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A parallel to the null ideal for inaccessible λ: Part I

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Dedicated to the memory of James E. Baumgartner (1943–2011)

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Abstract It is well known how to generalize the meagre ideal replacing $\aleph_0$ by a (regular) cardinal $\lambda > \aleph_0$ and requiring the ideal to be ($<\lambda$)-complete. But can we generalize the null ideal? In terms of forcing, this means finding a forcing notion similar to the random real forcing, replacing $\aleph_0$ by λ. So naturally, to call it a generalization we require it to be ($<\lambda$)-complete and λ^+-c.c. and more. Of course, we would welcome additional properties generalizing the ones of the random real forcing. Returning to the ideal (instead of forcing) we may look at the Boolean Algebra of λ-Borel sets modulo the ideal. Common wisdom have said that there is no such thing because we have no parallel of Lebesgue integral, but here surprisingly first we get a positive = existence answer for a generalization of the null ideal for a "mild" large cardinal λ—a weakly compact one. Second, we try to show that this together with the meagre ideal (for λ) behaves as in the countable case. In particular, we consider the classical Cichoń diagram, which compares several cardinal characterizations of those ideals. We shall deal with other cardinals, and with more properties of related forcing notions in subsequent papers (Shelah in The null ideal for uncountable cardinals; Iterations adding no λ-Cohen; Random λ-reals for inaccessible continued; Creature iteration for inaccesibles. Preprint; Bounding forcing with chain conditions for uncountable cardinals)

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Sh:1004

and Cohen and Shelah (On a parallel of random real forcing for inaccessible cardinals. arXiv:1603.08362 [math.LO]) and a joint work with Baumhauer and Goldstern.

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Contents

0 Introduction

0(A) Aim: for general audience

The ideals of null sets and of meagre sets on the reals are certainly central in mathematics. From the forcing point of view we speak of random real forcing and Cohen forcing. The Cohen forcing has natural generalizations (and relatives) when we replace $\mathscr{P}(\mathbb{N})$ by $\mathscr{P}(\lambda)$, or the set of the characteristic functions of subsets of λ, for a regular uncountable cardinal λ, replacing finite by "of cardinality $<\lambda$". But we lack a generalization of random real forcing to higher cardinals λ, replacing reals by λ-reals, e.g. members of ${}^{\lambda}2$. It has seemed that this lack is due to nature; the reason being that on the one hand the Baire category theorem generalizes naturally (when we are allowed to approximate in λ-steps and information of size $<\lambda$ instead finite; all this for regular λ), but on the other hand we know nothing remotely like Lebesgue measure.

Surprisingly, at least for me, there is a generalization: not of the Lebesgue measure, but of the ideal of null sets, i.e., the ones of Lebesgue measure zero. This is done here (i.e., in this part) for a mild large cardinal λ: weakly compact. The solution for more cardinals will be dealt with in a continuation (at some price). The present definition should be examined in two ways. First, we may list the well known properties of the null ideal (and of random real forcing) and try to prove (or disprove) them for our ideal. Second, random real forcing was used quite extensively in independence results; in particular for related cardinal invariants, so it is natural to try to generalize such applications.

The first issue is dealt with in Sect. 2 (assuming Definition 1.3 and intended for wider audience) and then Sects. 3–8 here. The second is treated in the continuation. Whereas success in the second issue should be easy to judge, concerning the first issue the reader may first list what are reasonable hopes and compare them with the discussion and description in Sect. 3. This is not done in the present section in order to help the reader to make a list of expectations independent of what we have done.

A set theoretically uninitiated reader may read the rest of Sect. 0(A) to see what are those large cardinals, look casually at Definition 1.3, just enough to see that the definition of $\mathbb{Q}_\kappa$, the parallel of the family of all closed subsets of $[0,1]_{\mathbb{R}}$ or ${}^\omega 2$ which are not Lebesgue null for κ strongly inaccessible, is natural and simple, then jump to Sect. 2 to see what we hope for and what is done.

Let us describe for the non-set-theoretic reader, what are these "large cardinals". Note that $\aleph_1$ is parallel in some respect to $\aleph_0$, whereas $\aleph_0$ is "the first infinite cardinal"; the number of natural numbers; $\aleph_1$ is the first uncountable cardinal, and is the number of countable ordinals (that is, isomorphism types of countable linear well orderings). Also both are so called regular: the union of less than $\aleph_\ell$ sets each of cardinality $< \aleph_\ell$ is $< \aleph_\ell$. But $\aleph_0$ is strong limit: $\kappa < \aleph_0 \Rightarrow 2^\kappa < \aleph_0$ whereas $\aleph_1$ is not. We can prove that there are strong limit cardinals: let $\beth_0 = \aleph_0, \beth_{n+1} = 2^{\beth_n}, \beth_\omega = \sum_{n<\omega} \beth_n$, now $\beth_\omega$ is a strong limit cardinal but alas is not regular. We say a cardinal λ is (strongly) inaccessible when λ is regular and strong limit, it is called "large cardinal" because we cannot prove its existence in ZFC but, modulo this, it is considered a very reasonable, small one. Similarly, the weakly compact ones which we now introduce: an uncountable cardinal is weakly compact when it is strongly inaccessible and satisfies the analog of the infinite Ramsey theorem: every graph with λ nodes has a subgraph with λ nodes which is complete or empty (alternatively, it satisfies the generalization of König lemma). So weakly compact cardinals are more similar to $\aleph_0$ than other cardinals, so it is not unnatural assumption when trying to generalize the null ideal.

0(B) For set theorists

In the present paper we prove that for a weakly compact cardinal λ there are (naturally defined) forcing notions adding a new $\eta \in {}^\lambda 2$ which have not few parallels (replacing "finite" by "of cardinality $<\lambda$") of the properties associated with random real forcing (and we define the relevant ideal). It seems natural to hope this will enable us to understand better related problems, in particular cardinal invariants of λ; on cardinal invariants for $\lambda = \aleph_0$, i.e. the continuum see Blass [2]; in higher cases see Cummings and Shelah [5]; in particular on strongly inaccessible see Rosłanowski and Shelah [19–21,23] and also [27].

In Sect. 1 we show for λ weakly compact that there is a (non-trivial) λ-bounding λ^+-c.c. ($<\lambda$)-strategically complete forcing notion and even a λ-complete one, see 0.4. We also generalize the construction for adding a member of $\prod_{\varepsilon<\lambda} \theta_\varepsilon$.

In the second section we discuss desirable properties of the ideal. In Sects. 3–8 we try to deal systematically with parallels of properties of the null ideal.

The ideal $\mathrm{id}(\mathbb{Q}_\kappa)$ (of subsets of ${}^\kappa 2$) determined by our forcing notion $\mathbb{Q}_\kappa$ is introduced in Sect. 3. There we also study the properties of κ-Borel subsets of ${}^\kappa 2$ related to this ideal.

Cardinal characteristics of the ideal $\mathrm{id}(\mathbb{Q}_\kappa)$ and their relations to $\mathfrak{b}_\kappa$, $\mathfrak{d}_\kappa$ and the characteristics of the κ-meagre ideal are investigated in Sects. 4 and 5. We present a parallel of Cichoń Diagram in Theorem 5.9.

In Sect. 6 we compare $\mathbb{Q}_\kappa$ and Cohen_κ. We note that forcing with one makes the set of ground model κ-reals small in the dual sense. We also investigate the class S_{awc} of all inaccessible cardinals κ for which $\mathbb{Q}_\kappa$ adds a Cohen real.

In the next section we introduce a parallel to "amoeba forcing"—a forcing notion $\mathbb{Q}_\kappa^{\mathrm{am}}$ adding a generic condition $\tilde{p}_\kappa \in \mathbb{Q}_\kappa$. And then, in Sect. 8, we investigate κ-Borel and κ-stationary-Borel sets and show that some relations associated with $\mathbb{Q}_\kappa$ are absolute.

We shall continue in successive papers, things delayed for various reasons. In particular in Cohen and Shelah [4] we shall eliminate the assumption "λ is weakly compact" and in [24, §1] we will investigate non-inaccessible case. A work with Baumhaver and Goldstern (see [28]) will deal with consistency results complimentary to the ZFC implications (i.e., inequalities) here. In [24, §1] we investigate adding many "λ-randoms". Further research concerning consistency results using iteration of creature forcing will be presented in [25]. We will also consider there constructions starting not with Cohen but other nice forcing notions and more.

0(C) Preliminaries

Definition 0.1 (0) We say η is a λ-real when $\eta \in {}^\lambda 2$.
(1) We define when $\mathbf{B} \subseteq {}^\lambda 2$ is a λ-Borel set naturally (see [38]), that is $X \subseteq {}^\lambda 2$ is a basic λ-Borel set if there exists $\nu \in {}^{\lambda>}2$ such that $X = ({}^\lambda 2)^{[\nu]} = \{\eta \in {}^\lambda 2 : \nu \triangleleft \eta\}$. The family of λ-Borel sets is the closure of the basic ones under unions and intersections of at most λ members, hence also by complements.

Note: actually $\mathbf{B}$ is an absolute definition of a subset of ${}^\lambda 2$ so $\mathbf{B}^{\mathbf{V}}$, "$\mathbf{B}$ as interpreted in the universe $\mathbf{V}$", is well defined for suitable $\mathbf{V}$.
(2) "F is a λ-Borel function" is defined similarly.
(3) $\mathbf{B} \subseteq {}^\lambda 2$ is a $\Sigma^1_1(\lambda)$-set when $\mathbf{B} = \{\langle \eta(2\alpha) : \alpha < \lambda\rangle : \eta \in \mathbf{B}_1\}$ for some λ-Borel set $\mathbf{B}_1$.
(4) $\mathbf{B} \subseteq {}^\lambda 2$ is a λ-stationary Borel set when for some λ-Borel function $F : {}^\lambda 2 \to \mathscr{P}(\lambda)$ we have $\eta \in B \Leftrightarrow F(\eta)$ is stationary.
(5) A set $X \subseteq {}^\lambda \mathscr{H}(\lambda)$ is λ-nowhere stationary Borel iff there is a λ-Borel function $\mathbf{B}$ from ${}^\lambda \mathscr{H}(\lambda)$ to $\mathscr{P}(\lambda)$ such that for every $\eta \in {}^\lambda \mathscr{H}(\lambda)$ we have: $\eta \in X$ iff $F(\eta)$ is a nowhere stationary subset of λ (see 0.6(2)). The complements of such X are λ-somewhere stationary sets.
(6) Similarly replacing ${}^{\lambda>}2$ by other trees with λ levels and λ nodes.

Definition 0.2 (1) We say that a set $B \subseteq {}^\lambda 2$ is λ-closed when:

- $\eta \in {}^\lambda 2 \wedge (\forall \alpha < \lambda)(\exists \nu \in B)(\eta \restriction \alpha = \nu \restriction \alpha) \Rightarrow \eta \in B$, equivalently

- for some sub-tree $T \subseteq {}^{\lambda>}2$ we have

$$B = \lim_\lambda(T) \stackrel{\text{def}}{=} \{\eta : \eta \text{ a sequence of length } \lambda \text{ such that } \alpha < \lambda \Rightarrow \eta{\restriction}\alpha \in T\}.$$

(2) Let $\mathbb{Q}$ be a family of subtrees of ${}^{\lambda>}2$ (or a quasi order with such set of elements). We say that $B \subseteq {}^{\lambda}2$ is a $\mathbb{Q}$-basic set <u>when</u> $B = \lim_\lambda(p)$ for some $p \in \mathbb{Q}$.
(3) Similarly replacing ${}^{\lambda>}2$ by other trees, as in 0.1(6).

Definition 0.3 (1) We say that a forcing notion $\mathbb{P}$ is α-strategically complete <u>when</u> the player COM has a winning strategy in the following game $\eth_\alpha(p, \mathbb{P})$ for each $p \in \mathbb{P}$.

The game $\eth_\alpha(p, \mathbb{P})$ involves two players, COM and INC. A play lasts α moves; in the β-th move, first the player COM chooses $p_\beta \in \mathbb{P}$ such that $p \le_{\mathbb{P}} p_\beta$ and $\gamma < \beta \Rightarrow q_\gamma \le_{\mathbb{P}} p_\beta$ and second the player INC chooses $q_\beta \in \mathbb{P}$ such that $p_\beta \le_{\mathbb{P}} q_\beta$.

The player COM wins a play if it has a legal move for every $\beta < \alpha$.
(2) We say that a forcing notion $\mathbb{P}$ is $(<\lambda)$-strategically complete <u>when</u> it is α-strategically complete for every $\alpha < \lambda$.

Remark 0.4 The difference between "$\mathbb{P}$ is λ-strategically complete" and "λ-complete" is not real, i.e., when we do not distinguish between equivalent forcing, those properties are very close (as in [34, Ch.XIV]), and here the difference does not matter, see e.g. 1.5(2).

Definition 0.5 (1) The λ-Cohen forcing is $({}^{\lambda>}2, \triangleleft)$.
(2) A forcing notion $\mathbb{Q}$ is λ-bounding or ${}^{\lambda}\lambda$-bounding <u>when</u> $\Vdash_{\mathbb{Q}}$ "for every function f from λ to λ there is $g \in ({}^{\lambda}\lambda)^{\mathbf{V}}$ such that $f \le g$, i.e., $\alpha < \lambda \Rightarrow f(\alpha) \le g(\alpha)$".
(3) We say that a $\mathbb{Q}$-name $\underset{\sim}{\eta} \in {}^{\alpha}\beta$ is a generic of $\mathbb{Q}$ <u>when</u> for some sequence $\langle \tau_p : p \in \mathbb{Q}\rangle$, τ_p an absolute function definable in $\mathbf{V}$ (or even a $(|\alpha| + |\beta|)$-Borel one) from ${}^{\alpha}\beta$ into $\{0, 1\}$ we have $\Vdash$ "$p \in \underset{\sim}{\mathbf{G}}$ iff $\tau_p(\underset{\sim}{\eta}) = 1$".

Definition 0.6 (1) Let S_{inac} be the class of all (strongly) inaccessible cardinals and let $S^\kappa_{\text{inac}} = \{\partial : \partial < \kappa \text{ is inaccessible}\}$.
(2) We say "S is nowhere stationary" <u>when</u> S is a set of ordinals, and for every ordinal δ of uncountable cofinality, $S \cap \delta$ is not a stationary subset of δ.
(3) For a set p of sequences of ordinals and η let $p^{[\eta]} = \{\nu \in p : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu\}$ and $p^{[\ge\eta]} = \{\nu \in p : \eta \trianglelefteq \nu\}$.

Definition 0.7 For an ideal $\mathbb{I}$ of subsets of X, including all singletons for simplicity, we define "the four basic cardinal invariants of the ideal":

(a) $\text{cov}(\mathbb{I})$, the covering number is $\min\{\theta$: there are $A_i \in \mathbb{I}$ for $i < \theta$ whose union is $X\}$,
(b) $\text{add}(\mathbb{I})$, the additivity of $\mathbb{I}$ is $\min\{\theta$: there are $A_i \in \mathbb{I}$ for $i < \theta$ whose union is not in $\mathbb{I}\}$,
(c) $\text{cf}(\mathbb{I})$, the cofinality of $\mathbb{I}$ is $\min\{\theta$: there are $A_i \in \mathbb{I}$ for $i < \theta$ such that $(\forall A \in \mathbb{I})(\exists i)(A \subseteq A_i)\}$,
(d) $\text{non}(\mathbb{I})$, the uniformity of $\mathbb{I}$ is $\min\{|Y| : Y \subseteq X \text{ but } Y \notin \mathbb{I}\}$.

Remark 0.8 We may use, e.g., $\text{cov}(\text{meagre}_\lambda)$ and $\text{cov}(\text{Cohen}_\lambda)$, they denote the same number.

Observation 0.9 *For any ideal* $\mathbb{I}$*:*

(*a*) $\text{add}(\mathbb{I}) \leq \text{cov}(\mathbb{I}) \leq \text{cf}(\mathbb{I})$,
(*b*) $\text{add}(\mathbb{I}) \leq \text{non}(\mathbb{I}) \leq \text{cf}(\mathbb{I})$

1 Like random real forcing for weakly compact κ

We consider the following question.

Question 1.1 (1) Is there a non-trivial forcing notion which is λ^+-c.c., $(<\lambda)$-strategically complete and which does not add a λ-Cohen sequence from ${}^\lambda 2$?
(2) Moreover is λ-bounding?

Recall that for $\lambda = \aleph_0$, "random real forcing" is such forcing notion but we do not know to generalize measure to λ with λ-completeness or so, whereas for Cohen forcing and many other definable forcing notions which add a Cohen real we know how to generalize.

We have wondered about this a long time, see [27] and some papers of Rosłanowski and Shelah [18,19,21,23]. Up to recently, we were sure that the answer was negative. Surprisingly for λ weakly compact there is a positive answer, a posteriori a straightforward one.

We will define a forcing notion $\mathbb{Q}_\kappa$ by induction on the inaccessible κ. Now, for κ the first inaccessible $\mathbb{Q}_\kappa$ is the κ-Cohen forcing. In fact, if κ is inaccessible but not a limit of inaccessible cardinals, then $\mathbb{Q}_\kappa$ is equivalent to the κ-Cohen forcing. If κ is a limit of inaccessibles, the conditions are such that the generic $\eta \in {}^\kappa 2$ satisfies for many inaccessibles $\partial < \kappa$, that $\underset{\sim}{\eta}{\restriction}\partial$ is somewhat ∂-Cohen, e.g., if $\langle \mathscr{I}_\partial : \partial \in S\rangle$ is a sequence such that $\mathscr{I}_\partial$ is a dense open subset of ${}^\partial 2$ and $S = \{\partial < \kappa : \partial$ is the first strong inaccessible in (α, κ) for some $\alpha < \kappa\}$, <u>then</u> for every large enough $\partial \in S$ we have $\underset{\sim}{\eta}{\restriction}\partial \in \mathscr{I}_\partial$.

At first glance this may look ridiculous: η is made more Cohen-like, but still in the end, i.e., for κ weakly compact, it has an antithetical character.

1(A) Adding an $\eta \in {}^\kappa 2$

Notation 1.2 (1) Here ∂, κ will denote strongly inaccessible cardinals.
(2) For $\mathscr{T} \subseteq {}^{\alpha>}2$ and $\eta \in {}^{\alpha>}2$ let $\mathscr{T}^{[\eta]} = \{\nu : \nu \trianglelefteq \eta$ or $\eta \trianglelefteq \nu \in \mathscr{T}\}$.
(3) For $\mathscr{T} \subseteq {}^{\delta>}2$ let $\lim_\delta(\mathscr{T}) = \{\nu \in {}^\delta 2 : (\forall \alpha < \delta)(\nu{\restriction}\alpha \in \mathscr{T})\}$.

Definition 1.3 We define a forcing notion $\mathbb{Q}_\kappa = \mathbb{Q}^2_\kappa$ by induction on inaccessible κ:

(A) $p \in \mathbb{Q}_\kappa$ <u>iff</u> there is a witness $(\varrho, S, \bar{\Lambda})$ which means:
  (a) p is a subtree of ${}^{\kappa>}2$, i.e., a non-empty subset of ${}^{\kappa>}2$ closed under initial segments,
  (b) (α) $S \subseteq \kappa$ is not stationary, moreover

(β) for every strongly inaccessible $\partial \leq \kappa$ the set $S \cap \partial$ is not stationary,
(γ) every member of S is (strongly) inaccessible,

(c) $\varrho = \mathrm{tr}(p)$ is the trunk of p which means:
(α) $\varrho \in {}^{\kappa>}2$,
(β) $\alpha \leq \ell g(\varrho) \Rightarrow p \cap {}^{\alpha}2 = \{\varrho \restriction \alpha\}$, hence $\mathrm{tr}(p) \in p$,
(γ) both $\varrho\hat{\ }\langle 0\rangle$ and $\varrho\hat{\ }\langle 1\rangle$ belongs to p,

(d) if $\varrho \trianglelefteq \eta \in p$ then $\eta\hat{\ }\langle 0\rangle, \eta\hat{\ }\langle 1\rangle \in p$,

(e) [continuity] if $\delta \in \kappa\backslash S$ is a limit ordinal $> \ell g(\varrho)$ and $\eta \in {}^{\delta}2$ <u>then</u>

$$\eta \in p \text{ iff } (\forall \alpha < \delta)(\eta \restriction \alpha \in p),$$

(f) (α) $\bar{\Lambda} = \langle \Lambda_\partial : \partial \in S\rangle$,
(β) Λ_∂ is a set of $\leq \partial$ dense open subsets of $\mathbb{Q}_\partial$,

(g) if $\partial \in S$ and $\partial > \ell g(\varrho)$ and $\eta \in {}^{\partial}2$, then
(α) $p \cap {}^{\partial>}2 \in \mathbb{Q}_\partial$,
(β) $\eta \in p$ <u>iff</u> $(\forall \alpha < \partial)(\eta \restriction \alpha \in p)$ and $(\forall \mathscr{I} \in \Lambda_\partial)(\exists q \in \mathscr{I})[\eta \in \lim_\partial(q)]$.

(B) $\mathbb{Q}_\kappa \models$ "$p \leq q$" <u>iff</u> $p \supseteq q$.

(C) (a) Let $S_p = \{\delta < \kappa : \delta > \ell g(\mathrm{tr}(p)), \delta$ is a limit ordinal and $\neg(\forall \eta \in {}^{\delta}2)[\eta \in p \leftrightarrow (\forall \alpha < \delta)(\eta \restriction \alpha \in p)]\}$, so $S_p \subseteq S$ when $(\mathrm{tr}(p), S, \bar{\Lambda})$ is a witness.
(b) We say $(\mathrm{tr}(p), S, \bar{\Lambda}, E)$ is a full witness for $p \in \mathbb{Q}_\kappa$ if $(\mathrm{tr}(p), S, \bar{\Lambda})$ is a witness for $p \in \mathbb{Q}_\kappa$ and E is a club of κ disjoint to S and to $[0, \ell g(\mathrm{tr}(p)))$,

Claim 1.4 *(1) For any κ and $\eta \in {}^{\kappa>}2$ we have $({}^{\kappa>}2)^{[\eta]}$ is a member of $\mathbb{Q}_\kappa$ with $\mathrm{tr}(({}^{\kappa>}2)^{[\eta]}) = \eta$.*
(2) If $p \in \mathbb{Q}_\kappa$ and $\ell g(\mathrm{tr}(p)) < \partial < \kappa$ <u>then</u> $p \cap {}^{\partial>}2$ belongs to $\mathbb{Q}_\partial$.
(3) If $p \in \mathbb{Q}_\kappa$ and $\eta \in p$ <u>then</u> $p^{[\eta]} \in \mathbb{Q}_\kappa$ and $p \leq p^{[\eta]}$ and $\mathrm{tr}(p^{[\eta]})$ is η if $\ell g(\eta) \geq \ell g(\mathrm{tr}(p))$ and is $\mathrm{tr}(p)$ otherwise.
(4) ${}^{\kappa>}2$ is the minimal member of $\mathbb{Q}_\kappa$.
(5) If $(\mathrm{tr}(p), S, \bar{\Lambda})$ is a witness for $p \in \mathbb{Q}_\kappa$ and $\ell g(\mathrm{tr}(p)) \geq \sup(S)$ <u>then</u> $p = ({}^{\kappa>}2)^{[\mathrm{tr}(p)]}$.
(6) Any triple $(\varrho, S, \bar{\Lambda})$ is a witness for at most one p.
(7) If $(\varrho, S, \bar{\Lambda})$ satisfies clauses $(c)(\alpha), (b)(\alpha), (\beta), (\gamma), (f)(\alpha), (\beta)$ of Definition 1.3(A) <u>then</u> there is one and only one $p \in \mathbb{Q}_\kappa$ which it witnesses.
(8) If $(\varrho, S, \bar{\Lambda})$ witnesses $p \in \mathbb{Q}_\kappa$, <u>then</u> also $(\varrho, S_p, \bar{\Lambda} \restriction S_p)$ witnesses it recalling Definition 1.3(C)(a).
(9) For every $p \in \mathbb{Q}_\kappa$ there is a maximal antichain $\mathscr{I}$ to which p belongs and $q_1 \neq q_2 \in \mathscr{I} \Rightarrow \lim_\kappa(q_1) \cap \lim_\kappa(q_2) = \emptyset$ hence $\{q \in \mathbb{Q}_\kappa : p \leq_{\mathbb{Q}_\kappa} q$ <u>or</u> $\lim_\kappa(q) \cap \lim_\kappa(p) = \emptyset\}$ is dense open.

Proof (1) Let $S = \emptyset$. Then $(\eta, \emptyset, <>)$ is a witness.
(2) If $(\mathrm{tr}(p), S, \langle \Lambda_\theta : \theta \in S\rangle)$ witnesses $p \in \mathbb{Q}_\kappa$, then $(\mathrm{tr}(p), S \cap \partial, \langle \Lambda_\theta : \theta \in S \cap \partial\rangle)$ witnesses $p \cap {}^{\partial>}2 \in \mathbb{Q}_\partial$.
(3)–(8) Easy, too.
(9) Let $\mathscr{I} = \{({}^{\kappa>}2)^{[\rho]} : \rho \in {}^{\kappa>}2\backslash p$ and $\alpha < \ell g(\rho) \Rightarrow \rho \restriction \alpha \in p\} \cup \{p\}$. □

Claim 1.5 *(1) If $p \in \mathbb{Q}_\kappa$ and $\rho \in p$, <u>then</u> there is η such that $\rho \trianglelefteq \eta \in \lim_\kappa(p)$.*
(2) If $\bar{p} = \langle p_i : i < \delta\rangle$ is a sequence of members of $\mathbb{Q}_\kappa$, $\bar{p}$ is increasing or at least $i < j < \delta \Rightarrow \mathrm{tr}(p_j) \in p_i$, $\langle \mathrm{tr}(p_i) : i < \delta\rangle$ is $\trianglelefteq$-increasing and

($\odot$) $\alpha < \delta \Rightarrow \min\left(S_{p_\alpha} \backslash \sup\{\ell g(\mathrm{tr}(p_i) + 1 : i < \delta\}\right) > \delta$,

then $p_\delta = \bigcap\{p_i : i < \delta\}$ *is a* $\leq_{\mathbb{Q}_\kappa}$*-lub of* $\bar{p}$.

(3) If $\delta < \kappa$, $p_i \in \mathbb{Q}_\kappa$ *is* $\leq_{\mathbb{Q}_\kappa}$*-increasing with* $i < \delta$, $(\eta_i, S_i, \bar{\Lambda}_i, E_i)$ *is a full witness for* p_i *satisfying* $i < j < \delta \Rightarrow E_j \subseteq E_i \wedge \min(E_i) < \ell g(\mathrm{tr}(p_j))$, *then the sequence* $\langle p_i : i < \delta\rangle$ *has a* $\leq_{\mathbb{Q}_\kappa}$*-upper bound.*

(4) If $p \in \mathbb{Q}_\kappa$ *and* $\mathscr{I}_i$ *is a dense subset of* $\mathbb{Q}_\kappa$ *for* $i < i(*)$ *and* $i(*) < \kappa^+$ *and* $\rho \in p$ *then there is* η *such that* $\rho \lhd \eta \in \lim_\kappa(p)$ *and* $(\forall i < i(*))(\exists q \in \mathscr{I}_i)(\eta \in \lim_\kappa(q))$.

