  $x_A$ . Therefore  $c^* \cap A$  is a compact subset of  $A \setminus \{x\}$  and so is contained in  $a_\alpha^*$  for some  $\alpha \in \kappa$ . Thus  $(c \setminus a_\alpha)^*$  is disjoint from  $A$  and so  $c \cap (a_{\alpha+1} \setminus a_\alpha)$  is finite. This proves that  $\mathcal{A}_A$  is not reaped by  $c$ .

The reaping number  $\mathfrak{r}$  is the minimum cardinal of a family that cannot be reaped, see [11] and [18]. For any infinite set  $a \subset \mathbb{N}$ , let  $\text{next}(a, \cdot)$  be the function in  $\mathbb{N}^{\mathbb{N}}$  defined by  $\text{next}(a, k) = \min(a \setminus \{1, \dots, k\})$ . As usual, for  $f, g \in \mathbb{N}^{\mathbb{N}}$  we say that  $f <^* g$  if  $\{k : g(k) \leq f(k)\}$  is finite.

**Proposition 2.2** ([11]). *If  $\mathcal{A} \subset [\mathbb{N}]^{\mathbb{N}_0}$  and if there is some  $g \in \mathbb{N}^{\mathbb{N}}$  such that  $\text{next}(a, \cdot) <^* g$  for all  $a \in \mathcal{A}$ , then  $\mathcal{A}$  can be reaped. In particular,  $\mathfrak{b} \leq \mathfrak{r}$ .*

If  $\{a_\alpha : \alpha \in \kappa\}$  is strictly  $\subset^*$ -increasing and  $\subset^*$ -cofinal in  $\mathcal{I}_A$  for an almost clopen set  $A$ , then the family  $\{a_{\alpha+1} \setminus a_\alpha : \alpha \in \kappa\}$  is an example of a *converging* family of infinite sets.

**Definition 2.3.** Let  $\mathcal{A}$  be a family of infinite subsets of  $\mathbb{N}$ . We say that  $\mathcal{A}$  converges if there is an ultrafilter  $x$  on  $\mathbb{N}$  such that for each  $U \in x$ , the set  $\{a \in \mathcal{A} : a \setminus U \neq^* \emptyset\}$  has cardinality less than that of  $\mathcal{A}$ .

We say that  $\mathcal{A}$  is hereditarily unreapable if each reappable subfamily of  $\mathcal{A}$  has cardinality less than that of  $\mathcal{A}$ .

**Lemma 2.4.** *If for a cardinal  $\kappa$ ,  $\{a_\alpha : \alpha \in \kappa\}$  is a strictly  $\subset^*$ -increasing family of subsets of  $\mathbb{N}$ , and if  $\{a_{\alpha+1} \setminus a_\alpha : \alpha \in \kappa\}$  converges, then the closure of  $\bigcup\{a_\alpha^* : \alpha \in \kappa\}$  is a simple almost clopen set of type  $\kappa$ .*

PROOF: Suppose that  $\{a_{\alpha+1} \setminus a_\alpha : \alpha \in \kappa\}$  converges to  $x \in \mathbb{N}^*$  and let  $A = \{x\} \cup \bigcup \{a_\alpha^* : \alpha \in \kappa\}$ . It suffices to prove that  $A$  is closed. Choose any  $y \in \mathbb{N}^* \setminus A$  and let  $a$  be an element of  $x \setminus y$ . Choose  $\beta \in \kappa$  so that  $(a_{\alpha+1} \setminus a_\alpha) \subset^* a$  for all  $\beta \leq \alpha \in \kappa$ . By induction on  $\alpha < \kappa$ , it follows that  $a_\alpha \setminus a_\beta \subset^* a$ . Therefore  $A$  is contained in the clopen set  $(a_\beta \cup a)^* \subset \mathbb{N}^* \setminus \{y\}$ .  $\square$

An ultrafilter  $x$  of  $\mathbb{N}^*$  is said to be an almost  $P_\kappa$ -point (or pseudo- $P_\kappa$ -point) if each set of fewer than  $\kappa$  many members of  $x$  have a pseudo-intersection (an infinite set mod finite contained in each of them). Certainly a converging family is hereditarily unreachably and converges to a point that is an almost  $P_\kappa$ -point where  $\kappa$  is the cardinality of the family. Clearly the cardinality of any hereditarily unreachably family will have cofinality not larger than the splitting number  $\mathfrak{s}$ . Therefore if, for example,  $\mathfrak{s} = \aleph_1$  and  $\mathfrak{r} = \mathfrak{c} = \aleph_2$ , there will be no hereditarily unreachably family. If  $\mathfrak{s} = \mathfrak{c}$ , then there is a hereditarily unreachably family of cardinality  $\mathfrak{s}$ . In the Mathias model of  $\mathfrak{s} = \mathfrak{c} = \mathfrak{b} = \aleph_2$ , there is no converging unreachably family because there is no almost  $P_{\aleph_2}$ -point. In the Goldstern–Shelah model of  $\mathfrak{r} = \mathfrak{s} = \aleph_1 < \mathfrak{u}$ , see [11], there is (easily checked) no converging family of cardinality  $\mathfrak{r}$ .

If there is a simple almost clopen set of type  $\kappa$ , are there restrictions on the behavior of its almost clopen complement and can there be simple almost clopen sets of different types (including the complement)? These are the types of questions that stimulated this study. The most compelling of these has been answered.

**Theorem 2.5.** *If  $A$  is a simple almost clopen set of type  $\kappa$  and if the complementary almost clopen set  $B$  is simple, then it also has type  $\kappa$ .*

We defer the proof of this, and the next two theorems, until the next section. Similarly, there is restriction on what the type of a simple almost clopen set can be that is shared by simple  $P_\kappa$ -points (as shown, for simple  $P_\kappa$ -points, by Nyikos (unpublished), see [2]).

**Theorem 2.6.** *If  $A$  is a simple almost clopen set of type  $\kappa$ , then  $\kappa$  is one of  $\{\mathfrak{b}, \mathfrak{d}\}$ .*

Now that we understand the limits on the behavior of a complementary pair of simple almost clopen sets, we look to the properties of the complement  $B$  when it is not assumed to be simple. The topological properties of character and tightness of  $x_B$  in  $B$  are natural cardinal invariants to examine. These correspond to natural properties of  $\mathcal{I}_B$  as well. An indexed subset  $\{x_\beta : \beta < \lambda\}$  of a space  $X$  is said to be a free sequence if the closure of each initial segment is disjoint from the closure of its complementary final segment. A  $\lambda$ -sequence  $\{x_\beta : \beta < \lambda\}$  is converging if there is a point  $x$  such that every neighborhood of  $x$  contains a final segment of  $\{x_\beta : \beta < \lambda\}$ . A subset  $D$  of  $\mathbb{N}^*$  is said to be *strongly discrete*, see [10] and [21], if there is a family of pairwise disjoint clopen subsets of  $\mathbb{N}^*$  each containing a single point of  $D$ .

**Theorem 2.7.** *If  $\kappa < \lambda$  are regular cardinals with  $\mathfrak{c} \leq \lambda$ , then there is a countable chain condition (ccc) forcing extension in which there is a simple almost clopen set*

*A of type  $\kappa$  such that the almost clopen complement  $B$  contains a free  $\lambda$ -sequence  $\{x_\beta : \beta < \lambda\}$  that converges to  $x_A$ .*

We finish this section by formulating some open problems about almost clopen sets and possible asymmetries.

**Question 2.8.** Can there exist simple almost clopen sets of different types?

If  $A$  is a simple almost clopen set of type  $\kappa$ , then, as a point in  $A$ ,  $x_A$  is a  $P$ -point with a linear neighborhood base of clopen sets. Therefore the next two questions seem natural. See also the remark about Nyikos' result, see [2], about simple  $P_\kappa$ -points preceding Theorem 2.6.

**Question 2.9.** If there is a simple almost clopen set of type  $\kappa$  is there a point of  $\mathbb{N}^*$  of character  $\kappa$ ? Is there a simple  $P_\kappa$ -point?

