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THE LAZY MODEL-THEORETICIAN'S GUIDE TO STABILITY

by

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§ 0. *INTRODUCTION*

The main aim of this article is to make propaganda for [S 1]; hence novelty is not the intention; there are many explanations, definitions and theorems and few proofs. No previous knowledge of stability is required, but then you have to take some statements on faith. We have in mind mainly those who are interested in algebraically-minded model theory, i.e. in generic models, the class of e-closed (= existentially closed) models and universal-homogeneous models rather than elementary classes and saturated models. So our main point is that though stability theory was developed for the latter context, almost everything goes through in the wider context (with suitable changes in the definitions). We also examine some specific algebraic theories (1). An exact list of the theorems from [S 1] which generalize or do not generalize, will be included in the Ph. D. thesis of M. Abramsky.

At this stage I will define stability theory as an attempt to give a classification of, and structure/non-structure theorems for elementary classes, and other related classes. An ideal *structure theorem* is a characterization (up to isomorphism) of

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(1) It seems that we get new information on the number of modules over a fixed ring (see 8.7).

each model in the class, by invariants, which are cardinals or sets of cardinals etc. (e.g. the dimension for algebraically closed fields, or the Ulm theorem for countable torsion groups (see [Fu]), or the theory of one equivalence relation (a model $M = (|M|, E)$ is characterized by the function f, $f(\lambda) = |\{a/E: |a/E|=\lambda\}|$).

An ideal *structure/non-structure theorem* is the characterization of the classes which have a structure theorem, together with a proof of the complexity of the other classes. More explicitly the problem presents itself as determining the possible functions $I(\lambda, K)$ = the number of models $M \in K$, of cardinality λ up to isomorphism. We have now a complete solution only for the family of the classes of $\aleph_\varepsilon$-saturated [2] models of a complete theory (the theorem essentially says that any T which does not satisfy $I(\lambda, T) = 2^\lambda$, is like some T_α; T_α has equivalence relations E_i $(i < \alpha)$ where E_i refines E_j when $i < j$). Notice that the question may degenerate when $2^{\aleph_\alpha} = 2^{|\alpha|} + 2^{\aleph_0}$, which may hold for every $\aleph_\alpha$.

We consider a classification meaningful if (but not only if) it helps in proving theorems not mentioning it, or has applications or appears in a characterization .For example $\aleph_0$-stability helps in proving Morley's categoricity theorem and also in dealing with the differential closure of a differential field of characteristic zero. Stability helps to investigate Keisler's order ([K 1], [S 4]), to prove the uniqueness of prime differential closure of fields (8.7), to characterize $\{\lambda : K$ has a homogeneous-universal [saturated] model in $\lambda\}$ and so forth.

Notice, that it is usually not hard to find properties implying complexity or simplicity; the point is to have both. For example $\aleph_0$-stability is a very strong property; but there are theo-

(2) This notion is slightly stronger than $\aleph_0$-saturated. It means that every type almost over a finite set is realized, see § 6.

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ries T which are not $\aleph_0$-stable, but $I(\lambda, T) \leqslant 2^{2^{\aleph_0}}$, and have a structure theorem (the example is due to Morley). Hence we use superstability or stability more than $\aleph_0$-stability. When we deal with $|T|^+$-saturated models stability is more interesting; but for elementary classes superstability is better. By § 8 we can have a structure theorem for the class of R-modules, iff the theory of the class is superstable; but if we assume $|R|^+$-saturation, or compactness of appropriate kind, we can have it in general, as all modules are stable.

A *non-structure theorem* is a little vague; specifically we are proving $I(\lambda, K) = 2^\lambda$. However this does not *a priori* exclude the existence of a complete set of invariants. The main non-structure theorem of [S 1] is for K = Mod T when T is not superstable and it is presented in [S 3]. However from this proof it follows that we should use invariants of the form A/D_λ (where λ is regular, $A \subseteq \lambda$, D_λ the filter on λ generated by the closed unbounded subsets of λ) and even this is not sufficient (e.g. when V = L) (explained in [S 2] for separable reduced p-groups).

Eklof [Ek] devised a more rigorous version of the thesis that there is no structure theorem here, by Feferman-Sabbagh generalized κ-functors. Note that in general, the $L_{\infty,\lambda}$-theory of models (of cardinality λ) does not characterize them up to isomorphism when T is unsuperstable, $\lambda > \aleph_0$ is regular (3); and if $\mu < \kappa(T)$ (μ regular) also $L_{\infty,\lambda, G(\mu)}$ is not sufficient ($G(\mu)$ is the game quantifier $(\ldots \forall x_\alpha \exists y_\alpha \ldots)_{\alpha<\mu}$).

We can apply the non-structure theorem itself; but usually we have to check directly that the appropriate construction works. This is done in [S 2] for reduced separable p-groups and in [MS] for universal locally finite groups which are not ele-

(3) Added in Proof June 76: Meanwhile it was proved also for singular $\lambda = \lambda^{\aleph_0}$.

mentary classes). Usually we get the existence of 2^λ models, no one embeddable in another (the kind of embedding is determined by the kind of formulas exhibiting nonsuperstability). The problem of many non-isomorphic models is closely related to the existence of *rigid models*; [M is rigid iff $a \neq b \in M \Rightarrow (M, a) \not\cong (M, b)$] so we can apply the method to (complete) order and (complete) Boolean algebras (see [S 3]) (improving previous partial results [Jn] [Lz] [Mc] [Mm]). Another related problem is indecomposability. Here unsuperstability of the theory of modules over a ring R gives us for each regular $\lambda > |R|$, an R-module M of cardinality λ which is not the free sum of modules of smaller cardinality (see 8.6).

We could have looked at the problem differently: instead of building models from sets, i.e. cardinals, we could build then from ordered sets; e.g. to characterize a real closed field by its order. Unfortunately we know essentially nothing in this direction. We can reinterpret the problem as classifying the unstable theories by complexity. The concrete problem we have is the classification of countable (first-order) theories by Keisler's order (see [K 1], [S 4]).

Here the theory of linear order is maximal ([S 1], VI), the stable theories are classified ([S 4]), and the problem is to classify the unstable theories without the strict order property; (see [S 5] § 4, [S 1] II § 4, III § 7, VI). There are at least two classes, and it is clear they should have some positive properties exemplifying their simplicity.

Another related problem is e.g. categoricity over a predicate (see [G 1], [G 2]; this is quite complicated, (see [S 6]).

Notation: L will be a first-order language; T a first-order theory in L; φ, ψ, ϑ formulas; $\varphi(\bar{x})$ means all free variables of φ are in $\bar{x}$.
We identify L with its set of first-order formulas.

Let Δ, Φ, Ψ be sets of formulas $\varphi(\bar{x})$. Let $\wedge \Phi = \{\bigwedge_{i=1}^{n} \varphi_i : \varphi_i \in \Phi\}$ etc., and e.g. $\exists \wedge \Phi = \exists(\wedge \Phi)$. Sub Φ is the set of subformulas of Φ, including the formulas of Φ. M, N are models, $|M|$ the universe of M, a, b, c, d elements, $\bar{a}, \ldots$ finite sequences. We write $\bar{a} \in A$ instead of Range $(\bar{a}) \subseteq A$; do not differentiate between $\bar{a}$ and its range, and write $\bar{a} \in M$ instead of $\bar{a} \in |M|$.

Let ${}^{\alpha}\lambda = \{f : f : \alpha \rightarrow \lambda\}$, ${}^{\alpha>}\lambda = \bigcup_{\beta<\alpha} {}^{\beta}\lambda$, $\lambda^{<\varkappa} = \sum_{0 \leq \mu < \varkappa} \lambda^{\mu}$

Sketch Proof of the Nonstructure Theorem (λ regular)

First Approximation: Unsuperstability gives us many types p_η over a small set. In fact for each λ there is a tree ${}^{\omega}\lambda$, points $\{a_\nu : \nu \in {}^{\omega>}\lambda\}$, $\{b_\eta : \eta \in {}^{\omega}\lambda\}$ and formulas φ_n $(n < \omega)$ such that for each n the formulas $\{\varphi_n(x, a_\nu) : \nu \in {}^{n}\lambda\}$ are pairwise inconsistent but $\models \varphi_n(b_\eta ; a_{\eta\restriction n})$ $(\eta \in {}^{\omega}\lambda)$. (we have a_η and not $\bar{a}_\eta$, for notational simplicity only).

The types $p_\eta = \{\varphi_n(\bar{x}, \bar{a}_{\eta\restriction n}) : n < \omega\}$ $(\eta \in {}^{\omega}\lambda)$ are mutually contradictory and there are ${}^{\omega}\lambda$ of them. If we take Skolem Hulls of various sets of the $\{b_\eta\}$ we can hopefully realize various collections of p_η's while omitting others, and so get many non-isomorphic models. How shall we do this ?

Second Approximation: Our problem would seem simpler in the unstable case, where the tree ${}^{\omega\geq}\lambda$ would be replaced by a linear ordering containing all the points b_η. Then we could build Ehrenfeucht-Mostowski models over the $\{a_\nu\}$ and some of the $\{b_\eta\}$, and try in this way to realize some types and to omit some others. What we must do here is:

1. Develop a theory of trees of indiscernibles and Ehrenfeucht-Mostowski models over such trees.

This can be done in a very natural way and then p_η is realized in the Skolem Hull of $\{b_\eta : \eta \in X\} \cup \{a_\eta : \eta \in {}^{\omega>}\lambda\}$ iff $\eta \in X$.

Now we start again with a tree of indiscernibles $\{a_\nu : \nu \in {}^{\omega>}\lambda\} \cup \{b_\eta : \eta \in {}^{\omega}\lambda\}$. We will restrict our attention to sequences $\eta \in {}^{\omega}\lambda$ which are increasing, and we let

$$B_\alpha = \{b_\eta : \lim_{\rightarrow} \eta(n) = \alpha\} \cup \{a_\nu : \nu \in {}^{\omega>}\lambda\}$$

for α a limit ordinal. We will build Ehrenfeucht-Mostowski models over sets of the form $B_X = \bigcup_{\alpha \in X} B_\alpha$ for $X \subseteq \lambda$ a set of limit ordinals. Let $M_X = EM(B_X)$ (an Ehrenfeucht-Mostowski model). We would like to have the various models M_X non-isomorphic. This requires more care.

Third Approximation: All we have to do now is:

2. Choose the sets X so that the models M_X are non-isomorphic.

The trick is to use stationary subsets of λ.

Fact (Solovay [Sl]*)* (λ regular). There are λ disjoint stationary sets $S_\alpha \subseteq \lambda$ $(\alpha < \lambda)$.

Fact (Solovay [Sl]*)*. For λ regular, each stationary set can be partitioned into λ disjoint stationary sets, hence there are disjoint stationary $S_\alpha \subseteq \{\gamma < \lambda : \mathrm{cf}\ \gamma = \aleph_0\}$ $(\alpha < \lambda)$.

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Now for $W \subseteq \lambda$ let $X_W = \bigcup \{S_\alpha : \alpha \in W\}$, then the various M_{X_W} are non-isomorphic (this should be proved more carefully.)

Fourth Approximation: But if $\lambda^{\aleph_0} > \lambda$ then $|M_{X_W}| > \lambda$, hence for each $\gamma < \lambda$, $cf\gamma = \aleph_0$; choose an increasing η_γ converging to γ, and redefine B_α as

$$\{b_{\eta_\alpha}\} \cup \{a_\nu : \nu \in {}^{\omega>}\lambda\}$$

Now we summarize the content of each section.

§ 1. Here we review the results from [S 7] on stability for homogeneous models including the stability spectrum theorem, and the equivalence between unstability and order. A worth-while new theorem proved here is

Theorem 1.13: There is a $\mathcal{D}$-homogeneous model of cardinality λ if $\lambda = \lambda^{<\lambda} + |\mathcal{D}|$ or $\mathcal{D}$ is stable in λ (remember that $\mathcal{D}$ is stable in λ iff $\lambda = \lambda(\mathcal{D}) + \lambda^{<\kappa(\mathcal{D})}$).

This generalizes a similar theorem for saturated models (formulated in [S 5] by combining results of Morley and Vaught [MV], Shelah [S 7], Harnik [H 1] and Shelah [S 1]. Some specific instances were proved by: Eklof and Fisher [EF] (abelian groups), Boffa [Bo 1], [Bo 2], [Bo 3] (infinite generic skew fields).

§ 2. Here we discuss two additional assumptions, which make the stability theory more like that of an elementary class; prove that a class of infinite generic models satisfies one of them; generalize this; and show that the more general cases are not so far from the main example. This provides us also with a tool to construct examples of classes of infinite generic models.

Historical Note: The context 2.1-2.5 is of Robinson [Rb 2]; a generalization to infinitary languages $L_{\varkappa,\omega}$ is due to Wood [W 3], and a definition and investigation of K^n is in Cherlin [Cr] (for the category of embeddings), sketching the generalization to infinitary languages $L_{\lambda,\varkappa}$; so in 2.6-2.10(1) there is nothing essentially new. (4)

§ 3. Here we define appropriate ranks, prove some of their properties, and use them to investigate the spectrum of stability.

As the treatment of Morley [Mr] of ranks and types is related to Boolean algebras, our treatment is related to lattices (see Simmons paper in this volume).

H. Priestly [Pr] develops the right kind of Stone space for lattices. Fisher, Simmons and Wheeler [F S W] and McKenzie and Shelah [McS] prove some model theoretic results essentially equivalent to the fact that countable lattices have $\leqslant \aleph_0$ or $2^{\aleph_0}$ ultrafilters (in [F S W] this is done for the number of theories of e. closed models of T, finite-generic models of T, and infinite generic models of T, T countable; in [McS] for the number of ep-types realized over a countable set in appropriate $\aleph_1$-homogeneous models.

§ 4. Here we deal with Ψ-prime models. We describe the natural way to construct prime models, prove their existence. They are unique when our class is $\aleph_0$-stable. The most important theorem is that for a countable stable T, if over A there is a strictly prime model, it is unique; we prove it for $|A| \leqslant \aleph_1$ only.

§ 5 Here we show that Morley's categoricity theorem generalizes easily.

(4) Added in proof June 76: Independently, W. Forrset generalized some theorems on $\aleph_0$-stable theories to the context of assumption II (see § 1).

§ 6. In the context of assumption II(i.e. infinite generic model for a universal T with the amalgamation property) we develop «forking» and prove its important properties.

Some of this, mainly the symmetry lemma, was developed later and independently, from a different point of view by Lascar (see this volume and [Ls]). He concentrates on superstable T and complete types over models.

§ 7. We deal with some «negative» results. We quote Macintyre that any infinite field with an $\aleph_0$-stable theory is algebraically closed; and then prove that $\aleph_1$-categorical division rings are fields, using Baldwin theorem ($\alpha_T < \omega$). We then show how our general results include a theorem of Boffa [Bo 1], [Bo 2], [Bo 3], (on $\{\lambda$: there is an infinite generic division ring of card $\lambda\}$) and solve one of his problems (on the number of non-isomorphic infinite generic division rings).

Historical Note: (Lemma 7.5 (1) is from Macintyre [Mc 2] and 7.3-7.6 are from letters to Macintyre distributed in summer 73. Later Baldwin and Saxel [B S] reproved 7.5. There exist papers of Zilberg on $\aleph_1$-categoricity of rings, partially overlapping Cherlin and Reineke [C R] (5).

§ 8. We mainly deal with modules. They are always stable (Baur [Br] Fisher [Fi]) and we characterize $\kappa(T)$, and in particular superstability. We also give essentially, a structure/nonstructure theorem for the class of R-modules over a fixed R, and show that formulas in this theory are not too complicated (8.3).

We also deal with differential fields and with separably closed fields and conclude with a criterion for an operation on models to preserve stability.

§ 1. *HOMOGENEITY.*

We shall deal in this article with four kinds of classes, for which we have stability theory. They are essentially:
Kind I: Elementary classes.

(5) Added in Proof June 76: Zilberg also proved independently that $\aleph_1$-categorical division rings are fields.

Kind II: The e.c. (existentially closed) models of an universal theory with the J.E.P. (joint embedding property) and the amalgamation property.

Kind III: The e.c. models of a universal theory with the J.E.P.

Kind IV: The elementary submodels of some $(\mathcal{D}, \lambda)$ — homogeneous model. (6)

More complicated cases, like the class of models of some $\psi \in L_{\omega_1, \omega}$ or abstract elementary classes will be dealt with elsewhere. In kind II (but not III) we can deal with the class of models of T. (7)

In kind I embeddings are elementary.

Kind I will be the subject of the book [S 1], and kind IV was the subject of [S 7]; the other two constitute the (small) novelty of this article. Here we shall review the necessary facts from [S 7], in order to use them and to see how restricting the context adds to the information. We shall also note that kinds II and III can be replaced by a somewhat more general condition. Some of the results will be reproved in the context of kinds II and III.

We prove here one new theorem: *there is a $(\mathcal{D}, \lambda)$ — homogeneous model of cardinality λ iff $\lambda = \lambda^{<\lambda} + |\mathcal{D}|$ or $\mathcal{D}$ is stable in λ (for good $\mathcal{D}$).*

Definition 1.1.

We define the Δ-*type* of the sequence $\bar{a}$ (usually finite) in M over A by

$$tp_\Delta(\bar{a}, A, M) = \{\varphi(\bar{x}, \bar{b}) : \bar{b} \in A,\ M \models \varphi(\bar{a}, \bar{b}),\ \varphi(\bar{x}, \bar{y}) \in \Delta\}$$

(6) We assume there is a $(\mathcal{D}, \lambda)$-homogeneous model of cardinality $\geqslant \lambda$ for every λ.

(7) Note that in kind III, for every model M of the theory, the type of every $\bar{a} \in M$ in any universal - homogeneous model N, $M \subseteq N$, is uniquely determined iff M is e-closed; whereas in kind II this holds for any M. Hence it is natural in kind II to deal with the class of models of the theory and in kind III with the e-closed ones.

When Δ is L(M) = the set of first order formulas in the language of M, we omit it.

Definition 1.2.

The *finite diagram of* M, $\mathcal{D}$(M), is $\{tp(\bar{a}, \emptyset, M) : \bar{a} \in M, (\bar{a}$ finite$)\}$.

Let $\mathcal{D}$ denote such sets.

Definition 1.3.

M is λ-*homogeneous* if for every $a_i, b_i \in M$, $i < \alpha < \lambda$ and $a_\alpha \in M$ such that

$$tp(\langle a_i : i < \alpha \rangle, \emptyset, M) = tp(\langle b_i : i < \alpha \rangle, \emptyset, M)$$

there is a $b_\alpha \in M$ such that

$$tp(\langle a_i : i \leqslant \alpha \rangle, \emptyset, M) = tp(\langle b_i : i \leqslant \alpha \rangle, \emptyset, M).$$

Definition 1.4.