(5) In (2) we may replace the demand ($\odot$) *with*

($\otimes$) *(a)* $\sup\{\ell g(\mathrm{tr}(p_i)) : i < \delta\} \notin S_{p_\alpha}$ *for* $\alpha < \delta$,

(b) if $\langle\mathrm{tr}(p_i) : i < \delta\rangle$ *is eventually constant, say* ρ, *then* $\min\left(S_{p_\alpha} \backslash (\ell g(\rho)+1)\right) > \delta$.

Proof We prove by induction on the inaccessibles κ that the five parts of the claim hold.

(1) Let $(\mathrm{tr}(p), S, \bar{\Lambda})$ be a witness for p. By 1.4(3) without loss of generality $\rho \trianglelefteq \mathrm{tr}(p)$.

Case 1 In S *there is a last member* ∂ *and* $\partial > \ell g(\mathrm{tr}(p)) \geq \ell g(\rho)$.
By 1.4(2), $p_1 = p \cap {}^{\partial >}2$ *belongs to* $\mathbb{Q}_\partial$. *Apply the induction hypothesis 1.5(4) for* ∂ *with* $p \cap {}^{\partial >}2$, Λ_∂ *here standing for* p, $\langle \mathscr{I}_i : i < i(*)\rangle$ *there to find* ϱ *such that* $\rho \lhd \varrho \in p \cap {}^{\partial}2$. *Now* $p^{[\varrho]} = ({}^{\kappa >}2)^{[\varrho]}$ *by 1.4(5), so the rest should be clear.*

Case 2 $\sup(S) \leq \ell g(\mathrm{tr}(p))$.
By 1.4(5) we know that $p = ({}^{\kappa >}2)^{[\mathrm{tr}(p)]}$.

Case 3 Neither Case 1 nor Case 2, i.e., $\sup(S) > \ell g(\mathrm{tr}(p))$ *and* S *has no last element. Let* $\theta = \mathrm{cf}(\mathrm{otp}(S))$ *and let* $\langle \alpha_\varepsilon : \varepsilon < \theta\rangle$ *be increasing continuous with limit* $\sup(S)$. Without loss of generality $\alpha_0 = \ell g(\mathrm{tr}(p))$ *and* $\varepsilon < \theta \Rightarrow \alpha_{\varepsilon+1} \in S$ *and* $\omega\varepsilon < \theta \Rightarrow \alpha_{\omega\varepsilon} \notin S$*; recalling that every member of* S *is strongly inaccessible and* S *is nowhere stationary this is clear. Now we choose* $\eta_\varepsilon \in p \cap {}^{\alpha_\varepsilon}2$ *by induction on* $\varepsilon < \theta$ *such that* $\eta_0 = \mathrm{tr}(p)$ *and* $\zeta < \varepsilon \Rightarrow \eta_\zeta \trianglelefteq \eta_\varepsilon$.

If $\varepsilon < \theta$ *is limit, then we let* $\eta_\varepsilon = \bigcup\{\eta_\zeta : \zeta < \varepsilon\}$ *and we note that it belongs to* p *by clause (A)(e) of Definition 1.3 (because* $\alpha_\varepsilon \notin S$*).*

If $\varepsilon = \zeta + 1 < \theta$, *then we use the induction hypothesis of part (4) for* $\partial = \alpha_\varepsilon$, *because* $\alpha_\varepsilon \in S$, *a set of inaccessibles.*

After the inductive construction is carried out, if $\theta = \kappa$, *i.e.,* $\sup(S) = \kappa$ *then* $\eta_\theta := \bigcup\{\eta_\varepsilon : \varepsilon < \kappa\}$ *is as required. If* $\theta < \kappa$, *i.e.,* $\sup(S) < \kappa$ *then* $\eta_\theta := \bigcup\{\eta_\varepsilon : \varepsilon < \theta\} \in p \cap {}^{\sup(S)}2$ *(remember Definition 1.3(A)(e)) and again by 1.4(5) we have* $p^{[\eta_\theta]} = ({}^{\kappa >}2)^{[\eta_\theta]}$ *so we can easily finish.*

(2) Let $(\eta_i, S_i, \bar{\Lambda}_i)$ *be a witness for* $p_i \in \mathbb{Q}_\kappa$ *for* $i < \delta$, without loss of generality $S_i = S_{p_i}$, *see clause (C) of Definition 1.3 or Claim 1.4(8). By our assumptions the sequence* $\langle \eta_i : i < \delta\rangle$ *is* $\trianglelefteq$*-increasing and let* $\eta_\delta = \bigcup\{\eta_i : i < \delta\}$. *Now if* $i, j < \delta$ *and* $i < j$ *then* $\eta_j = \mathrm{tr}(p_j) \in p_i$ *and if* $j < i$ *then* $\eta_j \trianglelefteq \eta_i = \mathrm{tr}(p_i)$. *Hence* $\eta_i \in \bigcap\{p_j : j < \delta\} = p_\delta$ *for all* $i < \delta$. *Consequently, recalling* $i < \delta \Rightarrow \min(S_i \backslash \sup\{\ell g(\mathrm{tr}(p_j))+ 1 : j < \delta\}) > \delta$, *we get* $\eta_\delta \in p_i$ *for all* $i < \delta$ *and thus* $\eta_\delta \in p_\delta$.

Let $S := \bigcup\{S_i : i < \delta\} \backslash (\ell g(\eta_\delta) + 1)$ *and* $\bar{\Lambda}_i = \langle \Lambda_{i,\partial} : \partial \in S_i\rangle$ *and for* $\partial \in S$ *let* $\Lambda_\partial := \bigcup\{\Lambda_{i,\partial} : i < \delta$ *and* $\partial \in S_i\}$. *So clearly* Λ_∂ *is a set of* $\leq |\delta| \cdot \partial$ *dense*

subsets of $\mathbb{Q}_\partial$. *Also* $\partial \in S \Rightarrow \partial > \delta$ *because if* $\partial \in S$ *then for some* $i < \delta$, $\partial \in S_i$ *and by an assumption* $\min(S_i \setminus \sup\{\ell g(\mathrm{tr}(p_i)) + 1 : i < \delta\}) > \delta$ *hence* $\partial > \delta$. *It follows that* $|\Lambda_\partial| \leq \partial$. *Now one easily shows that* $\eta_\delta, S, \langle \Lambda_\partial : \partial \in S\rangle$ *witness that* $p_\delta = \bigcap\{p_i : i < \delta\}$ *belongs to* $\mathbb{Q}_\kappa$*; being a* $\leq_{\mathbb{Q}_\kappa}$*-lub of* $\bar{p}$ *is obvious by the definition of* $\leq_{\mathbb{Q}_\kappa}$.
(3) Without loss of generality δ *is a limit ordinal. The assumptions on* p_i, E_i *imply that* $\eta_i \lhd \eta_j$ *when* $i < j < \delta$ *and* $\delta \leq \sup\{\ell g(\eta_i) : i < \delta\} \in \bigcap_{\alpha<\delta} E_\alpha$. *Consequently,*

$$\min\left(S_{p_\alpha} \setminus \sup\{\ell g(\mathrm{tr}(p_i)) : i < \delta\}\right) > \sup\{\ell g(\mathrm{tr}(p_i)) : i < \delta\} \geq \delta$$

and we may apply part (2).
(4) Without loss of generality $\rho \trianglelefteq \mathrm{tr}(p)$ *(recalling 1.4(3)) and* $i(*) = \kappa$.

First, if $\kappa > \delta_* := \sup\{\partial : \partial < \kappa$ *inaccessible*$\}$ *then by part (1) which, for* κ*, was already proven there is* $\eta \in p$ *such that* $\ell g(\eta) > \delta_*, \ell g(\mathrm{tr}(p))$. *Then* $p \leq_{\mathbb{Q}_\kappa} p^{[\eta]} = ({}^{\kappa>}2)^{[\eta]}$ *and* $p^{[\eta]} \leq_{\mathbb{Q}_\kappa} q \Rightarrow q = \left({}^{\kappa>}2\right)^{[\mathrm{tr}(q)]}$. *Consequently, the claim becomes a case of the Baire category theorem for* ${}^\kappa 2$.

So we assume that $\delta_* = \kappa$ *and by induction on* $i < \kappa$ *we choose* $p_i, \eta_i, S_i, \bar{\Lambda}_i, E_i$ *such that:*

- *(a)* $p_i \in \mathbb{Q}_\kappa$ *and* $(\eta_i, S_i, \bar{\Lambda}_i, E_i)$ *is a full witness for this,*
- *(b)* $p \leq p_0$*, and* $i < j < \kappa \Rightarrow p_i \leq_{\mathbb{Q}_\kappa} p_j$,
- *(c)* $i < j < \kappa \Rightarrow E_j \subseteq E_i \wedge \min(E_i) < \ell g(\mathrm{tr}(p_j))$,
- *(d) for every* $i < \kappa$*, for some* $q_i \in \mathscr{I}_i$ *we have* $q_i \leq p_i$.

Why can we carry out the induction? At stage δ *of the construction we use part (3) which we have already proved to find an upper bound* q *to* $\{p_i : i < \delta\} \cup \{p\}$. *Then, as* $\mathscr{I}_\delta$ *is dense, we may pick* $q_\delta \in \mathscr{I}_\delta$ *stronger than* q. *Let* $\partial < \kappa$ *be an inaccessible cardinal larger than* $\ell g(\mathrm{tr}(q_\delta))$ *and* $\sup\{\min(E_i) + 1 : i < \delta\}$. *By part (1) which we have already proved there exists* $\eta_\delta \in q_\delta \cap {}^\partial 2$. *Now it should be clear that we may choose* $p_\delta, S_\delta, \bar{\Lambda}_\delta, E_\delta$ *such that* $(\eta_\delta, S_\delta, \bar{\Lambda}_\delta, E_\delta)$ *is a full witness for* $p_\delta \in \mathbb{Q}_\kappa$ *and* $q_\delta \leq p_\delta$ *and* $E_\delta \subseteq \bigcap_{i<\delta} E_i$.

Having carried out the induction, $\eta := \bigcup\{\mathrm{tr}(p_i) : i < \kappa\}$ *is as required.*
(5) It can be easily reduced to part (2), but let us elaborate. Without loss of generality $\delta = \mathrm{cf}(\delta)$ *and let* $\nu = \bigcup\{\mathrm{tr}(p_i) : i < \delta\}$. *For each* $i < \delta$*, we have* $j \in (i, \delta) \Rightarrow \mathrm{tr}(p_i) \trianglelefteq \mathrm{tr}(p_j) \in p_j$ *and* $j < i \Rightarrow \mathrm{tr}(p_i) \in p_j$*, so together we have* $\mathrm{tr}(p_i) \in \bigcap\{p_j : j < \delta)\}$. *Hence, remembering* $(\otimes)(a)$*, we have* $\nu \in \bigcap_{i<\delta} p_i$. *If* $\langle \mathrm{tr}(p_i) : i < \delta\rangle$ *is not eventually constant, then* $\lg(\nu) \geq \mathrm{cf}(\delta)$*, and hence* $(\odot)$ *of part (2) holds and we are done. If* $\langle \mathrm{tr}(p_i) : i < \delta\rangle$ *is eventually constant then also* $(\odot)$ *of part (2) holds so we are done too. By the last two sentences we are done.* □

Claim 1.6 *Assume*

- *(a)* $\alpha \leq \beta < \kappa$,
- *(b)* $\eta \in {}^\beta 2$,
- *(c)* $(\mathrm{tr}(p_i), S_i, \bar{\Lambda}_i)$ *witness* $p_i \in \mathbb{Q}_\kappa$ *for* $i < \alpha$,
- *(d)* $\mathrm{tr}(p_i) \trianglelefteq \eta \in p_i$,
- *(e)* $S = \bigcup\{S_i : i < \alpha\} \setminus (\ell g(\eta) + 1)$,

Sh:1004

(f) for $\partial \in S$ *we let* $\Lambda_\partial := \bigcup\{\Lambda_{i,\partial} : \partial \in S_i\}$ *(so it is a set of* $\leq \partial$ *dense subsets of* $\mathbb{Q}_\partial$*).*

Then $\bigcap\{p_i^{[\eta]} : i < \alpha\} \in \mathbb{Q}_\kappa$ *is a* $\leq_{\mathbb{Q}_\kappa}$*-lub of* $\{p_i^{[\eta]} : i < \alpha\}$ *and has the witness* $(\eta, S, \langle \Lambda_\partial : \partial \in S\rangle)$.

Proof Should be clear.

□

Observation 1.7 *1. If* $p, q \in \mathbb{Q}_\kappa$ *and* $\mathbb{Q}_\kappa \models$ *"*$p \nleq q$*"* *then for some* r*, we have* $q \leq_{\mathbb{Q}_\kappa} r$ *and* r, p *are incompatible (so* $\lim_\kappa(p), \lim_\kappa(r)$ *are disjoint).*

2. If $p_1, p_2 \in \mathbb{Q}_\kappa$ *then the following conditions are equivalent:*
(a) p_1, p_2 *are compatible,*
(b) the sets $\lim_\kappa(p_1), \lim_\kappa(p_2)$ *are not disjoint,*
(c) $\mathrm{tr}(p_1) \in p_2$ *and* $\mathrm{tr}(p_2) \in p_1$,
(d) $\mathrm{tr}(p_1) \trianglelefteq \mathrm{tr}(p_2) \in p_1$ *or* $\mathrm{tr}(p_2) \trianglelefteq \mathrm{tr}(p_1) \in p_2$.

3. If $p \in \mathbb{Q}_\kappa$*, then there is a maximal antichain above* p *of cardinality* κ.

4. The $\mathbb{Q}_\kappa$*-name* $\underset{\sim}{\eta}_\kappa = \bigcup\{\mathrm{tr}(p) : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_\kappa}\}$ *is a name for a* κ*-real which is generic for* $\mathbb{Q}_\kappa$*, i.e.,* $\underset{\sim}{\mathbf{G}}_{\mathbb{Q}_\kappa}$ *is computable from* $\underset{\sim}{\eta}_\kappa$ *over* $\mathbf{V}$.

Proof (1) As $p \nleq q$, by the definition of $\leq_{\mathbb{Q}_\kappa}$ we have $q \nsubseteq p$, so we can choose $\nu \in q \backslash p$. Let $r = q^{[\nu]}$, so $q \leq r$ by 1.4(3). Since $\mathrm{tr}(r) = \nu \notin p$, we are done by (2).
(2) First, $(a) \Rightarrow (b)$ as letting r be a common upper bound of p_1, p_2 we have $\lim_\kappa(r) \subseteq \lim_\kappa(p_1) \cap \lim_\kappa(p_2)$ and recall $r \in \mathbb{Q}_\kappa \Rightarrow \lim_\kappa(r) \neq \emptyset$ by 1.5(1).

Second, $(b) \Rightarrow (c)$ as $\eta \in \lim_\kappa(p_\ell) \Rightarrow \mathrm{tr}(p_\ell) \trianglelefteq \eta \wedge \{\eta \restriction \alpha : \alpha < \kappa\} \subseteq p_\ell$.

Third, $(c) \Rightarrow (d)$ trivially.

Fourth, $(d) \Rightarrow (a)$ as without loss of generality $\mathrm{tr}(p_1) \trianglelefteq \mathrm{tr}(p_2) \in p_1$, hence $p_1^{[\mathrm{tr}(p_2)]}, p_2$ are members of $\mathbb{Q}_\kappa$ with the same trunk so are compatible by 1.6. As $\mathbb{Q}_\kappa \models$ "$p_1 \leq p_1^{[\mathrm{tr}(p_2)]}$", we are done.
(3) Let $\eta \in \lim_\kappa(p)$ and for $\alpha \in [\ell g(\mathrm{tr}(p)), \kappa)$ let $\nu_\alpha = (\eta \restriction \alpha)\hat{\ }\langle 1 - \eta(\alpha)\rangle$. Then $\{p^{[\nu_\alpha]} : \alpha \in [\ell g(\mathrm{tr}(p)), \kappa)\}$ is as required.
(4) Should be clear.

□

Claim 1.8 *(1)* $\mathbb{Q}_\kappa$ *is* κ*-strategically closed.*
(2) $\mathbb{Q}_\kappa$ *satisfies the* κ^+*-c.c.*

Proof (1) Immediate by 1.5(3).
(2) Obviously

$(*)_1$ ${}^{\kappa>}2$ has cardinality κ (recall that κ is inaccessible), and
$(*)_2$ if $p_1, p_2 \in \mathbb{Q}_\kappa$ have the same trunk then they are compatible.

Together we are clearly done. □

Claim 1.9 *(1) If* κ *is weakly compact then* $\mathbb{Q}_\kappa$ *is* κ*-bounding, i.e. for every* $f \in ({}^\kappa\kappa)^{\mathbf{V}[\mathbb{Q}_\kappa]}$ *there is* $g \in ({}^\kappa\kappa)^{\mathbf{V}}$ *such that* $f \leq g$*, that is,* $\alpha < \kappa \Rightarrow f(\alpha) \leq g(\alpha)$.
(2) Moreover, if $p \Vdash_{\mathbb{Q}_\kappa}$ *"*$\underset{\sim}{f} \in {}^\kappa\kappa$*" and* $\beta < \kappa$ *then for some* $\bar{\beta}$ *and* $q \in \mathbb{Q}_\kappa$ *we have:*

Sh:1004

- $p \le q$,
- $p \cap {}^{\beta \ge}2 = q \cap {}^{\beta \ge}2$,
- $\bar{\beta} = \langle \beta(i) : i < \kappa \rangle$ *is increasing continuous,* $\beta(0) \ge \beta$, $\beta(i) < \kappa$,
- *if* $\nu \in q \cap {}^{\beta(i+1)}2$ *then* $q^{[\nu]}$ *forces a value to* $\underset{\sim}{f}(i)$.

Proof (2) Let $p \Vdash$ "$\underset{\sim}{f} \in {}^{\kappa}\kappa$". By induction on $i < \kappa$ we choose p_i, $\beta(i)$, ϱ_i, S_i, $\bar{\Lambda}_i$ and E_i such that

(i) $p_i \in \mathbb{Q}_\kappa$,
(ii) $\langle \beta(j) : j \le i \rangle$ is an increasing continuous sequence of ordinals $< \kappa$,
(iii) $p_0 = p$ and $\beta(0) = \max\{\beta, \ell g(\mathrm{tr}(p)) + 1\}$,
(iv) $(\varrho_i, S_i, \bar{\Lambda}_i, E_i)$ is a full witness for $p_i \in \mathbb{Q}_\kappa$,
(v) if $j < i$ then
  (α) $p_j \le_{\mathbb{Q}_i} p_i$,
  (β) $p_j \cap {}^{\beta(j)\ge}2 = p_i \cap {}^{\beta(j)\ge}2$ (hence $\varrho_i = \varrho_0$), and $S_j \cap \big(\beta(j)+1\big) = S_i \cap \big(\beta(j)+1\big)$, $\bar{\Lambda}_j \restriction \big(\beta(j)+1\big) = \bar{\Lambda}_j \restriction \big(\beta(j)+1\big)$,
  (γ) $\beta(i) \in E_j$,
  (δ) $E_i \subseteq E_j$ and if i is limit then $E_i = \bigcap_{\alpha < i} E_\alpha$,
(vi) if $i = j+1$ and $\nu \in p_i \cap {}^{\beta(i)}2$ then $p_i^{[\nu]}$ forces a value to $\underset{\sim}{f}(j)$.

For $i = 0$ choose a full witness $(\varrho_0, S_0, \bar{\Lambda}_0, E_0)$ for p, and use clause (iii) to define $p_0, \beta(0)$.

For a limit $i < \kappa$ work as in the proof of 1.5(2).

For a successor i, say $i = j+1$, we shall use the definition of "κ is weakly compact". Let $\langle q_{j,\beta} : \beta < \beta(*) \rangle$ be a maximal antichain of $\mathbb{Q}_\kappa$ such that $q_{j,\beta} \Vdash$ "$\underset{\sim}{f}(j) = \gamma$" for some $\gamma = \gamma_{j,\beta}$ and $q_{j,\beta}$ is $\le_{\mathbb{Q}_\kappa}$-above p_j or $\lim_\kappa(q_{j,\beta}) \cap \lim_\kappa(p_j) = \emptyset$, recalling 1.4(9). Since $\mathbb{Q}_\kappa$ satisfies the κ^+-c.c., see 1.8(2), we know that $\beta(*) \le \kappa$, so by 1.7(3) without loss of generality $\beta(*) = \kappa$. Recalling each $S_{q_{j,\beta}}$ is nowhere stationary, clearly there is a club E of κ such that

$$\beta < \delta \in E \Rightarrow \delta \in E_j \backslash S_{q_{j,\beta}} \text{ and hence also } \delta \notin S_{p_j}.$$

By the weak compactness there is a strongly inaccessible cardinal $\partial(j) > \beta(j)$ belonging to E such that $\{q_{j,\beta} \cap {}^{\partial(j)>}2 : \beta < \partial(j)\}$ is a pre-dense subset of $\mathbb{Q}_{\partial(j)}$. Let

$$\mathscr{I} = \big\{q \in \mathbb{Q}_{\partial(j)} : \text{ for some } \beta < \partial(j) \text{ we have } (q_{j,\beta} \cap {}^{\partial(j)>}2) \le_{\mathbb{Q}_{\partial(j)}} q\big\}.$$

Clearly, $\mathscr{I}$ is a dense open subset of $\mathbb{Q}_{\partial(j)}$. Let

$$\mathscr{X} = \big\{\eta \in p_j \cap {}^{\partial(j)}2 : \big(\exists \beta < \partial(j)\big)\big(\eta \in q_{j,\beta} \cap {}^{\partial(j)}2\big)\big\}.$$

For each $\rho \in \mathscr{X}$ there is $r_{j,\rho} \ge p_j$ such that $\mathrm{tr}(r_{j,\rho}) = \rho$ and $r_{j,\rho}$ forces a value to $\underset{\sim}{f}(j)$. Indeed, there is $\beta < \partial(j)$ such that $\rho \in q_{j,\beta} \cap {}^{\partial(j)}2$, so by our assumptions on the $q_{j,\beta}$'s necessarily $p_j \le q_{j,\beta}$, so $q_{j,\beta}^{[\rho]}$ can serve as $r_{j,\rho}$. Let $(\rho, S_{j,\rho}, \bar{\Lambda}_{j,\rho})$ witness $r_{j,\rho} \in \mathbb{Q}_\kappa$. Lastly, we let

(a) $p_i = \bigcup\{r_{j,\rho} : \rho \in \mathscr{X}\}$,

(b) $\beta(i) = \min\big(E_j \backslash (\partial(j)+1)\big)$,
(c) $S_i = S'_i \cup S''_i \cup \{\partial(j)\}$, where

$$S'_i = \bigcup\big\{S_{r_{j,\rho}} : \rho \in (p_i \cap {}^{\partial(j)}2)\big\}\backslash(\partial(j)+1) \quad \text{and} \quad S''_i = S_j \cap \partial(j),$$

(d) $\bar\Lambda_i = \langle \Lambda_{i,\partial} : \partial \in S_i\rangle$, where
  (α) $\Lambda_{i,\partial}$ is $\Lambda_{j,\partial}$ if $\partial \in S''_i$, and
  (β) $\Lambda_{i,\partial}$ is $\bigcup\{\Lambda_{j,\rho,\partial} : \rho \in p_i \cap {}^{\partial(j)}2$ and $\partial \in S_{r_{j,\rho}}\}$ if $\partial \in S'_i$,
  (γ) $\Lambda_{i,\partial(j)}$ is $\{\mathscr{I}\}$,
(e) E_i is $E\backslash(\beta(i)+1)$ or just a club of κ which is $\subseteq E_j\backslash\beta(i)$ and is disjoint to $S_{r_{j,\rho}}$ for every $\rho \in \mathscr{X}$.

It should be clear that the objects defined above have the desired properties.

So we can carry out the induction on $i < \kappa$. After it is completed we define

$(*)_1$ $q = \bigcap\{p_i : i < \kappa\}$,
$(*)_2$ $S = \bigcup\{S_i : i < \kappa\}$,
$(*)_3$ $\bar\Lambda = \langle \Lambda_\partial : \partial \in S\rangle$ where $\Lambda_\partial = \bigcup\{\Lambda_{i,\partial} : i < \kappa$ satisfies $\partial \in S_i\}$ and
$(*)_3$ $E = \{\delta < \kappa : \delta = \beta(\delta)$ is a limit ordinal such that $i < \delta \Rightarrow \delta \in E_i\}$.

It easily follows from conditions (i)–(vi) that:

$(\oplus)_1$ $q \in \mathbb{Q}_\kappa$ has trunk ϱ_0,
$(\oplus)_2$ $(\varrho_0, S, \bar\Lambda, E)$ is a full witness for $q \in \mathbb{Q}_\delta$,
$(\oplus)_3$ $p \leq_{\mathbb{Q}_\kappa} q$ and $p \cap {}^{\beta\geq}2 = q \cap {}^{\beta\geq}2$,
$(\oplus)_4$ if $\nu \in q \cap {}^{\beta(j+1)}2$, then $q^{[\nu]}$ forces a value to $\underset{\sim}{f}(j)$.

(1) Follows from (2) proven above: $(\oplus)_4$, that is the last bullet in 1.9(2), suffices for defining a function $g \in \mathbf{V}$ such that q forces that it bounds $\underset{\sim}{f}$, we are done. □

Conclusion 1.10 *(1) If κ is a weakly compact cardinal* *<u>then</u>* *there is a $(<\kappa)$-strategically complete, κ^+-c.c., κ-bounding forcing notion (hence not adding a κ-Cohen), and of course, adding a new $\eta \in {}^\kappa 2$.*
02 In fact, the forcing is κ-Borel and is κ-strategically complete and it is equivalent to a $(<\kappa)$-complete forcing notion (which necessarily is κ^+-c.c. κ-bounding adding a new subset to κ). Also, the forcing is definable even without parameters.