**Question 2.10.** Is a simple almost clopen set of type  $\aleph_1$  necessarily homeomorphic to  $\mathbb{N}^*$ ?

**Question 2.11.** If  $A$  is a simple almost clopen set of type  $\kappa$ , is there a simple almost clopen set  $B'$  contained in the almost clopen complement  $B$  of  $A$  such that  $x_A = x_{B'}$ ? Is there a family of  $\kappa$ -many members of  $\mathcal{I}_B$  that converges to  $x_A$ ?

### 3. Proofs

Our analysis of simple almost clopen sets depends on the connection between the type of the clopen set and the ultrafilter ordering of functions from  $\mathbb{N}$  to  $\mathbb{N}$ . For an ultrafilter  $x$  on  $\mathbb{N}$  the ordering  $<_x$  is defined on  $\mathbb{N}^{\mathbb{N}}$  by the condition that  $f <_x g$  if  $\{n \in \mathbb{N} : f(n) < g(n)\} \in x$ . Since  $x$  is an ultrafilter, a set  $F \subset \mathbb{N}^{\mathbb{N}}$  is cofinal in  $(\mathbb{N}^{\mathbb{N}}, <_x)$  if it is not bounded. Of course a subset of  $\mathbb{N}^{\mathbb{N}}$  that is unbounded with respect to the  $<_x$ -ordering is also unbounded with respect to the mod finite ordering  $<^*$ .

Fix a  $<^*$ -unbounded family  $\{f_\xi : \xi < \mathfrak{b}\} \subset \mathbb{N}^{\mathbb{N}}$  such that each  $f_\xi$  is strictly increasing and such that  $f_\eta <^* f_\xi$  for all  $\eta < \xi < \mathfrak{b}$ . The following well-known fact will be useful.

**Proposition 3.1.** *For each infinite  $b \subset \mathbb{N}$  and each unbounded  $\Gamma \subset \mathfrak{b}$ , the family  $\{f_\xi \upharpoonright b : \xi \in \Gamma\}$  is  $<^*$ -unbounded in  $\mathbb{N}^b$ .*

PROOF: For each  $\eta < \mathfrak{b}$ , there is a  $\xi \in \Gamma \setminus \eta$  such that  $f_\eta <^* f_\xi$ , hence  $\{f_\xi : \xi \in \Gamma\}$  is  $<^*$ -unbounded. If  $g \in \mathbb{N}^b$ , then  $g \circ \text{next}(b, \cdot) \in \mathbb{N}^{\mathbb{N}}$ . So there is a  $\xi \in \Gamma$  such that  $f_\xi \not<^* g \circ \text{next}(b, \cdot)$ . Since  $f_\xi$  is strictly increasing,  $f_\xi \upharpoonright b \not<^* g$ .  $\square$

**Lemma 3.2.** *If a family  $\mathcal{A} \subset [\mathbb{N}]^{\aleph_0}$  converges to an ultrafilter  $x$  and if  $\{f_\xi : \xi \in \mathfrak{b}\}$  is bounded mod  $<_x$ , then  $\mathcal{A}$  has cardinality  $\mathfrak{b}$ .*

PROOF: Choose  $g \in \mathbb{N}^{\mathbb{N}}$  so that  $f_\xi <_x g$  for all  $\xi < \mathfrak{b}$ . Since  $\mathcal{A}$  cannot be reaped, Proposition 2.2 implies that  $\mathfrak{b} \leq |\mathcal{A}|$ . For each  $\xi$ , let  $U_\xi = \{n \in \mathbb{N} : f_\xi(n) < g(n)\} \in x$ . If  $\mathfrak{b} < |\mathcal{A}|$ , then there is a  $b \in \mathcal{A}$  such that  $b \not<^* U_\xi$  for all  $\xi < \mathfrak{b}$

(i.e.  $x$  is an almost  $P_{\mathfrak{b}+}$ -point). However we would then have that  $f_\xi \upharpoonright b <^* g \upharpoonright b$  for all  $\xi < \mathfrak{b}$ , and by Proposition 3.1, there is no such set  $b$ . This completes the proof.  $\square$

Concerning this next result, we do not know if it is consistent to have a family in  $[\mathbb{N}]^{\aleph_0}$  of singular cardinality that is converging.

**Lemma 3.3.** *If a family  $\mathcal{A} \subset [\mathbb{N}]^{\aleph_0}$  of regular cardinality converges to an ultrafilter  $x$  and if  $\{f_\xi: \xi \in \mathfrak{b}\}$  is unbounded mod  $<_x$ , then  $\mathcal{A}$  has cardinality equal to  $\mathfrak{d}$ .*

PROOF: Since we are assuming that  $\{f_\xi: \xi \in \mathfrak{b}\}$  is  $<_x$ -unbounded, it is actually  $<_x$ -cofinal. Also, we check that the family  $\{f_\xi \circ \text{next}(a, \cdot): \xi < \mathfrak{b}, a \in \mathcal{A}\}$  is a  $<^*$ -dominating family. Take any strictly increasing  $g \in \mathbb{N}^{\mathbb{N}}$  and choose  $\xi < \mathfrak{b}$  such that  $U = \{n: g(n) < f_\xi(n)\} \in x$ . Since  $\mathcal{A}$  converges to  $x$ , there is an  $a \in \mathcal{A}$  such that  $a \subset^* U$ . Since  $g$  is strictly increasing, it is clear that  $g < f_\xi \circ \text{next}(U, \cdot) <^* f_\xi \circ \text{next}(a, \cdot)$ . Again, since  $\mathcal{A}$  cannot be reaped, we have  $\mathfrak{b} \leq |\mathcal{A}|$  and this implies that  $\mathfrak{d} \leq |\mathcal{A}|$ . Assume that  $\{g_\beta: \beta < \mathfrak{d}\} \subset \mathbb{N}^{\mathbb{N}}$  is a  $<^*$ -dominating family. For each  $a \in \mathcal{A}$ , there is a  $\beta_a < \mathfrak{d}$  such that  $\text{next}(a, \cdot) <^* g_{\beta_a}$ . Now since  $\mathcal{A}$  is hereditarily unrepeatable, Proposition 2.2 implies that the mapping  $a \mapsto \beta_a$  is  $<|\mathcal{A}|$ -to-1. Since we assume that  $|\mathcal{A}|$  is regular, this implies that  $|\mathcal{A}| \leq \mathfrak{d}$ .  $\square$

**Corollary 3.4.** *If  $A$  is a simple almost clopen set of type  $\kappa$ , then  $\kappa = \mathfrak{b}$  if  $\{f_\xi: \xi < \mathfrak{b}\}$  is  $<_{x_A}$ -bounded. Otherwise  $\kappa = \mathfrak{d}$ .*

PROOF: Let  $\{a_\alpha: \alpha \in \kappa\}$  be the family contained in  $\mathcal{I}_A$  witnessing that  $A$  has type  $\kappa$ . Set  $\mathcal{A}$  equal to the family  $\{a_{\alpha+1} \setminus a_\alpha: \alpha \in \kappa\}$  which converges to  $x_A$ . If  $\{f_\xi: \xi < \mathfrak{b}\}$  is  $<_{x_A}$ -bounded, then by Lemma 3.2,  $\kappa = \mathfrak{b}$ . Otherwise, since  $\kappa$  is a regular cardinal, we have by Lemma 3.3,  $\kappa = \mathfrak{d}$ .  $\square$

PROOF OF THEOREM 2.5: Assume that  $A$  and its complementary almost clopen set  $B$  are both simple and let  $x = x_A$ . If  $\{f_\xi: \xi < \mathfrak{b}\}$  is  $<_x$ -bounded then, by Corollary 3.4 they both have type  $\mathfrak{b}$ ; otherwise they both have type  $\mathfrak{d}$ .  $\square$

PROOF OF THEOREM 2.6: Immediate from Corollary 3.4.  $\square$

We can improve Theorem 2.5. First we recall that a family  $\mathcal{A} \subset [\mathbb{N}]^{\aleph_0}$  is a *splitting* family if for all infinite  $b \subset \mathbb{N}$ , there is an  $a \in \mathcal{A}$  such that  $|b \cap a| = |b \setminus a|$ . We say that  $b$  is split by  $a$ . The splitting number  $\mathfrak{s}$  is the least cardinality of a splitting family and  $\mathfrak{s} \leq \mathfrak{d}$ , see [11].