M is $(\mathcal{D}, \lambda)$ — *homogeneous* if it is λ- homogeneous and $\mathcal{D}(M) = \mathcal{D}$. M is $\mathcal{D}$- *homogeneous* if it is $(\mathcal{D}, \|M\|)$ — homogeneous.

Assumption I. $\mathfrak{C}$ is some fixed $(\mathcal{D}, \lambda_0)$- homogeneous model, for a fixed $\mathcal{D}$ and λ_0 sufficiently large. All models are elementary submodels of $\mathfrak{C}$, all elements, sequences of elements and sets $(a, \dots, \bar{a}, \dots, A, \dots)$ are from $\mathfrak{C}$. Satisfaction $\models$ will mean «in $\mathfrak{C}$».

Definition 1.5.

$S^m_{\mathcal{D}}(A) =. \{tp(\bar{a}, A, \mathfrak{C}) : \bar{a} \in \mathfrak{C}, \ell(\bar{a}) = m\}$. (If $A \subset M < \mathfrak{C}$, $S^m_{\mathcal{D}}(A)$ depends only m, M, A and $\mathcal{D}$).

Lemma 1.6.

(1) *A set p of formulas with parameters from A is in* $S^m_{\mathcal{D}}(A)$ *iff:*

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(i) *the variables* are $\bar{x} = \langle x_0, \ldots, x_{m-1} \rangle$
(ii) p *is complete finitely satisfiable*
(iii) *for each* $\bar{b} \in A$, $\{\varphi(\bar{x};\bar{y}) : \varphi(\bar{x};\bar{b}) \in p\} \in \mathcal{D}$

(2) M *is* $(\mathcal{D}, \lambda)$ — *homogeneous iff whenever* $A \subseteq M$, $|A| < \lambda$, *and* $p \in S^m_{\mathcal{D}}(A)$ *then* p *is realized*

(3) *If* M, N *have cardinality* λ, $\mathcal{D}(M) = \mathcal{D}(N)$, *and* M, N *are homogeneous then* $M \cong N$. *If only* M *is* λ — *homogeneous,* N *can be elementarily embedded into* M.

Lemma 1.6, (2), stresses the similarity of homogeneity and saturation.

Definition 1.7.

(1) $\mathcal{D}$ is *λ-stable* iff $m < \omega$ and $|A| \leqslant \lambda$ implies $|S^m_{\mathcal{D}}(A)| \leqslant \lambda$.

(2) $\mathcal{D}$ is *stable* if it is λ-stable for some λ.

Definition 1.8.

$\mathcal{D}$ has the λ- *order property* if there are $\bar{a}_i (i < \lambda)$ and $\varphi(\bar{x}, \bar{y})$ such that

$$\models \varphi(\bar{a}_i, \bar{a}_j) \text{ iff } i < j.$$

Theorem 1.9. (Stability vs. Order)

(1) *If* $\mathcal{D}$ *has the λ-order property for all* λ *then* $\mathcal{D}$ *is unstable (in all* λ).

(2) *If* $\mathcal{D}$ *is not λ-stable,* $\lambda = \lambda^{\varkappa} + 2^{2^{\varkappa}}$, $\varkappa \geqslant |L|$ *then* $\mathcal{D}$ *has the* $\varkappa^{+}$*- order property.*

(3) *If* $\mathcal{D}$ has *the* ϰ*-order property for every* $\varkappa < \beth_{(2^{|L|})^{+}}$, *then* $\mathcal{D}$ *has the* ϰ*-order property for all* ϰ.

Remark: Essentially this theorem says that unstability and order are equivalent. See [S 7], [S 8].

Theorem 1.10. (The stability spectrum theorem).

Always (1) $\mathcal{D}$ *is unstable*

or (2) *there exist cardinals* $\kappa(\mathcal{D}), \lambda(\mathcal{D}) < \beth_{(2^{|L|})^{+}}$

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such that: $\mathcal{D}$ is λ-*stable iff* $\lambda = \lambda(\mathcal{D}) + \lambda^{<\kappa(\mathcal{D})}$. $(\lambda^{<\varkappa} = \sum_{\mu<\varkappa} \lambda^{\mu})$.

Remarks: (1) In (1) we stipulate $\lambda(\mathcal{D}) = \kappa(\mathcal{D}) = \infty$; in (2) if we demand «$\kappa(\mathcal{D})$ is regular» then $\kappa(\mathcal{D})$ is unique.

(2) In [S 7] we prove more; that if there is a $(\mathcal{D}, \lambda^+)$ — homogeneous model of cardinality $\geqslant \lambda^+$ and $\mathcal{D}$ is λ- stable then there are $(\mathcal{D}, \varkappa)$ — homogeneous models for all $\varkappa$.

(3) For theories, and even for kind III, $\kappa(\mathcal{D}) \leqslant |T|^+$ and $\lambda(\mathcal{D}) \leqslant 2^{|L|}$.

Definition 1.11.

Let I be a set of sequences of fixed (finite) length from $\mathfrak{C}$. I is *indiscernible over* A iff for all distinct $\bar{a}_i \in I$ $(i < n)$, $tp(\bar{a}_0 {}^\frown \ldots {}^\frown \bar{a}_{n-1}, A, \mathfrak{C})$ is fixed.

Theorem 1.12.

(1) *If* $\mathcal{D}$ *is* λ-*stable,* $|A| \leqslant \lambda < |I|$, I *a set of sequences from* $\mathfrak{C}$, *then there exists* $I' \subseteq I$, $|I'| = \lambda^+$, I' *indiscernible over* A.

(2) *If* I *is indiscernible over* A, *for any* $\bar{a}$ *there is* $J \subseteq I$, $|J| < \kappa(\mathcal{D})$ *such that* $I - J$ *is indiscernible over* $A \cup J \cup \bar{a}$.

Remark.

By 1.12 (2) for any indiscernible set $I(\bar{a} \in I \Rightarrow l(\bar{a}) = m)$, $|I| \geqslant \kappa(\mathcal{D})$, and A, we can define

$Av(I, A) = \{\varphi(\bar{x}, \bar{b}) :$ for $\kappa(\mathcal{D})$ $\bar{a}$'s in I $\models \varphi(\bar{a}, \bar{b})\} \in S^m_{\mathcal{D}}(A)$.

For theories and even kind II it suffices to assume I is infinite. Question: what about kind III and IV ?

Theorem 1.13. (Homogeneity spectrum theorem). [8]

There is a $(\mathcal{D}, \lambda)$ — *homogeneous model of cardinality* λ *iff* $\lambda = \lambda^{<\lambda} + |\mathcal{D}|$ *or* $\mathcal{D}$ *is stable in* λ.

[8] In fact, it suffices to assume on $\mathcal{D}$ that there is a $(\mathcal{D}, \lambda)$-homogeneous model of cardinality $\geqslant \lambda$ and replace «$\mathcal{D}$ stable in λ» by «M stable in λ».

Remark.

Naturally this should have appeared in [S 7] but I did not know it at the time. The reader can skip it easily.

Proof. The proof is split into cases.

case 1: $\lambda = \lambda^{<\lambda} + |\mathcal{D}|$

It is easy to prove the existence (by the classical [MV]).

case 2: $\lambda < |\mathcal{D}|$.

The non-existence is trivial.

case 3: $\lambda < \lambda^{<\lambda}$, $\mathcal{D}$ unstable.

See [S 7], 6.5 (2), using the order property.

case 4: $\mathcal{D}$ stable in λ.

If λ is regular the proof is trivial.

So assume λ is singular. Now we start using [S 7] notions and theorems heavily.

As $\mathcal{D}$ is stable in λ, $\lambda = \lambda^{<\kappa(\mathcal{D})}$ hence $\kappa(\mathcal{D}) \leqslant \mathrm{cf}\,\lambda$.

We define an increasing sequence M_i $(i \leqslant \lambda^2)$, $\|M_i\| = \lambda$ such that each $q \in S^m_{\mathcal{D}}(M_i)$ is realized in M_{i+1} and $M_\delta = \bigcup_{i<\delta} M_i$ for limit δ.

Let $A_0 \subseteq M_{\lambda^2}$, $|A_0| < \lambda$, $p_0 \in S^m_{\mathcal{D}}(A_0)$ and we should prove that p_0 is realized in M_{λ^2}.

Let $p_0 \subseteq p \in S^m_{\mathcal{D}}(M_{\lambda^2})$, and choose $C \subseteq M_{\lambda^2}$, $|C| < \kappa(\mathcal{D})$ such that p does not split strongly over C (see [S 7] § 4). As $\mathrm{cf}\,\lambda \geqslant \kappa(\mathcal{D})$ for some $\alpha_0 < \lambda^2$, $C \subseteq M_{\alpha_0}$. As $\mathcal{D}$ is stable in λ, $\mathcal{D}$ does not satisfy $(*\,\lambda)$ (see [S 7], Def 2.3 (1) and Th. 2.6., 2.7), hence $|\{i < \lambda : p \restriction M_{\lambda(i+1)}$ splits over $M_{\lambda i}\}| < \lambda$, hence for some β, $\alpha_0 < \beta < \lambda^2$ and $p \restriction M_{\beta+\lambda}$ does not split over M_β.

Now we shall prove that p does not split over $M_{\beta+\lambda}$. Otherwise suppose $\bar{b}, \bar{c} \in M_{\lambda^2}$, $tp(\bar{b}, M_{\beta+\lambda}) = tp(\bar{c}, M_{\beta+\lambda})$, $\varphi(\bar{x}; \bar{b})$, $\neg\varphi(\bar{x}, \bar{c}) \in p$. As before there is $\gamma, \beta < \gamma < \beta + \lambda$ such that $tp(\bar{b}, M_{\gamma+\omega})$ does not split over M^λ ($\lambda > \omega$ as λ is singular). Choose $\bar{b}_n \in M_{\gamma+n+1}$ which realizes $tp(\bar{b}, M_{\gamma+n})$, so by [S 7], p. 83, $\{\bar{b}_n : n < \omega\}$ is indiscernible over M_λ, and so are $\{\bar{b}_n : n < \omega\} \cup \{\bar{b}\}$, $\{\bar{b}_n : n < \omega\} \cup \{\bar{c}\}$. Hence p splits strongly over M_γ; hence over C, contradiction. So p does not split over $M_{\beta+\lambda}$.

For each $\gamma, \beta + \lambda < \gamma < \lambda^2$ choose $\bar{a}_\gamma \in M_{\gamma+1}$ realizing $p \restriction M_\gamma$, hence, again by [S 7] p. 83, $I = \{\bar{a}_\gamma : \beta + \lambda < \gamma < \lambda^2\}$ is indiscernible.
For each $\bar{c} \in M_{\lambda^2}$ $I_{\bar{c}} = \{\bar{a} \in I : a \text{ realizes } p \restriction \bar{c}\}$ has cardinality λ, hence $|I - I_{\bar{c}}| < \kappa(\mathcal{D})$, hence $p \restriction A_0 = p_0$ is realized by some $\bar{a}_\gamma \in I$.

Before treating the final case, we prove

Lemma 1.14. *Suppose* $A \subseteq B$, *and each* $p \in S^m_{\mathcal{D}}(A)$ $(m < \omega)$ *is realized in* B. *Then* $|\Gamma| \leqslant |\bigcup S^m_{\mathcal{D}}(A)|^{\aleph_0} \leqslant |\mathcal{D}|^{|A|+\aleph_0}$ *where* $\Gamma = \{p : p \in S^m_{\mathcal{D}}(B), m < \omega, p \text{ does not split over } A\}$.

Remark: This improve [S 7], 2.5.
Proof: For each $p \in \Gamma$ we define by induction $\bar{a}^i_p$ $(i < \kappa(\mathcal{D}))$ such that $tp(\bar{a}^i_p, B \cup \{\bar{a}^j_p : j < i\}$ extends p and does not split over A.

This is possible by [S 7] 2.2.(2), and $I_p = \{\bar{a}^i_p : i < \kappa(\mathcal{D})\}$ is

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indiscernible over B. Now

(*) if $\bar{b} \in B$, $tp(\bar{b}_1, A) = tp(\bar{b}, A)$, $q = tp(\bar{b}^\frown \bar{a}^\circ_p, A)$

then $|\{i < \kappa(\mathcal{D}) : q \neq tp(\bar{b}_1{}^\frown \bar{a}^i_p, A)\}| < \kappa(\mathcal{D})$.

(because we can define $\bar{a}^i_p$ $(\kappa(\mathcal{D}) \leqslant i < \kappa(\mathcal{D})^+)$ such that $tp(\bar{a}^i_p, B \cup \{\bar{a}^j_p : j < i\} \cup \bar{b}_1)$ extends p and does not split over A. Again this is possible and $I' = \{\bar{a}^i_p : i < \kappa(\mathcal{D})^+\}$ is indiscernible. Clearly for $i \geqslant \kappa(\mathcal{D})$, $tp(\bar{b}_1{}^\frown \bar{a}^i_p, A) = tp(\bar{b}^\frown \bar{a}^i_p, A) = tp(\bar{b}^\frown \bar{a}^\circ_p, A)$. So $|\{a \in I' : q = tp(\bar{b}_1{}^\frown \bar{a}, A)\}| > \kappa(\mathcal{D})$; hence, by 1.12(2), $|\{\bar{a} \in I' : tp(\bar{b}^\frown \bar{a}, A) \neq tp(\bar{b}_1{}^\frown \bar{a}, A)\}| < \kappa(\mathcal{D})$. So (*) follows.).

So if $p \neq q \in \Gamma$, there is no automorphism F of $\mathfrak{C}$, with $F \restriction A =$ the identity and $F(\bar{a}^i_p) = \bar{a}^i_q$ $(i < \kappa(\mathcal{D}))$. Hence $tp(\langle \bar{a}^i_p : i < \kappa(\mathcal{D}) \rangle, A) \neq tp(\langle \bar{a}^i_q : i < \kappa(\mathcal{D}) \rangle, A)$ hence by the indiscernibility $tp(\langle \bar{a}^i_p : i < \omega \rangle, A) \neq tp(\langle \bar{a}^i_q : i < \omega \rangle, A)$. Simple computation gives our result.

Continuation of the proof of 1.13.

Case 5. $\mathcal{D}$ is not stable in λ, but $\mathcal{D}$ is stable, $\lambda \geqslant |\mathcal{D}|$ and $\lambda < \lambda^{<\lambda}$.

If $\lambda^{<\kappa(\mathcal{D})} > \lambda$, this is easy by [S 7] § 6. So we assume $\lambda = \lambda^{<\kappa(\mathcal{D})}$. Similarily we can assume that $2^{\mu} > \lambda$ implies not $(*\mu)$, and let $\lambda_*(\mathcal{D})$ be the first cardinal μ such that not $(*\mu)$; so $2^{<\lambda_*(\mathcal{D})} \leqslant \lambda$. We can also assume $|A| < \lambda$ implies $|S^m_{\mathcal{D}}(A)| \leqslant \lambda$.

Suppose M is a $(\mathcal{D},\lambda)$ — homogeneous model of cardinality λ, so by hypothesis $|S^m_{\mathcal{D}}(M)| > \lambda$. Let $M = \bigcup_{i<cf\lambda} M_i$, M_i is increasing, $\|M_i\| < \lambda$. Now each $p \in S^m_{\mathcal{D}}(M)$ does not split over

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some M_i (If λ is regular because $\lambda_*(\mathcal{D}) \leqslant \lambda$; if λ is singular choose for each $i < cf\lambda$ $\bar{a}_i, \bar{b}_i \in M$, $tp(\bar{a}_i, M_i) = tp(\bar{b}_i, M_i)$ but $\varphi_i(\bar{x}, \bar{a}_i) \wedge \neg \varphi_i(\bar{x}, \bar{b}_i) \in p$. Now let $A = \bigcup_{i<cf\lambda} \bar{a}_i{}^\frown \bar{b}_i \subseteq |M|$, so $|A| < \lambda$ and $p \restriction A \in S^m_{\mathcal{D}}(A)$ is not realized in M; contradiction.)

So for some $\alpha < cf\lambda$, $m < \omega$, $\Gamma = \{p \in S^m_{\mathcal{D}}(M)$: p does not split over M } has cardinality $> \lambda$. As $|\bigcup_{m<\omega} S^m(M_i)| \leqslant \lambda$ (because $\|M_i\| < \lambda$ and the assumption), by lemma 1.14, we have necessarily that $\lambda^{\aleph_0} > \lambda$; hence by assumption $\kappa(\mathcal{D}) = \aleph_0$.

Let $\bar{a}^i_p$ $(i \leqslant \omega)$ be defined as in lemma 1.14. for each $p \in \Gamma$ with M_α for A and M for B. So $p \neq q \in \Gamma$ implies that for some n, $tp(<\bar{a}^i_p : i < n>, M_i) \neq tp(<\bar{a}^i_q : i < n>, M_i)$. Now we define by induction on $i \leqslant \omega$, $\bar{b}^i_p \in M$ for each $p \in \Gamma$ such that

(i) $tp(<\bar{b}^j_p : j \leqslant i>, M_\alpha) = tp(<\bar{a}^j_p : j \leqslant i>, M_\alpha)$

(ii) If $tp(<\bar{a}^j_p : j \leqslant i>, M_\alpha) = tp(\bar{a}^j_q : j \leqslant i>, M_\alpha)$ then $\bar{b}^j_p = \bar{b}^j_q$ for $j \leqslant i$.

Clearly this is possible (as $\lambda > \aleph_0$ by the hypothesis $\lambda < \lambda^{<\lambda}$, and $p \neq q$ implies $\bar{b}^\omega_p \neq \bar{b}^\omega_q$ because there is a minimal $n < \omega$ such that $tp(<\bar{a}^j_p : j \leqslant n>, M_\alpha) \neq tp(<\bar{a}^j_q : j \leqslant n>, M_\alpha)$, hence $\bar{b}^j_p = \bar{b}^j_q$ for $j < n$ but for $r = p, q$

$tp(<\bar{a}^0_r {}^\frown \ldots {}^\frown \bar{a}^{n-1}_r, \bar{a}^\omega_r>, M_\alpha) = tp(<\bar{a}^j_r : j \leqslant n>, M_\alpha)$

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So we get a contradiction : $\lambda \geqslant \|M\| = |\{\bar{b}_p^{\omega} : p \in \Gamma\}| > \lambda$.
Similarly we can prove

Theorem 1.15.

Suppose M_i *is* $(\mathcal{D}, \lambda)$ — *homogeneous,* $i < \delta$, cf $\delta \geqslant \kappa(\mathcal{D})$, $\lambda_*(\mathcal{D}) \leqslant \lambda$, M_i *increasing. Then* $\bigcup_{i<\delta} M_i$ *is* $(\mathcal{D}, \lambda)$ — *homogeneous.*

The categoricity theorem is

Theorem 1.16.