Proof (1) See above.
(2) Note that when κ is not weakly compact, $\mathbb{Q}_\kappa$ is not κ-Borel because "nowhere stationary" is not. However, if we replace the conditions by full witnesses of conditions with the natural order, this becomes easy. □

1(B) Adding a dominating member of $\prod_{\varepsilon<\lambda} \theta_\varepsilon$

Here we present a variant of the forcing from Sect. 1(A), this time dealing with sequences from $\prod_{\varepsilon<\lambda}\theta_\varepsilon$ instead of ${}^\lambda 2$ and we have an $|\varepsilon|^+$-complete filter D_ε on θ_ε for $\varepsilon < \lambda$. The main case is $D_\varepsilon = \{a \subseteq \theta_\varepsilon : |\theta_\varepsilon\backslash a| < \theta_\varepsilon\}$, so we write only this case, but the changes needed for the general case are minor. This is also true for

Sh:1004

$\langle \theta_\eta, D_\eta : \eta \in \mathscr{T} \rangle$ and $\mathscr{T} = \{\nu : \varepsilon < \ell g(\nu) \Rightarrow \nu(\varepsilon) < \theta_{\nu \restriction \varepsilon}\}$. So our starting point, e.g. the forcing for the first κ, is not the κ-Cohen forcing but $\mathbb{Q}_\theta$ of [27], which is the parallel for κ of the forcing of [32] for $\lambda = \aleph_0$.

Note that Definitions 1.12, 1.13 are used in [28], too. Also note that $\mathbb{Q}_{\bar\theta}$ is the "one step" forcing on which we shall build later.

The reader may ignore the version with $\bar{\mathscr{P}}$, i.e., use the default $\mathscr{P}_\kappa = \mathscr{P}(\mathscr{H}(\kappa))$.

Remark 1.11 For $\bar\theta = \langle \theta_\alpha : \alpha < \kappa \rangle$, $\mathbb{Q}_{\bar\theta} = \mathbb{Q}^1_\kappa$ was designed to make the old κ-reals κ-meagre, we still have to expect it to behave like random real forcing and do this indeed.

Definition 1.12 (1) Recall the weakly compact ideal on λ is $I^{\rm wc}_\lambda = \{A \subseteq \lambda$: for some first order formula $\varphi(X, Y)$ and $B \subseteq \mathscr{H}(\lambda)$ we have $(\forall X \subseteq \mathscr{H}(\lambda))(\mathscr{H}(\lambda) \models \varphi[X, B])$ but for no strongly inaccessible $\kappa \in A$ do we have $(\forall X \subseteq \mathscr{H}(\kappa))(\mathscr{H}(\kappa) \models \varphi[X, B \cap \mathscr{H}(\kappa)]\}$.
(2) $\diamondsuit_{S_*, I^{\rm wc}_\lambda}$ means that some $\bar A = \langle A_\alpha : \alpha \in S_* \rangle$ is an $I^{\rm wc}_\lambda$-diamond sequence, which means: for every $A \subseteq \mathscr{H}(\lambda)$ the set $\{\kappa \in S_* : A \cap \mathscr{H}(\kappa) = A_\kappa\}$ is $\neq \emptyset \mod I^{\rm wc}_\lambda$.
(3) We say $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha \in S_* \rangle$ is $I^{\rm wc}_\lambda$-positive <u>when</u> $S_* \in (I^{\rm wc}_\lambda)^+$ and $(\mathscr{P}_\alpha, \alpha, \in)$ and $(\mathscr{P}(\alpha), \alpha, \in)$ have the same first order theory, and moreover $(a) \Rightarrow (b)$ where

- (a) $\varphi(X, Y)$ is first order, $A \subseteq \mathscr{H}(\lambda)$ satisfies $X \subseteq \mathscr{H}(\lambda) \Rightarrow (\mathscr{H}(\lambda), \in) \models \varphi[X, A]$,
- (b) $(\exists^{I^{\rm wc}_\lambda} \kappa \in S_*)[A \cap \mathscr{H}(\kappa) \in \mathscr{P}_\kappa$ and $X \subseteq \mathscr{H}(\kappa) \Rightarrow (\mathscr{H}(\kappa), \in) \models \varphi[X, A \cap \kappa]]$.

(4) The default value of $\bar{\mathscr{P}}$ is $\langle \mathscr{P}(\mathscr{H}(\kappa)) : \kappa \in S_{\mathfrak{x}} \rangle$.

Definition 1.13 (1) We say $\mathfrak{x}$ is a 1-ip when $\mathfrak{x}$ consists of:

- (A) a weakly compact cardinal λ,
- (B) a sequence $\bar\theta = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$, where

$$\varepsilon < \lambda \Rightarrow (2 \le \theta_\varepsilon < \aleph_0) \vee (\varepsilon < \theta_\varepsilon = \mathrm{cf}(\theta_\varepsilon) < \lambda),$$

- (C) a stationary set $S_{\mathfrak{x}} \subseteq \lambda$ of strongly inaccessible cardinals satisfying

$$\zeta < \kappa \in S_{\mathfrak{x}} \Rightarrow \prod_{\varepsilon < \zeta} \theta_\varepsilon < \kappa,$$

- (D) (a) $\diamondsuit_{S_{\mathfrak{x}}, I^{\rm wc}_\lambda}$, i.e. diamond on $S_{\mathfrak{x}}$ holds even modulo the weakly compact ideal, or just
  - (b) $\bar{\mathscr{P}} = \langle \mathscr{P}_\kappa \subseteq \mathscr{P}(\mathscr{H}(\kappa)) : \kappa \in S_{\mathfrak{x}} \rangle$ is $I^{\rm wc}_\lambda$-positive, see Definition 1.12(3) above, so necessarily $S_{\mathfrak{x}} \in (I^{\rm wc}_\lambda)^+$; the default value is $\mathscr{P}_\kappa = \mathscr{P}(\mathscr{H}(\kappa))$,
- (E) $S^*_{\mathfrak{x}} := \{\kappa \le \lambda : \kappa$ weakly compact and $S_{\mathfrak{x}} \cap \kappa \in (I^{\rm wc}_\kappa)^+$ moreover the sequence $\bar{\mathscr{P}} \restriction (S_{\mathfrak{x}} \cap \kappa)$ is $I^{\rm wc}_\kappa$-positive (see 1.12(3))$\}$.

(2) If $\kappa \in S^*_{\mathfrak{x}}$ we may say "κ is $\mathfrak{x}$-weakly compact".
(3) Let $\bar\theta = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$ be as in clause 1(B) (we will fix it for this sub-section). Define $\mathbf{T}_\alpha = \prod_{\varepsilon < \alpha} \theta_\varepsilon$ for $\alpha < \lambda$ and $\mathbf{T}_{<\alpha} = \bigcup \{\mathbf{T}_\beta : \beta < \alpha\}$ for $\alpha \le \lambda$.

Convention 1.14 For this subsection
(0) $\mathfrak{x}$ is as in Definition 1.13.
(1) Let κ, ∂ denote members of $S_{\mathfrak{x}}$.
(2) Always p is a subtree of $\mathbf{T}_{<\kappa}$, for some $\kappa \leq \lambda$, typically it belongs to $\mathbb{Q}^1_\kappa$ for some $\kappa \leq \lambda$ and for $\eta \in p$ let $p^{[\eta]} = \{\nu \in p : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu\}$.

Definition 1.15 We define the forcing notion $\mathbb{Q}^1_\kappa$ by induction on κ (so $\kappa \in S_{\mathfrak{x}}$) as follows:

(A) $p \in \mathbb{Q}^1_\kappa$ <u>iff</u> some $S \subseteq \kappa \cap S_{\mathfrak{x}}$ witnesses it, which means
 (a) p is a subtree of $\mathbf{T}_{<\kappa}$,
 (b) p has trunk $\mathrm{tr}(p) \in \mathbf{T}_{<\kappa}$ that is
  - $\beta \leq \ell g(\mathrm{tr}(p)) \Rightarrow p \cap \mathbf{T}_\beta = \{\mathrm{tr}(p)\restriction\beta\}$ but
  - $(\exists^{\geq 2}\alpha)(\mathrm{tr}(p)\hat{\ }\langle\alpha\rangle \in p)$,
 (c) if $\eta \in p \wedge \ell g(\mathrm{tr}(p)) \leq \ell g(\eta) < \beta < \kappa$ then $(\exists\nu)(\eta \triangleleft \nu \in p \cap \mathbf{T}_\beta)$, follows from the rest,
 (d) if $\eta \in p$ and $\ell g(\mathrm{tr}(p)) \leq \ell g(\eta) < \kappa$ then[1]
  - if $\theta_{\ell g(\eta)} \geq \aleph_0$ <u>then</u> $(\forall^\infty i < \theta_{\ell g(\eta)})[\eta\hat{\ }\langle i\rangle \in p]$,
  - if $\theta_{\ell g(\eta)} < \aleph_0$ <u>then</u> $(\forall i < \theta_{\ell g(\eta)})(\eta\hat{\ }\langle i\rangle \in p)$,
 (e) if $\delta \in \kappa\backslash S$ is a limit ordinal and $\eta \in \mathbf{T}_\delta := \prod_{\varepsilon<\delta}\theta_\varepsilon$, then $\eta \in p \Leftrightarrow (\forall\beta < \delta)(\eta\restriction\beta \in p)$,
 (f) if $\partial \in \kappa \cap S$ hence $\partial \in S_{\mathfrak{x}}$ so is strongly inaccessible, <u>then</u> $p \cap \mathbf{T}_{<\partial} \in \mathbb{Q}^1_\partial$ and for some predense subsets $\mathscr{I}_i$ of $\mathbb{Q}^1_\partial$ for $i < i_* \leq \partial$, [if we have $\bar{\mathscr{P}}$ also $\mathscr{I}_i \in \mathscr{P}_\kappa$] for every $\eta \in \mathbf{T}_\partial$ we have:
  - $\eta \in p$ iff $(\forall\beta < \partial)(\eta\restriction\beta \in p)$ and $(\forall i < i_*)(\exists q \in \mathscr{I}_i)(\forall\beta < \partial)(\eta\restriction\beta \in q)$,
 (g) $S \subseteq \kappa \cap S_{\mathfrak{x}}$ is not stationary in any inaccessible $\partial \leq \kappa$, even if $\partial \notin S_{\mathfrak{x}}$ (yes also for $\partial = \kappa$), equivalently for any limit $\delta \leq \kappa$ as $S_{\mathfrak{x}}$ is a set of inaccessibles and $S \subseteq S_{\mathfrak{x}}$.

(B) $\leq_{\mathbb{Q}^1_\kappa}$ is the inverse inclusion.

Claim 1.16 *(1) $\mathbf{T}_{<\kappa}$ belongs to $\mathbb{Q}^1_\kappa$ and*

- *$p \in \mathbb{Q}^1_\kappa \Rightarrow \mathbb{Q}^1_\kappa \models$ "$\mathbf{T}_{<\kappa} \leq p$", and*
- *$\eta \in p \in \mathbb{Q}^1_\kappa \Rightarrow p \leq_{\mathbb{Q}^1_\kappa} p^{[\eta]} \in \mathbb{Q}^1_\kappa$.*

(2) For $p \in \mathbb{Q}^1_\kappa$ and $\alpha < \kappa$ the set $\{p^{[\eta]} : \eta \in p \cap \mathbf{T}_\alpha\}$ is predense in $\mathbb{Q}^1_\kappa$ above p.
(3) If $p \in \mathbb{Q}^1_\kappa$ and $\ell g(\mathrm{tr}(p)) < \partial < \kappa$ <u>then</u> $p \cap \mathbf{T}_{<\partial} \in \mathbb{Q}^1_\partial$. Moreover, if $p_\ell \in \mathbb{Q}^1_\kappa$, $\ell g(\mathrm{tr}(p_\ell)) < \partial < \kappa$ for $\ell = 1, 2$, then

$$p_1 \leq_{\mathbb{Q}^1_\kappa} p_2 \Rightarrow p_1 \cap \mathbf{T}_{<\partial} \leq_{\mathbb{Q}^1_\partial} p_2 \cap \mathbf{T}_{<\partial},$$

and

$$p_1 \perp_{\mathbb{Q}^1_\kappa} p_2 \Rightarrow p_1 \cap \mathbf{T}_{<\partial} \perp_{\mathbb{Q}^1_\partial} p_2 \cap \mathbf{T}_{<\partial}.$$

[1] Remember "$\forall^\infty i < \theta$" means "for all but boundedly many $i < \theta$".

(4) $\mathbb{Q}^1_\kappa$ *is a forcing notion and it satisfies the* κ^+*-c.c. Moreover, it is* κ^+*-centered as if* $p, q \in \mathbb{Q}^1_\kappa$ *have the same trunk* *<u>then</u>* p, q *are compatible, in fact,* $p \cap q$ *belongs to* $\mathbb{Q}^1_\kappa$ *and is a* $\le_{\mathbb{Q}^1_\kappa}$*-lub with the same trunk.*
(5) Suppose that $\nu \in \mathbf{T}_\gamma$ *and* $p_i \in \mathbb{Q}^1_\kappa$, $\mathrm{tr}(p_i) = \nu$ *for* $i < i(*)$ *and assume that*

($\boxdot$) *either* $i(*) \le \gamma$*, or*

$$(\forall \varepsilon)[\ell g(\nu) \le \varepsilon < \kappa \wedge \theta_\varepsilon \ge \aleph_0 \Rightarrow i(*) < \theta_\varepsilon] \quad \textit{and} \quad i(*) < \min(S_{\mathfrak{x}} \backslash (\ell g(\nu) + 1)).$$

Then $p = \bigcap \{p_i : i < i(*)\}$ *belongs to* $\mathbb{Q}^1_\kappa$*, has the trunk* ν *and is a* $\le_{\mathbb{Q}^1_\kappa}$*-lub of* $\{p_i : i < i(*)\}$*.*
(6) $p, q \in \mathbb{Q}^1_\kappa$ *are incompatible* *<u>iff</u>* $\mathrm{tr}(p) \notin q \vee \mathrm{tr}(q) \notin p$*.*
(7) If $\nu \in \mathbf{T}_\gamma$*,* $p_i \in \mathbb{Q}^1_\kappa$*, and* $\mathrm{tr}(p_i) \trianglelefteq \nu \in p_i$ *for* $i < i(*)$ *and* ($\boxdot$) *of part (5) holds,* *<u>then</u>* $p = \bigcap \{p_i^{[\nu]} : i < i(*)\}$ *is a lub of* $\{p_i^{[\nu]} : i < i(*)\}$ *in* $\mathbb{Q}^1_\kappa$ *and has trunk* ν*.*
(8) $\underset{\sim}{\eta} = \bigcup \{\mathrm{tr}(p) : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}^1_\kappa}\}$ *is a* $\mathbb{Q}^1_\kappa$*-name of a member of* $\prod_{\varepsilon<\kappa} \theta_\varepsilon$*.*
(9) If $\nu \in \prod_{\varepsilon<\kappa} \theta_\varepsilon$ *<u>then</u>* $\Vdash_{\mathbb{Q}^1_\kappa}$ *"for arbitrarily large* $\varepsilon < \kappa$ *we have* $\underset{\sim}{\eta}(\varepsilon) \ne \nu(\varepsilon)$ *and for every* $\varepsilon < \kappa$ *large enough* $\theta_\varepsilon \ge \aleph_0 \Rightarrow \underset{\sim}{\eta}(\varepsilon) > \nu(\varepsilon)$*".*
(10) $\underset{\sim}{\eta}$ *is a new branch of* $\mathbf{T}_{<\kappa}$ *and is generic for* $\mathbb{Q}^1_\kappa$*, i.e.* $\underset{\sim}{\mathbf{G}} = \{p \in \mathbb{Q}^1_\kappa : \underset{\sim}{\eta}$ *is a branch of* $p\}$*.*
(11) $\mathbb{Q}^1_\kappa$ *is* $(< \kappa)$*-strategically complete.*

Proof (1), (2), (3) Straightforward (for the second sentence of (3) use part (6)).
Concerning parts (4), (5) and (6), see more in 1.18 and 1.19.
(4) By (7) and the number of possible trunks of $p \in \mathbb{Q}^1_\kappa$ is $|\mathbf{T}_{<\kappa}| = \kappa$.
(5) By (7).
(6) Clearly if $\mathrm{tr}(p) \notin q$ then p, q are incompatible, and similarly if $q \notin \mathrm{tr}(p)$ so the implication "if" holds. For the other direction assume $\mathrm{tr}(p) \in q \wedge \mathrm{tr}(q) \in p$, and we shall prove that p, q are compatible. By symmetry without loss of generality $\ell g(\mathrm{tr}(p)) \le \ell g(\mathrm{tr}(q))$, let $\nu = \mathrm{tr}(q)$. Now $p^{[\nu]}$ and $q = q^{[\nu]}$ have the same trunk, so we are done by part (4).
(7) Let S_i be a witness for $p_i \in \mathbb{Q}^1_\kappa$, and let $S = \bigcup \{S_i : i < i(*)\} \backslash (\ell g(\nu) + 1)$. We shall prove that S witnesses that $p = \bigcap \{p_i^{[\nu]} : i < i(*)\}$ belongs to $\mathbb{Q}^1_\kappa$, then we are done as obviously $i < i(*) \Rightarrow p \subseteq p_i^{[\nu]}$ by the choice of p.

If $\partial \le \ell g(\nu)$ then $\partial \cap S = \emptyset$ and if $\ell g(\nu) < \partial < \kappa$, <u>then</u> each $S_i \cap \partial$ is not a stationary subset of ∂ for $i < i(*)$. Also $i(*) < \partial$.

[Why? If $i(*) \le \ell g(\nu)$ clear, if $i(*) > \ell g(\nu)$, <u>then</u> $S \cap [\ell g(\nu), i(*)] = \emptyset$ by assumption as $\partial > \ell g(\nu)$ clearly $i(*) < \partial$.] Together also $S = \bigcup \{S_i : i < i(*)\}$ is not stationary in ∂; that is, clause (g) of 1.15(A) holds.

Now obviously p is a subtree of $\mathbf{T}_{<\kappa}$, i.e. (a) of 1.15(A) holds. Also obviously $\alpha \le \ell g(\nu) \Rightarrow p \cap \mathbf{T}_\alpha = \{\nu \restriction \alpha\}$ and $p \cap \mathbf{T}_{\ell g(\nu)+1} \subseteq \{\nu\hat{\ }\langle\iota\rangle : \iota < \theta_{\ell g(\nu)}\}$. To prove clauses (b), (d) assume that $\eta \in p \cap \mathbf{T}_\varepsilon$ and $\nu \trianglelefteq \eta$. If $\theta_\varepsilon < \aleph_0$ then clearly $n < \theta_\varepsilon \wedge i < i(*) \Rightarrow \eta\hat{\ }\langle n\rangle \in p_i$ hence $\{\eta\hat{\ }\langle\iota\rangle : \iota < \theta_\varepsilon\} \subseteq p \cap \mathbf{T}_{\ell g(\eta)+1}$ so equality holds. Hence clause (d) holds in this case, and for $\varepsilon = \ell g(\nu)$ so $\eta = \nu$ then ν is indeed the trunk of p and 1.15(A)(b) holds.

If $\theta_\varepsilon \ge \aleph_0$ then $\theta_{\ell g(\eta)} = \mathrm{cf}(\theta_{\ell g(\eta)}) > i(*)$. Now, for each $i < i(*)$ there is $\iota(i) < \theta_\varepsilon$ such that $\{\eta\hat{\ }\langle\iota\rangle : \iota \in [\iota(i), \theta_\varepsilon)\} \subseteq p_i$ and hence $\iota(*) = \sup\{\iota(i) : i < i(*)\} < \theta_\varepsilon$.

Thus $\{\eta\hat{}\langle\iota\rangle : \iota \in [\iota(*), \theta_\varepsilon)\} \subseteq p$ and again clause (d) holds in this case, and for $\varepsilon = \ell g(\nu)$ so $\eta = \nu$, clearly $\mathrm{tr}(p)$ is well defined and equal to ν, so 1.15(b) holds.

The proof of clause 1.15(A)(c) follows from the rest.

The proofs of clauses (e), (f) are straightforward and clause (g) holds by the choice of S.

(8)–(11) Left to the reader. □

Observation 1.17 *If $p \leq_{\mathbb{Q}^1_\kappa} q$ and S is a witness for q and $\mathrm{tr}(p) = \mathrm{tr}(q)$ then S is a witness for p.*

Definition 1.18 Let $\kappa \in S_{\mathfrak{r}}$.
(1) For $\gamma < \kappa$ let $\mathbf{S}^{\mathrm{incr}}_{\kappa,\gamma}$ be the set of sequences $\langle (p_\alpha, q_\alpha, E_\alpha) : \alpha < \gamma\rangle$ satisfying[2]

(a) $p_\alpha \in \mathbb{Q}^1_\kappa$,
(b) $q_\alpha \in \mathbb{Q}^1_\kappa$.
(c) $\beta < \alpha \Rightarrow q_\beta \leq_{\mathbb{Q}^1_\kappa} p_\alpha$,
(d) E_α is a club of κ disjoint to some witness for $q_\beta \in \mathbb{Q}^1_\kappa$ for every $\beta < \alpha$,
(e) $p_\alpha \leq_{\mathbb{Q}^1_\kappa} q_\alpha$,
(f) $\ell g(\mathrm{tr}(p_\alpha)) \geq \alpha$,
(g) $\ell g(\mathrm{tr}(p_\alpha)) \in \bigcap\{E_\beta : \beta < \alpha\}$.

(2) For $\gamma \leq \kappa$ let $\mathbf{S}^{\mathrm{incr}}_{\kappa,<\gamma} = \bigcup\{\mathbf{S}^{\mathrm{incr}}_{\kappa,\beta} : \beta < \gamma\}$ and $\mathbf{S}^{\mathrm{incr}}_{\kappa} = \mathbf{S}^{\mathrm{incr}}_{\kappa,<\kappa}$.
(3) For $\gamma \leq \kappa$ let $\mathbf{S}^{\mathrm{pr}}_{\kappa,\gamma}$ be the set of sequences $\langle (p_\alpha, q_\alpha, E_\alpha) : \alpha < \gamma\rangle$ such that

(a) $p_\alpha, q_\alpha \in \mathbb{Q}^1_\kappa$ have trunks $\mathrm{tr}(p_0)$,
(b) E_α is a club of κ disjoint to $\ell g(\mathrm{tr}(p_0))$ such that for every $\beta < \alpha$, E_α is disjoint to some witness of $q_\beta \in \mathbb{Q}^1_\kappa$,
(c) $\min(E_\alpha) \geq \alpha$ is increasing (for transparency),
(d) $p_\alpha \leq_{\mathbb{Q}^1_\kappa} q_\alpha$,
(e) $q_\beta \leq_{\mathbb{Q}^1_\kappa} p_\alpha$ when $\beta < \alpha$,
(f) if $\beta < \alpha$ then $q_\beta \cap \mathbf{T}_{\min(E_\beta)} \subseteq p_\alpha$,
(g) if $\delta < \gamma$ is a limit ordinal then

$$p_\delta = \bigcap\left\{p_\alpha : \alpha < \delta\right\} \text{ and } p_\delta \cap \mathbf{T}_{\min(\cap\{E_\alpha:\alpha<\delta\})} \subseteq q_\beta \text{ for } \beta \in [\delta, \gamma).$$

(4) $\mathbf{S}^{\mathrm{pr}}_{\kappa,<\gamma} = \bigcup\{\mathbf{S}^{\mathrm{pr}}_{\kappa,\beta} : \beta < \gamma\}$ and $\mathbf{S}^{\mathrm{pr}}_{\kappa} = \bigcup\{\mathbf{S}^{\mathrm{pr}}_{\kappa,\gamma} : \gamma < \kappa\}$.

Claim 1.19 *(1) For every $p \in \mathbb{Q}^1_\kappa$ the sequence $\langle (p, p, \kappa)\rangle$ belongs to $\mathbf{S}^{\mathrm{incr}}_{\kappa}$.*
(2) $\mathbf{S}^{\mathrm{incr}}_{\kappa}$ is closed under unions of $\triangleleft$-increasing chains of length $< \kappa$.
(3) If $\bar{\mathbf{x}} = \langle (p_\alpha, q_\alpha, E_\alpha) : \alpha < \beta\rangle \in \mathbf{S}^{\mathrm{incr}}_{\kappa}$ then for some p_β we have: $\alpha < \beta \Rightarrow q_\alpha \leq p_\beta$ and if $p_\beta \leq q_\beta$ and E_β is a club of κ disjoint to some witness of q_β or just of p_β or just of q_γ for every $\gamma < \beta$ then $\bar{\mathbf{x}}\hat{}\langle (p_\beta, q_\beta, E_\beta)\rangle \in \mathbf{S}^{\mathrm{incr}}_{\kappa}$.

Proof (1) For $\gamma = 1$ we have $\langle (p, p, \kappa)\rangle \in \mathbf{S}^{\mathrm{incr}}_{\kappa,\gamma}$ (note that clause (d) of Definition 1.18(1) is trivially satisfied) and $\mathbf{S}^{\mathrm{incr}}_{\kappa,\gamma} \subseteq \mathbf{S}^{\mathrm{incr}}_{\kappa}$.

[2] May add: (h) if $\delta < \gamma$ is a limit ordinal then $p_\delta = \cap\{p_\alpha : \alpha < \delta\}$, we do not use this.