**Proposition 3.5.** *There is no almost  $P_{\mathfrak{s}+}$ -point in  $\mathbb{N}^*$ .*

PROOF: Let  $\mathcal{A}$  be a splitting family of cardinality  $\mathfrak{s}$ . We may assume that  $\mathcal{A}$  is closed under complements. Let  $x$  be any point of  $\mathbb{N}^*$ . It is easily seen that any pseudointersection of  $x \cap \mathcal{A}$  is not split by any member of  $\mathcal{A}$ . Since  $\mathcal{A}$  is splitting,  $x \cap \mathcal{A}$  has no pseudointersection, and so  $x$  is not an almost  $P_{\mathfrak{s}+}$ -point.  $\square$

Now we improve Theorem 2.5.

**Theorem 3.6.** *If  $A$  is a simple almost clopen set of type  $\kappa$  then  $x_A$  is not an almost  $P_{\kappa^+}$ -point.*

PROOF: We first note that by Proposition 3.5 we must have that  $\kappa < \mathfrak{d}$ . Therefore, by Lemma 3.3,  $\{f_\xi : \xi < \mathfrak{b}\}$  is  $<_{x_A}$ -bounded. Choose any  $g \in \mathbb{N}^{\mathbb{N}}$  so that  $f_\xi <_{x_A} g$  for all  $\xi < \mathfrak{b}$ . For each  $\xi$ , let  $U_\xi = \{n \in \mathbb{N} : f_\xi(n) < g(n)\}$ . By Proposition 3.1, we have that the collection  $\{U_\xi : \xi < \mathfrak{b}\} \subset x$  has no pseudointersection. By Theorem 2.5,  $\mathfrak{b} \leq \kappa$  and this proves the theorem.  $\square$

PROOF OF THEOREM 2.7: We first prove the easier special case when  $\kappa = \aleph_1$ . An  $\alpha$ -length finite support iteration sequence of posets, denoted  $(\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{Q}_\beta : \beta < \alpha \rangle)$ , will mean that  $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$  is an increasing chain of posets,  $\dot{Q}_\beta$  is a  $\mathbb{P}_\beta$ -name of a poset for each  $\beta < \alpha$ , and members  $p$  of  $\mathbb{P}_\alpha$  will be functions with domain a finite subset,  $\text{supp}(p)$ , of  $\alpha$  satisfying that  $p \upharpoonright \beta \in \mathbb{P}_\beta$  forces that  $p(\beta) \in \dot{Q}_\beta$ . As usual,  $p_2 < p_1$  providing  $p_2 \upharpoonright \beta \Vdash_{\mathbb{P}_\beta} "p_2(\beta) < p_1(\beta)"$  for all  $\beta \in \text{supp}(p_1)$ . Since  $\mathbb{P}_0$  is the trivial poset, we will allow ourselves to simply specify a poset  $Q_0$  in such an iteration sequence rather than the  $\mathbb{P}_0$ -name of that poset.

**Definition 3.7.** Let  $\mathcal{A} = \{a_\beta : \beta < \alpha\}$  be a  $\subset^*$ -increasing chain of subsets of  $\omega$ , and let  $\mathcal{I}$  be an ideal contained in  $\mathcal{A}^\perp$ . We define the poset  $Q = Q(\mathcal{A}; \mathcal{I})$  where  $q \in Q$  if  $q = (F_q, \sigma_q, b_q)$  where

- (1)  $F_q \in [\omega]^{< \aleph_0}$ ;
- (2)  $b_q \in \mathcal{I}$  is disjoint from  $F_q$ ;
- (3)  $\sigma_q : H_q \rightarrow \omega$  and  $H_q \in [\alpha]^{< \aleph_0}$ ;
- (4) for each  $\beta \in H_q$ ,  $a_\beta \setminus \sigma_q(\beta)$  is disjoint from  $b_q$ .

For  $r, q \in Q$  we define  $r < q$  providing  $F_r \supset F_q$ ,  $\sigma_r \supset \sigma_q$ , and  $b_r \supset b_q$ .

**Definition 3.8.** If  $Q$  is  $Q(\mathcal{A}; \mathcal{I})$  for some  $\subset^*$ -increasing chain of subsets of  $\omega$  and ideal  $\mathcal{I} \subset \mathcal{A}^\perp$ , then the  $Q$ -generic set  $\dot{a}_Q$  is defined as the natural name  $\{(\check{n}, q) : q \in Q \text{ and } n \in F_q\}$ , i.e. for each  $Q$ -generic filter  $G$ ,  $\text{val}_G(\dot{a}_Q)$  is equal to the union of the family  $\{F_q : q \in G\}$ .

**Lemma 3.9.** *Let  $\mathcal{A} = \{a_\beta : \beta < \alpha\}$  and  $\mathcal{I}$  be as given in the definition of  $Q(\mathcal{A}; \mathcal{I})$  and let  $Q = Q(\mathcal{A}; \mathcal{I})$ . Then  $1 \Vdash_Q " \dot{a}_Q \in \mathcal{I}^\perp "$ , and for all  $q \in Q(\mathcal{A}; \mathcal{I})$  and  $\beta \in \text{dom}(\sigma_q)$ ,  $q \Vdash_Q " a_\beta \setminus \sigma_q(\beta) \subset \dot{a}_Q "$ .*

PROOF: Let  $q \in Q$ ,  $\beta \in \sigma_q$  and  $b \in \mathcal{I}$ . Since  $q_b = (F_q, \sigma_q, b_q \cup (b \setminus F_q))$  forces that  $\dot{a}_Q$  is disjoint from  $b$ , this proves that  $1 \Vdash_Q " \dot{a}_Q \in \mathcal{I}^\perp "$ . Choose any  $k \in a_\beta \setminus \sigma_q(\beta)$  and let  $r \leq q$ . By condition (4),  $k \notin b_r$ . It is easily checked that  $r' = (F_r \cup \{k\}, \sigma_r, b_r) < r$  and that  $r' \Vdash_Q " k \in \dot{a}_Q "$ .  $\square$

For constructing our models, we will need instances of  $Q(\mathcal{A}; \mathcal{I})$  that are ccc. This first result is sufficient to produce simple almost clopen sets of type  $\aleph_1$ .

**Lemma 3.10.** *Let  $\mathcal{A} = \{a_\beta : \beta < \alpha\}$  be a  $\subset^*$ -increasing chain of subsets of  $\omega$  and let  $\mathcal{I}$  be an ideal contained in  $\mathcal{A}^\perp$ . If  $\alpha$  is countable or if  $\mathcal{I} = [\omega]^{< \aleph_0}$ , then  $Q(\mathcal{A}; \mathcal{I})$  is  $\sigma$ -centered.*



PROOF: Let  $b, F \in [\omega]^{<\aleph_0}$  and  $H \in [\alpha]^{<\aleph_0}$  and  $\sigma: H_q \rightarrow \omega$ . Then each of the sets  $\{q \in Q(\mathcal{A}; \mathcal{I}): F_q = F \text{ and } \sigma_q = \sigma\}$  and  $\{q \in Q(\mathcal{A}; \mathcal{I}): F_q = F \text{ and } b_q = b\}$  is centered.  $\square$

**Lemma 3.11.** *If  $\lambda$  is a regular cardinal with  $\mathfrak{c} \leq \lambda$ , then there is a ccc forcing extension in which there is a simple almost clopen set  $A$  of type  $\aleph_1$  such that there is a strongly discrete free  $\lambda$ -sequence converging to  $x_A$ .*

PROOF: There are ccc posets of cardinality  $\lambda$  that add a strictly  $\subset^*$ -increasing sequence  $\{b_\zeta: \zeta < \lambda\}$  of infinite subsets of  $\omega$  (e.g. [17, II Example 22]). Alternatively, by Definition 3.7 and Lemma 3.10, we could let  $Q_0$  be a  $\lambda$ -length finite support sequence of posets of the form  $Q(\{b_\beta: \beta < \zeta\}; [\omega]^{<\omega})$  and recursively let  $b_\zeta$  be the resulting  $\dot{a}_Q$  as in Definition 3.8. In each of these models we obtain that  $\mathfrak{c} = \lambda$ .