Suppose the class $\{M : M < \mathfrak{C}\}$ *is categorical in one* $\lambda > |L|$. *Then there is* $\lambda_1 < \beth_{(2^{|L|})^+}$ *such that*

(1) *the class is categorical in every* $\lambda \geqslant \lambda_1$; *moreover* $M < \mathfrak{C}$, $\|M\| \geqslant \lambda_1$ *implies* M *is* $\mathcal{D}$*-homogeneous.*

(2) *For every* μ, $|L| < \mu < \lambda_1$ *the class is not categorical in* μ; *moreover there is a non* — $|L|^+$ — *homogeneous model* $M < \mathfrak{C}$ *of cardinality* μ.

Remark: See [S 7] for better results, i.e. applicable to more cases. But we do not know whether the bound on λ_1 can be improved. Related questions: suppose there is a $(\mathcal{D}, \lambda^+)$ — homogeneous model of cardinality λ; is $\lambda \leqslant 2^{|L|}$?

For kind IV, can $\kappa(T)$ be $> (2^{|L|})^+$?

For which λ does the existence of a $(\mathcal{D}, \lambda)$ — homogeneous model imply the existence of a $(\mathcal{D}, \mu)$ — homogeneous model for every μ ? [9]

§ 2. *GENERIC MODELS*

Now we can state more exactly what extra assumptions we use in kinds II, III, and explain why the generic models satisfy them.

[9] Notice that 1.14 has new consequences concerning the question of Keisler and Morley, on the number of homogeneous models of T in cardinality λ, when G.C.H. fails.

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Assumption for kind III: there exists a set Φ of formulas such that (where $\mathfrak{C}$, $\mathfrak{D}$, are as before)

(i) *if p_1, p_2 are distinct m-types in $\mathfrak{D}$, then there are $\varphi_1(\bar{x}) \in p_1$, $\varphi_2(\bar{x}) \in p_2$ which are in Φ and are contradictory, that is $\mathfrak{C} \models \neg(\exists \bar{x})(\varphi_1(\bar{x}) \wedge \varphi_2(\bar{x}))$*

(ii) *any set of formulas from Φ; with parameters from $\mathfrak{C}$, of cardinality $< \|\mathfrak{C}\|$ which is finitely satisfiable in $\mathfrak{C}$, is satisfiable in $\mathfrak{C}$.*

Assumption for kind II: as the previous one, when we add

(iii) *every $\varphi(\bar{x}) \in \Phi$ has a negation in Φ, that is a $\psi(\bar{x}) \in \Phi$ such that $\mathfrak{C} \models (\forall \bar{x})(\varphi \bar{x}) \equiv \neg \psi(\bar{x}))$.*

Remark: In the assumption for kind III, we can assume Φ is closed under existential quantification, and for kind II, under negation. In any case we can assume it is closed under conjunctions and disjunctions.

We deviate from the assumption of § 1, by letting M be any submodel of $\mathfrak{C}$, such that for any formula $\varphi(\bar{x})$ which is in sub Φ, $\bar{a} \in M \Rightarrow \mathfrak{C} \models \varphi[\bar{a}] \leftrightarrow M \models \varphi[\bar{a}]$. (M is called Φ-closed; or $M <_{\text{sub } \Phi} \mathfrak{C}$).

In this section we prove that the assumptions defined above are satisfied by the appropriate kinds.

We let now T denote a fixed universal theory with the J E P.

Definition 2.1. (A) T^{ω} is the unique class of models of T such that

(1) T^{ω} is cofinal in T i.e. in Mod(T) (= the class of models of T); that is every model of T is embeddable in a model of T^{ω} (the usual notation is G_T, Cherlin [Cr] uses T^{∞}).

(2) T^{ω} is model complete, i.e. if $M_1, M_2 \in T^{\omega}$ and $M_1 \subseteq M_2$, then $M_1 < M_2$.

(3) If $M_1 < M_2 \in T^{\omega}$, then $M_1 \in T^{\omega}$.

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Note that T^ω is closed under unions of increasing chains.

Definition 2.1. (B) M is *infinitely generic* (for T) if $M \in T^\omega$.

Definition 2.2.

(A) $\mathcal{D}_\Delta(T) = \{tp_\Delta(\bar{a}, \emptyset, M) : \bar{a} \in M,$ M a model of T$\}$,

$\mathcal{D}_T = \mathcal{D}_L(T)$.

(B) $M <_\Delta N$ if $\bar{a} \in M$, $\varphi \in \Delta$, implies $M \models \varphi[\bar{a}] \Leftrightarrow N \models \varphi[\bar{a}]$ and $M \subseteq N$.

(C) In $\mathcal{D}_\Delta$, tp_Δ, etc., when Δ is the set of existential, positive-existential, quantifier-free, atomic formulas, we write instead of Δ, e, ep (= pe), qf, a resp. and denote Δ itself by Φ_e, Φ_{ep}, Φ_{qf}, Φ_a resp.

Lemma 2.3. (1) *If* M *is a model of* T, α *an ordinal,*

$$tp_e(<a_i : i < \alpha> \emptyset, M) \subseteq tp_e(<b_j : i < \alpha>, \emptyset, M)$$

then for every $a_\alpha \in M$ *there are* N *and* $b_\alpha \in N$ *such that* $M \subseteq N$, *and*

$$tp_e(<a_i : i \leqslant \alpha>, \emptyset, N) \subseteq tp_e(<b_i : i \leqslant \alpha>, \emptyset, N)$$

We can replace $\subseteq$ *by* = (for the types).

(2) *For any model* M *of* T *and* $\varkappa$, *there is a model* N, $M \subseteq N$ *such that*

(*) *For any* $a_i, b_i \in N$ $(i < \alpha < \varkappa)$ *such that*

$$tp_e(<a_i : i < \alpha>, \emptyset, N) = tp_e(<b_i : i < \alpha>, \emptyset, N)$$

for any $a_\alpha \in N$ *there is* $b_\alpha \in N$ *such that*

$$tp_e(<a_i : i \leqslant \alpha>, \emptyset, N) = tp_e(<b_i : i \leqslant \alpha>, \emptyset, N).$$

(3) *If in* (2) *in* M *every* $p \in \mathcal{D}_e(T)$ *is realized, then* (*) *is equivalent to* "N *is* $(\mathcal{D}^e(T), \varkappa)$*-homogeneous*" *where* $\mathcal{D}^e(T) = \{tp(\bar{a}, \emptyset, M) : \bar{a} \in M_0\}$, *where* M_0 *realizes every* $p \in \mathcal{D}_e(T)$ *and satisfies* (*).

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(4) *Every* $(\mathscr{D}^e(T), \aleph_0)$*-homogeneous model belongs to* T^ω.

(5) *If* T *has the amalgamation property, we can replace in* (1) - (4), e *by* qf.

Conclusion 2.4. $\mathscr{D}^e(T)$ *satisfies the assumption for kind III (with* Φ_e*). If* T *has the amalgamation property, then* $\mathscr{D}^e(T)$, Φ_{qf} *satisfy the assumption for kind II.*

Proof: Easy.

Remark: Note that by using $\mathfrak{C}$, the existential type of each $\bar{a}$ is determined. Note

Claim 2.5. $M \in T^\omega$ iff $M < N$ for some $(\mathscr{D}^e(T), \lambda)$-homogeneous models iff $\mathscr{D}(M) \subseteq \mathscr{D}^e(T)$.

* * *

We can present genericity in a more general way.

Context: let L be a language, and K a category of L-models such that

(1) the morphisms are some functions from one model to another

(2) direct limits exist

(3) for every atomic $R(\bar{x}) \in L$ and $\bar{a} \in M \in K$ there are g, $N \in K$, $g: M \to N$ such that for all $g': N \to N'$ in K, the truth value of $R(g' g(\bar{a}))$ (in N') is the same.

We say K has the J M P (joint mapping property) if for every $M_1, M_2 \in K$ there are $g_\ell : M_\ell \to N$ $(\ell = 1, 2)$ in K.

Definition 2.6. Let Ψ be a set of formulas from $L_{\infty,\omega}$ closed under subformulas, and let $\Psi_\alpha = \{\varphi \in \Psi : \varphi$ has quantifier depth $\leqslant \alpha\}$.

We define full subcategories K^α $(= K^\alpha(\Psi))$ of K, and for each $M \in K$, $\varphi(\bar{x}) \in \Psi$, when $M \Vdash \varphi[\bar{a}]$. (If Ψ is not mentioned, we let $\Psi = L_{\infty,\omega}$).

(1) for $\alpha = 0$, $M \in K^\alpha$ iff for each atomic formula $R(\bar{x})$ and $g: M \to N$ in K, and $\bar{a} \in M$, $M \models R[\bar{a}] \Leftrightarrow N \models R(g(\bar{a}))$

(2) we define $M \mid\vdash \varphi[a]$ by induction on φ, where $M \in K$, $\varphi \in \Psi$, $a \in M$

(A) for atomic φ, $M \mid\vdash \varphi[a]$ iff $M_1 \models \varphi[g(a)]$ for each $g : M \to M_1$ in K

(B) $M \mid\vdash \bigwedge_i \varphi_i$ iff $M \mid\vdash \varphi_i$ for each i

(C) $M \mid\vdash \exists x\, \varphi[x, a]$ iff there is $b \in M$ such that $M \mid\vdash \varphi[b, a]$

(D) $M \mid\vdash \neg\varphi(a)$ iff there are no $g : M \to N$ in K such that $N \mid\vdash \varphi[g(\bar{a})]$

(3) $M \in K^{\alpha}$ iff for each $\varphi \in \Psi_{\alpha}$, $a \in M$, $M \models \varphi[a] \Leftrightarrow M \mid\vdash$ $\mid\vdash \varphi[a]$

(4) K^{ω} is called the class of K-generic models.

Lemma 2.7.

(1) K^{α} *is decreasing.*

(2) *Each* K^{α} *is unbounded in* K, *i.e. for every* $M \in K$ *there is* $g : M \to N$ *in* K, $N \in K^{\alpha}$. *Also each* K^{α} *is closed under direct limits.*

(3) If $g : M \to N$ *is in* K, $a \in M$, $M \mid\vdash \varphi[a]$ *then* $N \mid\vdash \varphi[a]$.

(4) *If* K *has the amalgamation property,* Ψ *closed under negation then for every* $\varphi(x) \in \Psi$, $a \in M \in K$, $K \mid\vdash \neg\varphi(x)$ or $K \mid\vdash \neg\neg\varphi(x)$.

Remark: We could do the same for $\Psi \subseteq L_{\infty,\varkappa}$ but then K^{α} is closed only for direct limits of cofinality $\geqslant \varkappa$.

Definition 2.8. (A) Let T be a theory in L, Φ a set of L-formulas. $K(T, \Phi)$ will be the following category:

(i) $M \in K(T, \Phi)$ iff M is a model of T (an L-model, of course)

(ii) $g : M \to N$ is a morphism if $\varphi(x) \in \Phi$, $a \in M$, $M \models \varphi[a]$ implies $N \models \varphi[g(a)]$ (in short g is a Φ-morphism) ([10])

([10]) Note that $M <_{\Phi} N$ is not equivalent to he identity on $|M|$ being a Φ-morphism into N.

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(B) $K^{\alpha}(T, \Phi)$ is $K^{\alpha}(\text{sub } \Phi)$ where $K = K(T, \Phi)$.

Definition 2.9. $\mathscr{D}(T, \Phi) = \{ tp(\bar{a}, \Phi, M) : \bar{a} \in M \in K^{\omega}(T, \Phi) \}$.

Lemma 2.10.

(1) $K(T, \Phi)$ *satisfies the assumptions before definition* 2.6 *when*
   (A) $T \subseteq \forall \exists \Phi$.
   (B) Φ *is closed under subformulas except possibly the subformulas of a negation of an atomic subformula.*
   (C) *For every atomic* $R(\bar{x})$, $R(\bar{x}) \in \Phi$ *or* $\neg R(\bar{x}) \in \Phi$.

(2) *If in addition* $K(T, \Phi)$ *has the* J M P, *then* $\mathscr{D} = \mathscr{D}(T, \Phi)$ *satisfies the assumption from § 1, which corresponds to kind IV.*

(3) *Assumption III is satisfied in* $\mathscr{D}(T, \Phi)$ *for* $\exists \wedge \Phi$.

(4) *If* $K(T, \Phi)$ *has the amalgamation property,* T *universal,* $\Phi_{qf} = \Phi$, *then in (3) we can replace* $\exists \wedge \Phi$ *by* Φ_a, *and assumption II is satisfied.*

(5) *For a universal theory with the J E P,* $\mathscr{D}^e(T) = \mathscr{D}(T, \Phi_{qf})$; $T^{\omega} = K^{\omega}(T, \Phi_{qf})$. *If* $T_{\ell} \subseteq \forall \exists \Phi_{qf}$, T_1, T_2 *have the same universal consequences, then for each* $\alpha > 0$, $K^{\alpha}(T_1, \Phi_{qf}) = K^{\alpha}(T_2, \Phi_{qf})$. [11]

Remarks: Having proved that kinds II, III satisfy the appropriate assumptions, we will now prove conversely that any example (of kind IV) which satisfies assumption II or III is essentially of kind II or III. In 2.11 by changing the language, we get to «infinite positive forcing», and in 2.12 by also adding more elements we get to «infinite forcing».

Theorem 2.11. In assumption III we can restrict ourselves to $\mathscr{D}(T, \Phi_a)$ *for a universal-negative* T *with the J M P,* $K(T^{\circ}, \Phi_a)$ *having the amalgamation property. More exactly, let* $\mathfrak{C}$, Φ *be as in assumption III, and let*

[11] We can prove it as before, or add names for the appropriate formulas.

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(i) $L^* = \{R_{\varphi(\bar{x})} : \varphi(\bar{x}) \in \Phi\}$ *where* $R_{\varphi(\bar{x})}$ *is a predicate with* $l(\bar{x})$*-places and* $R_{x=y}(x_1, y_1)$ is $x_1 = y_1$.

(ii) $\mathfrak{C}^* = (|\mathfrak{C}|, \ldots, R_\varphi^{\mathfrak{C}^*}, \ldots)$ *where* $R_\varphi^{\mathfrak{C}^*} = \{\bar{a} \in \mathfrak{C} : \mathfrak{C} \models \varphi[\bar{a}]\}$

(iii) T^* *is the negative universal theory of* $\mathfrak{C}^*$.

Then

(1) $\mathfrak{C}^*$ *is* $\mathcal{D}(T^*, \Phi_a)$*-homogeneous,* $K(T^*, \Phi_a)$ *has the JMP.*

(2) $tp(\bar{a}, \emptyset, \mathfrak{C}) = tp(\bar{b}, \emptyset, \mathfrak{C})$ *iff* $tp(\bar{a}, \emptyset, \mathfrak{C}^*) = tp(\bar{b}, \emptyset, \mathfrak{C}^*)$.

(3) $A \subseteq |\mathfrak{C}|$ *is the universe of a* $(\mathcal{D}, \lambda)$*-homogeneous elementary submodel of* $\mathfrak{C}$, *iff it is the universe of a* $(\mathcal{D}(T^*, \Phi_a, \lambda)$*-homogeneous elementary submodel of* $\mathfrak{C}^*$.

(4) $\mathcal{D}, \Phi$ *satisfy assumption II iff* $\mathcal{D}(T^*, \Phi_a), \Phi_a$ *satisfy assumption II.*

(5) *If* $\Phi = \exists \wedge \Phi_1$, *we can replace* $\mathfrak{C}^*$ *by* $\mathfrak{C}^*_1 = (\mathfrak{C}, \ldots, R_\varphi, \ldots)_{\varphi \in \Phi_1}$. *If, in addition,* $\mathcal{D} = \mathcal{D}(T, \Phi_1)$ *as in lemma* 2.10 (1), *then* $\mathfrak{C} \restriction A <_{\exists \wedge \Phi_1} \mathfrak{C}$ *iff* $\mathfrak{C}^*_1 \restriction A <_{ep} \mathfrak{C}^*_1$ *iff* $\mathfrak{C}^* \restriction A <_{ep} \mathfrak{C}^*$.

Theorem 2.12. In 2.10 (1) *the case «*T *universal,* $\Phi = \Phi_a$*» is a particular case of «*T *universal,* $\Phi = \Phi_{qf}$*», except that equality is not standard. More exactly, let* T *be universal,* $\Phi = \Phi_a$, $\mathcal{D} = \mathcal{D}(T, \Phi)$, $\mathfrak{C}$ *is* $(\mathcal{D}, \lambda_0)$*-homogeneous* $(\lambda_0 = \|\mathfrak{C}\|)$. *We define* $\mathfrak{C}^+$ *as follows:*

(i) $|\mathfrak{C}^+| = |\mathfrak{C}| \times \lambda_0 \cup \{<\bar{a}, R, \bar{\alpha}> : R(\bar{x})$ *is an atomic formula,* $\mathfrak{C} \models R[\bar{a}]$, $\bar{\alpha} \in {}^{l(\bar{x})}\lambda_0\} \cup \lambda_0$ *(remember* $\lambda_0 = \{\alpha : \alpha < \lambda_0\}$)

(ii) $P^{\mathfrak{C}^+} = |\mathfrak{C}| \times \lambda_0$ *(a one place relation)*

(iii) $P_R^{\mathfrak{C}^+} = \{<\bar{a}, R, \bar{\alpha}> : \mathfrak{C} \models R(\bar{a}), \bar{\alpha} \in {}^{l(\bar{a})}\lambda_0\}$ *(a one place relation)*

(iv) $Q_R^{\mathfrak{C}^+} = \{<<a_0, \alpha_0>, \ldots, <a_{n-1}, \alpha_{n-1}>, <<a_0, \ldots, a_{n-1}>,$

$R, \bar{\alpha}>> : \alpha_l < \lambda_0, \mathfrak{C} \models R[a_0, \ldots, a_{n-1}], \bar{\alpha} = <\alpha_0, \ldots, \alpha_{n-1}>\}$

(v) *let* T^+ *be the universal theory of* $\mathfrak{C}^+$.

Then (1) $\mathfrak{C}^+$ *is* $(\mathscr{D}(T^+, \Phi_{qf}), \lambda_0)$*-homogeneous*

(2) *for* $\bar{a}, \bar{b} \in |\mathfrak{C}|$, $tp(\bar{a}, \emptyset, \mathfrak{C}) = tp(\bar{b}, \emptyset, \mathfrak{C})$ *iff* $tp(\bar{a}^*, \emptyset, \mathfrak{C}^+) = tp(\bar{b}^*, \emptyset, \mathfrak{C}^+)$ where $a^* = <a, o>$, and $<a_0, \ldots>^* = <a_o^*, \ldots>$.

(3) *for* $A \subseteq \mathfrak{C}^+$, *the following are equivalent*

(α) $\mathfrak{C}^+ \restriction A < \mathfrak{C}^+$

(β) $A' = \{a : <a, \alpha> \in A$ *for some* $\alpha < \lambda$, *and* $a \in \mathfrak{C}\}$ *is the universe of an elementary submodel of* $\mathfrak{C}$ *and if* $a_i^* = <a_i, \alpha_i> \in A$, $\mathfrak{C} \models R[a_0, \ldots]$ *then* $<<a_0, \ldots>, R, \bar{\alpha}> \in A$ *for some* $\bar{\alpha}$ *and* $<<a_0, \ldots>, R, \bar{\alpha}> \in A$ *implies* $a_0, \ldots \in A'$.