Sh:1004

(2) Obvious.
(3) If β is a successor ordinal this is easier, so we assume β is a limit ordinal. Let $\nu_\alpha = \mathrm{tr}(q_\alpha)$ for $\alpha < \beta$ hence $\langle \nu_\alpha : \alpha < \beta \rangle$ is a $\trianglelefteq$-increasing sequence of members of $\mathbf{T}_{<\kappa}$ and $\ell g(\nu_\alpha) \geq \alpha$. Hence $\nu_\beta := \cup\{\nu_\alpha < \beta\} \in \mathbf{T}_{\leq\kappa}$ has length $\geq \beta$. As $\beta < \kappa$ and κ is regular, necessarily $\ell g(\nu_\beta) < \kappa$ so $\nu_\beta \in \mathbf{T}_{<\kappa}$. Also recall $\alpha_1 < \alpha_2 < \beta \Rightarrow \ell g(\nu_{\alpha_2}) \in E_{\alpha_1}$, but E_{α_1} is a club of κ hence $\alpha_1 < \beta \Rightarrow \ell g(\nu_\beta) \in E_{\alpha_1}$. As $\alpha_1 + 1 < \alpha_2 < \beta \Rightarrow \nu_{\alpha_2} \in q_{\alpha_1}$ and E_{α_1+1} is disjoint to a witness for q_{α_1} and by the previous sentence $\ell g(\nu_\beta) \in E_{\alpha_1+1}$ we can deduce $\nu_\beta = \bigcup\{\nu_{\alpha_2} : \alpha_2 \in (\alpha_1 + 1, \beta)\} \in q_{\alpha_1}$. So clearly $\nu_\beta \in \bigcap_{\alpha<\beta} q_\alpha$ hence $\langle q_\alpha^{[\nu_\beta]} : \alpha < \beta \rangle$ is an increasing sequence of members of $\mathbb{Q}^1_\kappa$ with fixed trunk ν_β of length $\geq \beta$ as $\alpha < \beta \Rightarrow \ell g(\nu_\beta) \geq \ell g(\nu_\alpha) = \ell g(\mathrm{tr}(q_\alpha)) \geq \alpha$, see 1.18(1)(f). So by 1.16(5) we have $p_\beta := \bigcap\{q_\alpha^{[\nu_\beta]} : \alpha < \beta\} \in \mathbb{Q}^1_\kappa$ has trunk ν_β and is equal to $\left(\bigcap\{q_\alpha : \alpha < \beta\}\right)^{[\nu_\beta]}$. Let $E_\beta = \bigcap\{E_\alpha : \alpha < \beta\}$ and clearly p_β, E_β are as required. □

Claim 1.20 *(1) For every $p \in \mathbb{Q}^1_\kappa$ the sequence $\langle (p, p, \kappa) \rangle$ belongs to $\mathbf{S}^{\mathrm{pr}}_\kappa$.*
2) If $\gamma < \kappa$ and $\bar{\mathbf{x}} = \langle (p_\alpha, q_\alpha, E_\alpha) : \alpha < \gamma \rangle \in \mathbf{S}^{\mathrm{pr}}_{\kappa,\gamma}$ then there are (p_γ, E) with E a club of κ and $p_\gamma = \bigcap\{p_\alpha : \alpha < \gamma\}$ such that:
if $p_\gamma \leq q_\gamma, \beta < \gamma \Rightarrow q_\beta \cap \mathbf{T}_{\leq \min(E_\gamma)} \subseteq q_\gamma$ and $E_\gamma \subseteq E$ is a club of κ,
then $\bar{\mathbf{x}}\hat{\ }\langle (p_\gamma, q_\gamma, E_\gamma) \rangle \in \mathbf{S}^{\mathrm{pr}}_\kappa$.
(3) The union of a $\triangleleft$-increasing sequence of members of $\mathbf{S}^{\mathrm{pr}}_\kappa$ of length $< \kappa$ belongs to $\mathbf{S}^{\mathrm{pr}}_\kappa$.
(3A) If $\langle \bar{\mathbf{x}}_\beta : \beta < \delta \rangle$ is $\trianglelefteq$-increasing, $\bar{\mathbf{x}}_\beta = \langle (p_\alpha, q_\alpha, E_\alpha) : \alpha < \gamma_\beta \rangle \in \mathbf{S}^{\mathrm{pr}}_\kappa$ and $\langle \gamma_\beta : \beta < \delta \rangle$ is $\leq$-increasing and $\gamma := \bigcup\{\gamma_\beta : \beta < \delta\} < \kappa$ then $\langle (p_\alpha, q_\alpha, E_\alpha) : \alpha < \gamma \rangle \in \mathbf{S}^{\mathrm{pr}}_{\kappa,\gamma}$.
(3B) If in (3A), $\gamma = \kappa$ then $p_\kappa = \bigcap\{p_\alpha : \alpha < \kappa\}$ belongs to $\mathbb{Q}^1_\kappa$ and is a $\leq_{\mathbb{Q}^1_\kappa}$-lub of $\{p_\alpha, q_\alpha : \alpha < \kappa\}$.

Proof Straightforward. □

Crucial Claim 1.21 *If $\kappa = \lambda$ or just $\kappa \in S^*_{\mathfrak{r}}$ (see 1.13), $\gamma < \kappa$, $\bar{\mathbf{x}} = \langle (p_\alpha, q_\alpha, E_\alpha) : \alpha \leq \gamma \rangle \in \mathbf{S}^{\mathrm{pr}}_{\kappa,\gamma+1}$ and $\underset{\sim}{\tau}$ is a $\mathbb{Q}^1_\kappa$-name of a member of $\mathbf{V}$ then we can find $(p_{\gamma+1}, q_{\gamma+1}, E_{\gamma+1})$ such that*

- *(a)* $\bar{\mathbf{x}}\hat{\ }\langle (p_{\gamma+1}, q_{\gamma+1}, E_{\gamma+1}) \rangle \in \mathbf{S}^{\mathrm{pr}}_\kappa$,
- *(b) if $\eta \in q_{\gamma+1} \cap \mathbf{T}_{\min(E_{\gamma+1})}$ then $q_{\gamma+1}^{[\eta]}$ forces a value to $\underset{\sim}{\tau}$.*

Proof Let

$(*)_1$ $\mathscr{Y} = \{\mathrm{tr}(p) : p \in \mathbb{Q}^1_\kappa$ forces a value to $\underset{\sim}{\tau}$ and $\mathrm{tr}(p)$ has length $> \min(E_\gamma)\}$.

For $\eta \in \mathscr{Y}$ let p^*_η exemplify $\eta \in \mathscr{Y}$, i.e.

$(*)_2$ $\mathrm{tr}(p^*_\eta) = \eta$ and p^*_η forces a value to $\underset{\sim}{\tau}$, necessarily $\ell g(\eta) > \min(E_\gamma)$.

Clearly

$(*)_3$ (a) $\mathscr{Y} \subseteq \mathbf{T}_{<\kappa}$,
(b) if $p \in \mathbb{Q}^1_\kappa$ then for some $\eta \in \mathscr{Y}$ we have $\mathrm{tr}(p) \trianglelefteq \eta \in p$.

By Convention 1.14, there is $\partial \in S_{\mathfrak{x}} \cap \kappa \cap E_\gamma$ but $> \min(E_\gamma)$ such that letting $\mathscr{Y}_\partial = \mathscr{Y} \cap \mathbf{T}_{<\partial}$ we have

$(*)_4$ (a) $\ell g(\mathrm{tr}(p_\gamma)) < \partial$,
(b) if $p \in \mathbb{Q}^1_\partial$ then $\{\eta : \mathrm{tr}(p) \trianglelefteq \eta \in p\} \cap \mathscr{Y}_\partial \neq \emptyset$,
(c) recalling 1.13(D)(b), $\{(\eta, \nu) : \eta \in \mathscr{Y} \cap \mathbf{T}_{<\partial}$ and $\nu \in p^*_\eta \cap \mathbf{T}_{<\partial}\} \in \mathscr{P}_\partial$.

Define:

- $p_{\gamma+1} = \{\eta \in p_\gamma$: if $\ell g(\eta) \geq \partial$ and $\{\eta \restriction \varepsilon : \varepsilon < \partial\} \cap \mathscr{Y} \neq \emptyset$ and $\zeta < \partial$ is minimal such that $\eta \restriction \zeta \in \mathscr{Y}$ then $\eta \in p^*_{\eta \restriction \zeta}\}$,
- $q_{\gamma+1} = p_{\gamma+1}$,
- $E_{\gamma+1} \subseteq E_\gamma \backslash (\partial + 1)$ is a club of κ such that if $\eta \in q_{\gamma+1} \cap \mathbf{T}_{<\partial}$ then $E_{\gamma+1}$ is disjoint to some witness for p^*_η.

Clearly $(p_{\gamma+1}, q_{\gamma+1}, E_{\gamma+1})$ is as required. □

Claim 1.22 *If $\kappa \in S_{\mathfrak{x}}$ then $\mathbb{Q}^1_\kappa$ is κ-bounding, i.e. $\Vdash_{\mathbb{Q}^1_\kappa}$ "$({}^\kappa\kappa)^{\mathbf{V}}$ is $\leq_{J^{\mathrm{bd}}_\kappa}$-cofinal in ${}^\kappa\kappa$".*

Proof By 1.21 and Claim 1.20. □

2 What are the desired properties of the ideal

Our original aim was to disprove the existence of a forcing notion for λ having the properties of random real forcing equivalently, finding for an uncountable cardinal λ, a λ-complete ideal on $\mathscr{P}({}^\lambda 2)$ parallel to the ideal on null sets on ${}^{\mathbb{N}}2$. Having constructed one raises hopes for generalizing independence results about reals to ${}^\lambda 2$, so deriving independence results on λ-cardinality invariants.

In this section we try systematically to go over basic properties of the null ideal (and its relation with the meagre ideal). This results in a list of possible test problems for our ideal. Some of these questions are addressed in the present work, some are left for further research. The case of $\mathbb{Q}_{\bar\theta} = \mathbb{Q}^1_\kappa$ (of Sect. 1(B)) is similar and we intend to comment on it in Part II, i.e. [29].

On the meagre and null ideals (for $\lambda = \aleph_0$) see Oxtoby [14]. On the measure algebra and random reals see Fremlin's treatise [6] or Bartoszyński and Judah [1].

How do we measure success? The main properties of the null ideal which come to my mind are:

⊞ (a) an $\aleph_1$-complete ideal (with no atoms),
(b) the quotient Boolean Algebra satisfies the c.c.c., i.e. there is no uncountable family of non-null pairwise disjoint Borel sets,
(c) the forcing is bounding: this means the quotient Boolean Algebra is $(\aleph_0, \infty)$-distributive, that is if for each n, $\langle \mathbf{B}_{n,k} : k \in \mathbb{N}\rangle$ is a Borel partition of a non-null Borel set $\mathbf{B}$ then for some function $f : \mathbb{N} \to \mathbb{N}$, the set $\bigcap_n \bigcup_{k<f(n)} \mathbf{B}_{n,k}$ is not null.

A priori, for the set theoretic purposes, generalizing (a), (b), (c) was the aim. But for the ideal itself, a prominent property of the null ideal, and a very nice one, is

(d) the Fubini theorem: for a Borel set $A \subseteq [0, 1] \times [0, 1]$ the following are equivalent:

Sh:1004

(i) for all but null many x, for all but null many y we have $(x, y) \in A$,
(ii) for all but null many y, for all but null many x we have $(x, y) \in A$.

But alas, this fails, see Claim 6.6.

Maybe it is helpful to stress, that

$\boxtimes$ we are looking for λ^+-complete, λ^+-c.c., ideal with no atoms.

Below we make a list of statements generalizing the null ideal case, including the natural analogs of the properties listed above, delaying a try on some further properties.

A reader who goes first to this section can note just that

$\oplus$ (a) the forcing notion $\mathbb{Q}_\lambda$ is a set of subtrees of ${}^{\lambda>}2$ representing λ-closed subsets $\lim_\lambda(p)$ of ${}^\lambda 2$, where $\lim_\lambda(p) = \{\eta \in {}^\lambda 2 : (\forall \zeta < \lambda)(\eta \restriction \zeta \in p)\}$, parallel to the closed subsets of $[0, 1]_{\mathbb{R}}$ with positive Lebesgue measure, partially ordered by inverse inclusion,
(b) ${}^\lambda 2$ is the set of functions from λ to $2 = \{0, 1\}$.

Definition 2.1 Let λ be an inaccessible cardinal and let $\mathbb{Q}_\lambda = \mathbb{Q}^2_\lambda$ be the forcing notion introduced in Sect. 1(A).

1. For $\eta \in {}^\lambda 2$ and $\mathscr{I} \subseteq \mathbb{Q}_\lambda$, saying η fulfills $\mathscr{I}$ means $(\exists q \in \mathscr{I})(\eta \in \lim_\lambda(q))$.
2. For $\mathscr{I} \subseteq \mathbb{Q}_\lambda$ let $\mathrm{set}(\mathscr{I}) = \{\eta \in {}^\lambda 2 : \eta$ fulfills $\mathscr{I}\}$ and for a set Λ of subsets of $\mathbb{Q}_\lambda$ let $\mathrm{set}(\Lambda) = \bigcap\{\mathrm{set}(\mathscr{I}) : \mathscr{I} \in \Lambda\}$.
3. We define $\mathrm{id}(\mathbb{Q}_\lambda) = \{A \subseteq {}^\lambda 2$: there are $i(*) \le \lambda$ and dense open subsets $\mathscr{I}_i$ of $\mathbb{Q}_\lambda$ for $i < i(*)$ such that $\eta \in A \wedge i < i(*) \Rightarrow \eta$ does not fulfill $\mathscr{I}_i\}$.
4. A λ-real is $\eta \in {}^\lambda 2$.

Convention 2.2 $\lambda, \partial, \kappa$ vary on inaccessibles.

We have consulted several people on additional properties to be examined. For instance T. Bartoszyński suggested (P), (S), (U) of the first list below.

2(A) Desirable properties: first list

In this subsection we list various desirable properties and questions and sometimes give a relevant reference (in this paper) <u>but</u> we do not prove anything (whereas Sect. 3 on contain proofs).

(A) (α) The ideal $\mathrm{id}(\mathbb{Q}_\lambda)$ is λ^+-complete, i.e. closed under union of $\le \lambda$ sets.
(β) The forcing notion $\mathbb{Q}_\lambda$ is λ-complete (or at least λ-strategically complete, depending on the choice of the order).
(γ) The Boolean Algebra of λ-Borel subsets of ${}^\lambda 2$ modulo the ideal $\mathrm{id}(\mathbb{Q}_\lambda)$ satisfies the λ^+-c.c., see 3.9(2). Note that modulo $\mathrm{id}(\mathbb{Q}_\lambda)$, $\mathbb{Q}_\lambda$ is dense in this Boolean Algebra, this is (E) below.
(δ) The forcing notion $\mathbb{Q}_\lambda$ is λ-bounding, see 0.5(2), Sect. 1, when λ is a weakly compact cardinal.

(B) The definability of $\mathbb{Q}_\lambda$, i.e., $\mathbb{Q}_\lambda$ is nicely definable (with no parameters), see the definition by induction in Sect. 1; if λ is weakly compact then $\mathbb{Q}_\lambda$ is λ-Borel, the ideal is similarly definable, see 8.1; for other inaccessible cardinals λ the "nowhere stationary" is $\Sigma^1_1(\lambda)$ but by a somewhat cumbersome definition giving an equivalent forcing it is λ-Borel, see the proof of 1.10.

(C) Generalizing "adding (forcing) a Cohen real makes the set of old reals null", see 6.3.

(D) Generalizing "adding (i.e. forcing) a random real makes the old real meagre", see 6.1.

(E) Modulo the ideal $\mathrm{id}(\mathbb{Q}_\lambda)$, every λ-Borel set is equal to a union of at most λ sets of the form $\lim_\lambda(p)$, $p \in \mathbb{Q}_\lambda$, see 3.9.

(F) Can we define integral? We do not know; may we replace $[0,1]_{\mathbb{R}}$ as a set of values by some complete linear order, e.g. by "nice" ordered fields? Are symmetrically complete real closed fields relevant (see [39])? If we waive linearity does it help?

(G) Modulo the ideal, every λ-Borel function can be approximated by "steps function of level α" for many (so unboundedly) many $\alpha < \lambda$; where "step function" is being interpreted as: $f(\eta){\restriction}\alpha$ is determined by $\eta{\restriction}\alpha$ for $\eta \in {}^\lambda 2$, see 3.10.

(H) The Lebesgue density theorem, see 3.13, (it means: if the λ-Borel set $\mathbf{B} \subseteq {}^\lambda 2$ is $\mathrm{id}(\mathbb{Q}_\kappa)$-positive, then for some $\mathbf{B}_1 \in \mathrm{id}(\mathbb{Q}_\lambda)$ for every $\eta \in \mathbf{B}\backslash\mathbf{B}_1$ for some $\alpha < \lambda$ we have $({}^\lambda 2)^{[\eta{\restriction}\alpha]}\backslash\mathbf{B} \in \mathrm{id}(\mathbb{Q}_\lambda)$).

(I) The Fubini theorem, symmetry, unfortunately fails, see 6.6. However we intend to present some weak versions of symmetry in a continuation.

(J) The translation invariance, see 3.7(1).

(K) The permutation invariance (i.e. for permutations of λ): this works only for a variation on our forcing.

(L) Generalizing "if A is a Borel subset of $[0,1]_{\mathbb{R}} \times [0,1]_{\mathbb{R}}$ of positive measure then A contains a perfect rectangle (even half square)". But what is perfect? Not a copy of ${}^\lambda 2$ but λ-closed set, e.g. the λ-limit of a λ-Kurepa tree, actually one with "little pruning in limit levels"; specifically it is $\lim_\lambda(p)$ for some $p \in \mathbb{Q}_\lambda$, so λ-closed.

(M) Generalize the random algebra on ${}^\chi 2$ for χ possibly $> \lambda$. This will be addressed in a continuation, see [24, §1], [25].

(N) Generalize "modulo the null ideal every Borel set is equal to a union of $\leq \lambda$ sets, each λ-closed" see (E) above and see 3.9.

(O) Generalize "the set of reals is a union of a null set and a meagre set", see 3.8.

(P) Generalize Erdös–Sierpiński theorem: if $2^\lambda = \lambda^+$ or suitable cardinal invariants are equal to λ^+ then there is a permutation of ${}^\lambda 2$ interchanging the null and meagre ideal.

In fact, this is not hard now:

$(*)_1$ Assume that for $\ell = 1, 2$:
  (a) J_ℓ is an ideal of subsets of I,
  (b) J_ℓ is $|I|$-complete and generated by a family of $\leq |I|$ sets,
  (c) if $A_1 \in J_\ell$ then for some $A_2 \in J_\ell$ we have $|A_2\backslash A_1| = |I|$, and
  (d) there is $A \in J_1$ such that $I\backslash A \in J_2$.
  Then there is a permutation of I interchanging J_1 with J_2.

$(*)_2$ If $2^\lambda = \lambda^+$ and $I = {}^\lambda 2$ then the λ-meagre ideal and $\mathrm{id}(\mathbb{Q}_\lambda)$ satisfy (a)–(d) of $(*)_1$.

[Why? Clause (d) here holds by 3.8.]

(Q) Generalize the Borel conjecture: though not connected to random. Now consider:
- (α) the equivalence of the "for every $\langle \varepsilon_n : n \rangle$ the set is covered by $\bigcup_n I_n$, I_n is an interval of length $\leq \varepsilon_n$" and "the set can be translated away from any meagre set",
- (β) the ε_n's version has an obvious generalization,
- (γ) try shooting through a normal ultrafilter

(R) The dual Borel conjecture might be adressed in Part II. Now the question is:
- (∗) We are given an old set X of λ-reals of cardinality λ^+, say $X = \{\nu_\alpha : \alpha < \lambda^+\}$. View Cohen_λ as adding a λ-null set: e.g., for $\bar{p} = \langle p_\eta : \eta \in {}^{\lambda >}2 \rangle$, $p_\eta \in \mathbb{Q}_\lambda$, $\mathrm{tr}(p_\eta) = \eta$, and clearly p_η is a nowhere-dense cone, but we shall need more.

(S) (Selectors) Every Σ^1_1-relation have a reasonably definable, e.g. λ-Borel, choice function on a positive closed set even in any positive Borel set.

(T) The Hausdorff paradox and even Banach–Tarski paradox hold for $\mathbb{R}^3$. Do they hold for ${}^\lambda 2 \times {}^\lambda 2 \times {}^\lambda 2$?

(U) We know that "for every meagre set A there is a meagre set B such that: every $\leq \lambda$ translates of A can be covered by one translates of B", but fail for null, even for ${}^\omega\mathbb{Z}$. Generalize to λ.

On raising further problems see [29], concerning characters, differentiability, monotonicity (of functions) and going back to the case $\lambda = \aleph_0$.

We have not looked at clauses (L), (Q), (S)–(U).

2(B) Desirable properties: second list

Next we consider generalizing results more set theoretic in nature, related to forcing (maybe (B)(c), (d) from Sect. 2(A) should be here; from the problems listed below, (A) is treated here, on the others see part II, if at all)

(A) Cichoń's Diagram

This diagram sums up the provable inequalities between the basic cardinal invariants of the null ideal, the meagre ideal, $\mathfrak{d}$ (the dominating number) and $\mathfrak{b}$ (the unbounding number). The basic cardinal invariants of an ideal are the covering number, the additivity number, the cofinality and the non(= uniformity) of the ideal, see 0.7.

The diagram gives the provable inequalities among any two invariants (and two equalities each on three invariants). Moreover, under $2^{\aleph_0} \leq \aleph_2$ there are no more connections. Here we generalize the ZFC part (for λ inaccessible limit of inaccessibles), but the situation is different, e.g., there are more inequalities connecting 3 of the cardinal invariants, see 5.9.

We will deal with the complementary consistency results (about inequalities of any pair) in continuations, [28] and others.

(B) Generalizing the amoeba forcing

The amoeba forcing is the one adding a measure zero set including all the old ones; the conditions are closed subsets of $[0, 1]_{\mathbb{R}}$ of measure $> \frac{1}{2}$.

This is natural as the amoeba forcing has been important in set theory of the reals and is closely related to measure, see Sect. 7.

(C) The consistency of "every $A \in \mathscr{P}(\mathbb{R})^{\mathbb{L}[\mathbb{R}]}$ is Lebesgue measurable" (from $\chi > \lambda$ inaccessible).

Solovay [47] classical work proved for $\lambda = \aleph_0$ that if we Levy collapse the first inaccessible cardinal to being $\aleph_1$, this holds.

The problem is: we have names $\underset{\sim}{\eta}$ of λ-reals such that $\text{Levy}(\lambda, <\chi)/\underset{\sim}{\eta}$ is not $\text{Levy}(\lambda, <\chi)$ when λ is uncountable. Another formulation of the problem: there are $\text{Levy}(\lambda, <\chi)$-names $\underset{\sim}{\eta}_1, \underset{\sim}{\eta}_2$ of λ-Cohen reals and no automorphisms of the completion of $\text{Levy}(\lambda, <\chi)$ mapping one to the other.

This certainly occurs for λ-Cohen reals and probably for any other; that is we may add a λ-Cohen $\underset{\sim}{\eta} \in {}^{\lambda}2$ and compose it with a forcing shooting a club through $\underset{\sim}{\eta}^{-1}\{\ell\}$.

A possible avenue is to consider only "nice $\text{Levy}(\lambda, <\chi)$-names", i.e. such that the quotient is $\text{Levy}(\lambda, < \chi)$. In this case there is a "positive" set of λ-reals such that for subsets of it our aim is achieved. We can even define this set of reals. The question is whether we consider this is a "reasonable" or a "forced, artificial" solution?

Alternatively we may replace λ-Cohen by another forcing (or ideal) and/or change the collapse; in particular should check the failure for $\mathbb{Q}_\lambda$. We also may change the notion of a λ-real, e.g. replace it by A/(the non-stationary ideal) or use a filter generated by $\leq \lambda$ subsets of λ! All this is delayed for later parts. We should also check what occurs to sweetness in our present case (see [16,17]).

We may consider $\{\eta \in {}^{\lambda}2 : \eta$ is $(\mathbb{Q}, \underset{\sim}{\eta})$-generic over $\mathbf{V}_0$ such that every subset of λ from $\mathbf{V}_0[\eta]$ which is stationary in it, is also stationary in $\mathbf{V}\}$, or more. A related question is the complexity of maximal antichains, see 8.4, maybe use measurable cardinals.

What about $\mathscr{P}(\lambda)$ for λ singular strong limit of cofinality $\aleph_0$?

(D) Can we characterize Cohen_λ and $\mathbb{Q}_\lambda$ among (nicely definable) λ-Borel ideals? Recall Solecki–Kechris characterization of Cohen and random (for the ideals). We have not looked at it; there are limitations even for $\lambda = \aleph_0$, see e.g., [15].

(E) In [33] we showed that: for any Suslin c.c.c. forcing, if it adds an undominated real, it adds a Cohen real.

Subsequently some works show relatives (for other properties), on this see [35,40]. Related to this, by Shelah [38], the only "nice" c.c.c. forcing commuting with Cohen is Cohen itself. Do we have a parallel?

For a broader generalization of the case of $\aleph_0$ we may consider forcing, ultrafilters and forcing notions definable from ultrafilters.

(F) We know much on ultrafilters on $\mathbb{N}$. Also we have considerable knowledge about λ-complete ultrafilters on λ or higher cardinals when λ is a measurable cardinal. After the seventies there were set theoretic advances on non-regular ultrafilters, but not much set theoretic work was done on regular ultrafilter. However, in recent years there were studies of reasonable ultrafilters in [41], Rosłanowski and Shelah [20,22] and recently on ultrafilters related to saturation of ultra-powers and Keisler order, see Malliaris and Shelah [11,12] on cuts and $\mathfrak{p} = \mathfrak{t}$.

On characters of ultrafilters on $\mathbb{N}$ see Brendle and Shelah [3] and later [42], [44]; for an ultrafilter D on λ recall that $\chi(D)$ is the character = minimal cardinality of a subset generating it, $\pi\chi(D)$ pseudo-character = minimal cardinality of $\mathscr{A} \subseteq [\lambda]^\lambda$ such that $(\forall B \in D)(\exists A \in \mathscr{A})[A \subseteq B]$, note that $A \in \mathscr{A}$ is not necessarily in D! As in [20,22] dealing with the so called reasonable ultrafilters we may consider the Borel version (i.e. the minimal number of Borel subsets of D which generate it) and λ-real version. Then as in "reasonable ultrafilter", can we show CON(for every uniform ultrafilter D on λ, $\pi\chi_{\lambda-\mathrm{real}}(D) = \lambda^+ < 2^\lambda$)?

What about the ultrafilter forcing? Can reasonable ultrafilters on λ be generated by $< 2^\lambda$ sets? We can force a creature condition diagonalizing a uniform ultrafilter on λ.

(G) Related is Galvin–Prikry theorem which says that for any Borel (or even Σ^1_1) subset $\mathbf{B}$ of $\mathscr{P}(\mathbb{N})$ for some set $A \in [\mathbb{N}]^{\aleph_0}$, the set $[A]^{\aleph_0}$ is included in or disjoint from $\mathbf{B}$. Concerning a relative using a group from [30], generalizations to λ are considered by the author in some later works: [36,37,45,46], see also [7,43], less related [8–10]

(H) The consistency of Moore conjecture; so we should consider a topological space X which is λ-first countable (analog of first countable). Of course we can prove it using Dow lemma which holds for adding many λ-Cohens, so not clear how interesting.

(I) Preserving "η is $\mathbb{Q}_\lambda$-generic over N" parallel to [31, Ch.XVIII,§3], [31, Ch.VI,§3].

(J) (a) Try to connect $\mathrm{cf}(\mathbb{Q}_\lambda)$ and Cichoń's diagram and number of reasonable generators of an ultrafilter, see [41].

(b) Note that for the number of generators of an ultrafilter we have the following bounds.