For convenience we now work in such a ccc forcing extension in which  $\lambda = \mathfrak{c}$  and we construct a finite support ccc iteration sequence of cardinality  $\lambda$  and length  $\omega_1$  that will add a strictly  $\subset^*$ -increasing sequence  $\{a_\alpha: \alpha \in \omega_1\}$  of infinite subsets of  $\omega$  so that the closure  $A$  of  $\bigcup\{a_\alpha^*: \alpha \in \omega_1\}$  is almost clopen. Suppose that we do this in such a way that  $\{b_\zeta: \zeta < \lambda\}$  is contained in  $\{a_\alpha: \alpha \in \omega_1\}^\perp$  and for all  $U \in x_A$  and all  $\zeta < \lambda$  there is an  $\eta < \lambda$  such that  $U \cap (b_\eta \setminus b_\zeta)$  is infinite. We check that there is then a strongly discrete free  $\lambda$ -sequence converging to  $x_A$ . Let  $\{U_\zeta: \zeta < \lambda\}$  enumerate the members of  $x_A$ . Recursively define a strictly increasing function  $g$  from  $\lambda$  into  $\lambda$  satisfying that  $U_\zeta \cap (b_{g(\delta)} \setminus b_\delta)$  is infinite for all  $\zeta < \delta \in \lambda$ . Choose any cub  $C \subset \lambda$  satisfying that the family  $\{U_\xi: \xi < \delta\}$  is closed under finite intersections. For each  $\delta \in C$ , the family  $\{U_\zeta \cap (b_{g(\delta)} \setminus b_\delta): \zeta < \delta\}$  is closed under finite intersections, so we may choose an ultrafilter  $x_\delta$  extending it. Pass to a cub subset  $C_1 \subset C$  satisfying that  $g(\eta) < \delta$  for all  $\eta < \delta$  in  $C_1$ . Since the family  $\{b_\zeta: \zeta < \lambda\}$  is mod finite increasing and  $b_\delta \setminus b_\eta \in x_\eta$  for all  $\eta < \delta$  in  $C_1$ , it follows that  $\{x_\delta: \delta \in C_1\}$  is strongly discrete and free. Similarly, the sequence converges to  $x_A$  since  $U_\zeta \in x_\delta$  for all  $\zeta < \delta \in C_1$ .

Now we construct the iteration sequence to define the  $\subset^*$ -increasing chain  $\{a_\alpha: \alpha \in \omega_1\}$  that will be cofinal in  $\mathcal{I}_A$ . We will use iterands of the form  $\dot{Q}_\alpha = Q(\{\dot{a}_\beta: \beta < \alpha\}; \dot{\mathcal{I}}_\alpha)$  for  $0 < \alpha < \omega_1$ , and will recursively let  $\dot{a}_\alpha$  be the standard  $\mathbb{P}_{\alpha+1}$ -name for  $a_{\dot{Q}_\alpha}$  (as in Definition 3.8). Clearly the only choices we have for the construction are the definition of  $a_0$  and, by recursion, the definition of  $\dot{\mathcal{I}}_\alpha$ . By recursion on  $\gamma < \alpha < \omega_1$  we ensure that  $\mathbb{P}_\alpha$  forces each of the following:

- (1)  $\dot{\mathcal{I}}_{\gamma+1} \subset \dot{\mathcal{I}}_\alpha \subset \{\dot{a}_\beta: \beta < \alpha\}^\perp$ ;
- (2)  $\{b_\zeta: \zeta < \lambda\}$  is a  $\subset^*$ -unbounded subset of  $\dot{\mathcal{I}}_{\gamma+1}$ ;
- (3) if  $\alpha = \gamma + 1$ , then  $\dot{\mathcal{I}}_\alpha \cup \{\dot{a}_\beta: \beta < \alpha\}$  generates a proper maximal ideal.

To start the recursion, using Lemma 1.6, we can choose a proper maximal ideal  $\mathcal{I}_1$  extending the ideal generated by  $\{b_\zeta: \zeta < \lambda\} \cup \{b_\zeta: \zeta < \lambda\}^\perp$ . There is no  $\subset^*$ -bound for  $\{b_\zeta: \zeta < \lambda\}$  in  $\mathcal{I}_1$  since the complement of any such bound is in  $\mathcal{I}_1$ . Very likely  $\{b_\zeta: \zeta < \lambda\}^\perp$  is simply  $[\omega]^{<\omega}$ , so we let  $a_0$  be exceptional and equal the emptyset. We now have our definition (working in the extension by  $Q_0$ )

of  $Q_1 = Q(\{a_0\}; \mathcal{I}_1)$  and the generic set  $\dot{a}_1 = \dot{a}_{Q_1}$  is forced to be almost disjoint from every member of  $\mathcal{I}_1$  (it is a pseudointersection of the ultrafilter dual to  $\mathcal{I}_1$ ). Now assume that  $\alpha < \omega_1$  and that we have defined  $\dot{\mathcal{I}}_\beta$  for all  $\beta < \alpha$ . For the definition of  $\dot{\mathcal{I}}_\alpha$  we break into two cases.

If  $\alpha$  is a limit ordinal, then we define  $\dot{\mathcal{I}}_\alpha$  to be the  $\mathbb{P}_\alpha$ -name of the ideal  $\{\dot{a}_\beta : \beta < \alpha\}^\perp$ . By induction we have for  $\gamma < \alpha$  that  $\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\mathcal{I}}_\gamma \subset \{\dot{a}_\beta : \beta < \alpha\}^\perp = \dot{\mathcal{I}}_\alpha\text{”}$  as required in (1). Condition (2) holds by induction and (3) is vacuous.

In the case that  $\alpha = \gamma + 1$ , we note that, by the genericity of  $\dot{a}_\gamma$  and the induction hypothesis (1),  $\mathbb{P}_\alpha$  forces that the family  $\{\dot{a}_\gamma\} \cup \bigcup\{\dot{\mathcal{I}}_{\eta+1} : \eta < \gamma\}$  generates a proper ideal  $\dot{\mathcal{J}}_\alpha$ . Furthermore, hypotheses (2) (and the definition of  $\dot{a}_\gamma$ ) ensures that  $\mathbb{P}_\alpha$  forces that  $\{b_\zeta : \zeta < \lambda\}$  is  $\subset^*$ -unbounded in  $\dot{\mathcal{J}}_\alpha$ . Again using Lemma 1.6, we can choose  $\dot{\mathcal{J}}'_\alpha$  to be the  $\mathbb{P}_\alpha$ -name of any proper maximal ideal that contains  $\dot{\mathcal{J}}_\alpha \cup (\dot{\mathcal{J}}_\alpha)^\perp$ . Just as in the first inductive step,  $\mathbb{P}_\alpha$  again forces that  $\{b_\zeta : \zeta < \lambda\}$  is  $\subset^*$ -unbounded in  $\dot{\mathcal{J}}'_\alpha$ . The definition of  $\dot{\mathcal{I}}_\alpha$  is the  $\mathbb{P}_\alpha$ -name of  $\{b \in \dot{\mathcal{J}}'_\alpha : b \cap \dot{a}_\gamma =^* \emptyset\}$ . The set  $\dot{a}_\alpha \setminus \dot{a}_\gamma$  will be forced by  $\mathbb{P}_{\alpha+1}$  to be an infinite pseudointersection of the ultrafilter that is dual to  $\dot{\mathcal{J}}'_\alpha$ . For convenience, let  $\dot{y}_\alpha$  denote the  $\mathbb{P}_\alpha$ -name of this ultrafilter.

This completes the definition of the poset  $\mathbb{P}_{\omega_1}$ . Now we establish some additional properties. It will be convenient to observe that for each  $\alpha = \gamma + 1 < \omega_1$  the  $\mathbb{P}_\alpha$ -name for  $\{\omega \setminus (b \cup \dot{a}_\gamma) : b \in \dot{\mathcal{I}}_\alpha\}$  is forced to be a base for  $\dot{y}_\alpha$ . Let  $\dot{A}$  denote a  $\mathbb{P}_{\omega_1}$ -name of the closure in  $\omega^*$  of  $\bigcup\{\dot{a}_\alpha^* : \alpha \in \omega_1\}$ .