(4) *In (3) we can replace* $<$ *by* $<_e$ *in* (α) *and* $<_{ep}$ *in* (β).

Remark: Hence we can give examples for $K^\omega(T, \Phi_a)$ instead $K^\omega(T, \Phi_{qf})$, and almost always the relevant properties are preserved (except e.g. categoricity).

Remark: Normed spaces can be treated in this context. Let T_{NS} say that under + the model is an abelian group, and letting $R_a^b(x)$ to mean $a \leqslant \|x\| \leqslant b$, and $F_a(x)$ be ax (scalar multiplication) (a, b real numbers) that the natural conditions hold (e.g. $R_o^a(x) \wedge R_o^b(y) \rightarrow R_o^{a+b}(x+y)$ and $[a, b] = [a_1, b_1] \cup [a_2, b_2]$ implies $R_b^a(x) \equiv R_{b_1}^{a_1}(x) \vee R_{b_2}^{a_2}(x)$). Then the normed spaces are essentially $K^o(T_{NS}, \Phi_a)$. For the class of L_p-spaces we get stability (by an unpublished result of Krivine), and «forgetting completeness», categoricity for Hilbert space.

However, for Banach spaces, we should look only at, essentially, algebraically closed models, and replace cardinality by

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density character; and then for L_p-spaces we get $\aleph_0$-stability. Somewhat more complicated is the case of injective hull. Those variations will be dealt with in Abramsky Ph. D. thesis. Notice that $(\mathcal{D}(T_{NS}, \Phi_a), \aleph_1)$-homogeneous models are Banach spaces.

Remark. Note that if T, Φ_ℓ ($\ell = 1, 2$) satisfies the hypothesis of 2.10 (1), (2), and $\Phi_1 \subseteq \Phi_2$, then: if $\mathcal{D}(T, \Phi_2)$ is stable in λ, then $\mathcal{D}(T, \Phi_1)$ is stable in λ. So if (every completion of T) is λ-stable, $\mathcal{D}(T, \Phi_1)$ is λ-stable.

§ 3. *Ranks and Stability*

We work here in the context of assumption III, so L, $\mathfrak{C}$, $\mathcal{D}$, Φ are given and are fixed.

We let $\varphi(\bar{x}; \bar{y})$ denote a pair $\langle \underline{\varphi}_0(\bar{x}; \bar{y}), \varphi_1(\bar{x}; \bar{y}) \rangle$ where $\varphi_0, \varphi_1 \in \Phi$ are contradictory. Let $\underline{\varphi}(\bar{x}; \bar{y})^\ell = \varphi_\ell(\bar{x}; \bar{y})$ for $\ell = 0, 1$ and $M \models \underline{\varphi}[a, b]$ mean $M \models \varphi_0 [a, b]$.

Let $\underline{\Delta}$ denote a set of pairs $\underline{\varphi}$. A $\underline{\Delta}$-m-type is a (onsistent) set of formulas $\underline{\varphi}(\bar{x}, \bar{a})^t$ ($t \in \{0, 1\}$, $\bar{x} = \langle x_0, \ldots, x_{m-1} \rangle$.) Let $\underline{\Phi}^1 = \{\underline{\varphi} : \varphi_\ell \in \overline{\Phi}^1$ contradictory, $\ell = 0, 1\}$ [12].

Definition 3.1. We define the *rank* $R^m(p, \underline{\Delta}, \lambda)$ as an ordinal or ∞ or -1, where p is an m-type (or a set of m-formulas), and $\underline{\Delta}$ a set of pairs $\underline{\varphi}$ and $\lambda \geqslant 2$ is a cardinal or ∞. We stipulate $-1 < \alpha < \infty$ for any ordinal α, and define the rank as follows:

(i) $R^m(p, \underline{\Delta}, \lambda) \geqslant 0$ iff p is consistent, or equivalently finitely satisfiable in $\mathfrak{C}$.

(ii) $R^m(p, \underline{\Delta}, \lambda) \geqslant \delta$ (δ a limit ordinal) iff for any $\alpha < \delta$ $R^m(p, \underline{\Delta}, \lambda) \geqslant \alpha$.

(iii) $R^m(p, \underline{\Delta}, \lambda) \geqslant \alpha + 1$ iff for any finite $q \subseteq p$ and any $\mu < \lambda$ there are $\underline{\Delta}$-m-types $r_j (j \leqslant \mu)$ such that:

(a) $R^m(q \cup r_j, \underline{\Delta}, \lambda) \geqslant \alpha$

[12] For kind II there is no need for $\underline{\Phi}^1$, and we do not distinguish strictly between it and Φ.

and (b) the r_j are pairwise explicitly contradictory, that is for any distinct $i, j \leqslant \mu$ there is $\underline{\varphi}^t \in r_i$ such that $\underline{\varphi}^{1-t} \in r_j$.

Remarks. (1) $j \leqslant \mu$ occurs in (iii) (rather than $j < \mu$) so that $\lambda = 2$ (and not $\lambda = 3$) is the first interesting case.

(2) The most important cases for $\underline{\Delta}$ are $\underline{\Delta} = \underline{\Phi}$ (all possible pairs) and $\underline{\Delta}$ finite or with one pair (which are essentially equivalent). The latter case is interesting because the splitting types r_j will consist only of instances (using different parameters) of finitely many formulas of Φ, and hence we can apply compactness arguments.

(3) The most interesting cases for λ in what follows will be $\lambda = 2$ and $\lambda = \infty$. When we assume Kind II (section 6, on forking), $\lambda = \aleph_0$ is interesting too. Notice that in these cases the definition is absolute.

Lemma 3.2. (1) *If* $p_1 \subseteq p_2$ [or even $p_2 \vdash p_1$; i.e. for any finite $q_1 \subseteq p_1$ there is a finite $q_2 \subseteq p_2$ such that $\mathfrak{C} \models (\forall \bar{x}) (\wedge q_2 \rightarrow \wedge q_1)$] *and* $\underline{\Delta}_1 \supseteq \underline{\Delta}_2$ *and* $\lambda_1 \leqslant \lambda_2$, *then* $R^m(p_1, \underline{\Delta}_1, \lambda_1) \geqslant (R^m(p_2, \underline{\Delta}_2, \lambda_2)$.

(2) *For every* $p, \underline{\Delta}, \lambda$ *there is a finite* $q \subseteq p$ *such that* $R^m(q, \underline{\Delta}, \lambda) = R^m(p, \underline{\Delta}, \lambda)$.

Proof. (1) Check the definition.

(2) If $R^m(p, \underline{\Delta}, \lambda) = \infty$, choose $q = \emptyset$. If $R^m(p, \underline{\Delta}, \lambda) = -1$, p is inconsistent, so choose some finite inconsistent $q \subseteq p$. So let $R^m(p, \underline{\Delta}, \lambda) = \alpha < \infty$; then $R^m(p, \underline{\Delta}, \lambda) \not\geqslant \alpha + 1$, hence by the definition there is a finite $q \subseteq p$ for which there are no suitable r_j's. As q is a finite subset of q, also $R^m(q, \underline{\Delta}, \lambda) \not\geqslant \alpha + 1$; but, by (1), $R^m(q, \underline{\Delta}, \lambda) \geqslant R^m(p, \underline{\Delta}, \lambda)$, so the conclusion follows.

We note the following trivial but important claim:

Claim 3.3. If $R^m(p, \underline{\Delta}, 2) = \alpha < \infty$, *then there is no* $\underline{\varphi} \in \underline{\Delta}$ *and* $\bar{a}$ *such that* $R^m(p \cup \{\underline{\varphi}(\bar{x}; \bar{a})^t\}, \underline{\Delta}, 2) = \alpha$ *for* $t = 0,1$.

Remark. This claim explains why the rank for $\lambda = 2$ is important. The extension property (i.e. for each p which is realized and A, there is a complete type q over A such that $R^m(p, \underline{\Delta}, \lambda) = R^m(p \cup q, \underline{\Delta}, \lambda)$ and $p \cup q$ is realized) fails in general, but it holds for $\underline{\Delta}$ finite, $\lambda = \infty$, or (as noted by Hinkis) for $\lambda \geqslant \aleph_0$, p a Φ-m-type, when $\mathscr{D}$, Φ satisfy assumption II. If we waive the demand «p ∪ q is realized in $\mathfrak{C}$», there will be no problem, but not much interest too. Notice that for Φ-types, being consistent is equivalent to being realized.

Remark. There are quite accurate results specifying when $R^m(p, \underline{\Delta}, \lambda) \geq \alpha \Rightarrow R^m(p, \underline{\Delta}, \lambda) = \infty$ and when $R^m(p, \underline{\Delta}, \lambda) = R^m(p, \underline{\Delta}, \mu)$.

Lemma 3.4. For every finite p, $R^m(p, \underline{\varphi}, 2) \geq n$ *iff there are* $\bar{a}_\eta$ $(\eta \in {}^{n>}2)$ *such that for any* $\eta \in {}^n 2$, $p \cup \{\underline{\varphi}(\bar{x}, \bar{a}_{\eta \restriction \ell})^{\eta(\ell)} : \ell < n\}$ *is consistent* (notice we write $\underline{\varphi}$ for $\{\underline{\varphi}\}$).

Proof. $\Leftarrow$: let $p_\eta = p \cup \{\varphi(\bar{x}, \bar{a}_{\eta \restriction \ell})^{\eta(\ell)} : \ell < \ell(\eta)\}$ and now prove that $R^m(p_\eta, \underline{\varphi}, 2) \geq n - \ell(\eta)$ by downward induction on $\ell(\eta)$.

$\Rightarrow$: merely define $\bar{a}_\eta$ by induction on $\ell(\eta)$ such that $\eta \in {}^\ell 2$ implies $R^m(p_\eta, \underline{\varphi}, 2) \geq n - \ell(\eta)$.

Remark. The requirement that p be finite can be eliminated by a compactness argument when p is a Φ-m-type.

Lemma 3.5. If $R^m(\emptyset, \underline{\varphi}, 2) \geq \omega$, *then* $\mathscr{D}$ *is unstable* (in every cardinal λ).

Proof. Let $\mu = \min\{\mu : 2^\mu > \lambda\}$, so $\sum_{\alpha<\mu} 2^{|\alpha|} \leq \lambda$.

Let $\Gamma_\mu = \{\underline{\varphi}(\bar{x}_\eta ; \bar{y}_{\eta \restriction \alpha})^{\eta(\alpha)} : \eta \in {}^\mu 2, \alpha < \mu\}$. Now Γ_μ is finitely satisfiable in $\mathfrak{C}$ as it is sufficient to check each Γ_n, which is consistent by the previous lemma and the hypothesis. [13] Let $A = \cup\{\bar{y}_\eta : \eta \in {}^{\mu>}2\}$, and then the $\bar{x}_\eta$ $(\eta \in {}^\mu 2)$ realize differ-

[13] As Γ is a set of Φ-formulas, some assignment satisfies it.

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ent types over A (in $\mathfrak{C}$). Now $|A| \leqslant \aleph_0 \sum_{\alpha<\mu} 2^{|\alpha|} \leqslant \lambda <$ $2^{\mu} \leqslant |\{tp(\bar{x}_{\eta}, A) : \eta \in {}^{\mu}2\} \leqslant |S^{m}_{\mathcal{D}}(A)|$

Theorem 3.6. The following are equivalent:
(1) $\mathcal{D}$ *is stable,*

(2) $\mathcal{D}$ *is stable in every* $\lambda = \lambda^{|L|}$,

(3) *For every* $\underline{\varphi}$, $R^m(\emptyset, \underline{\varphi}, 2) < \omega$,

(4) *There are no* $\underline{\varphi}$ *and* $\bar{a}_n$ *such that* $\models \underline{\varphi}[\bar{a}_n, \bar{a}_m]^{\text{if}(n<m)}$ *where* $\text{if}(n<m) = \begin{cases} 0 & \textit{when } n < m \\ 1 & \textit{otherwise.} \end{cases}$

Remark. Only after this theorem stability makes sense: as by (1), (2) it is not an accident (i.e. «a priori» it is possible that $\mathcal{D}$ is stable just in one cardinal, and this will not give enough information), (3) enables us to prove theorems on stable $\mathcal{D}$, and (4) on unstable $\mathcal{D}$.

Proof. (1) $\Rightarrow$ (4) $\Rightarrow$ (2) : by 1.9.
(2) $\Rightarrow$ (1) : trivial
(1) $\Rightarrow$(3) : by the previous lemma
(3) $\Rightarrow$ (2) : Suppose $|A| \leqslant \lambda = \lambda^{|L|}$. For each $p \in S^m_{\mathcal{D}}(A)$ and $\underline{\varphi}$ choose a finite $p_{\underline{\varphi}} \subseteq p$ such that $R^m(p_{\underline{\varphi}}, \underline{\varphi}, 2) = R^m(p, \underline{\varphi}, 2)$. Let $p^* = \bigcup_{\underline{\varphi}} p_{\underline{\varphi}}$, so $|p^*| \leqslant |L|$. Clearly for every $\underline{\varphi}$, $R^m(p, \underline{\varphi}, 2)$ $= R^m(p^*, \underline{\varphi}, 2)$.
Now suppose $p, q \in S^m_{\mathcal{D}}(A)$, $p \neq q$, then there is $\bar{a} \in A$ such that $p \restriction \bar{a} \neq q \restriction \bar{a}$; hence $p_1 \neq q_1$ where $p_1 = \{\varphi(\bar{x}, \bar{y}) : \underline{\varphi}(\bar{x}, \bar{a}) \in p\}$, $q_1 = \{\underline{\varphi}(\bar{x}, \bar{y}) : \underline{\varphi}(\bar{x}, \bar{a}) \in q\}$. So there are contradictory $\varphi_{\ell} \in \Phi$ such that $\varphi_0(\bar{x}, \bar{y}) \in p_1$, $\varphi_1(\bar{x}, \bar{y}) \in p_2$. So letting $\underline{\varphi} = \langle\varphi_0(\bar{x}, \bar{y}); \varphi_1(\bar{x}, \bar{y})\rangle$, $\underline{\varphi}(x, \bar{a})^0 \in p$, $\underline{\varphi}(x, \bar{a})^1 \in q$. Hence, by

claim 3.3, it is impossible that $p^* = q^*$, Hence $|S^m_{\mathcal{D}}(A)| \leqslant |\{p^* : p \in S^m_{\mathcal{D}}(A)\}| \leqslant |A|^{|L|} \leqslant \lambda^{|L|} = \lambda$.

We can prove similarly

Theorem 3.7. The following conditions are equivalent:

(1) *$\mathcal{D}$ is stable in every $\lambda \geqslant 2^{|L|}$,*

(2) *$\mathcal{D}$ is stable in some λ, $\lambda^{\aleph_0} > \lambda$,*

(3) $R^m(\emptyset, \Phi, \infty) < \infty$.

If $\mathcal{D}$ satisfies any of the above, it is called *superstable.*

Theorem 3.8. The following conditions are equivalent when $|L| < 2^{\aleph_0}$

(1) *$\mathcal{D}$ is stable in every $\lambda \geqslant |L|$,*

(2) *$\mathcal{D}$ is stable in some $\lambda < 2^{\aleph_0}$,*

(3) $R^m(\emptyset, \Phi, 2) < \infty$.

If $\mathcal{D}$ satisfies (3), it is called *totally transcendental.*

§ 4. *Prime models*

We again work in the context of assumption III.

Définition 4.1.

Let K be a class of models $M \subseteq \mathfrak{C}$.
We call M K-prime over A if

(1) $A \subseteq M \in K$

(2) If $A \subseteq M' \in K$ then there is an embedding $f : M \rightarrow M'$ such that $f \restriction A$ is the identity and f is elementary in $\mathfrak{C}$ (that is can be extended to an automorphism of $\mathfrak{C}$).

Remark

The natural way to build a prime model over A is to add to it elements a_i one-by-one, so that each a_i realizes over $A_i = A \cup \{a_j : j < i\}$ a type which is realized in every M, $A_i \subseteq M \in K$. In the tractable cases this will be an isolated type.

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There is an axiomatic way (see [S 1] ch. IV) to describe the construction of a model M over a set A, by adding step by step, each time one element realizing a type of a specific sort (usually isolated in a proper sense).

Tractable interesting cases here are e.g. the e.c. models of T (T from § 2), the infinite generic models, the $(\mathcal{D}^e(T),\lambda)$-homogeneous models, the λ-$\exists_n$-compact structures (i.e. the set of models M, such that any $\exists_n$-type of cardinality $<\lambda$ over M, is realized in M). To prove existence, uniqueness etc., we have to assume usually, some stability assumptions.

This axiomatic treatment not only includes all previous cases and in it we can prove the known results (even the uniqueness of the constructed model) and see what hypothesis is needed for which theorem, but also shows that we can replace isolation by e.g. definability (e.g. the anti-prime models). See remarks to 6.10 [14].

Definition 4.2.

Let p be a type over A. We say that a formula $\varphi(\bar{x};\bar{b})$ *isolates* p if $\varphi(\bar{x};\bar{b}) \in p$ and $\models (\forall \bar{x})\ ((\varphi(\bar{x};\bar{b}) \rightarrow \psi)$ for all $\psi \in p$.
We say p is *isolated* if some formula isolates it.

Lemma 4.3.

Suppose $\mathcal{D}$ is totally transcendental. Then, for each Ψ $(*)_\Psi$ for every B, $\bar{b} \in B$ and $\varphi \in L$, if $\models (\exists x)\, \varphi(x,\bar{b})$ then there is $\varphi_1(x,\bar{b}_1)$ $(\varphi_1 \in \Phi, \bar{b}_1 \in B)$ and a^* such that $\models (\exists x)[\varphi(\bar{x},\bar{b}) \wedge \varphi_1(x,\bar{b}_1)]$ and $\varphi(x,\bar{b}) \wedge \varphi_1(x,\bar{b}_1) \vdash tp_\Phi(a^*, B)$.

Proof.
Choose $\varphi_1(x,\bar{b}_1)$, a Φ-formula, such that

[14] Added in proof (June 76): This can be applied also to the following. Let M be an e.c. group such that: if $H \subseteq L$, K are finitely generated subgroups of M, the free product with amalgamation $L \underset{H}{*} K$ is a subgroup of M. Let N be a group, all whose finitely generated subgroups "appear" in M. We can find a "canonical" (e.g. unique) closure of N to a group $L_{\infty,\omega}$-equivalent to M, by the above method.

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(i) $\models (\exists x)[\varphi(x, \bar{b}) \wedge \varphi_1(x, \bar{b}_1)]$
(ii) relative to (i), $\alpha = R^1(\varphi_1(x, \bar{b}_1), \Phi, 2)$ is minimal.

By 3.8 $\alpha < \infty$, and by 3.3 $\varphi(x, \bar{b}) \wedge \varphi_1(x, \bar{b}_1)$ isolates some type $tp_\Phi(a^*, B)$.