Claim 2.3 *(1) Letting η_λ be the $\mathbb{Q}_\lambda$-name of the generic, for $\alpha < \lambda$ we have that $\Vdash_{\mathbb{Q}_\lambda}$ "there is $\mathbf{G}' \subseteq \mathbb{Q}_\lambda$ such that: $\mathbf{G}'$ is a generic subset of $\mathbb{Q}_\lambda$ over $\mathbf{V}$, $\mathbf{V}[\mathbf{G}'] = \mathbf{V}[\underset{\sim}{\mathbf{G}}]$ and $\underset{\sim}{\eta}_\lambda[\mathbf{G}'] = \underset{\sim}{\eta}_{\lambda,\alpha}[\underset{\sim}{\mathbf{G}}]$" where $\eta_{\lambda,\alpha} \in {}^\lambda 2$ is defined by:*

$$\underset{\sim}{\eta}_{\lambda,\alpha}(i) = \begin{cases} \underset{\sim}{\eta}_\lambda(i) & \text{if } i < \alpha \\ 1 - \underset{\sim}{\eta}_\lambda(i) & \text{if } i \in [\alpha, \lambda). \end{cases}$$

(2) Similarly when for some $A \in \mathscr{P}(\lambda)^{\mathbf{V}}$

$$\underset{\sim}{\eta}_{\lambda,\alpha}(i) = \begin{cases} \underset{\sim}{\eta}_\lambda(i) & \text{if } i \in A \\ 1 - \underset{\sim}{\eta}_\lambda(i) & \text{if } i \in \lambda \backslash A. \end{cases}$$

(3) $\Vdash_{\mathbb{Q}_\lambda}$ "$\underset{\sim}{\eta}_\lambda \restriction A \neq_{J^{\mathrm{bd}}_A} i_A$ for $i = 0, 1$ for any $A \in ([\lambda]^\lambda)^{\mathbf{V}}$".

(4) $\chi(\lambda) := \min\{\mathrm{gen}(D) : D$ a uniform ultrafilter on $\lambda\}$ is $\geq \mathrm{cov}(\mathbb{Q}_\lambda), \mathrm{cov}(\mathrm{Cohen}_\lambda)$.

But we can still hope to find a relative of $\mathbb{Q}_\lambda$ such that adding λ^{++} such λ-reals (e.g. as in [28]) we get a universe $\mathbf{V}_1$ with $2^\lambda = \lambda^{++}$+ there is a uniform ultrafilter D on λ with $\chi(D) = \lambda^+$.

Sh:1004

(K) Here we start with λ-Cohen forcing (for χ inaccessible not limit of inaccessibles). We can start with $\mathbb{Q}_{\bar\theta\restriction\lambda}$ or with other definable λ^+-c.c. forcing; see part II.

3 On $\mathbb{Q}_\kappa$, κ-Borel sets and id($\mathbb{Q}_\kappa$)

In this and the following sections we analyze the ideal $\mathrm{id}(\mathbb{Q}_\kappa)$. A general frame including 2.1 is the following.

Definition 3.1 (1) Let $\mathrm{id}(\mathrm{Cohen}_\kappa)$ be the family of all κ-meagre subsets of ${}^\kappa 2$, i.e., it is the collection of all $A \subseteq {}^\kappa 2$ such that $A \subseteq \bigcup\{\lim_\kappa(\mathscr{T}_i)$ for $i < \kappa\}$, where each $\mathscr{T}_i$ is a nowhere dense subtree of ${}^{\kappa>}2$, i.e., $({}^{\kappa>}2, \lhd)$.

(2) We say $\mathbf{i} = (\kappa, \mathbb{Q}, \underset{\sim}{\eta}) = (\kappa_\mathbf{i}, \mathbb{Q}_\mathbf{i}, \underset{\sim}{\eta}_\mathbf{i})$ is an ideal case <u>when</u>:
- (a) κ is a regular cardinal,
- (b) $\mathbb{Q}$ is a forcing notion not adding bounded subsets of κ,
- (c) $\underset{\sim}{\eta}$ is a $\mathbb{Q}$-name of a member of ${}^\kappa 2$,
- (d) (α) each $p \in \mathbb{Q}$ is a subtree of $({}^{\kappa>}2, \trianglelefteq)$ and let $\mathbf{B}_p = \mathbf{B}_{\mathbf{i},p} = \lim_\kappa(p)$, and $p \Vdash$"$\underset{\sim}{\eta} \in \mathbf{B}_{\mathbf{i},p}$", <u>or</u> at least
  - (β) we have a mapping $p \mapsto \mathbf{B}_p = \mathbf{B}_{\mathbf{i},p}$ such that
    - $\mathbf{B}_{\mathbf{i},p}$ is a κ-Borel subset of ${}^\kappa 2$,
    - $p \le q \Rightarrow \mathbf{B}_{\mathbf{i},p} \supseteq \mathbf{B}_{\mathbf{i},q}$, and
    - $p \Vdash$ "$\underset{\sim}{\eta} \in \mathbf{B}_{\mathbf{i},p}$";

    so really the function $p \mapsto \mathbf{B}_p$ is part of $\mathbf{i}$.

Below let $\mathbf{i} = (\kappa, \mathbb{Q}, \underset{\sim}{\eta})$ be an ideal case.

(3) We let $\mathrm{id}^1_\mathbf{i} = \mathrm{id}_1(\mathbf{i})$ be

$$\{A \subseteq {}^\kappa 2 : \text{ for some } \kappa\text{-Borel set } \mathbf{B} \text{ we have } A \subseteq \mathbf{B} \text{ and } \Vdash_\mathbb{Q} \text{“}\underset{\sim}{\eta} \notin \mathbf{B}\text{”}\};$$

we may omit the 1.

(4) For a subset $\mathscr{I}$ of $\mathbb{Q}_\mathbf{i}$, we say that $\eta \in {}^\kappa 2$ fulfills $\mathscr{I}$ <u>when</u> $(\exists p \in \mathscr{I})(\eta \in \mathbf{B}_p)$.

(5) We define $\mathrm{id}^2_\mathbf{i} = \mathrm{id}_2(\mathbf{i})$ to be the collection of all sets $A \subseteq {}^\kappa 2$ such that there are pre-dense subsets $\mathscr{I}_i$ of $\mathbb{Q}_\mathbf{i}$ for $i < \kappa$ such that

$$A \subseteq \{\eta \in {}^\kappa 2 : \text{ for some } i < \kappa, \eta \text{ does not fulfill } \mathscr{I}_i\}.$$

Claim 3.2 *Let* $\mathbf{i}$ *be an ideal case.*

(1) Both $\mathrm{id}_1(\mathbf{i})$ *and* $\mathrm{id}_2(\mathbf{i})$ *are* κ^+*-complete ideals on* ${}^\kappa 2$*. Also* ${}^\kappa 2 \notin \mathrm{id}_1(\mathbf{i})$ *and if* $\mathbf{i}$ *is* κ*-complete then*[3] ${}^\kappa 2 \notin \mathrm{id}_1(\mathbf{i})$.

(2) In Definition 3.1(5) we can replace "pre-dense" by "dense open" or by "maximal antichain".

(3) If $\mathbb{Q}_\mathbf{i}$ *satisfies the* κ^+*-c.c.* <u>*then*</u> $\mathrm{id}_2(\mathbf{i}) \subseteq \mathrm{id}_1(\mathbf{i})$.

1. A sufficient condition for $\mathrm{id}_1(\mathbf{i}) \subseteq \mathrm{id}_2(\mathbf{i})$ *is:*

(∗) *(a) if* $p, q \in \mathbb{Q}_\mathbf{i}$ *are incompatible then* $\mathbf{B}_{\mathbf{i},p} \cap \mathbf{B}_{\mathbf{i},q} = \emptyset$*, and*

[3] Recall Prikry forcing.

Sh:1004

(b) if $\mathbf{B}$ *is a* κ*-Borel set* *then*

$$\left\{p \in \mathbb{Q}_{\mathbf{i}} : \ p \Vdash_{\mathbb{Q}_{\mathbf{i}}} \text{“}\underset{\sim}{\eta} \in \mathbf{B}\text{” or } \mathbf{B}_p \cap \mathbf{B} \in \mathrm{id}_2(\mathbf{i})\right\}$$

is a dense open subset of $\mathbb{Q}_{\mathbf{i}}$*.*

(3) *Let* κ *be strongly inaccessible and* $\mathbb{Q}_\kappa$ *and* $\underset{\sim}{\eta}$ *be as defined in 1.3 and 1.7(4), respectively. Then the triple* $\mathbf{i} = \mathbf{i}_\kappa = (\kappa, \mathbb{Q}_\kappa, \underset{\sim}{\eta})$ *is an ideal case and* $\mathrm{id}_1(\mathbf{i}) = \mathrm{id}_2(\mathbf{i})$.

(4) *The triple* $\mathbf{i} = \mathbf{i}_\kappa^{\mathrm{Cohen}} = (\kappa, \mathrm{Cohen}_\kappa, \underset{\sim}{\eta})$ *is an ideal case and we have* $\mathrm{id}_1(\mathbf{i}) = \mathrm{id}_2(\mathbf{i})$ *and it is closed under translations (cf 3.7).*

Remark 3.3 If in Definition 3.1(2)(d), $\mathbf{B}_p$ is just a Borel set, then 3.2 still holds.

Proof (1), (2) Obvious by the definitions.

(3) Assume $A \subseteq {}^\kappa 2$ belongs to $\mathrm{id}_2(\mathbf{i})$. Then by (2) we may find maximal antichains $\mathscr{I}_i \subseteq \mathbb{Q}_{\mathbf{i}}$ (for $i < \kappa$) such that

$$\eta \in A \ \Rightarrow \quad \text{for some } i < \kappa,\ \eta \text{ does not fulfill } \mathscr{I}_i.$$

Since we are assuming that $\mathbb{Q}_{\mathbf{i}}$ satisfies the κ^+-c.c., $\mathscr{I}_i$ has cardinality $\le \kappa$ for every $i < \kappa$. Let $\langle p_{i,\varepsilon} : \varepsilon < \varepsilon_i\rangle$ list $\mathscr{I}_i$, $\varepsilon_i \le \kappa$. Then

$$A \subseteq \mathbf{B} := \bigcup_{i<\kappa} \left({}^\kappa 2 \setminus \bigcup \{\mathbf{B}_{\mathbf{i}, p_{i,\varepsilon}} : \varepsilon < \varepsilon_i\}\right).$$

Clearly $\mathbf{B}$ is a κ-Borel set. Also, since each $\mathscr{I}_i$ is a maximal antichain, for all $i < \kappa$ we have

$$\Vdash_{\mathbb{Q}_{\mathbf{i}}} \text{“ } \mathscr{I}_i \cap \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_{\mathbf{i}}} \ne \emptyset \text{ and hence } \underset{\sim}{\eta} \in \mathbf{B}_{\mathbf{i}, p_{i,\varepsilon}} \text{ for some } \varepsilon < \varepsilon_i\text{”},$$

and hence $\Vdash_{\mathbb{Q}_{\mathbf{i}}}$ “$\underset{\sim}{\eta} \notin \mathbf{B}$”. Consequently $\mathbf{B} \in \mathrm{id}_1(\mathbf{i})$ but $A \subseteq \mathbf{B}$ hence $A \in \mathrm{id}_1(\mathbf{i})$, so we are done.

(4) Assume $\mathbf{B}$ is a κ-Borel set and it belongs to $\mathrm{id}_1(\mathbf{i})$. We shall prove $\mathbf{B} \in \mathrm{id}_2(\mathbf{i})$, clearly this suffices.

Let $\mathscr{I} = \{p : p \text{ forces } \underset{\sim}{\eta} \in \mathbf{B} \text{ or forces } \mathbf{B}_p \cap \mathbf{B} \in \mathrm{id}_2(\mathbf{i})\}$, so by the assumption $(*)$(b) the set $\mathscr{I}$ is an open dense subset of $\mathbb{Q}_{\mathbf{i}}$. Let $\mathscr{I}' \subseteq \mathscr{I}$ be a maximal antichain and let $\mathscr{I}\text{''} = \{p \in \mathscr{I}' : p \nVdash_{\mathbb{Q}_{\mathbf{i}}} \text{“}\underset{\sim}{\eta} \in \mathbf{B}\text{”}\}$. Since we assumed $\mathbf{B} \in \mathrm{id}_1(\mathbf{i})$, necessarily $\mathscr{I}\text{''} = \mathscr{I}'$. So for each $p \in \mathscr{I}\text{''}$, $\mathbf{B}_p \cap \mathbf{B} \in \mathrm{id}_2(\mathbf{i})$ and there is a sequence $\langle \mathscr{I}_{p,i} : i < \kappa\rangle$ witnessing it. Without loss of generality if $i < \kappa$, $p \in \mathscr{I}\text{''}$ then $\mathscr{I}_{p,i}$ is a maximal antichain of $\mathbb{Q}_{\mathbf{i}}$ and for every $q \in \mathscr{I}_{p,i}$ we have $(p \le q) \vee (p, q$ are incompatible). For $i < \kappa$ let

$$\mathscr{I}^i = \left\{q \in \mathbb{Q}_{\mathbf{i}} : \text{ for some } p \in \mathscr{I}'' \text{ we have } (p \le q) \wedge q \in \mathscr{I}_{p,i}\right\}.$$

Clearly, each $\mathscr{I}^i$ is a maximal antichain. Easily $\{\mathscr{I}^i : i < \kappa\}$ witnesses $\mathbf{B}$ is included in some member of $\mathrm{id}_2(\mathbf{i})$, so we are done.

(5) For being an ideal case, in Definition 3.1(2), clauses (a), (b), (c) are obvious (remember Claim 1.8 and Observation 1.7(4)) and clause (d) is easy, too. It suffices to prove that $\mathrm{id}_2(\mathbf{i}) \subseteq \mathrm{id}_1(\mathbf{i})$ and $\mathrm{id}_1(\mathbf{i}) \subseteq \mathrm{id}_2(\mathbf{i})$.

Concerning "$\mathrm{id}_2(\mathbf{i}) \subseteq \mathrm{id}_1(\mathbf{i})$" note that $\mathbb{Q}_\kappa$ satisfies the κ^+-c.c., so by 3.2(3) we deduce the inclusion.

Let us argue that $\mathrm{id}_1(\mathbf{i}) \subseteq \mathrm{id}_2(\mathbf{i})$. Suppose that $\mathbf{B}$ is a κ-Borel subset of ${}^\kappa 2$ and $\Vdash_{\mathbb{Q}_\kappa}$"$\underset{\sim}{\eta} \notin \mathbf{B}$". We may find $\mathscr{T}$ and $\bar{\mathbf{B}}$ such that

(⊛) (a) $\mathscr{T}$ is a subtree of ${}^{\omega>}\kappa$ with no infinite branch,
(b) for every $\rho \in \mathscr{T}$, either $\mathrm{suc}_{\mathscr{T}}(\rho) = \emptyset$, or $\mathrm{suc}_{\mathscr{T}}(\rho) = \{\rho^\frown\langle 0\rangle\}$ or $\mathrm{suc}_{\mathscr{T}}(\rho)$ is infinite,
(c) $\bar{\mathbf{B}} = \langle \mathbf{B}_\rho : \rho \in \mathscr{T}\rangle$ is a system of κ-Borel subsets of ${}^\kappa 2$,
(d) $\mathbf{B}_{\langle\rangle} = \mathbf{B}$,
(e) if $\rho \in \mathscr{T}$ and $\mathrm{suc}_{\mathscr{T}}(\rho) = \emptyset$, then for some $i_\rho < \kappa$ and $c_\rho < 2$ we have $\mathbf{B}_\rho = \{\nu \in {}^\kappa 2 : \nu(i_\rho) = c_\rho\}$,
(f) if $\rho \in \mathscr{T}$ and $|\mathrm{suc}_{\mathscr{T}}(\rho)| = 1$, then $\mathbf{B}_\rho = {}^\kappa 2 \setminus \mathbf{B}_{\rho^\frown\langle 0\rangle}$,
(g) if $\rho \in \mathscr{T}$ and $\mathrm{suc}_{\mathscr{T}}(\rho)$ is infinite, then $\mathbf{B}_\rho = \bigcap\{\mathbf{B}_\varrho : \varrho \in \mathrm{suc}_{\mathscr{T}}(\rho)\}$.

Then by induction on $\ell g(\rho)$ for each $\rho \in \mathscr{T}$ we choose $\mathscr{I}_\rho$ and $\bar{t}^\rho$ so that for each $\rho \in \mathscr{T}$:

(⊗) (a) $\mathscr{I}_\rho$ is a maximal antichain of $\mathbb{Q}_\kappa$ and $\bar{t}^\rho = \langle t^\rho_p : p \in \mathscr{I}_\rho\rangle$, $t^\rho_p < 2$ for each $p \in \mathscr{I}_\rho$,
(b) if $t^\rho_p = 1$, then $p \Vdash$"$\underset{\sim}{\eta} \in \mathbf{B}_\rho$" and if $t^\rho_p = 0$, then $p \Vdash$"$\underset{\sim}{\eta} \notin \mathbf{B}_\rho$",
(c) if $\mathrm{suc}_{\mathscr{T}}(\rho) = \emptyset$ and $p \in \mathscr{I}_\rho$, then $\ell g(\mathrm{tr}(p)) > i_\rho$ (see (⊛)(e) above),
(d) if $|\mathrm{suc}_{\mathscr{T}}(\rho)| = 1$, then $\mathscr{I}_\rho = \mathscr{I}_{\rho^\frown\langle 0\rangle}$ and $t^\rho_p = 1 - t^{\rho^\frown\langle 0\rangle}_p$ for $p \in \mathscr{I}_\rho$,
(e) if $\mathrm{suc}_{\mathscr{T}}(\rho)$ is infinite, $p \in \mathscr{I}_\rho$ and $t^\rho_p = 0$, then $p \Vdash$"$\underset{\sim}{\eta} \notin \mathbf{B}_\varrho$" for some $\varrho \in \mathrm{suc}_{\mathscr{T}}(\rho)$,
(f) if $\rho \lhd \varrho \in \mathscr{T}$ and $q \in \mathscr{I}_\varrho$, then there is unique $p \in \mathscr{I}_\rho$ such that $p \leq q$.

Now let $Y = \bigcap_{\rho\in\mathscr{T}} \mathrm{set}(\mathscr{I}_\rho)$ (see 2.1(2)) and note that ${}^\kappa 2\setminus Y \in \mathrm{id}_2(\mathbf{i})$. By induction on $\mathrm{dp}(\rho, \mathscr{T})$ we are going to argue that for $\rho \in \mathscr{T}$:

$(\heartsuit)_\rho$ for each $\nu \in Y$ we have

$$\nu \in \mathbf{B}_\rho \iff (\exists p \in \mathscr{I}_\rho)(\nu \in \lim_\kappa(p) \wedge t^\rho_p = 1).$$

Case 1 $\mathrm{suc}_{\mathscr{T}}(\rho) = 0$.
Since $\nu \in Y$ *there is unique* $p \in \mathscr{I}_\rho$ *such that* $\nu \in \lim_\kappa(p)$, *recalling that for* $p, q \in \mathbb{Q}_\kappa$

$$(p, q \textit{ are incompatible}) \Rightarrow (\mathrm{tr}(p) \notin q \vee \mathrm{tr}(q) \notin p) \Rightarrow \lim_\kappa(p) \cap \lim_\kappa(q) = \emptyset.$$

We know that $\mathbf{B}_\rho = \{\nu \in {}^\kappa 2 : \nu(i_\rho) = c_\rho\}$ *(see* (⊛)*(e)) and* $\ell g(\mathrm{tr}(p)) > i_p$ *(see* (⊗)*(c)), so*

$$\nu \in \mathbf{B}_\rho \iff \mathrm{tr}(p)(i_p) = c_p \iff t^\rho_p = 1.$$

Case 2 $|\mathrm{suc}_{\mathscr{T}}(\rho)| = 1$.
Let p be the unique element of $\mathscr{I}_\rho = \mathscr{I}_{\rho\hat{}\langle 0\rangle}$ such that $\nu \in \lim_\kappa(p)$. Then

$$\nu \in \mathbf{B}_\rho \iff \nu \notin \mathbf{B}_{\rho\hat{}\langle 0\rangle} \iff t_p^{\rho\hat{}\langle 0\rangle} = 0 \iff t_p^\rho = 1.$$

Case 3 $\mathrm{suc}_{\mathscr{T}}(\rho)$ *is infinite.*
Let p be the unique element of $\mathscr{I}_\rho$ such that $\nu \in \lim_\kappa(p)$.
First, assume $t_p^\rho = 1$. Thus $p \Vdash$ "$\underset{\sim}{\eta} \in \mathbf{B}_\rho = \bigcap\{\mathbf{B}_\varrho : \varrho \in \mathrm{suc}_{\mathscr{T}}(\rho)\}$". Suppose that $\varrho \in \mathrm{suc}_{\mathscr{T}}(\rho)$ and let q be the unique element of $\mathscr{I}_\varrho$ such that $\nu \in \lim_\kappa(q)$. Then, by $(\otimes)(f)$, $p \leq q$ and hence $q \Vdash$ "$\underset{\sim}{\eta} \in \mathbf{B}_\rho \subseteq \mathbf{B}_\varrho$", so $t_q^\varrho = 1$. By the inductive hypothesis we get $\nu \in \mathbf{B}_\varrho$. Since $\varrho \in \mathrm{suc}_{\tilde{\mathscr{T}}}(\rho)$ was arbitrary we conclude that $\nu \in \bigcap\{\mathbf{B}_\varrho : \varrho \in \mathrm{suc}_{\mathscr{T}}(\rho)\} = \mathbf{B}_\rho$.
Second, assume $t_p^\rho = 0$. By $(\otimes)(e)$ we know that $p \Vdash$ "$\underset{\sim}{\eta} \notin \mathbf{B}_\varrho$" for some $\varrho \in \mathrm{suc}_{\mathscr{T}}(\rho)$. Let $q \in \mathscr{I}_\varrho$ be the unique element such that $\nu \in \lim_\kappa(q)$. Then $p \leq q$ and hence $t_q^\varrho = 0$. By the inductive hypothesis we get $\nu \notin \mathbf{B}_\varrho$ and hence also $\nu \notin \mathbf{B}_\rho$.

Finally note that our assumption "$\Vdash \underset{\sim}{\eta} \notin \mathbf{B}$" implies that $t_p^{\langle\rangle} = 0$ for all $p \in \mathscr{I}_{\langle\rangle}$. Therefore, $(\heartsuit)_{\langle\rangle}$ implies $Y \cap \mathbf{B} = \emptyset$, so $\tilde{\mathbf{B}} \in \mathrm{id}_2(\mathbf{i})$.
(6) This is similar but easier. □

Definition 3.4 (1) For $\mathbf{i}$ as in 3.1, we define $\mathrm{cov}(\mathbf{i})$, $\mathrm{add}(\mathbf{i})$, $\mathrm{non}(\mathbf{i})$, $\mathrm{cf}(\mathbf{i})$ as those numbers for the ideal $\mathrm{id}(\mathbf{i})$, see 0.7.
(2) If $\kappa_{\mathbf{i}}, \underset{\sim}{\eta}_{\mathbf{i}}$ are clear from $\mathbb{Q}_{\mathbf{i}}$ we may write $\mathbb{Q}_{\mathbf{i}}$ instead of $\mathbf{i}$ and write $\mathrm{id}(\mathbb{Q}_{\mathbf{i}})$ etc. In particular we will be using this convention for $\mathbb{Q}_\kappa$ from Sect. 1 and for Cohen_κ.

Recalling $S^\kappa_{\mathrm{inac}} = \{\partial : \partial < \kappa \text{ is inaccessible}\}$, note that for low inaccessible κ's, $\mathbb{Q}_\kappa$ is like κ-Cohen, that is,

Claim 3.5 *(1) If $\kappa > \sup(S^\kappa_{\mathrm{inac}})$ <u>then</u> for some open dense subsets $\mathscr{I}_1, \mathscr{I}_2$ of $\mathbb{Q}_\kappa$, Cohen_κ respectively, we have $\mathbb{Q}_\kappa \restriction \mathscr{I}_1 \cong \mathrm{Cohen}_\kappa \restriction \mathscr{I}_2$.*
(2) If $S \subseteq S^\kappa_{\mathrm{inac}}$ is bounded in κ <u>then</u> $\mathbb{Q}_{\kappa,S}$ satisfies the conclusion of part (1), where $\mathbb{Q}_{\kappa,S}$ is naturally defined as $\mathbb{Q}_\kappa \restriction \{p : S_p \subseteq S\}$.

Proof (1) Let $\mu = \sup(S^\kappa_{\mathrm{inac}})$, so $\mu < \kappa$.
Let $\mathscr{I}_1 = \{p \in \mathbb{Q}_\kappa : \ell g(\mathrm{tr}(p)) \geq \mu\}$, let $\mathscr{I}_2 = \{\eta \in \mathrm{Cohen}_\kappa : \ell g(\eta) \geq \mu\}$ and $F : \mathscr{I}_1 \to \mathscr{I}_2$ be $F(p) = \mathrm{tr}(p)$.
(2) Similarly. □

Claim 3.6 *(1) $\mathrm{id}(\mathbb{Q}_\kappa)$ is a κ^+-complete ideal on ${}^\kappa 2$ and also $\mathrm{id}(\mathrm{Cohen}_\kappa)$ is.*
(2) If κ is weakly compact and $\mathscr{I}_\alpha \subseteq \mathbb{Q}_\kappa$ is pre-dense for $\alpha < \alpha_* < \kappa^+$ <u>then</u> the sets $\mathscr{I}_1^*, \mathscr{I}_2^*$ are dense open subsets of $\mathbb{Q}_\kappa$ where

$$\mathscr{I}_1^* = \big\{p \in \mathbb{Q}_\kappa : \text{for every } \alpha < \alpha_* \text{ there is } \partial < \kappa \text{ such that } [\eta \in p \cap {}^\partial 2 \Rightarrow p^{[\eta]} \text{ is above some } q \in \mathscr{I}_\alpha]\big\}.$$

and

$$\mathscr{I}_2^* = \big\{p \in \mathbb{Q}_\kappa : \lim_\kappa(p) \subseteq \bigcap_{\alpha<\alpha_*} \mathrm{set}(\mathscr{I}_\alpha)\big\}$$

Sh:1004

(see 2.1(2)).

(3) Assume κ is weakly compact. Suppose that $p \in \mathbb{Q}_\kappa$ as witnessed by $(\mathrm{tr}(p), S_p, \bar{\Lambda}_p)$, *$\alpha < \kappa$ and let $\mathbf{B} \subseteq {}^\kappa 2$ be a κ-Borel set. Then there is $q \in \mathbb{Q}_\kappa$ such that:*

(i) $p \leq q$, $\mathrm{tr}(p) = \mathrm{tr}(q)$,
(ii) $S_p \cap \alpha = S_q \cap \alpha$, $\bar{\Lambda}_p \restriction \alpha = \bar{\Lambda}_q \restriction \alpha$ *and*
(iii) for some $\beta \in (\alpha, \kappa)$, if $\nu \in q \cap {}^\beta 2$ then
either $q^{[\nu]} \Vdash$ "$\underset{\sim}{\eta} \in \mathbf{B}$" *and* $\lim_\kappa(q^{[\nu]}) \subseteq \mathbf{B}$,
or $q^{[\nu]} \Vdash$ "$\underset{\sim}{\eta} \notin \mathbf{B}$" *and* $\lim_\kappa(q^{[\nu]}) \cap \mathbf{B} = \emptyset$.