**Claim 1.** *For each  $\beta < \alpha < \omega_1$ ,  $\mathbb{P}_{\alpha+1}$  forces that  $\dot{a}_\alpha \setminus \dot{a}_\beta$  is a pseudointersection of the filter  $\dot{y}_{\beta+1}$ .*

PROOF: We prove the claim by induction on  $\alpha \geq \beta + 1$ . For  $\alpha = \beta + 1$ ,  $\dot{a}_\alpha$  is almost disjoint from each member of  $\dot{\mathcal{I}}_\alpha$ , and so  $\dot{a}_\alpha \setminus \dot{a}_\beta$  is almost disjoint from every member of  $\dot{\mathcal{J}}'_\alpha$ . Thus  $\dot{a}_\alpha \setminus \dot{a}_\beta$  is forced to be mod finite contained in every member of the dual filter, namely  $\dot{y}_{\beta+1}$ . Similarly, for  $\alpha > \beta + 1$ ,  $\dot{a}_\alpha$  is forced to be almost disjoint from each member of  $\dot{\mathcal{I}}_\alpha$ . This means that  $\dot{a}_\alpha$  is almost disjoint from each member of  $\dot{\mathcal{I}}_{\beta+1}$ , and so  $\dot{a}_\alpha \setminus \dot{a}_{\beta+1}$  is also almost disjoint from every member of  $\dot{\mathcal{J}}'_{\beta+1}$ .  $\square$

**Claim 2.** *The family  $\{\dot{y}_{\beta+1} : \beta < \omega_1\}$  is a family of  $\mathbb{P}_{\omega_1}$ -names and the union is forced to generate an ultrafilter  $\dot{x}_A$  that is indeed the unique boundary point of  $\dot{A}$ .*

PROOF: Since, for each  $\beta < \alpha < \omega_1$ ,  $\dot{\mathcal{I}}_{\beta+1}$  is contained in  $\dot{\mathcal{I}}_{\alpha+1}$  and  $\dot{a}_\beta$  is forced to be a mod finite subset of  $\dot{a}_\alpha$ , the filter dual to the ideal generated by  $\{\dot{a}_\beta\} \cup \dot{\mathcal{I}}_{\beta+1}$  is contained in the filter dual to the ideal generated by  $\{\dot{a}_\alpha\} \cup \dot{\mathcal{I}}_{\alpha+1}$ . This implies that  $\mathbb{P}_{\omega_1}$  forces that  $\bigcup\{\dot{y}_{\beta+1} : \beta < \omega_1\}$  is a filter. Furthermore, since  $\mathbb{P}_{\omega_1}$  is ccc, every  $\mathbb{P}_{\omega_1}$ -name of a subset of  $\omega$  is forced to be equal to a  $\mathbb{P}_\beta$ -name for some  $\beta < \omega_1$ . Since  $\mathbb{P}_{\beta+1}$  forces that  $\dot{y}_{\beta+1}$  is an ultrafilter, this shows that  $\mathbb{P}_{\omega_1}$  forces that  $\dot{x}_A$  is an ultrafilter. Finally, it follows from the previous claim and Lemma 2.4 that  $\dot{x}_A$  is the unique boundary point of  $\dot{A}$ .  $\square$

We now verify the final property required of the construction.

**Claim 3.** *Let  $\mathbb{P}_{\omega_1}$  are forces that for each  $U \in \dot{x}_{\dot{A}}$  and each  $\zeta < \lambda$ , there is an  $\eta < \lambda$  such that  $U \cap (b_\eta \setminus b_\zeta)$  is infinite.*

PROOF: Let  $\dot{U}$  be any  $\mathbb{P}_{\omega_1}$ -name and assume that  $p \in \mathbb{P}_{\omega_1}$  forces that  $\dot{U} \in \dot{x}_{\dot{A}}$ . Since  $P_{\omega_1}$  is ccc and  $p \Vdash_{\mathbb{P}_{\omega_1}} \text{“}\dot{U} \subset \omega\text{”}$ , we may suppose there is an  $\alpha < \omega_1$  such that  $p \in \mathbb{P}_{\alpha+1}$  and  $\dot{U}$  is a  $\mathbb{P}_{\alpha+1}$ -name. By Claim 2,  $p \Vdash_{\mathbb{P}_{\alpha+1}} \text{“}\dot{U} \in \dot{y}_{\alpha+1}\text{”}$ . Let  $\dot{W}$  be the  $\mathbb{P}_{\alpha+1}$ -name of  $\omega \setminus (\dot{U} \cup \dot{a}_\alpha)$ , and note that  $p \Vdash_{\mathbb{P}_{\alpha+1}} \text{“}\dot{W} \in \dot{z}_{\alpha+1}\text{”}$ . Therefore, by induction hypothesis (2),  $p \Vdash_{\mathbb{P}_{\alpha+1}} \text{“}(\exists \eta < \lambda) b_\eta \setminus (b_\zeta \cup \dot{W}) \neq^* \emptyset\text{”}$ . Since  $\mathbb{P}_{\alpha+1}$  is ccc and  $\lambda$  has uncountable cofinality, we may choose an  $\eta < \lambda$  such that  $p \Vdash_{\mathbb{P}_{\alpha+1}} \text{“}b_\eta \setminus (b_\zeta \cup \dot{W}) \neq^* \emptyset\text{”}$ . Finally we have that  $p \Vdash_{\mathbb{P}_{\alpha+1}} \text{“}(b_\eta \setminus b_\zeta) \cap \dot{U} \neq^* \emptyset\text{”}$  since  $p \Vdash_{\mathbb{P}_{\alpha+1}} \text{“}\dot{a}_\alpha \cap b_\eta =^* \emptyset\text{”}$ .  $\square$

This completes the proof of Lemma 3.9.  $\square$

Now we return to the complete proof of Theorem 2.7.

**Definition 3.12.** We say that  $\mathcal{A}$  is a pre-ccc sequence if  $\mathcal{A} = \{a_\beta : \beta < \alpha\} \subset [\omega]^{\aleph_0}$  is  $\subset^*$ -increasing and for each sequence  $\{b_\xi : \xi \in \omega_1\} \subset \mathcal{A}^\perp$ , and sequence  $\{\gamma_\xi : \xi \in \omega_1\} \subset \alpha$  there are  $\xi < \eta$  such that either  $a_{\gamma_\xi} \cap b_\xi \neq \emptyset$  or  $(a_{\gamma_\xi} \cup a_{\gamma_\eta}) \cap (b_\xi \cup b_\eta) = \emptyset$ .

**Lemma 3.13.** *If  $\mathcal{A}$  is a pre-ccc sequence, then  $Q(\mathcal{A}; \mathcal{I})$  is ccc for any ideal  $\mathcal{I} \subset \mathcal{A}^\perp$ .*

PROOF: Let  $\mathcal{A} = \{a_\beta : \beta < \alpha\} \subset [\omega]^{\aleph_0}$  be  $\subset^*$ -increasing. Let  $\{q_\xi : \xi \in \omega_1\} \subset Q = Q(\mathcal{A}; \mathcal{I})$ . By passing to a subcollection we can suppose there is a single  $F \in [\omega]^{<\aleph_0}$  such that  $F_{q_\xi} = F$  for all  $\xi$ . For each  $\xi$ , let  $b_\xi = b_{q_\xi}$ ,  $\sigma_\xi = \sigma_{q_\xi}$ , and  $H_\xi = \text{dom}(\sigma_\xi)$ . By passing to an uncountable subcollection, we can assume that the family  $\{H_\xi : \xi \in \omega_1\}$  is a  $\Delta$ -system with root  $H$ . Similarly, we can assume that for all  $\xi, \eta$ ,  $\sigma_\xi \upharpoonright H = \sigma_\eta \upharpoonright H$ . For each  $\xi$ , let  $\gamma_\xi$  be the maximum element of  $H_\xi$ .