Definition 4.4.

Let $A \subseteq B$. Let $\Psi \subseteq L$ be closed under subformulas, $\Psi = \exists \Psi$. By a *Ψ-isolating sequence* of B over A, we mean a sequence $<<a_j, \varphi_j(x, \bar{b}_j)> : j < \alpha>$ such that $B = A \cup \{a_j : j < \alpha\}$, $b_j \subseteq A_j = A \cup \bigcup_{i<j} a_i$ and $\varphi_j(x, \bar{b}_j)$ is a Ψ-formula such that $\varphi_j(x, \bar{b}_j) \vdash tp_\Phi(a_j, A_j)$ and $\models \varphi_j[a_j, \bar{b}_j]$ (equivalently $\varphi_j(x, \bar{b}_j)$ isolates $tp(a_j, A_j)$, since $tp_\Phi(a_j, A_j)$ determines $tp(a_j, A_j)$).

We say M is *strictly Ψ-prime over* A if M has a Ψ-isolating sequence over A and M is Ψ-closed: i.e. for every $\varphi(\bar{x}) \in \Psi = \text{Sub}\,\Psi$, $M \models \varphi[\bar{a}] \Leftrightarrow \mathfrak{C} \models \varphi[\bar{a}]$ for each $\bar{a} \in M$.
M is *Ψ-prime over* B, if it is prime among the Ψ-closed models. If $\Psi = \exists \wedge \Phi$ we omit it.

Theorem 4.5. (The existence of prime models)
(1) *If M is strictly Ψ-prime over A, then it is Ψ-prime over A, i.e. prime among the Ψ-closed $M \subseteq \mathfrak{C}$.*
(2) *If $\mathcal{D}$ is totally transcendental, then over any A there is a strictly Ψ-prime model.*
(3) *In (2), it suffices to assume that $(*)_\Psi$ of 4.3 holds.*

Proof.
(1) *If* $<<a_i, \varphi_i(x, \bar{b}_i)> : <\alpha>$ is a Ψ-isolating sequence of M over A, N is Ψ-closed, $A \subseteq N$, then we define $c_i \in N$, by induction such that $tp(<a_j : j \leqslant i>, A) = tp(<c_j : j \leqslant i>, A)$. If we have defined for each $j < i$, $\bar{b}_i = \bar{b}_i^1{}^\frown <a_{j(i,1)}, \ldots>$, $\bar{b}_i^1 \in A$, $j(i, l) < i$, choose c_i to satisfy $\varphi_i(x, \bar{b}_i^1, c_{j(i,1)}, \ldots)$.
(2), (3) Use 4.3 (*) repeatedly.

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Lemma 4.6.

If L *is countable and* $\Psi = \exists \wedge \Phi$ *then* $(*)_\Psi$ *in lemma* 4.3 *is actually equivalent to:*
$(**)_\Psi$ *over every* A *there is a* Ψ*-prime model of* T.

Proof.
$(*)_\Psi \Rightarrow (**)_\Psi$ essentially this is the previous theorem.
$(**)_\Psi \Rightarrow (*)_\Psi$: suppose $(*)_\Psi$-fails and B, $\varphi(x, \bar{a})$ $(\bar{a} \in B, \varphi \in \Psi)$

is a counterexample; that is no $\varphi_1(x, \bar{b}_1)$ is suitable. So for every $\varphi_1 \in \Psi$, $\bar{b}_1 \in B$, $\models \neg(\exists x)[\varphi_1(x, \bar{b}_1) \wedge \varphi(x, \bar{b})]$ or there are $\vartheta_1, \vartheta_2 \in \Phi$, $\bar{a}_1\, \bar{a}_2 \in B$ such that $\models (\exists x)(\varphi(x, \bar{b}) \wedge \varphi_1(x, \bar{b}_1) \wedge \vartheta_\ell(x, \bar{a}_\ell))$ $(\ell = 1{,}2)$ and $\vartheta_\ell(x, \bar{a}_\ell)$ $(\ell = 1{,}2)$ are contradictory.
By a Löwenhein-Skolem argument, we can assume B is countable.
Suppose M is Ψ-prime over B, so M is countable.
Let $\Gamma = \{tp_\Phi(a^*, B) : a^* \in M,\ a^* \notin B, \models \varphi(a^*, \bar{b})\}$.

It is easy to find a Ψ-closed model M_1, $B \subseteq M_1$, in which no $p \in \Gamma$ is realized.

Lemma 4.7.

If M *is strictly* Ψ*-prime over A then*
(1) *For every* $\bar{c} \in M$, *there is an* $(\exists \wedge \Psi)$*-formula* $\psi(\bar{x}, \bar{a})$, $(\bar{a} \in A)$, *which isolates* $tp_\Phi(\bar{c}, A)$ *and which* $\bar{c}$ *satisfies.*

(2) *We can find a* Ψ*-isolating sequence of* M *over* A *of length* $\|M\|$.

Remark.
For $\mathcal{D}$ totally transcendental we can characterize the Ψ-prime models as in [S 9], hence

Theorem 4.8.

For $\mathcal{D}$ totally transcendental $\Phi \subseteq \Psi = \exists \Psi$ closed under subformulas, the Ψ-closed prime model over A *is unique over* A. If $L = \Phi = \Psi$, $|L| = \aleph_0$, $\mathcal{D}$ is stable (so $\mathcal{D} = \mathcal{D}(T, L)$) then we can prove uniqueness when there is a strictly Ψ-prime model, but we have no characterization. But even for countable theories (i.e. $\mathcal{D} = \mathcal{D}(T, L)$) which satisfy 4.3 (*), we do not know if uniqueness necessarily holds ([15]). We think the answer is negative, and the following proof may give us an idea how a counterexample might look. The situation is the same for uncountable T.

Aid.

The proof of the next theorem will use stationary sets. The main facts we need are as follows.

Let $\varkappa$ be a fixed cardinal of cofinality $> \aleph_0$. Then $X \subseteq \varkappa$ is *closed unbounded* if $\varkappa = \sup X$ and for each $\alpha < \varkappa$, $\alpha = \sup (X \cap \alpha)$ implies $\alpha \in X$.

We call $S \subseteq \varkappa$ *stationary* if S meets every closed unbounded set.

Fact A: The intersection of two (or even $\lambda < \mathrm{cf}\varkappa$) closed unbounded subsets of $\varkappa$, is closed and unbounded.

Fact B: If $X \subseteq \varkappa$ is closed unbounded and $F : X \to \varkappa$ is normal (i.e. strictly increasing and continuous), then the set of fixed points of F (i.e. $\{\alpha : F(\alpha) = \alpha\}$) is a closed unbounded set.

Fact C (Fodor's Theorem): If S is a stationary subset of $\varkappa$, and $F : S \to \varkappa$ is regressive (i.e. $\alpha \in S \Rightarrow F(\alpha) < \alpha$) and $\varkappa$ is regular (i.e. $\mathrm{cf}\,\varkappa = \varkappa$) then F is constant on some stationary $S^* \subseteq S$. (See e.g. Juhasz [J 1]). We use those facts freely.

The Uniqueness Theorem 4.9.

Suppose L *is countable, $\mathcal{D}$ is stable and* $\Phi = L$. *If over* A *there is a strictly prime model* M, *then any prime model over* A *is isomorphic to it.*

([15]) Added in proof June 76: There are countable (complete) theories T, which satisfy $(*)_L$, but the uniqueness fails.

Proof:

We prove it only when $|A| \leqslant \aleph_1$. If $|A| < \aleph_0$ then Vaught's argument works. Henceforth let $|A| = \aleph_1$, and let $<<a_j, \varphi_j(x, \bar{a}_j)> : j < \omega_1>$ be an isolating sequence for M over A, and let N be prime over A, so we can assume $A \subseteq N \subseteq M$. Let M_i $(i < \omega_1)$ be countable models such that $\bigcup_{i<\omega_1} M_i = M$

There is a closed unbounded $X \subseteq \omega_1$ such that for each $i \in X$

(i) $M_i < M, M_i \cap N < N$
(ii) $M_i \cap \{a_j : j < \omega_1\} = \{a_j : j < i\}$
(iii) $j < i$ implies $\bar{a}_j \in M_i$
(iv) for each $\bar{c} \in M_i$, $q = tp(\bar{c}, A)$ is isolated by some $\varphi(\bar{x}) \in q \restriction (A \cap M_i)$.

We define
$S_1 = \{i \in X : \text{for some } \bar{b} \in N,\ q = tp(\bar{b}, A \cup (N \cap M_i)) \text{ is not isolated}\}$.

Case 1: S_1 is not stationary.
We shall show that in this case, N is isomorphic to M over A. There is by assumption a closed unbounded set X_1 disjoint from S_1. Hence there is a closed unbounded $X^* \subseteq X_1 \cap X$, such that for each $i, j \in X^*$, $i < j$ and $\bar{b} \in N \cap M_j$, $q = tp(\bar{b}, A \cup (N \cap M_i))$ is isolated by some $\varphi(\bar{x}, \bar{c}) \in q$, $\bar{c} \in (A \cap M_j) \cup (N \cap M_i)$.
We shall define, by induction on $i \in X^*$, elementary maps $F_i : A \cup (N \cap M_i) \to A \cup M_i$ such that:
— $F_i \restriction A$ is the identity
— $i < j$; $i, j \in X^*$ implies that F_j extends F_i.
For i a limit (in X^*) $F_i = \bigcup_{j<i, j \in X^*} F_j$. Now suppose $i, j \in X^*$, j the sucessor of i in X^* and F_i is defined. We shall define F_j, just by Vaught's argument, because $M_j \cap N$, M_j are prime over $(M_j \cap A) \cup (M_i \cap N)$, $(M_j \cap A) \cup M_i$ resp.. For the first $i \in X^*$ we define F_i similarly.
Clearly $\bigcup_{i<\omega_1} F_i$ **gives the required isomorphism from** N **onto** M over A.

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Case 2: S_1 is stationary

We shall show that this implies a contradiction. For all $i \in S_1$, there is $\bar{b}_i \in N$ such that $tp(\bar{b}_i, A \cup (N \cap M_i))$ is not isolated. However, since M has an L-isolating sequence, for each $i \in S_1$, $q_i = tp(\bar{b}_i, A \cup M_i)$ is isolated by some $\varphi_i(\bar{x}, \bar{a}_i, \bar{c}_i) \in q_i$ where $\bar{a}_i \in A$, $\bar{c}_i \in M_i$. By Fodor's theorem there are a stationary set $S_2 \subseteq S_1$, a formula φ and a sequence $\bar{c}$ such that for each $i \in S_2$, $\varphi_i = \varphi$ and $\bar{c}_i = \bar{c}$.

Choose $i_0 < i_1 < \ldots < i_k < \ldots$ $(k < \omega)$ in S_2 such that $a_{i_k}, b_{i_k} \in M_{i_{k+1}}$.

For each $n < \omega$ we can find $\bar{c}^*_n \in N \cap M_{i_n}$ such that $tp(\bar{c}^*_n, A \cup \bigcup_{\ell < n} \bar{b}_{i_\ell}) = tp(\bar{c}, A \cup \bigcup_{\ell < n} \bar{b}_{i_\ell})$ (if we replace A by $A \cap M_{i_n}$, this holds as M_{i_n} is strictly prime over $A \cap M_{i_n}$; and the same $\bar{c}^*_n$ is good by the choice of X, and as $S_2 \subseteq S_1 \subseteq X$).

Now note.

First Fact: If $k \leqslant n$ then $\models \varphi[\bar{b}_{i_k}, \bar{a}_{i_k}, \bar{c}^*_n]$.

This holds by the choice of $\bar{c}^*_n$.

Second Fact: If $k > n$ then $\models \neg\varphi[\bar{b}_{i_k}, \bar{a}_{i_k}, \bar{c}^*_n]$. As in case 1, there is an elementary map F, Dom F $= A \cup M_{i_k}$, $F \restriction A =$ the identity, F maps $\bar{b}_{i_\ell}$, $\bar{c}$ $(\ell < n)$ to $\bar{b}_{i_\ell}$, $\bar{c}^*_n$ $(\ell < n)$ and F maps M_{i_k} onto M_{i_k}.

Hence $\varphi(\bar{x}, \bar{a}_{i_k}, \bar{c}^*_n)$ isolates a complete type over $A \cup M_{i_k}$. So if $\models \varphi[\bar{b}_{i_k}, \bar{a}_{i_k}, \bar{c}^*_n]$ then it isolates $tp(\bar{b}_{i_k}, A \cup M_{i_k})$ hence $tp(\bar{b}_{i_k},$

$A \cup (N \cap M_{i_k}))$, contradicting the choice of $\bar{b}_{i_k}$

From the two facts we infer that for k, $n < \omega$, $\models \varphi[\bar{b}_{i_k}, \bar{a}_{i_k}, \bar{c}^*_n]$

iff $k \leqslant n$. We get a contradiction. ($\mathscr{D}$) is stable by hypothesis, but we just proved it has the order property (see 3.6)).

§ 5. *Categoricity*

We work in the context of assumption III.

We prove the parallel of Morley's categoricity theorem which is quite similar to the standard case. We could have used 1.15 to shorten the proof. For simplicity, we restrict ourselves to countable L.

Definition 5.1.

(1) Let K be a class of structures and λ a cardinal. K is *λ-categorical* (or *categorical in* λ) iff any two structures in K of cardinality λ are isomorphic
(2) Ψ *is categorical in* λ if the class of Ψ-closed models is categorical in λ (where $\Phi \subseteq \Psi$)

Theorem 5.2. Let L *be countable. If* Ψ *is categorical in one* $\lambda > \aleph_0$, *then it is categorical in every* $\lambda > \aleph_0$; *assuming* $\Psi =$ Sub $\Psi = \exists \Psi$.

Proof: we prove the theorem in three steps:
Step 1: $\mathscr{D}$ is $\aleph_0$-stable.
Step 2: In every λ there is a $(\mathscr{D}, \lambda)$-homogeneous model
Step 3: If there is a Ψ-closed model of cardinality $\lambda > \aleph_0$ which is not $(\mathscr{D}, \lambda)$-homogeneous, then in every $\mu > \aleph_0$ there is such a model.

We can conclude that every Ψ-closed model of cardinality $\lambda > \aleph_0$ is $(\mathscr{D}, \lambda)$-homogeneous. By 1.6(3), it follows that Ψ is categorical in λ.

Step 1.

The idea is the following: first build one model of cardinality λ in which there are few types realized over each countable subset. If $\mathscr{D}_T$ is $\aleph_0$-unstable, we build a second model of cardinality λ in which there are many types realized over some countable set. This contradicts the λ-categoricity, so $\mathscr{D}_T$ must be $\aleph_0$-stable.

We will first build the second model, which is easy: If $\mathscr{D}$ is $\aleph_0$-unstable, this means there is some countable subset A of $\mathfrak{C}$ and $\mathfrak{C}$ realizes $\geqslant \aleph_1$ different types over A. We may choose an elementary substructure M of $\mathfrak{C}$ satisfying:

$$\|M\| = \lambda \text{ and M realizes } \geqslant \aleph_1 \text{ types over } A \subseteq |M|$$

Next we will build a model N of cardinality λ which realizes just countably many types over every countable subset of N. We use Ehrenfeucht-Mostowski models EM(I) generated by a sequence I of indiscernibles. We will review this notion in more detail after finishing the proof.

We expand $\mathfrak{C}$ to $\mathfrak{C}^s$ by adding Skolem-functions. Because $K' = \{M \mid M \models \text{Th}\,\mathfrak{C}^s, \text{M is } \Psi\text{-closed}\}$ has models of arbitrarily large cardinality, it is possible to find a structure in K' generated by indiscernibles I (because K' is definable in $L_{(2^{\aleph_0})^+,\aleph_0}$ this requires an argument based on the Erdös-Rado theorem and compactness). Replacing I by the ordinal λ and taking N to be the corresponding Ehrenfeucht-Mostowski model N = EM(λ) we see that N has cardinality λ, and we claim:

(*) for any countable subset A of N, N realizes only countably many types over A.

The argument proving (*) may be reconstructed from the observation that the ordered set λ realizes only countably many types over any countable subset.

Now M, N cannot be isomorphic, contradicting the λ-categoricity. Thus M cannot exist, so $\mathscr{D}_T$ is $\aleph_0$-stable.

Review of Ehrenfeucht-Mostowski Models:

Whenever $A \subseteq \mathfrak{C}^S$ we define the hull H(A) as the closure of A under Skolem functions in $\mathfrak{C}^S$, considered as a submodel of

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$\mathfrak{C}$. Then H(A) is determined by A and the types of n--tuples of elements of A in $\mathfrak{C}$.

If I is a sequence of indiscernibles in $\mathfrak{C}^S$, then we write EM(I) instead of H(I). EM(I) can be reconstructed from

1. the order type of I;
2. the types p_n of increasing n-tuples from I (there is one p_n for each n). The set $\{p_n : \text{all } n\}$ is called the *character* **Γ**.

If I is a sequence of indiscernibles in $\mathfrak{C}^S$ with character Γ and if J is any ordered set, we may construct an Ehrenfeucht-Mostowski model $EM^1(J, \Gamma)$ generated by J as a sequence of indiscernibles of character Γ, and let $EM(J, \Gamma)$ be its L-reduct. Obviously the models $EM(J, \Gamma)$ are infinitely generic (Once indiscernibles have been constructed, they may be taken to lie in $\mathfrak{C}$).

Step 2.

Obvious by 1.13.

The main point is:

Step 3.

If $\lambda_1, \lambda_2 > \aleph_0$ and we have a Ψ-closed non-$\mathcal{D}$-homogeneous model M_2 of cardinality λ_2, then we can find such a model in λ_1.

Proof: This is a Lowenheim-Skolem theorem. We will shrink M_2 to a model M_0 and expand M_0 to the desired model M_1. Fix any M_2 of cardinality λ_2 which is not $(\mathcal{D}_T, \lambda_2)$-homogeneous. Fix a subset $A_2 \subseteq |M_2|$ of cardinality less than λ_2 such that some type p_2 in $S_{\mathcal{D}}(A_2)$ is not realized in M_2.

Going down: We will make a countable structure M_0 resembling M_2. By the $\aleph_0$-stability of $\mathcal{D}$ and 1.10, 1.12, there is in M an indiscernible set I_2 of cardinality $(\aleph_0 + |A_2|)^+$ over A_2. Let M'_2 be Ψ-prime over $A_2 \cup I_2$, and embed M'_2 in M_2. In particular M'_2 omits p_2. Thus:

(*) There is no Ψ_1-formula φ with parameters in $A_2 \cup I_2$ which isolates p_2. ($\Psi_1 = \exists \wedge \Psi$).

That is:

(**) for every Ψ_1-formula φ over $A_2 \cup I_2$, there is a formula $\underline{\varphi} \in \Phi$, $\underline{\varphi}^1 \in p_2$ and $\varphi \wedge \underline{\varphi}^o$ is consistent.