Proof (1) By 3.2(1).
(2) By 1.9(2), pedantically by its proof.
(3) We prove this by the induction on the depth γ of (the κ-Borel representation; see the proof of 3.2(5)) of $\mathbf{B}$.

Case 1 $\gamma = 0$ *so* $\mathbf{B} = \{\nu \in {}^\kappa 2 : \nu(i) = c\}$ *for some* $i < \kappa$, $c < 2$.
Obvious.

Case 2 $\mathbf{B}$ *is the complement of a κ-Borel set $\mathbf{B}_1$ of depth $< \gamma$.*
Obvious by the phrasing of (3)(iii).

Case 3 $\mathbf{B} = \bigcap_{\alpha<\alpha(*)} \mathbf{B}_\alpha$, *where $\alpha(*) \leq \kappa$ and $\mathbf{B}_\alpha$ are κ-Borel sets of depth $< \gamma$. Let $\mathscr{I}^1_\alpha = \{q \in \mathbb{Q}_\kappa : q$ satisfies (3)(iii) for $\mathbf{B}_\alpha$ and α with $\beta = \beta_{q,\alpha} < \kappa\}$. By the induction hypothesis $\mathscr{I}^1_\alpha$ is dense open in $\mathbb{Q}_\kappa$. Let*

$$\mathscr{I}_2 = \Big\{q \in \mathbb{Q}_\kappa : \underline{\text{ either }}\ q \Vdash \text{“}\underset{\sim}{\eta} \notin \mathbf{B}_\alpha^{\mathbf{V}[\mathbb{Q}_\kappa]}\text{” for some } \alpha = \alpha(q) < \alpha_* \ \underline{\text{ or }}\ q \Vdash \text{“}\underset{\sim}{\eta} \in \mathbf{B}^{\mathbf{V}[\mathbb{Q}_\kappa]}\text{”}\Big\}.$$

Clearly $\mathscr{I}_2$ is dense open. Let

$$\mathscr{I}_{3,1} = \left\{q \in \mathscr{I}_2 : q \Vdash \text{“}\underset{\sim}{\eta} \notin \mathbf{B}''_{\alpha(q)} \text{ and } q \in \mathscr{I}^1_{\alpha(q)}\right\}.$$

Then for $q \in \mathscr{I}_{3,1}$ we have $(\exists\beta)(\forall\nu \in q \cap {}^\beta 2)(\lim_\kappa(q^{[\nu]}) \cap \mathbf{B}_{\alpha(q)} = \emptyset)$ *and hence* $\lim_\kappa(q) \cap \mathbf{B}_{\alpha(q)} = \emptyset$ *for $q \in \mathscr{I}_{3,1}$. We let*

$$\mathscr{I}_{3,2} = \{q \in \mathbb{Q}_\kappa : q \Vdash \text{“}\underset{\sim}{\eta} \in \mathbf{B}'' \text{ and } \lim_\kappa(q) \subseteq \mathbf{B}\}$$

and finally we set $\mathscr{I}_3 = \mathscr{I}_{3,1} \cup \mathscr{I}_{3,2}$.
Next consider:

($\circledast$) *for every $q_0 \in \mathbb{Q}_\kappa$ there is $q \in \mathscr{I}_3$ above q_0.*

Why is ($\circledast$) sufficient? First note that for every $q \in \mathscr{I}_3$ the demand (3)(iii) hold for the pair $(q, \mathbf{B})$. Indeed, by the definition of $\mathscr{I}_3$ we have to check the two possibilities: $q \in \mathscr{I}_{3,1}$ and $q \in \mathscr{I}_{3,2}$. If $q \in \mathscr{I}_{3,1}$, then $\alpha(q)$ is well defined and $\lim_\kappa(q) \cap \mathbf{B}_{\alpha(q)} = \emptyset$, *so $\beta = 0$ is as required. If $q \in \mathscr{I}_{3,2}$ then also $\beta = 0$ is as required. Now we may use ($\circledast$) and 1.9(2) to get $q \in \mathbb{Q}_\kappa$ satisfying (i)–(iii) of (3).*

Sh:1004

Why does (⁎) hold? Let $q_0 \in \mathbb{Q}_\kappa$ be given. We may find q_1 above q_0 such that either $q_1 \Vdash \underset{\sim}{\eta} \in \mathbf{B}$ or $q_1 \Vdash \underset{\sim}{\eta} \notin \mathbf{B}$. First assume that the latter is true. Then for some $\alpha < \alpha(*)$ and $q_2 \geq q_1$ we have $q_2 \Vdash \underset{\sim}{\eta} \notin \mathbf{B}_\alpha$. By the inductive hypothesis there is $q_3 \geq q_2$ satisfying (3)(iii) for $\mathbf{B}_\alpha$ and α. Since $q_3 \Vdash \underset{\sim}{\eta} \notin \mathbf{B}_\alpha$, this implies $\lim_\kappa(q_3) \cap \mathbf{B}_\alpha = \emptyset$ and therefore $q_3 \in \mathscr{I}_{3,1} \subseteq \mathscr{I}_3$.

Second, assume $q_1 \Vdash \underset{\sim}{\eta} \in \mathbf{B}$, i.e., $q_1 \Vdash$ "$\underset{\sim}{\eta} \in \mathbf{B}_\alpha$ for every $\alpha < \alpha(*)$". Let

$$\mathscr{I}_{3,2,\alpha} = \big\{r \in \mathbb{Q}_\kappa : r \text{ is incompatible with } q_1 \text{ or } q_1 \leq r \text{ and } \lim_\kappa(r) \subseteq \mathbf{B}_\alpha\big\};$$

by the inductive hypothesis it is an open dense set. By 3.6(2) we may find $q_4 \geq q_1$ such that

$$\big(\forall\alpha < \alpha(*)\big)\big(\exists\partial < \kappa\big)\big(\eta \in q_4 \cap {}^{\partial}2 \Rightarrow (q_4)^{[\eta]} \in \mathscr{I}_{3,2,\alpha}\big).$$

Since $(q_4)^{[\eta]} \in \mathscr{I}_{3,2,\alpha}$ implies $\lim_\kappa((q_4)^{[\eta]}) \subseteq \mathbf{B}_\alpha$ (as $q_4 \geq q_1$), we conclude $\lim_\kappa(q_4) \subseteq \mathbf{B}_\alpha$ for all $\alpha < \alpha(*)$. Hence $q_4 \in \mathscr{I}_{3,2} \subseteq \mathscr{I}_3$. □

Claim 3.7 *Considering ${}^\kappa 2$ as an Abelian Group (with addition $\oplus$ modulo 2, coordinatewise), the ideal $\mathrm{id}(\mathbb{Q}_\kappa)$ is closed under translation, i.e. if $\mathbf{B} \subseteq {}^\kappa 2$ and $\eta \in {}^\kappa 2$ then $\mathbf{B} \in \mathrm{id}(\mathbb{Q}_\kappa) \Leftrightarrow \eta \oplus \mathbf{B} \in \mathrm{id}(\mathbb{Q}_\kappa)$ where $\eta \oplus \mathbf{B} := \{\eta \oplus \nu : \nu \in \mathbf{B}\}$.*

Proof Straightforward. □

Claim 3.8 *If κ is an inaccessible limit of inaccessibles, then ${}^\kappa 2$ can be partitioned to two sets A_0, A_1 such that A_0 is in $\mathrm{id}(\mathrm{Cohen}_\kappa)$ and A_1 is in $\mathrm{id}(\mathbb{Q}_\kappa)$.*

Proof Let $\langle\kappa_i : i < \kappa\rangle$ list the inaccessibles $< \kappa$ in the increasing order and let

$$\mathscr{I}_{\kappa_{i+1}} = \big\{q \in \mathbb{Q}_{\kappa_{i+1}} : \ell g(\mathrm{tr}(q)) > \kappa_i \text{ and } \mathrm{tr}(q)\restriction[\kappa_i, \ell g(\mathrm{tr}(q)) \text{ is not constantly zero }\big\}.$$

Clearly, $\mathscr{I}_{\kappa_{i+1}}$ is an open dense subset of $\mathbb{Q}_{\kappa_{i+1}}$. Now, for $\eta \in {}^{\kappa>}2$ let $p_\eta \in \mathbb{Q}_\kappa$ be witnessed by $(\eta, \{\kappa_{i+1} : \kappa_i > \ell g(\eta)\}, \langle\Lambda_{\kappa_{i+1}} : \kappa_i > \ell g(\eta)\rangle)$ where $\Lambda_{\kappa_{i+1}} = \{\mathscr{I}_{\kappa_{i+1}}\}$. Then

(a) p_η indeed belongs to $\mathbb{Q}_\kappa$,
(b) $\mathrm{tr}(p_\eta) = \eta$,
(c) p_η is a nowhere-dense subtree of ${}^{\kappa>}2$.

Let $A_0 = \bigcup\{\lim_\kappa(p_\eta) : \eta \in {}^{\kappa>}2\}$, $A_1 = {}^\kappa 2 \setminus A_0$. Let us argue that they are as required.

First, why does A_1 belong to $\mathrm{id}(\mathbb{Q}_\kappa)$? Clearly A_1 is κ-Borel and for $p \in \mathbb{Q}_\kappa$ we shall prove $p \nVdash$ "$\eta \in A_1$", this suffices. Let $\nu = \mathrm{tr}(p)$, hence p, p_ν are compatible so let $q \in \mathbb{Q}_\kappa$ be a common upper bound. Then $q \Vdash$ "$\underset{\sim}{\eta} \in \lim_\kappa(q) \subseteq \lim_\kappa(p_\nu) \subseteq A_0 = {}^\kappa 2 \setminus A_1$".

Second, why does $A_0 \in \mathrm{id}(\mathrm{Cohen}_\kappa)$? Because it is the union of $|{}^{\kappa>}2| = \kappa$ nowhere dense sets (remember clause (c)). □

Claim 3.9 *(1) [κ weakly compact] Any κ-Borel set $\mathbf{B}$ is equal modulo $\mathrm{id}(\mathbb{Q}_\kappa)$ to the union of $\leq \kappa$ sets, each is κ-closed and even $\mathbb{Q}_\kappa$-basic, see Definition 0.2(2).*
(2) $\mathrm{Borel}_\kappa/\mathrm{id}(\mathbb{Q}_\kappa)$ is a κ^+-c.c. Boolean Algebra.

Sh:1004

Proof (1) We have $\mathrm{id}_1(\mathbb{Q}_\kappa) = \mathrm{id}_2(\mathbb{Q}_\kappa)$ by 3.2(5). As $\mathbb{Q}_\kappa$ satisfies the κ^+-c.c. it is enough to show that for a dense set of $p \in \mathbb{Q}_\kappa$, we have that $\lim_\kappa(p) \subseteq \mathbf{B}$ or $\lim_\kappa(p)$ is disjoint from $\mathbf{B}$. But this easily holds by 3.6(3).
(2) Should be clear. ☐

Claim 3.10 (κ weakly compact) *Assume F is a κ-Borel function from ${}^\kappa 2$ to ${}^\kappa 2$. For a dense set of $p \in \mathbb{Q}_\kappa$, the function F can be read continuously on $\lim_\kappa(p)$, i.e. for some club C of κ and $\bar{h} = \langle h_\alpha : \alpha \in C\rangle$ we have:*

(i) $h_\alpha : p \cap {}^\alpha 2 \longrightarrow {}^\alpha 2$,
(ii) if $\eta \in p \cap {}^\alpha 2$, $\nu \in p \cap {}^\beta 2$, $\eta \triangleleft \nu$ and $\{\alpha, \beta\} \subseteq C$ then $h_\alpha(\eta) \triangleleft h_\beta(\nu)$,
(iii) if $\eta \in \lim_\kappa(p)$ then $F(\eta) = \bigcup\{h_\alpha(\eta\restriction\alpha) : \alpha \in C\}$.

Remark 3.11 This is parallel to "every Borel function $F : [0, 1] \longrightarrow [0, 1]$ can be approximated by step functions, that is functions such that for some finite partitions of $[0, 1]$ to intervals, it is constant on each interval".

Proof By 1.9(2), the set

$$\mathscr{I} = \left\{q \in \mathbb{Q}_\kappa : (\forall\alpha < \kappa)(\exists\beta < \kappa)(\forall\nu \in q \cap {}^\beta 2)(q^{[\nu]} \text{ forces a value to } F(\underset{\sim}{\eta})\restriction\alpha)\right\}$$

is an open dense subset of $\mathbb{Q}_\kappa$.

Let us fix $q \in \mathscr{I}$. Then by the definition of $\mathscr{I}$ there are an increasing sequence $\langle\beta(q,\alpha) : \alpha < \kappa\rangle$ of ordinals below κ and a sequence $\langle g(q,\alpha) : \alpha < \kappa\rangle$ of functions such that for each $\alpha < \kappa$ we have

$$g(q,\alpha) : {}^{\beta(q,\alpha)}2 \longrightarrow {}^\alpha 2 \quad \text{and} \quad \nu \in q \cap {}^{\beta(q,\alpha)}2 \Rightarrow q^{[\nu]} \Vdash \text{“}F(\underset{\sim}{\eta})\restriction\alpha = g(q,\alpha)(\nu)\text{”}.$$

Let $E_q = \{\delta < \kappa : \delta \text{ is a limit ordinal and } (\forall\alpha < \delta)(\beta(q,\alpha) < \delta)\}$; clearly it is a club of κ. For $\delta \in E_q$ we define a function $h_{q,\delta} : q \cap {}^\delta 2 \longrightarrow q \cap {}^\delta 2$ by:

$$h_{q,\delta}(\nu) = \bigcup\left\{g(q,\alpha)(\nu\restriction\beta(q,\alpha)) : \alpha < \delta\right\} \quad \text{for } \nu \in q \cap {}^\delta 2.$$

Clearly, for every $\delta \in E_q$ and $\nu \in {}^\delta 2$ we have

$$(\boxtimes)\quad q^{[\nu]} \Vdash \text{“ } F(\underset{\sim}{\eta})\restriction\delta = \bigcup_{\alpha<\delta}(F(\underset{\sim}{\eta})\restriction\alpha) = \bigcup_{\alpha<\delta} g(q,\alpha)(\underset{\sim}{\eta}\restriction\beta(q,\alpha)) = h_{q,\delta}(\nu)\text{ ”}.$$

For $\delta \in E_q$ and $\nu \in {}^\delta 2$ consider the set

$$Y_{\delta,\nu} = \left\{\eta \in \lim_\kappa(q) : \nu \triangleleft \eta \text{ and } F(\eta)\restriction\delta \neq h_{q,\delta}(\nu)\right\}.$$

It is a κ-Borel set which (by ($\boxtimes$)) belongs to $\mathrm{id}_1(\mathbb{Q}_\kappa) = \mathrm{id}_2(\mathbb{Q}_\kappa)$. Hence

$$Y := \bigcup\left\{Y_{\delta,\nu} : \delta \in E_q \text{ and } \nu \in {}^\delta 2\right\} \in \mathrm{id}(\mathbb{Q}_\kappa).$$

Let $q^* \geq q$ be such that $\lim_\kappa(q^*) \cap Y = \emptyset$ (exists by the proof of 3.9(1)). Then $q^*, E_q, \langle h_{q,\delta} : \delta \in E_q\rangle$ have the properties required in (i)–(iii) and the Claim follows. ☐

Remark 3.12 For κ which is not weakly compact we may get a weaker result for $\mathrm{id}_1(\mathbb{Q}_\kappa) = \mathrm{id}_2(\mathbb{Q}_\kappa)$. For each $\alpha < \kappa$ let $\mathscr{I}_\alpha$ be a maximal antichain of $\mathbb{Q}_\kappa$ such that

$$q \in \mathscr{I}_\alpha \Rightarrow q \text{ forces a value to } F(\underset{\sim}{\eta}){\restriction}\alpha.$$

Without loss of generality

$(*)_0$ $\alpha < \beta \wedge q \in \mathscr{I}_\beta \Rightarrow (\exists p \in \mathscr{I}_\alpha)(p \leq q)$

Let $\langle q_{\alpha,i} : i < i(\alpha) \leq \kappa\rangle$ list $\mathscr{I}_\alpha$ and let $\nu_{\alpha,i}$ be such that $q_{\alpha,i} \Vdash$"$F(\underset{\sim}{\eta}){\restriction}\alpha = \nu_{\alpha,i}$". Then clearly $\mathrm{tr}(q_{\alpha,j}) \trianglelefteq \mathrm{tr}(q_{\alpha,i}) \in q_{\alpha,j} \Leftrightarrow i = j$. Let $Y_\alpha = \bigcup_{i<i(\alpha)} \lim(q_{\alpha,i})$ and note that:

$(*)_1$ (a) $Y_\alpha = {}^\kappa 2 \mod \mathrm{id}(\mathbb{Q}_\kappa)$ decreases with α, and
  (b) $\langle \lim_\kappa(q_{\alpha,i}) : i < i(\alpha)\rangle$ is a partition of Y_α.

Define $H_\alpha : Y_\alpha \longrightarrow {}^\alpha 2$ by $H_\alpha(\eta) = \nu_{\alpha,i}$ if $\eta \in \lim_\kappa(q_{\alpha,i})$. Then

$(*)_3$ (a) H_α is continuous on Y_α in the sense that $H_\alpha(\eta)$ is the value of $H'_\alpha(\eta{\restriction}j)$ for every large enough $j < \kappa$, where
  (b) we let $H'_\alpha : {}^{\kappa>}2 \longrightarrow {}^{\kappa>}2$ be

$$H'_\alpha(\nu) = \begin{cases} \nu_{\alpha,i} & \text{if } \mathrm{tr}(q_{\alpha,i}) \trianglelefteq \nu \in q_{\alpha,i}, \\ \langle (0)_\alpha \rangle & \text{if there is no such } i. \end{cases}$$

Now consider

$(*)_4$ (a) $Y = \bigcap_{\alpha<\kappa} Y_\alpha$ and note $Y = {}^\kappa 2 \mod \mathrm{id}(\mathbb{Q}_\kappa)$, and
  (b) let $H : Y \longrightarrow {}^\kappa 2$ be defined by $H(\eta) = \lim\langle H_\alpha(\eta) : \alpha < \kappa\rangle$.

Concerning Lebesgue Density Theorem:

Conclusion 3.13 (κ weakly compact) *If $X \subseteq {}^\kappa 2$ is κ-Borel, then for some $Y \in \mathrm{id}(\mathbb{Q}_\kappa)$ for every $\eta \in X \backslash Y$ for every $\alpha < \kappa$ large enough $(2^\kappa)^{[\eta{\restriction}\alpha]} \cap X$ includes $\lim_\kappa(p)$ for some $p \in \mathbb{Q}_\kappa$.*

Remark 3.14 So this holds also for the complement of X.

Proof By 3.6(3) there is a maximal antichain $\langle p_i : i < i_*\rangle$ of members of $\mathbb{Q}_\kappa$ and $S \subseteq i_*$ such that $i \in S \Rightarrow \lim_\kappa(p_i) \subseteq X$ and $i \in i_* \backslash S \Rightarrow \lim_\kappa(p_i) \cap X = \emptyset$. Then $i_* < \kappa^+$ and let $Y = {}^\kappa 2 \backslash \bigcup\{\lim_\kappa(p_i) : i < i_*\}$, so clearly $Y \in \mathrm{id}(\mathbb{Q}_\kappa)$. If $\eta \in X\backslash Y$, then by the choice of Y for some $i < i_*$, $\eta \in \lim_\kappa(p_i)$ and necessarily $i = i(\eta)$ is unique and $i \in S$. Let $\alpha(\eta)$ be $\ell g(\mathrm{tr}(p_{i(\eta)}))$. Clearly we are done. □

Claim 3.15 *If $\mathscr{I} \subseteq \mathbb{Q}_\kappa$ is dense open and $W \subseteq \kappa = \sup(W)$ then for some $\bar{p} = \langle p_\rho : \rho \in \Omega\rangle$ we have:*

(a) $\Omega \subseteq {}^{\kappa>}2$, moreover $\Omega \subseteq \bigcup\{{}^\alpha 2 : \alpha \in W\}$,
(b) $p_\rho \in \mathscr{I} \subseteq \mathbb{Q}_\kappa$ has trunk ρ for every $\rho \in \Omega$,
(c) if $\rho \triangleleft \nu \in p_\rho$ then $\nu \notin \Omega$,
(d) $\{p_\rho : \rho \in \Omega\}$ is a predense subset of $\mathbb{Q}_\kappa$, moreover is a maximal antichain,

Sh:1004

(e) *letting* $(\rho_\rho, S_\rho, \bar\Lambda_\rho)$ *witness* $p_\rho \in \mathbb{Q}_\kappa$ *we have: if* $\rho_1, \rho_2 \in \Omega$ *and* $\ell g(\rho_1) \le \alpha = \ell g(\rho_2)$ *then* $S_{\rho_2} = S_{\rho_1} \backslash (\alpha+1)$, $\bar\Lambda_{\rho_2} = \bar\Lambda_{\rho_1} \restriction S_{\rho_2}$.

Proof Let $\Omega_1 = \{\mathrm{tr}(p) : p \in \mathscr{I}\}$ and for $\rho \in \Omega_1$ choose $p^1_\rho \in \mathscr{I}$ such that $\mathrm{tr}(p^1_\rho) = \rho$ and let $(\rho, S^1_\rho, \bar\Lambda^1_\rho)$ witness $p^1_\rho \in \mathbb{Q}_\kappa$ with $\min(S^1_\rho) > \ell g(\rho)$. Note that

$$\rho \in \Omega_1 \ \wedge\ \rho \trianglelefteq \nu \in p^1_\rho \ \Rightarrow\ \nu \in \Omega_1$$

because $\mathscr{I}$ is open dense. Let $S_* = \bigcup\{S^1_\rho : \rho \in \Omega_1\}$ and note that S_* is a nowhere stationary subset of κ. Let $\bar\Lambda = \langle \Lambda_\partial : \partial \in S_*\rangle$ where

$$\Lambda_\partial = \bigcup\{\Lambda^1_{\rho,\partial} : \rho \text{ satisfies } \rho \in \Omega_1 \cap {}^{\partial>}2 \text{ and } \partial \in S^1_\rho\}.$$

Easily, if $\partial \in S_*$ then Λ_∂ is a set of $\le \partial$ dense subsets of $\mathbb{Q}_\partial$.

Next, for $\rho \in \Omega_1$ let $p^2_\rho \in \mathbb{Q}_\kappa$ be witnessed by $(\rho, S_*, \bar\Lambda)$. Now we define $\Omega_{2,\alpha}$ by induction on $\alpha \in W$ such that

$$\Omega_{2,\alpha} = \left\{\rho \in {}^\alpha 2 : \rho \in \Omega_1 \text{ and if } \beta \in W \cap \alpha \wedge \varrho \in \Omega_{2,\beta} \wedge \varrho \triangleleft \rho \text{ then } \rho \notin p^2_\varrho\right\}.$$

Lastly, let $\Omega = \bigcup_{\alpha \in W} \Omega_{2,\alpha}$ and $p_\rho = p^2_\rho$ for $\rho \in \Omega$. Now check. □

Claim 3.16 *Assume that* κ *is inaccessible limit of inaccessibles and* $W_\varepsilon \subseteq \kappa = \sup(W_\varepsilon)$ *for* $\varepsilon < \kappa$ *are pairwise disjoint. If* $A \in \mathrm{id}(\mathbb{Q}_\kappa)$ <u>*then*</u> *for some* $(S, \bar\Lambda), \bar p, \bar{\mathscr{I}}$*:*

(a) $\bar p = \langle p_\rho : \rho \in {}^{\kappa>}2\rangle$, $p_\rho \in \mathbb{Q}_\kappa$ *is defined by* $(\rho, S\backslash(\ell g(\rho)+1), \bar\Lambda \restriction (S\backslash(\ell g(\rho)+1)))$,
(b) $\bar{\mathscr{I}} = \langle \mathscr{I}_\varepsilon : \varepsilon < \kappa\rangle$,
(c) $\mathscr{I}_\varepsilon \subseteq \{p_\rho : \rho \in {}^{\kappa>}2 \ \wedge \ell g(\rho) \in W_\varepsilon\}$ *is a predense set and even a maximal antichain of* $\mathbb{Q}_\kappa$*,*
(d) $A \subseteq \bigcup\{{}^\kappa 2\backslash \mathrm{set}(\mathscr{I}_\varepsilon) : \varepsilon < \kappa\}$.

Proof Follows by the proof of 3.15 but we give details. Let $A \in \mathrm{id}(\mathbb{Q}_\kappa)$, hence there are a maximal antichains $\mathscr{I}_\varepsilon$ of $\mathbb{Q}_\kappa$ such that $A \subseteq \bigcup_{\varepsilon<\kappa}\left({}^\kappa 2\backslash \mathrm{set}(\mathscr{I}_\varepsilon)\right)$. As $\mathbb{Q}_\kappa$ satisfies the κ^+-c.c. clearly $|\mathscr{I}_\varepsilon| \le \kappa$.

Recalling $\kappa = \sup(S^\kappa_{\mathrm{inac}})$ hence without loss of generality each $p \in \mathscr{I}_\varepsilon$ is nowhere-dense (see the proof of 3.8) and hence $|\mathscr{I}_\varepsilon| = \kappa$. Let $\mathscr{I}_\varepsilon = \{p_{\varepsilon,i} : i < \kappa\}$ and suppose that each $p_{\varepsilon,i}$ is defined by $(\eta_{\varepsilon,i}, S_{\varepsilon,i}, \bar\Lambda_{\varepsilon,i})$. Without loss of generality $\partial \in S_{\varepsilon,i} \Rightarrow \ell g(\eta_{\varepsilon,i}) < \partial$. Let

$(*)_1$ $S = \{\partial \in S^\kappa_{\mathrm{inac}}$: for some $\varepsilon, i < \partial$ we have $\partial \in S_{\varepsilon,i}\}$.

Clearly,

$(*)_2$ S is a nowhere stationary subset of S^κ_{inac}.

Let

$(*)_3$ $\bar\Lambda = \langle \Lambda_\partial : \partial \in S\rangle$ where for $\partial \in S$ we let

$$\Lambda_\partial = \bigcup\left\{\Lambda_{\varepsilon,i,\partial} : \varepsilon < \partial,\ i < \partial \text{ and } \partial \in S_{\varepsilon,i}\right\}.$$

Sh:1004

Clearly,

$(*)_4$ $(\langle\rangle, S, \bar{\Lambda})$ defines a condition $p_* \in \mathbb{Q}_\kappa$, as $S \subseteq \kappa$ is nowhere stationary and if $\partial \in S$ then Λ_∂ is a set of $\leq \partial$ pre-dense subsets of $\mathbb{Q}_\partial$.