Next, we choose an integer  $\bar{m}$  sufficiently large so that there is again an uncountable  $I \subset \omega_1$  and a subset  $\bar{b}$  of  $\bar{m}$  such that for all  $\xi \in I$  and all  $\beta \in H_\xi$   $\sigma_\xi(\beta) < \bar{m}$ ,  $a_\beta \setminus \bar{m} \subset a_{\gamma_\xi}$ , and  $b_\xi \cap \bar{m} = \bar{b}$ . Now we apply the pre-ccc property for the family  $\{\gamma_\xi : \xi \in I\}$  and the sequence  $\{b_\xi \setminus \bar{m} : \xi \in I\}$ . Since  $a_{\gamma_\xi}$  is disjoint from  $b_\xi \setminus \bar{m}$  for all  $\xi \in I$ , there must be  $\xi < \eta$  from  $I$  so that  $(a_{\gamma_\xi} \cup a_{\gamma_\eta})$  is disjoint from  $((b_\xi \cup b_\eta) \setminus \bar{m})$ .

We claim that  $r = (F, \sigma_\xi \cup \sigma_\eta, b_\xi \cup b_\eta)$  is in  $Q$  and is an extension of each of  $q_\xi$  and  $q_\eta$ . It suffices to prove that for  $\beta \in H_\xi$ ,  $a_\beta \setminus \sigma_\xi(\beta)$  is disjoint from  $b_\eta$ , and similarly, that  $a_\beta \setminus \sigma_\eta(\beta)$  is disjoint from  $b_\xi$  for  $\beta \in H_\eta$ . Since  $b_\xi \cap \bar{m} = b_\eta \cap \bar{m} = \bar{b}$ , it suffices to consider  $a_\beta \setminus \bar{m}$  in each case. For  $\beta \in H_\xi$ , we have  $a_\beta \setminus \bar{m} \subset a_{\gamma_\xi}$  and  $a_{\gamma_\xi}$  is disjoint from  $b_\eta \setminus \bar{m}$ . For  $\beta \in H_\eta$ , we similarly have that  $a_\beta \setminus \bar{m} \subset a_{\gamma_\eta}$  and  $a_{\gamma_\eta}$  is disjoint from  $b_\xi \setminus \bar{m}$ .  $\square$

For the remainder of the section,  $\kappa < \lambda$  are regular cardinals with  $\mathfrak{c} \leq \lambda$  (as in Theorem 2.7).

**Definition 3.14.**  $\mathfrak{A}$  is the class of triples  $(\mathfrak{P}, \mathcal{A}, \mathcal{J})$  such that, there is an ordinal  $0 < \alpha \leq \kappa$ , and the following holds for each  $\gamma < \beta < \alpha$ :

- (1)  $\mathfrak{P} = (\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{Q}_\beta : \beta < \alpha \rangle)$  is a finite support iteration sequence of ccc posets, and has cardinality  $\lambda$ ;
- (2)  $\mathcal{A}$  is an  $\alpha$ -sequence  $\{\dot{a}_\beta : \beta < \alpha\}$  and  $\dot{a}_0 = \emptyset$ ;
- (3)  $\mathcal{J}$  is an  $\alpha$ -sequence  $\{\dot{I}_\beta : \beta < \alpha\}$ ;
- (4)  $\dot{a}_\beta$  is a  $\mathbb{P}_{\beta+1}$ -name, and  $\Vdash_{\mathbb{P}_{\beta+1}} \text{“}\dot{a}_\gamma \subset^* \dot{a}_\beta \subset \omega\text{”}$ ;
- (5)  $\mathbb{P}_1$  forces that  $\dot{I}_1$  is an ideal on  $\omega$  that has a cofinal  $\subset^*$ -increasing chain of type  $\lambda$ ;
- (6)  $\dot{I}_\beta$  is a  $\mathbb{P}_\beta$ -name, and  $\Vdash_{\mathbb{P}_\beta} \text{“}\dot{I}_\beta \subset \{\dot{a}_\xi : \xi < \beta\}^\perp$  is an ideal”;
- (7)  $\Vdash_{\mathbb{P}_\beta} \text{“}\dot{I}_{\gamma+1} \subset \dot{I}_\beta\text{”}$ , and  $\Vdash_{\mathbb{P}_\beta} \text{“}\dot{I}_1$  is  $\subset^*$ -unbounded in  $\dot{I}_{\gamma+1}\text{”}$ ;
- (8)  $\Vdash_{\mathbb{P}_\beta} \text{“}\dot{Q}_\beta = Q(\{\dot{a}_\xi : \xi < \beta\}; \dot{I}_\beta)\text{”}$ ;
- (9)  $\mathbb{P}_{\gamma+1}$  forces that  $\{\dot{a}_\gamma\} \cup \dot{I}_{\gamma+1}$  generates a maximal ideal;
- (10)  $\dot{a}_\beta$  is the  $\mathbb{P}_{\beta+1}$ -name for  $\dot{a}_{Q(\{\dot{a}_\xi : \xi < \beta\}; \dot{I}_\beta)}$ ;
- (11) if  $\text{cf}(\beta) = \omega$ , then  $\Vdash_{\mathbb{P}_\beta} \text{“}\dot{I}_\beta = \bigcup \{\dot{I}_{\xi+1} : \xi < \beta\}\text{”}$ .

We omit the routine modifications of the proof of Lemma 3.11 needed to prove the following.

**Lemma 3.15.** *If  $(\mathfrak{P}, \mathcal{A}, \mathcal{J}) \in \mathfrak{A}$  and  $\mathfrak{P} = (\langle \mathbb{P}_\beta : \beta \leq \kappa \rangle, \langle \dot{Q}_\beta : \beta < \kappa \rangle)$ , and  $\mathcal{A} = \{\dot{a}_\beta : \beta < \kappa\}$ , then  $\mathbb{P}_\kappa$  is a ccc poset that forces that the closure  $A$  of  $\bigcup \{\dot{a}_\beta^* : \beta < \kappa\}$  is a simple almost clopen set of type  $\kappa$  and that there is a strongly discrete free  $\lambda$ -sequence converging to  $x_A$ .*

To finish the proof of Theorem 2.7 we have to prove there is an element of  $\mathfrak{A}$  of length  $\kappa$ . Towards doing so, we let  $<_{\mathfrak{A}}$  denote the obvious coordinatewise ordering on the elements of  $\mathfrak{A}$ . Using Zorn's lemma, we let  $\mathcal{C} \subset \mathfrak{A}$  be a maximal  $<_{\mathfrak{A}}$ -chain. It is easily checked that the maximality of  $\mathcal{C}$  implies that it has a maximal element which we denote

$$(\mathfrak{P}, \mathcal{A}, \mathcal{J}) = (\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{Q}_\beta : \beta < \alpha \rangle, \{\dot{a}_\beta : \beta < \alpha\}, \{\dot{I}_\beta : \beta < \alpha\})$$

and we prove that  $\alpha = \kappa$ .

**Lemma 3.16.** *If  $\alpha < \kappa$  then  $(\mathfrak{P}, \mathcal{A}, \mathcal{J})$  is not maximal.*

PROOF: In order to extend  $(\mathfrak{P}, \mathcal{A}, \mathcal{J})$  we have to define a  $\mathbb{P}_\alpha$ -name  $\dot{I}_\alpha$  and to then let  $\dot{Q}_\alpha$  be the  $\mathbb{P}_\alpha$ -name of  $Q(\mathcal{A}; \dot{I}_\alpha)$ . We then set  $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{Q}_\alpha$  and let  $\dot{a}_\alpha$  be the  $\mathbb{P}_{\alpha+1}$ -name for  $\dot{a}_{Q(\mathcal{A}; \dot{I}_\alpha)}$ . We must show that  $(\mathfrak{P} \frown \{(\langle \mathbb{P}_{\alpha+1} \rangle, \langle \dot{Q}_\alpha \rangle)\}, \mathcal{A} \frown \{\dot{a}_\alpha\}, \mathcal{J} \frown \{\dot{I}_\alpha\})$  is in  $\mathfrak{A}$  to complete the proof. We defer to the end of the proof that our choice of  $\dot{I}_\alpha$  will result in the fact that  $\mathbb{P}_\alpha$  forces that  $Q(\mathcal{A}; \dot{I}_\alpha)$  is ccc.

First we assume that  $\alpha$  is a limit ordinal and explain the construction of  $\dot{I}_\alpha$ . If  $\alpha$  has uncountable cofinality, we let  $\dot{I}_\alpha$  simply be the  $\mathbb{P}_\alpha$ -name for  $\mathcal{A}^\perp$ . If  $\alpha$  has countable cofinality, then define  $\dot{I}_\alpha = \bigcup \{\dot{I}_{\xi+1} : \xi < \alpha\}$  as required in item (11). These definitions ensure that items (2)–(11) in the definition of  $\mathfrak{A}$  are fulfilled.