We may choose countable sets $A_o \subseteq A_2$, $I_o \subseteq I_2$ such that the type $p_o = p_2 \restriction A_o$ satisfies the analog of (**), hence of (*), and I_o is infinite. Of course, I_o is indiscernible over A_o. Let M_o be Ψ-prime over $A_o \cup I_o$. Then M_o omits p_o. The next task is to expand M_o to a large model omitting p_o.
Going up: We assume that A_o, I_o, p_o are countable and satisfy

(*). Let I_1 be a set of indiscernibles over A_o such that $I_o \subseteq I_1$ and $|I_1| = \lambda_1$. Let M_1 be Ψ-prime over $A_o \cup I_1$. We still have:

(**)' There is no Ψ_1-formula φ with parameters in $A_o \cup I_1$ which isolates p_o.

Hence M_1 omits p_o. Since A_o is countable, M_1 is not even $(\mathcal{D}, \aleph_1)$-homogeneous. This proves the lemma, and completes the proof of the Łoś' Conjecture.

Remark.
To see that this theorem is interesting, we will give some examples of theories T such that E_T (the class of e-closed models of T) is categorical in $\aleph_1$ without being elementary. We want three such examples:

1. E_T categorical in $\aleph_1$ and not in $\aleph_0$;
2. E_T categorical in $\aleph_0$ and not in $\aleph_1$;
3. E_T categorical both in $\aleph_0$ and $\aleph_1$.

Example 1. (Macintyre)
We consider models which consist of pairs of algebraically closed fields $F_o \subseteq F_1$, of fixed characteristic. The existentially complete models are pairs (F_o, F_1 in which F_1 has transcendence degree 1 over F_o. Clearly this is an example of type 1. It is also an example of a complete inductive non model-complete theory.

Example 2.
T is the theory of undirected graphs without cycles. The exis-

tentially complete models are the connected graphs without cycles having infinite branching at each point.

Example 3.

Adjoin to example 2 a function symbol $f(x_1, x_2, x_3)$. We want $f(a_1, a_2, x)$ to be an automorphism $f_{a_1a_2}$ of our graph carrying a_1 to a_2. Take as a further axiom:

$$\text{(A)}\quad f_{a_2a_3}\, f_{a_1a_2} = f_{a_1a_3}.$$

$K^1 = K^1(T, \Phi_{qf}) = E_T$ consists of connected graphs without cycles having constant infinite branching at each point. Because we have axiom (A), the function symbol f does not affect the number of isomorphism types. The computation of E_T requires some care.

§ 6. *Forking*

In this section we work in the context of assumption II, assume $\mathscr{D}$ is stable, and for simplicity let $\Phi = \Phi_{qf}$, hence $tp_{qf}(\bar{a}, A)$ determines $tp(\bar{a}, A)$; so here a type will mean a q.f. type (though many theorems are true more generally). The important and positive notion will be «not forking over» which is closely related to «definable over», and sometimes plays a role similar to «isolated».

Forking was invented to deal with

A. A union of an increasing elementary chain of $\varkappa$-saturated models is $\varkappa$-saturated, when the cofinality of the length of the sequence is $\geqslant \kappa(T)$.
The difficult case is $\kappa(T) < \varkappa < |T|$.
Another use of it is in proving the stability spectrum theorem. For 1.10 we can use also strong splitting, but it is essential for

B. $\lambda(\mathscr{D})$ is $|\mathscr{D}|$ or $2^{\aleph_0}$.

Another use is

C. Constructing anti-prime models [S 1] (ch. IV). See remark to 6.10.

D. Proving that if a countable T has only universal models in $\aleph_1$, then it is categorical in $\aleph_1$ (affirming a conjecture of Keisler).

The relation of forking and ranks is determined in [S 1, III, § 4] and has applications to ranks.

If T is superstable, we can, in some cases ,use ranks instead of forking (when p is complete over B, $A \subseteq B$). This is natural, as «p does not fork over A» is equivalent to «p and $p \restriction A$ have the same rank $R(—, L, \infty)$»; and there are equivalent formulations.

All theorems here are from [S 1], most of them from ch. III. See also Lascar [Ls], and this volume.

The notion of «tp(a, B) does not fork over A, $A \subseteq B$» is similar to «the free algebras generated by A, B, $A \cup \{a\}$ are freely amalgamated in $\mathfrak{C}$», and many times in algebraic contexts this is what we get by explicating.

«Forking» was invented as a version of strong splitting (see [S 7]). However it can be developed only under more restricted assumptions.

Definition 6.1. A type p *forks over* A if there exists a set of indiscernibles $I_j = \{\bar{a}^j_\alpha \mid \alpha < \omega\}$ over A and q.f. formulas φ_j such that $p \vdash \bigvee_{j<n} [\varphi_j(\bar{x}, \bar{a}^j_0) \Leftrightarrow \neg \varphi_j(\bar{x}, \bar{a}^j_1)]$.

When $A = \emptyset$, we omit it.

The notion is not interesting for unstable theories with this definition.

Example: Let T be the theory of an equivalence relation E; $p = \{xEa\}$ does not fork if E has finitely many equivalence classes, otherwise yes.

Theorem 6.2.

(1) *If* $q \subseteq p$ (*or even* $p \vdash q$) *are types and* p *does not fork over* A, q *does not fork over* A.

(2) *If* p *is over* B *and* p *does not fork over* A, *we can com-*

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plete p *to a q.f.complete type over* B *which does not fork over* A. (This we call the «extension property» of (not-forking) types).

(3) *If* p *is over* A, p *does not fork over A.* (Remember $\mathcal{D}_T$ is stable).

Proof of (3). (as a sample proof)

Assume p forks over A. Then $p \vdash \bigvee_{j<n} [\varphi_j(x, \bar{a}_0^j) \Leftrightarrow \neg\varphi_j(x, \bar{a}_1^j)]$.

Let $\{\varphi_j : j \leqslant n\} \subseteq \Delta$, Δ finite. By stability we have $R^m(p, \Delta, \aleph_0) < \omega$. Extend p to a q.f. complete type q over $A \cup \bigcup_{m,j} \bar{a}_m^j$ of the same rank. So there is a j such that (w.l.o.g.) $p \vdash \varphi_j(x_j, \bar{a}_0^j) \wedge \neg\varphi_j(x_j, \bar{a}_1^j)$. By stability again $[R^m(p, \Delta, \aleph_0) < \omega$ is enough$]$ φ_j divides I_j in two sets, one big and one small. W.l.o.g. for most $\bar{a} \in I_j$, $\neg\varphi_j(x, \bar{a}) \in p$. Now for all $m_0 < \omega$, $p \cup \{\varphi_j(x_j, a_m^j)^{\text{if } (m=m_0)} : m \in \omega\}$ has the same ranks as p, which contradicts the definition of rank.

Remark: we need quantifier-free for completing p preserving the rank.

Definition 6.3.

$\varphi(\bar{x}, \bar{a})$ *is almost over* A iff $\{\varphi(\bar{x}, F(\bar{a})) : F$ is an automorphism of $\mathfrak{C}$ over $A\}$ has cardinality $< |\mathfrak{C}|$ (here this amounts to being finite).

Examples: 1. Algebraically closed fields, $A = \emptyset$.

$x^2 + 17x - 3 = 0$ is over $\emptyset$

$x^2 + \pi x - 3 = 0$ is not almost over $\emptyset$

$x^2 + \sqrt[7]{3}\, x - \sqrt[3]{7}$ is almost over $\emptyset$

(in fact it has 21 twinbrothers)

2. E is an equivalence relation with finitely many equivalence classes. Then x E a is almost over $\emptyset$.

Definition 6.4.
p *is stationary over* A iff
(i) p does not fork over A and
(ii) p has no two contradictory extensions which do not fork over A.

Definition 6.5.
The *strong type* $stp_{qf}(\bar{a}, A) = \{\varphi(\bar{x}, \bar{b}) : \models \varphi(\bar{a}, \bar{b})$ and $\varphi(\bar{x}, \bar{b})$ is almost over A and φ is qf$\}$.
(There is no cardinality problem in the definition: we can choose representatives for equivalent formulas in $stp_{qf}(\bar{a}, A)$; there are $\leqslant |A| + |T|$ such equivalence classes).
What we want to understand is: what are the possibilities of type extensions which do not fork; how much do we need for a type to be stationary ?

Theorem 6.6. (The finite equivalence relation theorem)
A strong type $stp_{qf}(\bar{a}, A)$ *is stationary over A.*
Intuitively this says: for each equivalence relation $\varphi(x, y, \bar{b})$, $b \in A$, with finitely many equivalence classes, $stp_{qf}(\bar{a}, A)$ says to which class it belongs i.e. to the class of $\bar{a}$.
Hence
Conclusion 6.7. If $tp_{qf}(\bar{a}, B)$ *does not fork over* $A \subseteq B$; $\bar{b}$, $\bar{c} \in B$, $stp_{qf}(\bar{b}, A) = stp_{qf}(\bar{c}, A)$, *then*

$$tp_{qf}(\bar{a} \frown \bar{b}, A) = tp_{qf}(\bar{a} \frown \bar{c}, A).$$

Theorem 6.8. Let $B \subseteq A$, a *a sequence,* $p = tp_{qf}(\bar{a}, A)$; p *does not fork over* B *iff* $stp_{qf}(\bar{a}, A)$ *does not fork over* B.
Proof ($\Leftarrow$ easy) Suppose $stp_{qf}(\bar{a}, A)$ does fork over B. As strong types are closed under conjunction some $\varphi(\bar{x}, \bar{b}) \in stp(\bar{a}, A)$ forks over B.

Hence $\varphi(\bar{x}, \bar{b}) \models \bigvee_{j \leqslant n} \varphi_j(x, \bar{a}^j_0) \Leftrightarrow \neg \varphi_j(x, \bar{a}^j_1)$ and $\{\bar{a}^j_\ell \mid \ell < \omega\}$

indiscernible over B.
Let $\varphi(\bar{x}, F_j(\bar{b}))$ $(j < k < \omega)$ be all «sisters» of $\varphi(\bar{x}, \bar{b})$ up to equivalence, where F_j is an automorphism of $\mathfrak{C}$ and $F_j \restriction A = id_A$.

Then $p \vdash \bigvee \varphi(\bar{x}, F_j(\bar{b})) \vdash \bigvee_{i,j} [\varphi_i(x, F_j(\bar{a}^j_0) \Leftrightarrow \neg \varphi_j(x, F_j(\bar{a}^j_i))]$

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and $\{F_i(\bar{a}^j_\ell) : \ell < \omega\}$ is indiscernible over B.

So tp $(\bar{a}, A)$ forks over B, a contradiction.

Lemma 6.9. If p *is a complete qf-m-type over* A, *it is stationary over* $B \subseteq A$ *iff it is stationary over* A *and does not fork over* B.

Remarks:

(1) For complete types this amounts to being stationary.
(2) If M is existentially closed, $tp_{qf}(a, M)$ is stationary.
Some important facts are summarized by

Theorem 6.10.

(i) *If* $B \subseteq A$, $tp_{qf}(\bar{a} \frown \bar{b}, A)$ *does not fork over* B, *then* $tp_{qf}(\bar{b}, A)$ *does not fork over* B.

(ii) (Symmetry) $tp_{qf}(\bar{a}, A \cup \{b\})$ *forks over* A *iff* $tp_{qf}(\bar{b}, A \cup \{\bar{a}\})$ *forks over* A.

(iii) (Transitivity) *Let* $C \subseteq B \subseteq A$. *If* $tp_{qf}(\bar{a}, A)$ *does not fork over* B *and* $tp_{qf}(\bar{a}, B)$ *does not fork over* C, *then* $tp_{qf}(\bar{a}, A)$ *does not fork over* C. (Also the converse holds).

(iv) *Let* $B \subseteq A$. *Then* $tp_{qf}(\bar{a} \frown \bar{b}, A)$ *does not fork over* B *iff* $tp_{qf}(\bar{a}, A)$ *does not fork over* B *and* $tp_{qf}(\bar{b}, A \cup \{\bar{a}\})$ *does not fork over* $B \cup \{\bar{a}\}$.

Proof (i) trivial

(ii) Define by induction on $i < \omega$ sequences $\bar{a}_i, \bar{b}_i$.

For $i = 0$, $\bar{a}_i = \bar{a}, \bar{b}_i = \bar{b}$; for $i > 0$, choose $\bar{a}_i \frown \bar{b}_i$ such that $stp_{qf}(\bar{a}_i \frown \bar{b}_i, A \cup \bigcup_{j<i} \bar{a}_j \frown \bar{b}^j)$ does not fork over A and extends $stp_{qf}(\bar{a} \frown \bar{b}, A)$. Suppose $tp_{qf}(\bar{a}, A \cup \bar{b})$ forks over A but $tp_{qf}(\bar{b}, A \cup \bar{a})$ does not fork over A.

By 6.7 if $j < i$ then $tp_{qf}(\bar{a}_j \frown \bar{b}_i, A) = tp_{qf}(\bar{a}_0 \frown \bar{b}_i, A)$.

By 6.6 if $j \geqslant i$ then $\bar{a}_j \frown \bar{b}_j$ realizes

$$tp_{qf}(\bar{a}_i \frown \bar{b}_i, A \cup \bigcup_{\alpha<i} \bar{a}_\alpha \frown \bar{b}_\alpha).$$

Easily for all $j < i$, $tp_{qf}(\bar{a}_j \frown \bar{b}_i, A) = tp_{qf}(\bar{a} \frown \bar{b}, A)$.

Similarly if $j > i > 0$ $tp_{qf}(\bar{a}_j \frown \bar{b}_i, A) = tp_{qf}(\bar{a}_j \frown \bar{b}_0, A)$

$$= tp_{qf}(\bar{a}_1 \frown \bar{b}_0, A) \neq tp_{qf}(\bar{a}_0 \frown \bar{b}_0, A)$$

(as $tp_{qf}(\bar{a}_\ell, A \cup \{\bar{b}_0\})$ forks over A iff $\ell = 0$).

This shows $\mathcal{D}$ has the order property, contradiction.

(iii) There is $\bar{a}'$ realizing $stp_{qf}(\bar{a}, C)$ such that $stp_{qf}(\bar{a}, A)$ does not fork over C (by 6.6, 6.8).

So $stp_{qf}(\bar{a}, B)$, $stp_{qf}(\bar{a}', B)$ extend $stp_{qf}(\bar{a}, C)$ and do not fork over C; hence they are equal. By trivial monotonicity properties $stp_{qf}(\bar{a}', A)$ does not fork over B, so similarly $stp_{qf}(\bar{a}', A) = stp_{qf}(\bar{a}, A)$, so the latter too does not fork over C; so we finish.

(iv) $\Rightarrow$: So suppose $tp_{qf}(\bar{a} {}^\frown \bar{b}, A)$ does not fork over B. Then by (i) $tp_{qf}(\bar{a}, A)$ does not fork over B. Let $\bar{c} \in A$, then $tp_{qf}(\bar{a}, {}^\frown \bar{b}, B \cup \{\bar{c}\})$ does not fork over B (by monotonicity of forking) hence $tp_{qf}(\bar{c}, B \cup \{\bar{a} {}^\frown \bar{b}\})$ does not fork over B (by (ii)), hence over $B \cup \bar{a}$, hence $tp_{qf}(\bar{b}, B \cup \bar{a} \cup \bar{c})$ does not fork over $B \cup \bar{a}$ (by (ii) again). As this holds for every $\bar{c}$, $tp_{qf}(\bar{b}, A \cup \bar{a})$ does not fork over $B \cup \bar{a}$.

$\Leftarrow$: So assume $tp_{qf}(\bar{a}, A)$, $tp_{qf}(\bar{b}, A \cup \bar{a})$ does not fork over B, $B \cup \bar{a}$ respectively. Let $\bar{b}_i (i < i_0)$ be a maximal list such that $i \neq j \Rightarrow stp_{qf}(\bar{b}_i, B \cup \bar{a}) \neq stp_{qf}(\bar{b}_j, B \cup \bar{a})$.

We can find (by a simple version of 6.2(2)) $\bar{b}'_i, \bar{a}'$ such that

$stp_{qf}(\bar{a}' {}^\frown \dots {}^\frown \bar{b}'_i {}^\frown \dots, A)$ extends $stp_{qf}(\bar{a} {}^\frown \dots {}^\frown \bar{b}_i {}^\frown \dots, B)$ and

does not fork over B (i.e. every finite subtype satisfies this). Then w.l.o.g. $\bar{a}' = \bar{a}$, hence for some i, $stp_{qf}(\bar{b}'_i, B \cup \bar{a}) =$

$stp_{qf}(\bar{b}, B \cup \bar{a})$, and as $stp_{qf}(\bar{b}'_i, A \cup \bar{a})$ does not fork over

$B \cup \bar{a}$ (by the "only if part" of (iv)), they are equal, so $stp_{qf}(\bar{a} {}^\frown \bar{b}, A)$ does not fork over B.

Remark: If $\mathcal{D}$ is stable, then for every A there is a model M (say existentially closed) with $A \subseteq |M|$, such that for every $\bar{a} \in |M|$, $tp_{qf}(\bar{a}, A)$ does not fork over some finite subset of A. If $\mathcal{D}$ is superstable, the above is trivially true: every M does it. This helps us to show that for unsuperstable T, $\lambda = |T| + \aleph_1$,

$I(\lambda, T) = 2^\lambda$, and to build models with absolute indiscernibles. The facts from the theorem above fit well into an

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axiomatic treatment of prime models. This is done in chapter IV of [S 1].

Definition 6.11.
$\kappa(\mathcal{D})$ is the first cardinal $\varkappa$ such that there are no A_i $(i < \varkappa)$ with $A_i \subsetneqq A_j$ for $i < j)$ and $tp_{qt}(\bar{a}, A_{i+1})$ forks over A_i (cf. the $\kappa(\mathcal{D})$ in theorem 12).

Theorem 6.12. (i) *$\mathcal{D}$ is superstable iff* $\kappa(D) = \aleph_0$
(ii) *For stable $\mathcal{D}$ and* p *over* A *there is a* $B \subseteq A$ *with* $|B| < \kappa(T)$ *s.t.* p *does not fork over* B.

Forking is defined only for stable theories. For unstable theories there are troubles in defining it generally, at least it doesn't make sense. But if we have Skolem-functions, the following lemma may well serve as an alternative definition.

Lemma 6.13. p *does not fork over* M *iff* p *is finitely satisfiable in* M (though p is not necessarily over M).

This lemma, as a definition for unstable theories, is useful in the following problem: how can we show that there are λ-universal but not λ^+-universal models for some unstable theory. If we are not interested in the sharp calculation of this λ (i.e. if $\lambda_1 < \lambda_2$ we are happy with λ_1-universal but not λ_2-universal) an ultrapower construction with Erhenfeucht-Mostowski model will do. If we are interested in the sharp λ, we need forking as defined by the previous lemma.
In particular, for every T we have either

(i) T is categorical in all $\lambda > |T|$ after adding $2^{|T|}$ constants (and hence all models are saturated and hence universal),

or

(ii) for all λ, μ, $\mu \geqslant 2^{\lambda}$, $\lambda \geqslant 2^{|T|}$, T has a λ-universal but not λ^+-universal model of cardinality μ.