Lastly,

$(*)_5$ (a) for $\rho \in {}^{\kappa>}2$ let $p_\rho = (\rho, S\backslash(\ell g(\rho)+1), \bar{\Lambda}\restriction(S\backslash(\ell g(\rho)+1)))$,
(b) for $\varepsilon < \kappa$ let

$$\mathscr{I}'_\varepsilon = \big\{p_\rho : \text{for some } i < \kappa \text{ we have } i, \varepsilon < \ell g(\rho) \in W_\varepsilon \text{ and } \eta_{\varepsilon,i} \trianglelefteq \rho \in p_{\varepsilon,i}\big\}.$$

Then

$(*)_6$ for each $\varepsilon < \kappa$
(a) $\mathscr{I}'_\varepsilon$ is a predense subset of $\mathbb{Q}_\kappa$, and
(b) $\mathrm{set}(\mathscr{I}'_\varepsilon) \subseteq \mathrm{set}(\mathscr{I}_\varepsilon)$.

[Why? For clause (a), if $q \in \mathbb{Q}_\kappa$ then some $p \in \mathscr{I}_\varepsilon$ is compatible with q and hence there is $r \geq q, p$. Let $i < \kappa$ be such that $p = p_{\varepsilon,i}$ and let $\rho \in r$ be such that $\ell g(\rho) > \varepsilon, i, \ell g(\mathrm{tr}(r))$ and $\ell g(\rho) \in W_\varepsilon$. Now, $p = p_{\varepsilon,i} \leq r$ implies $\eta_{\varepsilon,i} = \mathrm{tr}(p) \trianglelefteq \mathrm{tr}(r) \trianglelefteq \rho \in r \subseteq p_{\varepsilon,i}$. Hence $p_\rho \in \mathscr{I}'_\varepsilon$ has trunk ρ and hence it is compatible with r, so also with q. Concerning clause (b), assume $\eta \in \mathrm{set}(\mathscr{I}'_\varepsilon) \subseteq {}^\kappa 2$. Then for some $\rho \in {}^{\kappa>}2$ we have $p_\rho \in \mathscr{I}'_\varepsilon$ and $\eta \in \lim_\kappa(p_\rho)$. By the definition of $\mathscr{I}'_\varepsilon$, for some $i < \ell g(\rho)$ we have $\eta_{\varepsilon,i} \trianglelefteq \rho \in p_{\varepsilon,i}$. Hence $\mathrm{tr}(p_\rho) \in p_{\varepsilon,i}$. By the choice of p_ρ, clearly $\lim_\kappa(p_\rho) \subseteq \lim_\kappa(p^{[\rho]}_{\varepsilon,i}) \subseteq \lim_\kappa(p_{\varepsilon,i}) \subseteq \mathrm{set}(\mathscr{I}_\varepsilon)$, so we are done.]

To get "$\mathscr{I}_\varepsilon$ a maximal antichain" we choose $\Omega_{\varepsilon,j} \subseteq {}^j 2$ by induction on $j \in W_\varepsilon\backslash(\varepsilon+1)$ by:

$(*)_7$ $\Omega_{\varepsilon,j} = \big\{\rho \in {}^j 2 :$ for some $i \in W_\varepsilon \cap j\backslash(\varepsilon+1)$, $\eta_{\varepsilon,i} \trianglelefteq \rho \in p_{\varepsilon,i}$ but for no $i_1 \in W_\varepsilon \cap j\backslash(\varepsilon+1)$ and $\nu \in \Omega_{\varepsilon,i_1}$ do we have $\rho \in p_\nu\big\}$.

Then let

$(*)_8$ (a) $\Omega_\varepsilon = \bigcup\{\Omega_{\varepsilon,j} : j \in W_\varepsilon\backslash(\varepsilon+1)\}$,
(b) $\mathscr{I}''_\varepsilon = \{p_\rho : \rho \in \Omega_\varepsilon\}$.

Now (S, Λ), $\langle p_\rho : \rho \in \Omega_\varepsilon\rangle$ and $\langle \mathscr{I}''_\varepsilon : \varepsilon < \kappa\rangle$ are as required. □

4 On add$(\mathbb{Q}_\kappa)$ and cf$(\mathbb{Q}_\kappa)$

Definition 4.1 (1) For $\alpha < \kappa$, $\nu \in {}^\alpha 2$, $p \in \mathbb{Q}_\kappa$, $\eta \in p \cap {}^\alpha 2$ we let

$$p^{[\eta,\nu]} = \big\{\rho : \rho \trianglelefteq \nu \text{ or for some } \varrho \text{ we have } \eta\hat{\ }\varrho \in p \wedge \rho = \nu\hat{\ }\varrho\big\}.$$

(2) For $\mathscr{I} \subseteq \mathbb{Q}_\kappa$, $\alpha < \kappa$ and a permutation π of ${}^\alpha 2$ let

$$\mathscr{I}^{[\alpha,\pi]} = \big\{p^{[\eta,\nu]} : p \in \mathscr{I}, \eta \in p \cap {}^\alpha 2 \text{ and } \nu = \pi(\eta)\big\}.$$

Sh:1004

(3) Let Λ be a collection of subsets of $\mathbb{Q}_\kappa$ and let $\alpha < \kappa$. For a permutation π of ${}^\alpha 2$ we let

$$\Lambda^{[\alpha,\pi]} = \{\mathscr{I}^{[\alpha,\pi]} : \mathscr{I} \in \Lambda\}.$$

We also define

$$\Lambda^{[\alpha]} = \{\mathscr{I}^{[\alpha,\pi]} : \pi \text{ is a permutation of } {}^\alpha 2 \text{ and } \mathscr{I} \in \Lambda\}$$

and $\Lambda^{[<\alpha]} = \bigcup\{\Lambda^{[\beta]} : \beta < \alpha\}$, here we allow $\alpha = \kappa$.

Claim 4.2 *(1) If $\alpha < \kappa$ and $\mathscr{I} \subseteq \mathbb{Q}_\kappa$ is open/dense/predense/maximal antichain/of cardinality $\leq \kappa$ <u>then</u> so is $\mathscr{I}^{[\alpha,\pi]}$ in $\mathbb{Q}_\kappa$.*
(2) If $\alpha < \kappa$ and Λ is a collection of subsets of $\mathbb{Q}_\kappa$, <u>then</u>
- $\left(\Lambda^{[\alpha]}\right)^{[\alpha]} = \Lambda^{[\alpha]}$ *and* $|\Lambda^{[\alpha]}| \leq |\Lambda| + 2^{2^{|\alpha|}} + \aleph_0 \leq |\Lambda| + \kappa$,
- $\left(\Lambda^{[<\alpha]}\right)^{[<\alpha]} = \Lambda^{[<\alpha]}$ *and* $|\Lambda^{[<\alpha]}| \leq |\Lambda| + \Sigma\{2^{2^{|\beta|}} : \beta < \alpha\} \leq |\Lambda| + \kappa$.

Proof Easy. □

Definition 4.3 (1) For an inaccessible cardinal κ let $\Pr(\kappa)$ mean:
there are predense sets $\mathscr{I}_\varepsilon \subseteq \mathbb{Q}_\kappa$ for $\varepsilon < \kappa$ such that
if $p \in \mathbb{Q}_\kappa$ <u>then</u> $\lim_\kappa(p) \nsubseteq \bigcap_{\varepsilon<\kappa} \operatorname{set}(\mathscr{I}_\varepsilon)$.
(2) Let $S^\kappa_{\mathrm{pr}} = \{\partial < \kappa : \partial \in S^\kappa_{\mathrm{inac}} \wedge \Pr(\partial)\}$ and

$$\mathbf{nst}^{\mathrm{pr}}_\kappa = \mathbf{nst}_{\kappa,\mathrm{pr}} = \{S \subseteq S^\kappa_{\mathrm{inac}} : S \text{ is nowhere stationary and } S \subseteq S^\kappa_{\mathrm{pr}}\}.$$

Observation 4.4 *(1) If κ is inaccessible but it is not a Mahlo cardinal, then* $\Pr(\kappa)$.
(2) If κ is weakly compact, then $\neg\Pr(\kappa)$.
(3) If $\kappa = \sup(S^\kappa_{\mathrm{inac}})$, then $\kappa = \sup(S^\kappa_{\mathrm{pr}})$.
(4) If κ is Mahlo, i.e., S^κ_{inac} is a stationary subset of κ, then S^κ_{pr} is a stationary subset of κ.

Proof (1) First assume $\theta = \sup(S^\kappa_{\mathrm{inac}}) < \kappa$. For $\varepsilon < \kappa$ define

$$\mathscr{I}_\varepsilon = \left\{\left({}^{\kappa>}2\right)^{[\nu^\frown\langle 0\rangle]} : \nu \in {}^{\kappa>}2 \ \wedge\ \ell g(\nu) > \varepsilon\right\}.$$

It should be clear that each $\mathscr{I}_\varepsilon$ is a predense subset of $\mathbb{Q}_\kappa$ and we claim that they witness $\Pr(\kappa)$. So suppose that $p \in \mathbb{Q}_\kappa$ and pick $\nu \in p$ of length greater than θ and than $\ell g(\operatorname{tr}(p))$; note that then $p^{[\nu]} = ({}^{\kappa>}2)^{[\nu]}$. Let $\eta \in {}^\kappa 2$ be such that $\nu \lhd \eta$ and $\eta(i) = 1$ for $i \in [\ell g(\nu), \kappa)$. Clearly, $\eta \in \lim_\kappa(p)$ but $\eta \notin \operatorname{set}(\mathscr{I}_\varepsilon)$ for $\varepsilon > \ell g(\nu)$.

Second, assume $\kappa = \sup(S^\kappa_{\mathrm{inac}})$ but it is not Mahlo. Let E be a club of κ disjoint from S^κ_{inac} and let $\langle\alpha_i : i < \kappa\rangle$ be the increasing enumeration of E. For $\varepsilon < \kappa$ let

$$\mathscr{I}_\varepsilon = \left\{\left({}^{\kappa>}2\right)^{[\nu^\frown\langle 0\rangle]} : \nu \in {}^{\alpha_i}2 \ \wedge\ i > \varepsilon\right\}.$$

Clearly, each $\mathscr{I}_\varepsilon$ is a predense subset of $\mathbb{Q}_\kappa$. We will argue that they witness $\Pr(\kappa)$. Let $p \in \mathbb{Q}_\kappa$ and fix ε such that $\alpha_\varepsilon > \ell g(\operatorname{tr}(p))$. By induction on $i \in [\varepsilon, \kappa)$ choose $\nu_i \in {}^{\alpha_i}2 \cap p$ so that

Sh:1004

- if $\varepsilon \le j < i < \kappa$ then $\nu_j{}^\frown\langle 1\rangle \trianglelefteq \nu_i$.

(It is clearly possible; at successor stages remember 1.5(1) and at limit stages remember the choice of E.) Then $\eta := \bigcup\{\nu_i : \varepsilon \le i < \kappa\} \in \lim_\kappa(p)$ does not belong to $\mathrm{set}(\mathscr{I}_\varepsilon)$.
(2) Remember Claim 3.6(2).
(3, 4) Follow from part (1). □

Question 4.5 For which inaccessible cardinals κ do we have $\Pr(\kappa)$? See [28].

Claim 4.6 *The following are equivalent for κ:*

(a) $\neg\Pr(\kappa)$.
(b) If Λ is a set of $\le \kappa$ maximal antichains of $\mathbb{Q}_\kappa$ and $\alpha < \kappa$, <u>then</u> there is $p \in \mathbb{Q}_\kappa$ such that $\mathrm{tr}(p) = \langle\rangle$, $S_p \cap \alpha = \emptyset$ *and* $\lim_\kappa(p) \subseteq \mathrm{set}(\Lambda)$.

Proof (b) ⇒ (a) Straightforward by Definition 4.3(1).
(a) ⇒ (b) Suppose that $\Pr(\kappa)$ does not hold.

Assume Λ is a set of $\le \kappa$ maximal antichains of $\mathbb{Q}_\kappa$. Let $\Lambda_1 = \Lambda^{[<\kappa]}$ (see 4.1). Then $\Lambda_1 = (\Lambda_1)^{[<\kappa]}$ and $|\Lambda_1| \le \kappa$ (remember 4.2). Since $\Pr(\kappa)$ fails, there is a condition $q \in \mathbb{Q}_\kappa$ such that $\lim(q) \subseteq \mathrm{set}(\Lambda_1)$ and $\ell g(\mathrm{tr}(q)) > \alpha$, $S_q \cap \alpha = \emptyset$.

Let $S_p = S_q$ and for $\partial \in S_p$ let $\Lambda_\partial = \Lambda_{q,\partial}$. Put $\bar\Lambda = \langle\Lambda_\partial : \partial \in S_p\rangle$ and let p be the condition determined by $(\langle\rangle, S_p, \bar\Lambda)$.

Note that if $\eta \in q \cap {}^\beta 2$, $\beta < \kappa$, then for every $\nu \in {}^\beta 2$ also $q^{[\eta,\nu]}$ satisfies $\lim(q^{[\eta,\nu]}) \subseteq \mathrm{set}(\Lambda_1)$ by the choice of Λ_1. Therefore we also get $\lim(p) \subseteq \mathrm{set}(\Lambda_1) \subseteq \mathrm{set}(\Lambda)$, so p is as required. □

Claim 4.7 *Suppose that $p \in \mathbb{Q}_\kappa$, $\ell g(\mathrm{tr}(p)) < \alpha_* \le \beta_* \le \kappa$. <u>Then</u> there is $q \in \mathbb{Q}_\kappa$ such that*

(a) $p \le q$, $\mathrm{tr}(q) = \mathrm{tr}(p)$ *and*
(b) $S_q \setminus (\alpha_*, \beta_*) = S_p \setminus (\alpha_*, \beta_*)$ *and* $\gamma \in S_q \setminus (\alpha_*, \beta_*) \Rightarrow \Lambda_{q,\gamma} = \Lambda_{p,\gamma}$,
(c) $S_q \cap (\alpha_*, \beta_*) \subseteq S^\kappa_{\mathrm{pr}}$.

Proof We prove this by induction on β_*.

Case 0 $\alpha_* = \beta_*$ *or* $\alpha_* + 1 = \beta_*$
Trivial, as then $(\alpha_*, \beta_*) = \emptyset$.

Case 1 $\beta_* = \sup(\beta_* \cap S_p) + 1$ *but* $\sup(\beta_* \cap S_p) \notin S_p \setminus S^\kappa_{\mathrm{pr}}$.
Let $\gamma_* = \sup(\beta_* \cap S_p)$. *Use the inductive hypothesis for p and (α_*, γ_*) to get a condition q. It will satisfy the demands for (α_*, β_*) as well as either $\gamma_* \notin S_p$ or else $\gamma_* \in S^\kappa_{\mathrm{pr}}$.*

Case 2 $\beta_* > \sup(\beta_* \cap S_p) + 1$
Use the inductive hypothesis for $\gamma_* = \sup(\beta_* \cap S_p) + 1$, *proceeding like in Case 1.*

Case 3 $\beta_* = \sup(\beta_* \cap S_p)$, *so β_* is limit*
Pick an increasing continuous sequence $\bar\alpha = \langle\alpha_i : i \le \mathrm{cf}(\beta_*)\rangle$ *such that* $\alpha_0 = \alpha_*$, $\alpha_{\mathrm{cf}(\beta_*)} = \beta_*$ *and* $\alpha_i \notin S_p$ *for all* $0 < i < \mathrm{cf}(\beta_*)$. *By induction on* $i \le \mathrm{cf}(\beta_*)$ *choose* q_i *such that*

(a) $q_0 = p$, $\mathrm{tr}(q_i) = \mathrm{tr}(p)$,

(b) $S_{q_i} \setminus (\alpha_0, \alpha_i) = S_p \setminus (\alpha_0, \alpha_i)$ *and* $\gamma \in S_{q_i} \setminus (\alpha_0, \alpha_i) \Rightarrow \Lambda_{q_i,\gamma} = \Lambda_{p,\gamma}$,

(c) *if* $j < i$, *then* $q_j \le q_i$, $S_{q_i} \setminus (\alpha_j, \alpha_i) = S_{q_j} \setminus (\alpha_j, \alpha_i)$ *and* $\gamma \in S_{q_i} \setminus (\alpha_j, \alpha_i) \Rightarrow \Lambda_{q_i,\gamma} = \Lambda_{q_j,\gamma}$,

(d) *if* $i = j + 1$, *then* $S_{q_i} \cap (\alpha_j, \alpha_i) \subseteq S^\kappa_{\rm pr}$.

There are no problems in carrying out the inductive construction. Then $q_{{\rm cf}(\beta_*)}$ *is as required.*

Case 4 $\beta_* = \partial + 1$, $\partial \in S_p \setminus S^\kappa_{\rm pr}$ *and* $\partial > \alpha_*$
Here we use 4.6 for $\mathbb{Q}_\partial$, $\Lambda_{p,\partial}$ *and the ordinal* α_*. *So there is* $p_* \in \mathbb{Q}_\partial$ *such that*

- ${\rm tr}(p_*) = \langle\rangle$,
- $S_{p_*} \subseteq (\alpha_*, \partial)$, *and*
- $\lim(p_*) \subseteq {\rm set}(\Lambda_{p,\partial})$.

Now we define a condition q_1 *by letting:*

- ${\rm tr}(q_1) = {\rm tr}(p)$,
- $S_{q_1} = (S_p \setminus \{\partial\}) \cup S_{p_*}$,
- $\Lambda_{q_1,\theta}$ *is*
  - $\Lambda_{p,\theta}$ *if* $\theta \in S_p \setminus S_{p_*}$,
  - $\Lambda_{p_*,\theta}$ *if* $\theta \in S_{p_*} \setminus S_p$,
  - $\Lambda_{p,\theta} \cup \Lambda_{p_*,\theta}$ *if* $\theta \in S_{p_*} \cap S_p$.

Then we continue as in Case 1 with q_1, α_*, β_* *(as* $\partial \notin S_{q_1}$*).* □

Conclusion 4.8 *For any* $\alpha < \kappa$, *the set* $\{p \in \mathbb{Q}_\kappa : S_p \subseteq S^\kappa_{\rm pr} \setminus \alpha\}$ *is dense in* $\mathbb{Q}_\kappa$.

Note that if $\kappa > \sup(S^\kappa_{\rm inac})$, then ${\rm id}(\mathbb{Q}_\kappa) = {\rm id}({\rm Cohen}_\kappa)$. Therefore:

Hypothesis 4.9 For the rest of this section we assume that $\kappa = \sup(S^\kappa_{\rm inac})$ (so also $\kappa = \sup(S^\kappa_{\rm pr})$, remember 4.4(3)).

Definition 4.10 (1) Let ${\rm add}(\mathbf{nst}^{\rm pr}_\kappa)$ be the minimal cardinal μ such that there are $S_\zeta \in \mathbf{nst}^{\rm pr}_\kappa$ for $\zeta < \mu$ with the property that there is no $S \in \mathbf{nst}^{\rm pr}_\kappa$ satisfying

$$\zeta < \mu \Rightarrow S_\zeta \subseteq S \mod \text{bounded}.$$

Dually, ${\rm cf}(\mathbf{nst}^{\rm pr}_\kappa)$ is the minimal cardinal μ such that there are $S_\zeta \in \mathbf{nst}^{\rm pr}_\kappa$ for $\zeta < \mu$ with the property that for every $S \in \mathbf{nst}^{\rm pr}_\kappa$ there is $\zeta < \mu$ satisfying $S \subseteq S_\zeta$ mod bounded.

(2) For $S \subseteq S^\kappa_{\rm inac}$ we define:

(a) $\mathbb{Q}^*_{\kappa,S}$ is the subforcing of $\mathbb{Q}_\kappa$ consisting of all conditions $p \in \mathbb{Q}_\kappa$ satisfying $S_p \subseteq S$.

(b) ${\rm id}[\mathbb{Q}^*_{\kappa,S}]$ is the collection of all $A \subseteq {}^\kappa 2$ such that for some $\bar{\mathscr{J}} = \langle \mathscr{J}_\zeta : \zeta < \kappa\rangle$ we have

(i) each $\mathscr{J}_\zeta$ is predense subset (or maximal antichain) of $\mathbb{Q}_\kappa$,

(ii) $\mathscr{J}_\zeta \subseteq \mathbb{Q}^*_{\kappa,S}$ for each $\zeta < \kappa$, and

(iii) $A \subseteq \bigcup_{\zeta<\kappa} \left({}^\kappa 2 \setminus {\rm set}(\mathscr{J}_\zeta)\right)$,

(c) ${\rm add}({\rm id}[\mathbb{Q}^*_{\kappa,S}], {\rm id}(\mathbb{Q}_\kappa)) = \min\{|\mathscr{A}| : \mathscr{A} \subseteq {\rm id}[\mathbb{Q}^*_{\kappa,S}] \wedge \bigcup \mathscr{A} \notin {\rm id}(\mathbb{Q}_\kappa)\}$.

(3) $\mathrm{Add}^*_{\mathrm{pr},\kappa} = \min\left\{\mathrm{add}(\mathrm{id}[\mathbb{Q}^*_{\kappa,S}], \mathrm{id}(\mathbb{Q}_\kappa)) : S \in \mathbf{nst}^{\mathrm{pr}}_\kappa\right\}$.

Claim 4.11 *(1)* $\mathrm{add}(\mathbb{Q}_\kappa) = \min\left\{\mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa), \mathrm{Add}^*_{\mathrm{pr},\kappa}\right\}$.
(2) $\mathrm{cf}(\mathbb{Q}_\kappa) \geq \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$.

Proof (1) (Step 1) $\underline{\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)}$.
Let $S_\zeta \in \mathbf{nst}^{\mathrm{pr}}_\kappa$ for $\zeta < \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$ be such that

$$S \in \mathbf{nst}^{\mathrm{pr}}_\kappa \Rightarrow \bigvee_\zeta \kappa = \sup(S_\zeta \backslash S).$$

For $\partial \in S^\kappa_{\mathrm{pr}}$ let $\Lambda^*_\partial = \{\mathscr{I}^\partial_\varepsilon : \varepsilon < \partial\}$ witness $\partial \in S^\kappa_{\mathrm{pr}}$ (see Definition 4.3(1)). For $\zeta < \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$ let[4]

$$\mathbf{B}_\zeta = {}^\kappa 2 \backslash \left\{\eta \in {}^\kappa 2 : \left(\forall^\infty \partial \in S_\zeta\right)\left(\eta{\restriction}\partial \in \mathrm{set}(\Lambda^*_\partial)\right)\right\}.$$

Clearly $\mathbf{B}_\zeta \in \mathrm{id}(\mathbb{Q}_\kappa)$. Now it suffices to prove that $\mathbf{B} := \bigcup\left\{\mathbf{B}_\zeta : \zeta < \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)\right\} \notin \mathrm{id}(\mathbb{Q}_\kappa)$. So suppose towards a contradiction that $\mathbf{B} \in \mathrm{id}(\mathbb{Q}_\kappa)$ and let $(S, \bar{\Lambda}, \bar{p}, \bar{\mathscr{I}})$ be given by Claim 3.16 for $\mathbf{B}$. Next,

$(*)_1$ if $\varepsilon < \kappa, \alpha < \kappa$ and $\eta \in {}^\alpha 2$, $\underline{\text{then}}$ there are β, ν, ρ such that
  (a) $\alpha < \beta < \kappa$,
  (b) $\eta \triangleleft \nu \in {}^\beta 2$,
  (c) $p_\rho \in \mathscr{I}_\varepsilon$ and $\rho \trianglelefteq \nu$ and $\nu \in p_\rho$,
  (d) if $\partial \in S \cap (\alpha, \beta]$ then $\nu{\restriction}\partial \in \mathrm{set}(\Lambda_\partial)$.

[Why? Consider the triple $(\eta, S\backslash(\alpha+1), \langle\Lambda_\partial : \partial \in S\backslash(\alpha+1)\rangle)$. It defines the condition $p_\eta \in \mathbb{Q}_\kappa$ and we know that $\mathscr{I}_\varepsilon$ is a predense subset of $\mathbb{Q}_\kappa$. Hence for some $\rho \in {}^{\kappa>}2$, $p_\rho \in \mathscr{I}_\varepsilon$ and the conditions p_ρ, p_η are compatible in $\mathbb{Q}_\kappa$. Then there is $\nu \in {}^{\kappa>}2$ such that $\mathrm{tr}(p_\rho) \triangleleft \nu \in p_\rho, \mathrm{tr}(p_\eta) \triangleleft \nu \in p_\eta$. By the definition of p_η above, $\ell g(\nu), \nu, \rho$ satisfy all the requirements.]

Now,

$(*)_2$ For $\varepsilon < \kappa$ let $F^\varepsilon_1, F^\varepsilon_2 : {}^{\kappa>}2 \longrightarrow {}^{\kappa>}2$ be such that for each $\eta \in {}^{\kappa>}2$, the triple (β, ν, ρ) given by $\beta = \ell g(F^\varepsilon_1(\eta)), \nu = F^\varepsilon_1(\eta)$ and $\rho = F^\varepsilon_2(\eta)$, is as required above in $(*)_1$ for ε and η.
$(*)_3$ Let $E_1 = \{\delta < \kappa : \delta$ a limit ordinal and $(\varepsilon < \delta \wedge \eta \in {}^{\delta>}2) \Rightarrow F^\varepsilon_1(\eta) \in {}^{\delta>}2\}$.

By the choice of $\langle S_\zeta : \zeta < \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)\rangle$ there is $\zeta < \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$ such that $S_\zeta \backslash S$ is unbounded in κ. Easily we may choose an unbounded set $S' \subseteq S_\zeta \backslash S$ such that

- the closure E of S' is disjoint from S, and
- if $\gamma_0 \in E, \gamma_1 = \min(E\backslash(\gamma_0+1))$, then $(\gamma_0, \gamma_1) \cap E_1 \neq \emptyset$.

Let $\langle\gamma_i : i < \kappa\rangle$ list $E \cup \{0\}$ in the increasing order (so $\gamma_{i+1} \in S_\zeta \backslash S$ and $\gamma_i \notin S$; remember $\gamma_0 = 0 \notin S \subseteq S^\kappa_{\mathrm{inac}}$). By induction on $i < \kappa$ we choose $\eta_i \in {}^{\gamma_i}2$ such that

4 Recall that "$\forall^\infty \partial \in S$" means "for all but boundedly many $\partial \in S$".

Sh:1004

(a) $j < i < \kappa \Rightarrow \eta_j \triangleleft \eta_i$,
(b) if $i = j + 1$ then $\eta_i \notin \mathrm{set}(\Lambda^*_{\gamma_i})$ and $F_1^j(\eta_j) \trianglelefteq \eta_i$,
(c) if $\partial \in S \cap (\gamma_i + 1)$, $i > 0$, then $\eta_i \restriction \partial \in \mathrm{set}(\Lambda_\partial)$,
(d) if $j < i$ then $F_2^j(\eta_j) \trianglelefteq \eta_i \in p_{F_2^j(\eta_j)}$ (follows from (b)+(c) and $(*)_2$).