In the case that  $\alpha$  is a successor, let  $\alpha = \delta + 1$ . We proceed exactly as in the successor case in the proof of Lemma 3.11. By property (7) of  $\mathfrak{A}$ ,  $\mathbb{P}_\alpha$  forces that  $\dot{\mathcal{I}}_1$  is unbounded in  $\dot{\mathcal{I}}_{\gamma+1}$  for all  $\gamma < \delta$ . In this case,  $\mathcal{A}$  has a maximal element  $\dot{a}_\alpha$  and we follow the steps in Lemma 3.11 to define a  $\mathbb{P}_\alpha$ -name of an ideal  $\dot{\mathcal{I}}_\alpha$  that is forced to contain for each  $\gamma < \delta$ ,  $\dot{\mathcal{I}}_{\gamma+1}$  as an  $\subset^*$ -unbounded subset, and in such a way that every member is almost disjoint from  $\dot{a}_\alpha$  and the ideal generated by  $\{\dot{a}_\alpha\} \cup \dot{\mathcal{I}}_\alpha$  is a maximal ideal. This ensures that properties (2)–(11) of  $\mathfrak{A}$  will hold in this successor case.

Now to finish the proof, we have to prove that  $\mathbb{P}_\alpha$  forces that  $Q(\mathcal{A}; \dot{\mathcal{I}}_\alpha)$  is ccc. By Lemma 3.13, it suffices to prove that  $\mathbb{P}_\alpha$  forces that  $\mathcal{A}$  is a pre-ccc sequence. To prove this we consider, as in the definition of pre-ccc,  $\mathbb{P}_\alpha$ -names  $\{\dot{\gamma}_\xi: \xi \in \omega_1\}$  and  $\{\dot{b}_\xi: \xi \in \omega_1\}$  such that there is some  $p_0 \in \mathbb{P}_\alpha$  forcing that for each  $\xi < \omega_1$ ,  $\dot{\gamma}_\xi \in \alpha$ ,  $\dot{b}_\xi \in \dot{\mathcal{A}}^\perp$  and  $\dot{a}_{\gamma_\xi} \cap \dot{b}_\xi$  is empty.

Assume first that, by possibly passing to an extension of  $p_0$  and to an uncountable subsequence, there is some  $\beta < \alpha$  such that  $p_0$  forces that  $\dot{\gamma}_\xi < \beta$  for all  $\xi < \omega_1$ . For each  $0 < \xi < \omega_1$  choose  $p_\xi < p_0$ , an integer  $m_\xi$  and a  $\gamma_\xi < \beta$  so that  $a_{\gamma_\xi} \setminus a_\beta \subset m_\xi$  and  $p_\xi \Vdash_{\mathbb{P}_\alpha}$  “ $\dot{\gamma}_\xi = \gamma_\xi$  and  $a_\beta \cap \dot{b}_\xi \subset m_\xi$ ”. We may also assume that there is some  $d_\xi \subset m_\xi$  such that  $p_\xi \Vdash_{\mathbb{P}_\alpha}$  “ $\dot{b}_\xi \cap m_\xi = d_\xi$ ”. There is a pair  $m, d$  so that  $S = \{\xi: m_\xi = m \text{ and } d_\xi = d\}$  is uncountable. Since  $\mathbb{P}_\alpha$  is ccc, we may choose  $\xi < \eta$  from  $S$  so that  $p_\xi$  and  $p_\eta$  are compatible. Let  $q \in \mathbb{P}_\alpha$  be any common extension of  $p_\xi$  and  $p_\eta$ . Note that  $q$  forces that  $\dot{a}_{\gamma_\xi} \cup \dot{a}_{\gamma_\eta}$  is contained in  $a_\beta \cup m$  and that  $(\dot{b}_\xi \cup \dot{b}_\eta) \setminus m$  is disjoint from  $a_\beta$ . Since  $q$  also forces that  $\dot{b}_\xi \cap m = \dot{b}_\eta \cap m$ , it should now be clear that  $q$  forces that  $\dot{a}_{\gamma_\xi} \cup \dot{a}_{\gamma_\eta}$  is disjoint from  $\dot{b}_\xi \cup \dot{b}_\eta$ . This proves that  $p_0$  does not force this to be a violation of the pre-ccc property.

Now we may suppose that  $p_0$  forces that  $\{\dot{\gamma}_\xi: \xi \in \omega_1\}$  is strictly increasing and cofinal in  $\alpha$ . We may assume that  $p_0$  decides the value,  $\gamma_0$ , of  $\dot{\gamma}_0$ . For each  $\xi < \omega_1$ , choose any  $p_\xi < p_0$  that decides a value,  $\gamma_\xi$ , of  $\dot{\gamma}_\xi$  and that  $\dot{b}_\xi$  is a  $\mathbb{P}_\beta$ -name for some  $\beta \in \text{supp}(p_\xi)$ . Let  $g$  be a continuous strictly increasing function from  $\omega_1$  into  $\alpha$  with cofinal range. Since  $\mathbb{P}_\alpha$  is ccc we have, for each  $\delta \in \omega_1$ , the set  $\{\xi: \gamma_\xi < g(\delta)\}$  is countable. Therefore there is a cub  $C \subset \omega_1$  such that for all  $\delta \in C$ ,  $g(\delta) \leq \gamma_\delta$  and  $g(\delta)$  is a limit ordinal with countable cofinality. We may also arrange that for each  $\delta \in C$  and  $\xi < \delta$ ,  $\text{supp}(p_\xi) \subset g(\delta)$ .

For each  $\delta \in C$ , we may extend  $p_\delta$  so as to ensure that each of  $g(\delta)$  and  $\gamma_\delta$  are in  $\text{supp}(p_\delta)$  and such that there is a  $\beta \in \text{supp}(p_\delta)$  such that  $\dot{b}_\delta$  is a  $\mathbb{P}_\beta$ -name. We also extend each  $p_\delta$  so that we can arrange a list of special properties (referred to as “determined” in many similar constructions). Specifically, for each  $\beta \in \text{supp}(p_\delta)$

- (1) there are  $F_\beta^\delta \in [\omega]^{<\aleph_0}$ ,  $H_\beta^\delta \in [\beta]^{<\aleph_0}$ ,  $\sigma_\beta^\delta: H_\beta^\delta \rightarrow \omega$ , and a  $\mathbb{P}_\beta$ -name  $\dot{b}_\beta^\delta$  such that  $p_\delta \upharpoonright \beta \Vdash_{\mathbb{P}_\beta}$  “ $p_\delta(\beta) = (F_\beta^\delta, \sigma_\beta^\delta, \dot{b}_\beta^\delta)$ ”;
- (2) if  $g(\delta) < \gamma_\delta$ , then  $g(\delta) \in H_{\gamma_\delta}^\delta$ ;
- (3)  $H_\beta^\delta \subset \text{supp}(p_\delta)$ ;

- (4) if  $\beta$  is a limit with countable cofinality, then there is a  $\mu_\beta^\delta < \beta$  such that  $\dot{b}_\beta^\delta$  is a  $\mathbb{P}_{\mu_\beta^\delta}$ -name and  $\text{supp}(p_\delta) \cap \beta \subset \mu_\beta^\delta$ ;
- (5) if  $\iota < \beta$  is in  $\text{supp}(p_\delta)$ , then  $H_\iota^\delta \supset H_\beta^\delta \cap \iota$ .