From this one deduces, answering a question due to J. Keisler,

Theorem 6.14. If $|T| = \aleph_0$ *and every model of* T *of cardinality* $\aleph_1$ *is universal then* T *is categorical in* $\aleph_1$.

A natural question is whether restricting ourselves to kind II is merely incompetence.

Lemma 6.15. There is a countable universal T, *such that* $\mathcal{D} = \mathcal{D}(T, \Phi_a)$ *is superstable and there is a countable* $A \subseteq \mathfrak{C}$ *such that for each* $a \in \mathfrak{C}$, $tp_{qf}(a, A)$ *splits strongly over* $\emptyset$, i.e. there is an infinite indiscernible set $I \subseteq A$, and $b, c \in I$ and qf φ such that $\models \varphi[a, b] \wedge \neg\varphi[a, c]$.

Remark: By 2.12 we can find such an example for $\mathcal{D} = \mathcal{D}(T, \Phi_{qf})$, i.e. infinite generic models.

Proof. Let us define a model M. Its universe $|M|$ is the set of functions f such that (i) Dom $f = [n, \infty] = \{m < \omega \mid n \leqslant m\}$, Range $f \subseteq \omega$
(ii) f is eventually zero

The relations are $P_n = \{f : \text{Dom } f = [n, \infty]\}$

$$R_n = \{\langle f_1, f_2\rangle : f_1, f_2 \in P_n,\ f_1 \restriction [n+1, \infty] = f_2 \restriction [n+1, \infty] \text{ and } f_1(n) \neq f_2(n)\}.$$

The functions are G_n, defined by $G_n(f) = f \restriction [n, \infty]$.
Let T be the universal theory of M.
Now if $\mathcal{D} = \mathcal{D}(T, \Phi_a)$, $\mathfrak{C}$ a $(\mathcal{D}, \lambda_0)$-homogeneous model, then for any $a, b \in \mathfrak{C}$ for some n, m

$$G_n(a) = G_m(b).$$

The rest is left to the reader.

§ 7. *Algebra:* non-structure theorems

The problem here is to show that various algebraic theories are unstable etc... In § 7, § 8, for simplicity we concentrate on assumption I. Notice that by [S 1], e.g.

Theorem 7.1.

(1) *If* T *is not superstable,* $\lambda \geqslant \aleph_1 + |T|$, *then* $I(\lambda, T) = 2^\lambda$. In most cases this is true for pseudo-elementary classes, and

we get 2^λ models, no one elementarily embedded into another. (We can omit "elementarily" e.g. if the formulas showing unsuperstability are q.f..)

(2) *If* T *is countable (for simplicity) but not* $\aleph_0$*-stable, then for each* $\lambda > \aleph_0$, T *has min* $\{2^\lambda, 2^{2^{\aleph_0}}\}$ *non isomorphic models, no one elementarily embedded into another.*

(3) *If* T *is* $\aleph_0$*-stable not* $\aleph_1$*-categorical then* $I(\aleph_\alpha, T) \geqslant |\alpha + 1|$

(4) (Baldwin [Ba]) *If* T *is categorical,* $\alpha_T = R(\emptyset, L, \aleph_0) < \omega$.

Note: Bokut asked on the number of e.c. algebras (associative, non-associative) over a field F. This class is unstable, so for $\lambda \geqslant |F| + \aleph_1$, the number is 2^λ. Similar results holds of course for division rings. For e.c. groups Ziegler proved that for each e.c. group M, there is $N \equiv_{\infty,\omega} M$ in each cardinal $\geqslant \|M\|$; we improve this to: for each $\lambda \geqslant \|M\| + \aleph_1$ there are 2^λ such N's (both results should appear in Springer Lecture Notes by Ziegler).

FIELDS

Theorem 7.2.

(Macintyre [Mc 2]) *The only infinite fields with* $\aleph_0$*-stable theory are the algebraically closed fields.*

We still do not know whether there are stable or superstable fields or $\aleph_0$-stable division rings except the algebraically [or separably] closed fields (discarding the finite ones). (See 7.2, 8. 9) There is a small hope (hope, because then we can apply theorems on stability to such fields).

Macintyre, using 7.5 and elaborations of [Mc 2] proved: if a field F is superstable then for each n it has finitely many extensions of degree n. ([16])

([16]) Added in Proof June 76: Note that if $\tilde{F}$ is the algebraic closure of the field F, then Th(F) is λ-stable iff $Th(\tilde{F}, F)$ is λ-stable.

Theorem 7.3:

The only infinite division rings with $\aleph_1$-categorical theory are the algebraically closed (commutative) fields

Claim 7.4

Suppose $T = Th(M)$. *Suppose* $\tau(y_1, \ldots, y_n, \bar{z})$ *is a term in the language of* T; $\bar{b}$, $\bar{a}_i$ *are sequences of elements of* M, $\varphi_i(x, \bar{a}_i)$ *is realized by infinitely many elements of* M *and:*

$$M \models (\forall y_1, \ldots, y_n, y^1, \ldots, y^n \left[\bigwedge_{i=1}^{n} \varphi_i(y_i, \bar{a}_i) \wedge \right.$$

$$\left. \bigwedge_{j=1}^{n} \varphi_j(y^j, \bar{a}_j) \wedge \tau(y_1, \ldots, y_n, \bar{b}) = \tau(y^1, \ldots, y^n, \bar{b}) \rightarrow \bigwedge_{i=1}^{n} y_i = y^i \right]$$

then: $n \leqslant \alpha_T =^{df} R^n(\emptyset, L, \aleph_0)$

Proof: Easy by induction on n

Lemma 7.5 Suppose $M = \langle |M|, o, R_1, \ldots \rangle$, *M is infinite,* $\langle |M|, o \rangle$ *is a group,* $T = Th(M)$, *and* $X_n \subseteq |M|$ *are definable subgroups (with parameters). then:*

1) *If* $X_n \underset{\neq}{\supset} X_{n+1}$ *for each* n, *then* T *is not* $\aleph_0$*-stable;*
2) *If* $X_n \supseteq X_{n+1}$, $(X_n : X_{n+1})$ *is infinite, then* T *is not superstable;*
3) *If* $X_n \subseteq X_{n+1}$, $(X_{n+1} : X_n)$ *is infinite, then* T *is not* $\aleph_1$*-categorical. If in addition T is totally transcendantal,* $\alpha_T \geqslant \omega$.

Proof: Let $\varphi_n(x)$ define X_n (we ignore the parameters).

1) Let $E_n(x, y) = \varphi_n(x^{-1} y) \wedge \varphi_0(x) \wedge \varphi_0(y)$. E_n is an equivalence relation over X_0, because X_n is a group and $x \in X_0$, $z \in X_n$ implies $E_n(x, xz)$. Clearly E_{n+1} refines E_n as $X_{n+1} \subseteq X_n$. Now for each n, and $a \in X_0$, there are b, c, aE_nb, aE_nc but not $bE_{n+1}c$. For choose $x \in X_n - X_{n+1}$ then aE_nax, but not $aE_{n+1}ax$ (as $a^{-1}(ax) \notin X_{n+1}$); so choose $b = a$, $c = ax$. So we can define by induction on $l(\eta)$, $a_\eta \in M$ for each

$\eta \in {}^{\omega>}2$ such that $a_\eta \in X_0$, and if $\ell(\eta) = n$,

$a_\eta E_n a_{\eta\frown<0>} E_n a_{\eta\frown<1>}$ but not $a_{\eta\frown<0>} E_{n+1} a_{\eta\frown\{1\}}$:

Let $A = \{a_\eta : \eta \in {}^{\omega>}2\}$.

For each $\eta \in {}^{\omega}2$, $p_\eta = \{x E_n a_{\eta\frown<n>} : n < \omega\}$ is consistent and is over A and the p_η's are pairwise contradictory.

So $|S(A)| = 2^{\aleph_0}$.

2) Similar to 1). Only note that if M is λ-saturated then for each n, and $a \in X_0$, there are $b_i \in X_0$, $i < \lambda$, such that aE_nb_i but not $b_iE_{n+1}b_j$ for $i \neq j$

3) By 7.4 and 7.1(4).

Claim 7.6:

If D *is a division ring,* T = Th(D) *is* $\aleph_1$*-categorical and* $\varphi(x, \bar{a})$ *defines in* D *an infinite sub-division ring* R, *then, looking at* D *as a (say right) vector space over* R, *its dimension is finite.* (In fact it suffices to assume D is a ring with no zero divisors, but we do not gain as any such stable ring is a division ring)

Proof: Immediate by 7.4, 7.1(4)

Proof of 7.3: If D has an infinite center C, then by 7.6 [D: C] is finite, by 7.2, C is of course algebraically closed, hence by well known facts from algebra D = C, so we are through. So assume that C is finite. Suppose now that $a \in D$, and C(a) is infinite (C(a) — the sub-division ring of D generated by C and a).

Let $D^* = \{b \in C : a, b \text{ commute}\}$. Then D^*'s center is infinite ($\supseteq C(a)$), hence as in the previous argument D^* is an algebraically closed field. So by 7.6 [D: D*] is finite, hence D is isomorphic to a ring of matrices of elements of D* (but this embedding is not cannonical on D*), hence D satisfies an identity, hence by Kaplansky's theorem [D: C] is finite, but also C

is finite. Hence D is finite, contradiction. If there is no such a, assume w.l.o.g. D is $\aleph_1$-saturated.
Then for some n, $a^n = a$ for every $a \in D$, so D satisfies an identity and we finish as before.
So we finish 7.3

Rings

The results on stability and $\aleph_1$-categoricity (7.7 - 7.9) are summed up in Cherlin and Reineke [CR] (see (1), (2), (3)), which contains more material. Macintyre and Rosenstein [MR] classifying the $\aleph_0$-categorical rings with no nilpotent elements, Baldwin and Rose [BR] proving (4) (on $\aleph_0$-categoricity), give alternative proofs for some results of [CR], and some other results. Theorem 7.7 - 7.11 summarise some of their results. Baur, Macintyre and Cherlin, [*1], have characterized $\aleph_0$-categorical $\aleph_0$-stable theories of rings and groups with $\alpha_T<\omega$ (e.g. they are abelian by finite). Felgner, [*2], characterizes $\aleph_1$-categorical semi-simple rings which are not necessarily commutative; in [*3] and [*4] he makes considerable advance for $\aleph_0$-categorical stable groups.
Let J = J(R) be the Jacobson radical of R.

Theorem 7.7:

If R *is a ring with at least one non zero divisor, and* R *is* (λ)-*stable, then* R *has a unit,* J *is nilpotent, and* R/J *is a finite product of matrix rings of the form* $M_n(D)$, D *a* (λ)-*stable division ring.*

Theorem 7.8:

Suppose R *is a commutative noetherian ring, with at least one nonzero divisor, then* R *is* (λ)-*stable iff* J *is nilpotent and* R/J *is a finite product of* (λ)-*stable fields.*

Theorem 7.9:

Suppose R *is a commutative ring with at least one non-zero divisor and* R/J *is infinite; then* R *is* $\aleph_1$-*categorical iff* $R = R' \oplus H$ *where:*

i) H *is finite*
ii) R″ *is a noetherian local ring* (local means having a unique maximal ideal M), *this ideal* M *is nilpotent and* R′/M *is an algebraically closed field.*

Remark:

(1) If R has characteristic zero, R stable, then R/J is necessarily infinite.
(2) The case R/J finite is open, but if in addition R is stable and noetherian, then R is finite.

Theorem 7.10:

Suppose R *is noetherian, with at least one non-zero divisor. Then* R *is* $\aleph_0$*-categorical iff* R *is finite.*

Theorem 7.11:

Suppose R *has* 1 *and no nilpotent element. Then* R *is* $\aleph_0$*-categorical iff* R *is a finite direct product* $\prod_{1\leq i\leq n} R_i$ *where each* R_i *is of the form*

$C = C(X, F: X_1 \ldots X_n, F_1 \ldots F_n)$ *where:*

i) X *is a Boolean space with finitely many isolated points,* F *is a finite field,* $X_1 \ldots X_n$ *are closed subsets of* X, $F_1 \ldots F_n$ *are subfields of* F,
ii) C *is a ring of continuous functions* $f: X \to F$ *such that* $f(X_i) \subseteq F_i$ *and the dual structure* $(\hat{X}, \hat{X}_1, \ldots \hat{X}_n)$ *(in the sense of Stone duality) (a Boolean algebra with ideals* $\hat{X}_1 \ldots \hat{X}_n$*) is* $\aleph_0$*-categorical.* (Those structures are classified in [MR].)

Generic division rings

Let T be the (universal) theory of division rings. Cohn [CO] prove the JEM (for a fixed characteristic) and the amalgamation property. Boffa [BO1], [BO2], [BO3] want to find in which cardinal T has a universal homogeneous model. By 1.13 it suffices to note that $\mathcal{D}(T, \Phi_{qf})$ is unstable (which, in fact, he does). He asked for the number of infinite generic division rings, so 7.1 solves his problem (in cardinality $\lambda > \aleph_0$, there are 2^λ non-isomorphic ones)

Question:

Suppose D is a division ring, and T is the universal theory of $(D,a)_{a \in D}$, and it has the JEM. In which cases is the class of infinite generic models of T stable ? superstable ? $\aleph_0$-stable ? $\aleph_1$ categorical ?

§ 8. ALGEBRA AND STRUCTURE THEOREMS

Algebraically closed fields

It is well known that for each p, the theory of algebraically closed fields of characteristic p is $\aleph_1$-categorical. In fact this was the motivation of Łoś 'conjecture.

Modules.

For each ring R, we consider (always left) modules over R as additive groups with one-place functions F_a for each $a \in R$. Let R be fixed, $L = L_R$ the corresponding language and $T = T_R$ be the theory of such modules. By $|R|$ we mean $|R| + \aleph_0$. The first order theories of abelian groups were classified by Szmielew [Sz]. Macintyre [Mc 1] classified the $\aleph_1$-categorical theories of abelian groups. Sabbagh [Sb 2] reproved Smielew's theorem as well as Eklof and Fisher [EF] who further found in which cardinals a given complete theory of abelian groups has a saturated model. By 1.13 (and as mentioned in [S 5]) they essentially classified abelian groups by stability. In particular every abelian group is stable.

Theorem 8.1. (Mycielski [My], Sabbagh [Sb 2]) *If* M *is* $|R|^+$-*saturated,* $M < N$, *or even* $M <_{ep} N$, *then* M *is a direct summand of* N.

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Remark. In fact, we prove M is atomically compact (see e.g. Taylor [T 1]).

Proof. It suffices to show there is a homomorphism from N into M, which is the identity over M (as $N = M \oplus \mathrm{Ker}\, f$). For this it suffices to prove:
(*) If Γ is a set of ep-formulas, with parameters from M, finitely satisfiable in M, *then* Γ is satisfiable in M.

(Because let $|N| = \{a_i : i < \alpha\}$, $\Gamma = \{\varphi(x_{i_1}, \ldots, a_{j_1}, \ldots) : \varphi$ is qf, $N \models \varphi[a_{i_1}, \ldots, a_{j_1}, \ldots]$ and $a_{j_1} \ldots \in M\}$).

Clearly for proving (*) it suffices to prove it for Γ with one variable (replace variables appearing in Γ by elements in M one by one; in limit stages remember that «finite satisfiability» has a finite character; for a variable x let

$\Gamma' = \{\exists \bar{y}) \bigwedge_{i=1}^{n} \varphi_i(x, \bar{y}) : \varphi_i(x, \bar{y}) \in \Gamma\}$ and apply the assumption).

So let $\Gamma = \{\varphi_i(x) : i < \alpha\}$ and let, w.l.o.g.

$$\varphi_i(x) = (\exists y_1 \ldots y_{n(i)}) \bigwedge_{m<m(i)} \sum_{j=1}^{n(i)} r^{i,m}_j y_j + r^{i,m} x = a^{i,m} \text{ where}$$

$r^{i,m}_j$, $r^{i,m} \in R$, $a^{i,m} \in M$. (φ_j could have a disjunction inside, but by extending Γ we can get rid of it). Let us call $\varphi_{i(1)}$, $\varphi_{i(2)}$ similar if $n(i(1)) = n(i(2))$, $m(i(1)) = m(i(2))$, $r^{i(1),m}_j = r^{i(2),m}_j$, $r^{i(1),m} = r^{i(2),m}$. Clearly similarity is an equivalence relation, so let φ_i ($i < \alpha_0 \leqslant |R| + \aleph_0$) be a set of representatives. So $\Gamma_0 = \{\varphi_i(x) : i < \alpha_0\}$ is finitely satisfiable in M, $|\Gamma_0| \leqslant |R|$; so there is a b_0 realizing it. We shall show that b_0 realizes Γ. For φ_β there is $i < \alpha_0$ such that φ_i, φ_β are similar. So let $b \in M$ satisfies $\varphi_\beta \wedge \varphi_i$. So we can find $y^\beta_1, \ldots$ and $y^i_1, \ldots$ such that:

(i) $\sum_{j=1}^{n(\beta)} r^{\beta,m}_j y^\beta_j + r^{\beta,m} b = a^{\beta,m}$ for $m < m(\beta)$

(ii) $\sum_{j=1}^{n(\beta)} r_j^{\beta,m} y_j^i + r^{\beta,m} b = a^{i,m}$ for $m < m(\beta)$

(remember the similarity). Substracting, we get

$$\sum_{j=1}^{n(\beta)} r_j^{\beta,m} (y_j^\beta - y_j^i) = a^{\beta,m} - a^{i,m}.$$

Hence if y_j $(j < n(i))$ exemplifies $\varphi_i(b_o)$, then $y_j + y_j^\beta - y_j^i$ exemplifies $\varphi_\beta(b_o)$.

Theorem 8.2. (Baur [Br], Fisher [Fi 1] [Fi 2])
Every module has a stable theory; so clearly also $\mathcal{D}(T_R, \Phi_{qf})$ *is stable.*

Proof. If not, there are R-modules $M_1, M_2, A \subseteq M_1 \subset M_2$, $|A| = 2^{|R|}$,

$|\{tp(a, A, M_2) : a \in M_2\}| > \|M_1\|$. W.l.o.g., M_2 is $|R|^+$-saturated ,and $A = |M_1|$, $M_1 < M_2$, M_1 is $|R|^+$-saturated too. Now by Feferman-Vaught if $a_1^l, a_2^l, \ldots \in M_2$, $f : M_2 \to M_1$ is a homomorphism, $f \restriction M_1$ = the identity, $M_o = \text{Ker } f$, and $f(a_i^1) = f(a_i^2)$, $tp(\langle \ldots, a_i^1 - f(a_i^1), \ldots \rangle, \emptyset, M_o) = tp(\langle \ldots, a_i^2 - f(a_i^2), \ldots \rangle, \emptyset, M_o)$, then $tp(\langle \ldots, a_i^1, \ldots \rangle, \emptyset, M_2) = tp(\langle \ldots, a_i^2, \ldots \rangle, \emptyset, M_2)$. So we get a contradiction easily. The second part is left to the reader.

The following lemma is much related to Sabbagh [Sb 3]:

Lemma 8.3. Every formula in L_R *is equivalent (modulo* T'_R*) to a Boolean combination of pe-formulas where* $L'_R = L_R \cup \{c_i : i < |R| + \aleph_o\}$, T'_R *is a* Π_2*-theory in* L'_R, *extending* T_R, *such that every* L_R*-model of* T_R *can be expanded to a model of* T'_R.

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Proof. Left to the reader, remembering

Claim 8.4. (1) Let Ψ be a set of atomic formulas in L_R, Ψ_n the set of first-order formulas whose atomic subformulas belong to Ψ and their quantifier depth is $\leqslant n$. Let $th_n(M) = \{\psi \in \Psi_n : M \models \psi\}$ and note $|\Psi| < \aleph_0 \Rightarrow |\Psi_n| < \aleph_0$. *Then for each* n *there is* $k(n) < \omega$ *such that : if* $M = \bigoplus_{i \in I} M_i$ *(free sum of modules) then* $th_n(M)$ *is determined by the truth value of the following statements : for each* t *which is a formally possible* $th_n(N)$ *and cardinal* $\lambda, \lambda \in \{0, 1, \dots, k(n)\} \cup \{\aleph_0\}$, «$|\{i \in I : t = th_n(M_i)\}| \geqslant \lambda$».

(2) *Part* (1) *can be generalized to similar classes* (as abelian structures of Fisher [Fi 1]).

Remark. For such theorems see Feferman-Vaught [FV], Galvin [Ga]. Eklof and Fisher [EF] proved : $th(M_i) = th(N_i) \Rightarrow th(\bigoplus M_i) = th(\bigoplus N_i)$ and Barwise and Eklof [BE] had similar results for infinitary languages (which we can too). The case $\lambda = \aleph_0$ is needed above, for if B_i are Boolean rings, $\bigoplus_{i<\alpha} B_i$ has one iff $\alpha < \omega$. See also Waszkicwicz and Weglorz [WW].

The following theorem characterizes $\kappa_r(T_R)$ except when $\kappa(T_R) = \aleph_0$.

Theorem 8.5. The following conditions on R, $\varkappa$ *(where* cf $\varkappa > \aleph_0$*) are equivalent*

(1) $\varkappa \leqslant \kappa(T_R)$ *(where* $\kappa(T_R) = \sup\{\kappa(T) : T$ *a completion of* $T_R\}$*)*,

(2) *For each* $n < \omega$, *every submodule of* ${}_R(R^n)$ *is generated by* $< \varkappa$ *elements.*

(3) *Every ideal of* R *is generated by* $< \varkappa$ *elements.* [17]

Proof. (2) ⇒ (1). Essentially just like 8.1, 8.2 (M will be $\varkappa$-compact). In the proof of 8.1 we have to show that we can re-

[17] It seems that (2) ⇒ (1) does not work, so 8.5 is proved with qf-unstability in [1]. But 8.7 can be generalized for charactering $\varkappa < \kappa(\tau)$.

place $\Gamma_k = \{\varphi_i(x) : <\alpha_o, n(i), m(i) \leqslant k\}$ $(k < \omega)$ by a subset of cardinality $< \varkappa$, which is easy by our assumption.

not (2) $\Rightarrow$ *not (1)*. (Here cf $\varkappa > \aleph_0$ is not used). We can find $n < \omega$, and $\bar{a}^{\alpha} = (a_1^{\alpha}, \ldots, a_n^{\alpha}) \in R^n$ $(\alpha < \varkappa)$ such that $\bar{a}^{\alpha}$ does not belong to the submodule of R^n generated by $\{\bar{a}^{\beta} : \beta < \alpha\}$. For each λ, let M_λ be the module generated by: y_η^ℓ $(\eta \in {}^{\varkappa}\lambda$, $\ell = 1, \ldots, n)$, z_ν $(\nu \in {}^{\alpha}\lambda, \alpha < \varkappa)$ and subject only to the restriction: whenever ν is an initial segment of η, $\alpha \in \lambda$, $\sum_{\ell=1}^{n} a_\ell^{\alpha} y_\eta^\ell = z_\nu$. It suffices to show that $\sum_{\ell=1}^{n} a_\ell^{\alpha} y_\eta^\ell = z_\nu$ holds iff ν is an initial segment of η. Suppose η_0, ν is a counter-example, and let h be the following endomorphism of M, where

$\alpha = \min\{\alpha : \eta_0(\alpha) \neq \nu(\alpha)\} : \eta_1 \in {}^{\varkappa}\lambda$, ν an initial segment of η_1:

(i) for $\varrho \in {}^{\varkappa}\lambda$, $\varrho \upharpoonright (\alpha+1) = \nu \upharpoonright (\alpha+1)$, $h(y_\varrho^\ell) = y_{\eta_1}^\ell$,

(ii) for $\varrho \in {}^{\varkappa}\lambda$, $\varrho \upharpoonright (\alpha+1) \neq \nu \upharpoonright (\alpha+1)$, $h(y_\varrho^\ell) = y_{\eta_0}^\ell$

(iii) for $\varrho \in {}^{\varkappa>}\lambda$, $\varrho \upharpoonright (\alpha+1) = \nu \upharpoonright (\alpha+1)$, $h(z_\varrho) = z_{\eta_1 \upharpoonright \ell(\varrho)}$

(iv) for $\varrho \in {}^{\varkappa>}\lambda$, $\varrho \upharpoonright (\alpha+1) \neq \nu \upharpoonright (\alpha+1)$, $h(z_\varrho) = z_{\eta_0 \upharpoonright \ell(\varrho)}$.

Clearly h is a projection onto the submodule generated by $y_{\eta_0}^\ell$, $y_{\eta_1}^\ell$ $(\ell = 1, \ldots, n)$ subject to the conditions:

$$\Sigma a_\ell^{\beta} y_{\eta_0}^\ell = \Sigma a_\ell^{\beta} y_{\eta_1}^\ell \quad (\text{for } \beta < \alpha).$$

Easily this contradicts the assumption on $\bar{a}^{\gamma}$ $(\gamma < \varkappa)$.
(2) ⇒ (3). Trivial.
(3) ⇒ (1). We prove by induction on n. $R^{n+1} = R^n \oplus R$, so if M is a submodule of ${}_R(R^{n+1})$, then $R \oplus M/R$ (considering R as submodule of R^{n+1}) and $M \cap R^n$ are $< \varkappa$-generated; from this the conclusion follows.

Remark. Here we get the unstability by qf-formulas. In fact we proved (we can even refine 8.5 by adding in (2) $l(a) = n_0$, and restrict (2) to n_0):

Theorem 8.6. The following conditions on $\varkappa$ are equivalent (when cf $\varkappa > \aleph_0$):

(1) *If* $\lambda^{<\varkappa} = \lambda \geqslant 2^{|R|}$, $|A| \leqslant \lambda$, $A \subseteq M \models T_R$, *then* $|\{tp_{qf}(\bar{a}, A, M) : \bar{a} \in M\}| \leqslant \lambda$;
(2) *For each* $n < \omega$, *every submodule of* ${}_R(R^n)$ *is generated by* $< \varkappa$ *elements; and when* $\varkappa > \aleph_0$,
(3) *The conditions of 8.4.*

Remark. We can infer now that if R is not noetherian, T_R is not superstable (which can be inferred from «not artinian»); moreover, there are qf-formulas, hence (by 7.1) for every regular $\lambda > \aleph_0$ there are 2^{λ} R-modules, no one embeddable in another. Of course this conclusion holds under more general conditions.

Theorem 8.7. The following conditions are equivalent:

(1) T_R is superstable,
(2) *not: for every regular* $\lambda > |R|$ *there is an* R-*module of cardinality* λ *which is not the direct sum of* R-*modules of cardinality* $< \lambda$,
(3) *every* R-*module is the direct sum of countably generated* R-*modules,*
(4) *if* $M \underset{ep}{<} N$ *then there is a projection from* N *onto* M,
(5) *every* e p-1-*type over* M *which is finitely satisfiable in*

M, *has an equivalent finite subtype, or equivalently is realized in* M.

Note: It seems that the number of e.g. $|R|^+$-saturated modules in $\aleph_\alpha$ is $\leqslant 2^{|\alpha|}$, so we have a structure theorem.

Remark. Fisher [Fi] generalizes this (to abelian structures) and strengthens it to having a unique decomposition. ([18])

Proof (5) $\Rightarrow$ (4). Just like the proof of 8.1.

(4) $\Rightarrow$ (3). This is a standard fact. Indeed (4) means that every R-module is pure-injective, hence that every module is pure-projective hence that every module is a direct summand of a direct sum of finitely presented R-modules (see for all those implications Warfield [Wf]). A classical theorem of Kaplansky concludes the proof.

For completeness, let us prove that every R-module M is the direct sum of R-modules of cardinality $\leqslant |R|$. We prove it by induction on $\|M\|$. For $\|M\| \leqslant |R|$ there is nothing to prove. Otherwise let $\|M\| = \lambda > |R|$, and we define by induction on $\alpha < \lambda$ an increasing continuous sequence $M_\alpha <_{ep} M$, $\|M_\alpha\| < \lambda$, $\bigcup_{\alpha<\lambda} M_\alpha = M$, and projections h_α from M onto M_α. (We define by induction on α, M_α and then h_α using (3)). Let $N_\alpha = \operatorname{Ker} h_{\alpha+1} \restriction M_{\alpha+1}$, so $M_{\alpha+1} = M_\alpha \oplus N_\alpha$, hence $M = \sum_{\alpha<\lambda} N_\alpha$; so using the induction hypothesis on λ the conclusion of (3) follows.

(3) $\Rightarrow$ (2). Trivial.

not (1) $\Rightarrow$ not (2). Easy by the construction in [S 3] 1.1, see here § 0,7.1.

not (5) $\Rightarrow$ not (1). So there is an R-module M and ep-for-

([18]) In a proper sense. We can add: (6) $I(\aleph_\alpha, T_R) < 2^{\aleph_\alpha}$ for some $\aleph_\alpha > |T_R|$, (7) $I(\aleph_\alpha, T) \leqslant |\alpha|^{2^{|R|}}$ (and we can get the exact value; see [S 1], IX, § 2).

mulas $\varphi_n(x) = \varphi_n(x, a^i_1, \ldots, a^i_k)$ $(a^i_\ell \in M)$ such that $M \models (\exists x) \bigwedge_{\ell<n} \varphi_\ell(x)$ but $M \models (\exists x)(\bigwedge_{\ell \leqslant n} \varphi_\ell(x) \wedge \neg\varphi_n(x))$. By notational changes we can assume $\varphi_n(x) = (\exists y_1 \ldots y_{k(n)}) \bigwedge_{i<k(n)} \sum^i_{j=1} r^i_j y_j + s_i x = a_j$ and that for each n, $M \models \neg\varphi_{n+1}(c_n) \wedge \bigwedge_{i<k(n)} \sum^i_{i=1} r^i_j b^n_j + s_i c_n = a_j$. For notational simplicity let $k(n) = n$.

Let us find M^*, c^*, b_j $(j < \omega)$ such that $M < M^* \models \sum^i_{j=1} r^i_j b_j + s_i c^* = a_j$. Now for each λ, let N be an R-module generated by x^η, y^η_ℓ, z_ν $(\eta \in {}^\omega\lambda, \nu \in {}^{\omega>}\lambda, \ell < \omega)$ subject only to the following conditions: when $\eta \in {}^\omega\lambda$, $\bigwedge_{i<k(n)} \sum^i_{j=1} r^i_j y^\eta_j + s_i x^\eta = z_{\eta \restriction i}$. Clearly $N \models \varphi_n(x_\eta, z_{\eta \restriction 0}, \ldots, z_{\eta \restriction k(n)})$, and it is sufficient to prove that if $\eta \in {}^\omega\lambda$, $n < \omega$, $\nu = \eta \restriction n \frown \langle\alpha\rangle$, $\eta(n) \neq \alpha$, then $N \models \neg\varphi_{n+1}(x^\eta, z_{\nu \restriction 0}, \ldots, z_\nu)$. Suppose η, ν are a counterexample, and we shall define a homomorphism h from N into M^*:

(i) if $\eta^1 \in {}^\omega\lambda$, $\eta^1 \restriction (n+1) \neq \nu$ then $h(x^{\eta^1}) = c^*$ and $h(y^{\eta^1}_j) = b_j$,

(ii) if $\eta^1 \in {}^\omega\lambda$, $\eta^1 \restriction (n+1) = \nu$ then $h(x^{\eta^1}) = c_n$ and $h(y^{\eta^1}_j) = b_j$ for $j \leqslant k(n)$,

(iii) if $\eta^1 \in {}^\omega\lambda$, $\eta^1 \restriction (n+1) = \nu$, $j > k(n)$, then $h(y^{\eta^1}_j) = b_j$,

(iv) if $\varrho \in {}^m\lambda$, $m \leqslant n$ or $\varrho \restriction (n+1) \neq \nu$, then $h(z_\varrho) = a_m$,

(v) for other $\varrho \in {}^m\lambda$, choose any $\eta^1 \in {}^\omega\lambda$, $\varrho = \eta^1 \restriction m$ and

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$$\text{let } h(z_\varrho) = \sum_{j=1}^{m} r^i_j\, h(y_j^{\eta^1}) + s_m h(x^{\eta^1}).$$

The contradiction we get is clear (by the choice of η, ν, $N \models \varphi_{n+1}(x^\eta, z_{\nu \restriction o}, \ldots, z_\nu)$, so as h is ep-formula, $M^* \models \varphi_{n+1}(c_n, a_o, \ldots, a_{n+1})$ contradicting the choice of c_n.

Sabbagh noticed that our results imply that if R is commutative and countable, then T_R is superstable iff T_R is $\aleph_o$-stable.

Differentially closed fields.

A differential field is a natural generalization of a field: we add to the theory of fields of characteristic p a one-place function D, and the axioms $D(x + y) = Dx + Dy$, $D(xy) = (Dx)y + x(Dy)$, and get T^p_{df}. Using Seidenberg [Si], Robinson [Rb 1] showed that T^o_{df} has a model completion T^o_{do} (the theory of differentially closed fields).

Blum [Bl 1] gives a nice axiomatization: if P_1, P_2 are differential polynomials with one variable y, of orders m_1, m_2, $m_1 > m_2$, P_2 not identically zero, then: there is a solution for $P_1 = 0 \wedge P_2 \neq 0$, and for $P_2 = 0$. (19)

Later Wood [W 1] [W 2], again using [Si 1], proves similar results for T^p_{dc}, $p > 0$, when we add the p-th root as an operation; i.e. we add the axioms: $r(x) \neq 0 \rightarrow r(x)^p = x$, $r(x) = 0 \rightarrow (\forall y)\, y^p \neq y$ (note that $x = y^p \rightarrow Dx = 0$) and we get the theory of radical fields T^p_{rd} and its model completion the theory T^p_{rdc} of the radically differentially closed fields.

Blum [Bl 1] shows T^o_{dc} is $\aleph_o$-stable: quite naturally, the Morley rank of a 1-type is $\leqslant$ the minimal m such that it implies x_o

(19) Meanwhile S. Fakir in a mineograph asserts there is a mistake, and suggests a correction.

is a root of a differential polynomial of order m, and ω if there is no such m; hence by Morley [M 1], over any differential field, there is a prime differentially closed field, hence by [S 9] (here 4.5), it is unique up to isomorphism (but it is not minimal, see [Kl 3] [Rs 1] [S 10]).

Shelah [S 10] and Wood [W 2] prove, independently, that over any radical differential field there is a prime radical differentially closed field. In [S 10] it is proved that T^{p}_{rdc} is stable (it is not superstable as noted by [S 10] from a proof in [W 1] that it is not $\aleph_0$-stable).

Hence by 4.6,

Theorem 8.8. Over any radical differential field (in fact a field such that $Dx = 0 \rightarrow (\exists y)\, y^p = x$), *there is a unique prime radical differentially closed field.*

We should note that though T^{o}_{dc} is $\aleph_0$-stable, by [S 10] it has the dimensional order property (e.g. by adding the cardinality quantifier $\exists^{>\aleph_0}$ we get long orders in some models), hence it has non-structure theorem.

Macintyre and Shelah note

Theorem 8.9. If F *is a separably closed but not algebraically closed field, then* Th(F) *is stable but not superstable.*

Proof. In some cases it follows from the theorem on the stability of T^{p}_{rdc} ([S 10]) and always similarly. The unsuperstability follows from 7.5(2) as F^{p^n} is a strictly decreasing sequence of definable additive subgroups and $F^{p^n}/F^{p^{n+1}}$ is infinite (quotient as additive groups).

C. Wood, [*5], has written the proof in details. It seems that those theories has the dimensional order property, hence 2^{λ} modules in each $\lambda > \aleph_0$.

Operations preserving stability.

For a wide class of operations on models, we can prove that they preserve stability

Lemma 8.10. If $\mathrm{Op}(\ldots, M_i, \ldots)_{i<\alpha}$ *is an operation on models*

(i) *preserving elementary equivalence* (i.e. $M_i \equiv N_i \Rightarrow \mathrm{Op}(\ldots, M_i, \ldots) \equiv \mathrm{Op}(\ldots, N_i, \ldots)$),

(ii) *for each* λ, *it preserves* λ^+*-saturation* (i.e. if each M_i is λ^+-saturated then $\mathrm{Op}(\ldots, M_i, \ldots)$ is),

(iii) *if each* M_i *is a*λ_0*-stable model then* $\mathrm{Op}(\ldots, M_i, \ldots)$ *is, then* $\mathrm{Th}(M_i)$ *is* λ_0*-stable for each* i *implies* $\mathrm{Th}\ \mathrm{Op}(\ldots, M_i, \ldots)_{i<\alpha}$ *is* λ_0*-stable.*

Proof. Trivial (in fact, (i) is not needed).

The simplest application is that reducts preserve stability. Similar theorems hold for stability of classes we discuss here (generic, $\mathscr{D}$-homogeneous, etc...). The proof of the hypothesis is in many cases easy by Feferman-Vaught [FV], or maybe the refinements to autonomous systems of Galvin [G 1].

J. Wierzejewski [Wr] used a particular instance of 8.10 (for products using the lemma of J. Waszkiewicz and B. Weglorz [WW] that finite products preserve saturation). So he proved $\aleph_0$-stability, superstability, stability and also the negation of the finite-cover-property, the negation of the independence property, and the negation of the strict order property.

He also proved in [Wr] that the negations of those properties are not preserved even by squares. The author found when $M = M_1 + M_2$, M is stable iff M_1 and M_2 are stable; and similarly for the other properties.

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