Let us note that, by our assumption on  $C$ , we have defined  $\mu_{g(\delta)}^\delta < g(\delta)$ , as in (4) for all  $\delta \in C$ . By the pressing down lemma, there is a stationary set  $S \subset C$  and a  $\mu < \alpha$  such that  $\mu_{g(\delta)}^\delta < \mu$  for all  $\delta \in S$  and so  $g(\delta)$  is the minimum element of  $\text{supp}(p_\delta) \setminus \mu$ . Since  $\mathbb{P}_{\mu+1}$  is ccc, it follows by a standard argument that there is a  $\bar{p} \in \mathbb{P}_{\mu+1}$  with the property that for all cub  $\tilde{C} \subset C$ , the set  $\{p_\xi \upharpoonright \mu : \xi \in S \cap \tilde{C}\}$  is pre-dense below  $\bar{p}$ . Pick such a  $\bar{p}$  and let  $\bar{p} \in G_{\mu+1}$  be  $\mathbb{P}_{\mu+1}$ -generic filter. Now we work in  $V[G_{\mu+1}]$ . In effect,  $\bar{p}$  has forced that  $\{\xi \in S : p_\xi \upharpoonright \mu \in G_{\mu+1}\}$  is stationary. By passing to a stationary subset  $S_1$  of this set, we may also arrange that the values of the pair  $\{F_{g(\xi)}^\xi, F_{\gamma_\xi}^\xi\}$  is the same for all  $\xi \in S_1$ . For all  $\beta \in \mu+1$ , we let  $a_\beta$  denote the valuation of  $\dot{a}_\beta$  by  $G_{\mu+1}$ . By further shrinking  $S_1$  we may suppose there is an  $\bar{m} \in \omega$  and a  $\bar{b} \subset \bar{m}$ , satisfying that for all  $\delta \in S_1$

- (1) for all  $\beta \in \text{supp}(p_\delta)$ ,  $F_\beta^\delta \subset \bar{m}$ , and for all  $\iota \in H_\beta^\delta$ ,  $\sigma_\beta^\delta(\iota) \subset \bar{m}$ ;
- (2) for all  $\beta \in \text{supp}(p_\delta) \cap \mu$ ,  $a_\iota \setminus a_\mu \subset \bar{m}$ ;
- (3)  $\bar{b} = \bar{m} \cap b_{g(\delta)}^\delta$ , where  $b_{g(\delta)}^\delta$  is the valuation of  $\dot{b}_{g(\delta)}^\delta$  by  $G_{\mu+1}$ ;
- (4)  $b_{g(\delta)}^\delta \cap a_\mu \subset \bar{m}$ .

Fix any  $\xi < \eta$  from  $S_1$ . Define  $q_\xi$  so that  $\text{supp}(q_\xi) = \text{supp}(p_\xi) \setminus \mu + 1$ , and for  $\beta \in \text{supp}(q_\xi)$ ,

$$q_\xi(\beta) = \begin{cases} (F_{g(\xi)}^\xi, \sigma_{g(\xi)}^\xi \cup \{(\mu, \bar{m})\}, \dot{b}_{g(\xi)}^\xi \cup \dot{b}_{g(\eta)}^\eta) & \text{if } \beta = g(\xi), \\ (F_\beta^\xi, \sigma_\beta^\xi, \dot{b}_\beta^\xi \cup b_{g(\eta)}^\eta \setminus \bar{m}) & \text{if } g(\xi) < \beta. \end{cases}$$

We prove by induction on  $\beta \in \text{supp}(q_\xi)$ , that there is a condition  $r_\beta^\xi \in G_{\mu+1}$  such that  $r_\beta^\xi \cup (q_\xi \upharpoonright \beta + 1) \leq p_\xi \upharpoonright \beta + 1$ . Evidently, for the case  $\beta = g(\xi)$ ,  $F_{g(\xi)}^\xi$  and  $\bar{m} \cap a_\iota$  are disjoint from  $\bar{b}$  and so there is some condition in  $G_{\mu+1}$  that forces that, they are disjoint from  $\dot{b}_{g(\eta)}^\eta$ . Similarly, for  $\iota \in H_{g(\xi)}^\xi$ ,  $a_\iota \setminus \bar{m} \subset a_\mu$ , and since  $a_\mu \setminus \bar{m} \cap (b_{g(\xi)}^\xi \cup b_{g(\eta)}^\eta)$  is empty, there is a condition  $r$  in  $G_{\mu+1}$  that forces that  $q_\xi(g(\xi)) \in \dot{Q}_{g(\xi)}$  and that  $q_\xi(g(\xi)) < p_\xi(g(\xi))$ . In addition,  $r \cup q_\xi \upharpoonright g(\xi) + 1$  forces that  $\dot{a}_{g(\xi)}$  is disjoint from  $b_{g(\eta)}^\eta \setminus \bar{m}$ . Now, suppose that  $g(\xi) < \beta \in \text{supp}(p_\xi)$ , and that  $r \cup q \upharpoonright \beta$  is a condition in  $\mathbb{P}_\beta$  that is below  $p_\xi \upharpoonright \beta$ . We recall that  $H_\beta^\xi \subset \text{supp}(p_\xi)$ , and so it follows that  $r \cup q_\xi \upharpoonright \beta$  forces that  $\dot{a}_\iota$  is disjoint from  $b_{g(\eta)}^\eta \setminus \bar{m}$  for all  $\iota \in H_\beta^\xi$ . This is the only thing that needs verifying when checking that  $r \cup q_\xi \upharpoonright \beta + 1 < p_\xi \upharpoonright \beta + 1$ .

Now that we have that  $r \cup q_\xi$  forces that  $\dot{a}_{\gamma_\xi}$  is disjoint from  $b_{g(\eta)}^\eta \setminus \bar{m}$ , we can add  $\{(\gamma_\xi, \bar{m})\}$  to  $\sigma_{g(\eta)}^\eta$  and still have a condition. Similarly, for all  $\iota \in \text{supp}(p_\eta) \cap \mu$ ,  $a_\iota \setminus \bar{m}$  is contained in  $a_\mu$ , and  $r \cup q_\xi$  forces that  $a_\mu \setminus \bar{m}$ , being a subset of  $\dot{a}_{\gamma_\xi}$ , is disjoint from  $\dot{b}_\xi$ . This implies that  $r \cup q_\xi$  forces that  $(F_{g(\eta)}^\eta, \sigma_{g(\eta)}^\eta \cup \{(\gamma_\xi, \bar{m})\}, b_{g(\eta)}^\eta) \cup$

$(\dot{b}_\xi \setminus \overline{m})$  is a condition in  $\dot{Q}_{g(\eta)}$  and is less than  $p_\eta(g(\eta))$ . Now we define a condition  $q_\eta$  so that  $\text{supp}(q_\eta) = \text{supp}(p_\eta) \setminus \mu + 1$ , and for  $\beta \in \text{supp}(q_\eta)$ ,

$$q_\eta(\beta) = \begin{cases} (F_{g(\eta)}^\eta, \sigma_{g(\eta)}^\eta \cup \{(\gamma_\xi, \overline{m})\}, \dot{b}_{g(\eta)}^\eta \cup \dot{b}_{\gamma_\xi}) & \text{if } \beta = g(\xi), \\ (F_\beta^\eta, \sigma_\beta^\eta, \dot{b}_\beta^\eta \cup (\dot{b}_{\gamma_\xi} \setminus \overline{m})) & \text{if } g(\eta) < \beta. \end{cases}$$

It again follows, by induction on  $\beta \in \text{supp}(q_\eta)$ , that  $r \cup q_\xi$  forces that  $r \cup q_\xi \cup (q_\eta \upharpoonright \beta + 1)$  is a condition in  $\mathbb{P}_{\beta+1}$  and is below  $p_\eta \upharpoonright \beta + 1$ . Finally, we observe that  $r \cup q_\xi \cup q_\eta$  forces that  $\dot{a}_{\gamma_\xi} \subset \dot{a}_{\gamma_\eta}$  because it forces that  $\dot{a}_{\gamma_\xi} \cap \overline{m} = \dot{a}_{\gamma_\eta} \cap \overline{m}$  and that  $\dot{a}_{\gamma_\xi} \setminus \overline{m} \subset \dot{a}_{g(\eta)} \setminus \overline{m} \subset \dot{a}_{\gamma_\eta}$ . Similarly  $r \cup q_\xi \cup q_\eta$  forces that  $\dot{a}_{\gamma_\eta}$  is disjoint from  $\dot{b}_\xi$  because  $\dot{b}_\xi \cap \overline{m} = \dot{b}$  and  $\dot{a}_{\gamma_\eta}$  is disjoint from  $\dot{b}_\xi \setminus \overline{m}$ . This completes the proof that  $\mathcal{A}$  is pre-ccc and the proof of the Lemma 3.16.  $\square$

This finishes the proof of the main theorem, Theorem 2.7.  $\square$

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