If we succeed in carrying out the induction, then we may let $\eta = \bigcup_{i<\kappa} \eta_i$ and note that

- η belongs to $\mathbf{B}_\zeta$ because $\eta \restriction \gamma_i \notin \mathrm{set}(\Lambda^*_{\gamma_i})$ for all successor $i < \kappa$ by clause (b),
- η does not belong to $\mathbf{B}$ by clauses (c)+(d).

Consequently, η witnesses $\mathbf{B}_\zeta \nsubseteq \mathbf{B}$, a contradiction.
Why can we carry out the induction?
For $i = 0$ it is trivial.
For a limit $i < \kappa$ we let $\eta_i = \bigcup_{j<i} \eta_j$.
Let $i = j + 1$. First, $F_1^j(\eta_j)$ satisfies the requirements on η_i except that $\ell g(F_1^j(\eta_j))$ is not γ_i (and so "$\eta_i \notin \mathrm{set}(\Lambda^*_{\gamma_i})$" from (b) is meaningless): it is $< \gamma_i$ by the choices of E_1 and E.

Second, we use the definition of $S_\zeta \subseteq S^\kappa_{\mathrm{pr}}$ and $\gamma_{j+1} \in S_\zeta \backslash S$ for the condition with trunk $F_1^j(\eta_j)$ and $\langle \Lambda_\partial : \partial \in (\gamma_j, \gamma_{j+1}) \cap S \rangle$ and the choice of $\Lambda^*_{\gamma_i}$.

This completes the proof of "$\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$".
(Step 2) $\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathrm{Add}^*_{\mathrm{pr},\kappa}$.
It should be obvious that if $S \subseteq S^\kappa_{\mathrm{pr}}$ then $\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathrm{add}(\mathrm{id}[\mathbb{Q}^*_{\kappa,S}], \mathrm{id}(\mathbb{Q}_\kappa))$.
(Step 3) $\min \left\{\mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa), \mathrm{Add}^*_{\mathrm{pr},\kappa}\right\} \leq \mathrm{add}(\mathbb{Q}_\kappa)$.
Why? Assume $A_i \in \mathrm{id}(\mathbb{Q}_\kappa)$ for $i < i_* < \min \left\{\mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa), \mathrm{Add}^*_{\mathrm{pr},\kappa}\right\}$. For each i let $(S_i, \bar{\Lambda}_i, \bar{\mathscr{I}}_i, \bar{p}_i)$ be given by Claim 3.16 for A_i. By Conclusion 4.8 (and the proof of 3.16) we may also require that $S_i \in \mathbf{nst}^{\mathrm{pr}}_\kappa$ for all $i < i_*$. As $i_* < \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$ there is $S \in \mathbf{nst}^{\mathrm{pr}}_\kappa$ such that

$$i < i_* \Rightarrow S_i \subseteq S \mod J^{\mathrm{bd}}_\kappa.$$

Then easily $A_i \in \mathrm{id}[\mathbb{Q}^*_{\kappa,S}]$ for every $i < i_*$. Since $i_* < \mathrm{Add}^*_{\mathrm{pr},\kappa}$ we also have $i_* < \mathrm{add}(\mathrm{id}[\mathbb{Q}^*_{\kappa,S}], \mathrm{id}(\mathbb{Q}_\kappa))$ and hence $\bigcup_{i<i^*} A_i \in \mathrm{id}(\mathbb{Q}_\kappa)$ and we are done.
(2) In order to show $\mathrm{cf}(\mathbb{Q}_\kappa) \geq \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$ let us assume towards contradiction that $\mu := \mathrm{cf}(\mathbb{Q}_\kappa) < \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$. Let $\langle \mathbf{B}_\zeta : \zeta < \mu \rangle$ witness $\mu = \mathrm{cf}(\mathbb{Q}_\kappa)$ and let $S_\zeta, \bar{\Lambda}_\zeta$, $\bar{p}_\zeta = \langle p_{\zeta,\rho} : \rho \in {}^{\kappa >}2 \rangle$ and $\bar{\mathscr{I}}_\zeta = \{\mathscr{I}_{\zeta,i} : i < \kappa\}$ be given by 3.16 for $\mathbf{B}_\zeta$. Let $S \in \mathbf{nst}^{\mathrm{pr}}_\kappa$ be such that

$$\zeta < \mu \Rightarrow \kappa = \sup(S \backslash S_\zeta).$$

For each $\partial \in S$ let $\Lambda^*_\partial = \{\mathscr{I}^\partial_\varepsilon : \varepsilon < \partial\}$ witness $\partial \in S^\kappa_{\mathrm{pr}}$ (see Definition 4.3(1)) and let

$$\mathbf{B} := \{\eta \in {}^\kappa 2 : (\exists^\infty \partial \in S)(\exists \varepsilon < \partial)(\eta \restriction \partial \notin \mathrm{set}(\mathscr{I}^\partial_\varepsilon))\}.$$

Clearly $\mathbf{B} \in \mathrm{id}(\mathbb{Q}_\kappa)$, so for some $\zeta < \mu$ we have $\mathbf{B} \subseteq \mathbf{B}_\zeta$. Let $E \subseteq \kappa \setminus S_\zeta$ be a club and let $p \in \mathbb{Q}_\kappa$ be a condition determined by $(\langle\rangle, S_\zeta, \bar{\Lambda}_\zeta)$. By induction on $i < \kappa$ we choose $\alpha_i \in E$ and $\eta_i \in {}^{\alpha_i}2 \cap p$ so that

(i) $\langle \alpha_i : i < \kappa \rangle \subseteq E$ is increasing continuous,
(ii) $\langle \eta_i : i < \kappa \rangle$ is $\triangleleft$-increasing continuous,
(iii) for each $i < \kappa$, for some $\rho \in {}^{\kappa>}2$ we have $\rho \triangleleft \eta_{i+1}$ and $p_{\zeta,\rho} \in \mathscr{I}_{\zeta,i}$,
(iv) for each $i < \kappa$ there is $\partial \in (\alpha_i, \alpha_{i+1}) \cap S$ such that $\eta_{i+1} \restriction \partial \notin \bigcap_{\varepsilon<\partial} \mathrm{set}(\mathscr{I}^\partial_\varepsilon)$.

It should be clear how to carry out the construction. At the end, the sequence $\eta := \bigcup_{i<\kappa} \eta_i \in {}^\kappa 2$ belongs to $\mathbf{B}$ (by (iv)) but it does not belong to $\mathbf{B}_\zeta$ (by (iii)), contradicting the choice of $\zeta < \mu$. □

Claim 4.12 *If κ is Mahlo and there is a non-reflecting stationary set $S \subseteq S^\kappa_{\mathrm{pr}}$, then*

(1) $\mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa) \leq \mathfrak{b}_\kappa$,
(2) above we actually have $\mathrm{add}(\mathbf{nst}_{\kappa,S}) = \mathfrak{b}_\kappa$,
1. $\mathfrak{d}_\kappa \leq \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$.

Proof Straightforward, as for $S' \subseteq S$ we have:
$S' \in \mathbf{nst}^{\mathrm{pr}}_\kappa$ if and only if S' is non-stationary. □

5 The parallel of the Cichoń diagram

As before, $\lambda, \partial, \kappa$ vary on inaccessibles.
We have a characterization of κ-meagre sets similar to the one for the case of $\kappa = \aleph_0$. (Note: here κ inaccessible is used.)

Observation 5.1 *(1) If $X \subseteq {}^\kappa 2$ is κ-meagre and $A \subseteq \kappa$ is unbounded then there is an increasing sequence $\bar{\alpha}$ of members of A of length κ and $\eta \in {}^\kappa 2$ such that*

$$X \subseteq X_{\eta,\bar{\alpha}} := \{\nu \in {}^\kappa 2 : \textit{for every } i < \kappa \textit{ large enough, } \eta \restriction [\alpha_i, \alpha_{i+1}) \nsubseteq \nu\}.$$

Moreover, if A contains a club of κ then the sequence $\bar{\alpha}$ above can be increasing continuous.
(2) If $\eta \in {}^\kappa 2$ and $\bar{\alpha}$ is an increasing sequence of ordinals $< \kappa$ of length κ then the set $X_{\eta,\bar{\alpha}}$ defined above is a κ-meagre subset of ${}^\kappa 2$.

Proof (1) Let $X \subseteq \bigcup\{\lim_\kappa(\mathscr{T}_i) : i < \kappa\}$ where $\mathscr{T}_i$ is a nowhere dense subtree of ${}^{\kappa>}2$. For every infinite $\alpha \in A$ let $\langle (\eta_{\alpha,\varepsilon}, i_{\alpha,\varepsilon}) : \varepsilon < 2^{|\alpha|} \rangle$ list ${}^\alpha 2 \times \alpha$, and then we choose $\nu_{\alpha,\varepsilon}, \beta_{\alpha,\varepsilon}$ by induction on $\varepsilon \leq 2^{|\alpha|}$ such that:

(a) $\beta_{\alpha,\varepsilon} = \beta(\alpha, \varepsilon) < \kappa$ is increasing continuous with ε,
(b) $\nu_{\alpha,\varepsilon} \in {}^{\beta(\alpha,\varepsilon)}2$,
(c) $\zeta < \varepsilon \Rightarrow \nu_{\alpha,\zeta} \trianglelefteq \nu_{\alpha,\varepsilon}$,
(d) $\eta_{\alpha,\varepsilon} {}^\frown \nu_{\alpha,\varepsilon+1} \notin \mathscr{T}_{i_{\alpha,\varepsilon}}$.

Why we can? For $\varepsilon = 0$, let $\nu_{\alpha,\varepsilon} = \langle\rangle$, for limit ε let $\nu_{\alpha,\varepsilon} = \bigcup\{\nu_{\alpha,\zeta} : \zeta < \varepsilon\}$ recalling (by 2.2) that $\mathrm{cf}(\kappa) = \kappa > 2^{|\alpha|} \geq \varepsilon$ and for $\varepsilon = \zeta + 1$ use "$\mathscr{T}_{i_{\alpha,\varepsilon}}$ is nowhere dense subtree of ${}^{\kappa>}2$".

Now by induction on $i < \kappa$ we choose (α_i, ν_i) such that:

Sh:1004

(e) $\alpha_i \in A$ is infinite increasing with i, α_i minimal under these restrictions,
(f) $\nu_i \in {}^{\alpha_i}2$ is $\triangleleft$-increasing,
(g) if $i = j+1$ and $\gamma = 2^{|\alpha_j|}$ then $\alpha_i = \min\{\alpha \in A : \alpha > \alpha_j + \ell g(\nu_{\alpha_j,\gamma})\}$ and ν_i is a member of ${}^{\alpha_i}2$ such that $\nu_j{}^\frown\nu_{\alpha_j,\gamma} \triangleleft \nu_i$.

There is no problem to carry out the induction and $\langle \alpha_i : i < \kappa\rangle$, $\eta := \bigcup\{\nu_i : i < \kappa\}$ are as required.
(2) Should be clear. $\square$

Remark 5.2 The ideal id(Cohen$_\kappa$) is an ideal of subsets of ${}^\kappa 2$. It has a natural relative on ${}^\kappa\kappa$—the ideal of meagre subsets of ${}^\kappa\kappa$. The two ideals are isomorphic in a suitable sense and they have the same cardinal coefficients, cf [13, Section 4].

Claim 5.3 *(1)* $\mathrm{add}(\mathrm{Cohen}_\kappa) \le \mathfrak{b}_\kappa \le \mathrm{non}(\mathrm{Cohen}_\kappa)$.
(2) $\mathrm{cov}(\mathrm{Cohen}_\kappa) \le \mathfrak{d}_\kappa \le \mathrm{cf}(\mathrm{Cohen}_\kappa)$.
(3) $\mathrm{cf}(\mathrm{Cohen}_\kappa) = \max\{\mathfrak{d}_\kappa, \mathrm{non}(\mathrm{Cohen}_\kappa)\}$.
(4) $\mathrm{add}(\mathrm{Cohen}_\kappa) = \min\{\mathfrak{b}_\kappa, \mathrm{cov}(\mathrm{Cohen}_\kappa)\}$.

Proof Our arguments are similar to those for $\kappa = \aleph_0$.
(1) We will show that $\mathrm{add}(\mathrm{Cohen}_\kappa) \le \mathfrak{b}_\kappa$ (the inequality $\mathfrak{b}_\kappa \le \mathrm{non}(\mathrm{Cohen}_\kappa)$ should be clear; remember 5.2). Let $\mu = \mathfrak{b}_\kappa$ and let $\{g_\alpha : \alpha < \mu\} \subseteq {}^\kappa\kappa$ exemplify this. For each $\alpha < \mu$ let

$$E_\alpha = \big\{\delta < \kappa : \delta \text{ is a limit ordinal and } (\forall i < \delta)(g_\alpha(i) < \delta)\big\}.$$

Let $\bar\beta_\alpha = \langle \beta_{\alpha,i} : i < \kappa\rangle$ list E_α in the increasing order and let $\eta_\iota \in {}^\kappa 2$ be constantly ι for $\iota = 0, 1$. Then $\{X_{\eta_\iota,\bar\beta_\alpha} : \iota < 2 \text{ and } \alpha < \mu\}$ is a collection of μ many κ-meagre sets. Assume towards contradiction that their union $A = \bigcup\{X_{\eta_\iota,\bar\beta_\alpha} : \iota < 2 \text{ and } \alpha < \mu\}$ is meagre. Hence, by 5.1, there are $\eta \in {}^\kappa 2$ and an increasing continuous $\bar\beta \in {}^\kappa\kappa$ such that $A \subseteq X_{\eta,\bar\beta}$. Let $g \in {}^\kappa\kappa$ be defined by $g(j) = \beta_{j+1}$. Then for some $\alpha < \mu$ we have $\neg(g_\alpha \le_{J^{\mathrm{bd}}_\kappa} g)$. If $\beta_j < \beta_{\alpha,i} \le \beta_{j+1}$, then $j \le \beta_j < \beta_{\alpha,i}$ and hence $g_\alpha(j) < \beta_{\alpha,i} \le \beta_{j+1} = g(j)$, so the set

$$S = \{j < \kappa : (\beta_j, \beta_{j+1}] \cap \{\beta_{\alpha,i} : i < \kappa\} = \emptyset\}$$

is of size κ. Choose a subset $S_0 \subseteq S$ of size κ such that $j \in S_0 \Rightarrow j+1 \notin S_0$. Let $\nu \in {}^\kappa 2$ be such that $\nu{\restriction}[\beta_j, \beta_{j+1}) = \eta{\restriction}[\beta_j, \beta_{j+1})$ for $j \in S_0$ and $\nu(i) = 1$ whenever $i \notin \bigcup\{[\beta_j, \beta_{j+1}) : j \in S_0\}$. Then $\nu \in X_{\eta_0,\bar\beta_\alpha} \setminus X_{\eta,\bar\beta}$, contradicting $A \subseteq X_{\eta,\bar\beta}$.
(2) We will show that $\mathfrak{d}_\kappa \le \mathrm{cf}(\mathrm{Cohen}_\kappa)$. So let $\mu = \mathrm{cf}(\mathrm{Cohen}_\kappa)$ and let $\langle A_\alpha : \alpha < \mu\rangle$ list a cofinal subset of id(Cohen$_\kappa$). For each $\alpha < \mu$ we can find $(\nu_\alpha, \bar\beta_\alpha)$ as in 5.1 such that $A_\alpha \subseteq X_{\nu_\alpha,\bar\beta_\alpha}$. Let

$$E_\alpha = \big\{\delta < \kappa : \delta \text{ is a limit ordinal such that } (\forall i)\ (\beta_{\alpha,i} < \delta \Leftrightarrow i < \delta)\big\},$$

it is a club of κ. Towards contradiction assume $\mathfrak{d}_\kappa > \mu$. Then there is a club E of κ such that $\sup(E_\alpha \backslash E) = \kappa$ for all $\alpha < \mu$. Let $\nu \in {}^\kappa 2$ and the sequence $\bar\beta$ list E

in increasing order and consider the κ-meagre set $X_{\nu,\bar\beta}$. For some $\alpha < \mu$ we have $X_{\nu,\bar\beta} \subseteq A_\alpha \subseteq X_{\nu_\alpha,\bar\beta_\alpha}$. Easy contradiction to $\kappa = \sup(E_\alpha \setminus E)$.

The inequality $\mathrm{cov}(\mathrm{Cohen}_\kappa) \le \mathfrak{d}_\kappa$ should be clear (remember 5.2).

(3) Recall that $\mathrm{non}(\mathrm{Cohen}_\kappa) \le \mathrm{cf}(\mathrm{Cohen}_\kappa)$ by 0.9 and $\mathfrak{d}_\kappa \le \mathrm{cf}(\mathrm{Cohen}_\kappa)$ is proved in (2) above. So we are left with:

$$\mathrm{cf}(\mathrm{Cohen}_\kappa) \le \mathfrak{d}_\kappa + \mathrm{non}(\mathrm{Cohen}_\kappa).$$

Let $\mu = \mathrm{non}(\mathrm{Cohen}_\kappa)$; now

($\boxplus$) there is $\{\varrho_\beta : \beta < \mu\} \subseteq {}^\kappa\kappa$ such that for every $\nu \in {}^\kappa\kappa$ for some $\beta < \mu$ we have $\sup\{i < \kappa : \varrho_\beta(i) = \nu(i)\} = \kappa$.

[Why? For $\rho \in {}^\kappa 2$ let $\nu_\rho \in {}^\kappa\kappa$ be such that for $i < \kappa$, $\nu_\rho(i)$ is $\gamma_{\rho,i}$ when $\gamma_{\rho,i} < \kappa$ is the minimal $\gamma < \kappa$ such that, if possible, $\rho(i+\gamma) = 1$ (and if there is no such γ then it is 0). Let $\eta_0 \in {}^\kappa\kappa$ be constantly 0. Now if $\Lambda \subseteq {}^\kappa 2$ is non-meagre of cardinality μ then recalling 5.1 the set $\{\nu_\rho : \rho \in \Lambda\} \cup \{\eta_0\} \subseteq {}^\kappa\kappa$ is as required.]

Let $\langle E_\gamma : \gamma < \mathfrak{d}_\kappa\rangle$ be a sequence of clubs of κ such that for any club E of κ, for some γ, $E_\gamma \subseteq E$, this is a variant of the definition of $\mathfrak{d}_\kappa$. For $\gamma < \mathfrak{d}_\kappa$ let $\bar\alpha_\gamma = \langle \alpha_{\gamma,i} : i < \kappa\rangle$ list $E_\gamma \cup \{0\}$ in increasing order.

Let $\langle \rho_j : j < \kappa\rangle$ list $\bigcup\{{}^{[i,j)}2 : i < j < \kappa\}$ and for $(\beta,\gamma,\xi) \in \mu \times \mathfrak{d}_\kappa \times \mathfrak{d}_\kappa$ let $A_{\beta,\gamma,\xi} = X_{\varrho_{\beta,\gamma},\bar\alpha_\xi}$ from 5.1 where:

($\circledcirc$) for $\beta < \mu$ and $\gamma < \mathfrak{d}_\kappa$ let $\varrho_{\beta,\gamma} \in {}^\kappa 2$ be such that $\varrho_{\beta,\gamma} \restriction [\alpha_{\gamma,i}, \alpha_{\gamma,i+1})$ is equal to $\rho_{\varrho_\beta(i)}$ if $\rho_{\varrho_\beta(i)} \in {}^{[\alpha_{\gamma,i},\alpha_{\gamma,i+1})}2$ and is constantly zero otherwise.

So $\mathscr{A} = \{A_{\beta,\gamma_1,\gamma_2} : \beta < \mu,\ \gamma_1 < \mathfrak{d}_\kappa,\ \gamma_2 < \mathfrak{d}_\kappa\}$ is a subset of $\mathrm{id}(\mathrm{Cohen}_\kappa)$ and has cardinality $\le \mu + \mathfrak{d} + \mathfrak{d} = \max\{\mu, \mathfrak{d}\}$. Hence it suffices to prove that $\mathscr{A}$ is cofinal in $\mathrm{id}(\mathrm{Cohen}_\kappa)$. To this end let $A \in \mathrm{id}(\mathrm{Cohen}_\kappa)$, and let $\eta \in {}^\kappa 2$ and increasing $\bar\alpha \in {}^\kappa\kappa$ be such that $A \subseteq X_{\eta,\bar\alpha}$ (remember 5.1).

Now, $E := \{\alpha < \kappa : \alpha$ is limit and $(\forall i < \alpha)(\alpha_i < \alpha)\}$ is a club of κ, hence there is $\gamma(1) < \mathfrak{d}_\kappa$ such that $E \supseteq E_{\gamma(1)}$. Then $A \subseteq X_{\eta,\bar\alpha} \subseteq X_{\eta,\bar\alpha_{\gamma(1)}}$. Let $\varrho \in {}^\kappa\kappa$ be such that $i < \kappa \Rightarrow \eta\restriction[\alpha_{\gamma(1),i}, \alpha_{\gamma(1),i+1}) = \rho_{\varrho(i)}$ and let $\beta < \mu$ be such that $B = \{i < \kappa : \varrho(i) = \varrho_\beta(i)\}$ is an unbounded subset of κ. Pick $\gamma(2) < \mathfrak{d}$ such that

$$E_{\gamma(2)} \subseteq \left\{\alpha \in E_{\gamma(1)} : \alpha \text{ is limit and } (\forall i < \alpha)(\alpha_{\gamma(1),i} < \alpha)\right\}$$

and $[\alpha_{\gamma(2),i}, \alpha_{\gamma(2),i+1}) \cap B \ne \emptyset$ for every i. Now clearly it suffices to prove:

($*$) $A \subseteq A_{\beta,\gamma(1),\gamma(2)}$.

Why does ($*$) hold? Fix $\nu \in A$ and we shall prove that $\nu \in A_{\beta,\gamma(1),\gamma(2)}$. By the choice of $(\eta,\bar\alpha)$ we know $\nu \in X_{\eta,\bar\alpha}$, so for $i < \kappa$ large enough $\nu\restriction[\alpha_i,\alpha_{i+1}) \nsubseteq \eta$. Let $i^* < \kappa$ be such that $\nu\restriction[\alpha_i,\alpha_{i+1}) \nsubseteq \eta$ for all $i \ge i^*$.

Let $i \in [i^*,\kappa)$. By the choice of $\gamma(2)$ we can fix $i_1 \in B$ such that $\alpha_{\gamma(2),i} \le i_1 < \alpha_{\gamma(2),i+1}$. Then, by the definition of B, we have $\varrho(i_1) = \varrho_\beta(i_1)$ and by the choice of ϱ we have $\rho_{\varrho(i_1)} = \rho_{\varrho_\beta(i_1)} = \eta\restriction[\alpha_{\gamma(1),i_1}, \alpha_{\gamma(1),i_1+1}) \in {}^{[\alpha_{\gamma(1),i_1},\alpha_{\gamma(1),i_1+1})}2$. By the choice of $\varrho_{\beta,\gamma(1)}$ in ($\circledcirc$) we have

($\boxdot$) $\varrho_{\beta,\gamma(1)}\restriction[\alpha_{\gamma(1),i_1}, \alpha_{\gamma(1),i_1+1}) = \eta\restriction[\alpha_{\gamma(1),i_1}, \alpha_{\gamma(1),i_1+1})$.

Since $E_{\gamma(1)} \subseteq E$, we may find $i_2 < \kappa$ such that $[\alpha_{i_2}, \alpha_{i_2+1}) \subseteq [\alpha_{\gamma(1),i_1}, \alpha_{\gamma(1),i_1+1})$. Then necessarily $i_2 \geq i_1 \geq i^*$ and hence we have

$$\nu\restriction[\alpha_{i_2}, \alpha_{i_2+1}) \neq \eta\restriction[\alpha_{i_2}, \alpha_{i_2+1}) = \varrho_{\beta,\gamma(1)}\restriction[\alpha_{i_2}, \alpha_{i_2+1}),$$

and consequently $\nu\restriction[\alpha_{\gamma(1),i_1}, \alpha_{\gamma(1),i_1+1}) \neq \varrho_{\beta,\gamma(1)}\restriction[\alpha_{\gamma(1),i_1}, \alpha_{\gamma(1),i_1+1})$. Since $E_{\gamma(2)} \subseteq \{\alpha < \kappa : \alpha \text{ is limit and } (\forall j < \alpha)(\alpha_{\gamma(1),j} < \alpha)\}$, we know that

($\boxtimes$) $[\alpha_{\gamma(1),i_1}, \alpha_{\gamma(1),i_1+1}) \subseteq [\alpha_{\gamma(2),i}, \alpha_{\gamma(2),i+1})$ and thus $\nu\restriction[\alpha_{\gamma(2),i}, \alpha_{\gamma(2),i+1}) \neq \varrho_{\beta,\gamma(1)}\restriction[\alpha_{\gamma(2),i}, \alpha_{\gamma(2),i+1})$.

Now we easily finish concluding that $\nu \in X_{\varrho_{\beta,\gamma(1)},\bar{\alpha}_{\gamma(2)}} = A_{\beta,\gamma(1),\gamma(2)}$, as desired.
(4) It follows from 0.9 and 5.3(1) that $\mu := \text{add}(\text{Cohen}_\kappa) \leq \min\{\mathfrak{b}_\kappa, \text{cov}(\text{Cohen}_\kappa)\}$. In order to show the converse inequality assume towards contradiction that $\mu < \min\{\mathfrak{b}_\kappa, \text{cov}(\text{Cohen}_\kappa)\}$. Suppose that $\mathscr{A} = \{A_\gamma : \gamma < \mu\}$ is a family of members of $\text{id}(\text{Cohen}_\kappa)$ (and we will argue that $\bigcup \mathscr{A} \in \text{id}(\text{Cohen}_\kappa)$). For $\gamma < \mu$ let $(\eta_\gamma, \bar{\beta}_\gamma)$ be as in 5.1 and such that $A_\gamma \subseteq X_{\eta_\gamma,\bar{\beta}_\gamma}$ and let

$$E_\gamma = \{\alpha < \kappa : \alpha \text{ is limit and } (\forall i < \alpha)(\beta_{\gamma,i} < \alpha)\}$$

(it is a club of κ). As $\mu < \mathfrak{b}_\kappa$ we may find an increasing continuous sequence $\bar{\beta} = \langle \beta_j : j < \kappa \rangle$ of ordinals below κ such that for each γ and every sufficiently large j we have $\beta_j \in E_\gamma$. Then $X_{\eta_\gamma,\bar{\beta}_\gamma} \subseteq X_{\eta_\gamma,\bar{\beta}}$. Since $\mu < \text{cov}(\text{Cohen}_\kappa)$, by an easy dualization of ($\boxplus$) of (3), we have:

$(\boxplus)^*_\perp$ there is $\nu \in {}^\kappa 2$ such that for every $\gamma < \mu$ the set

$$Z_\gamma := \{j < \kappa : \eta_\gamma\restriction[\beta_j, \beta_{j+1}) = \nu\restriction[\beta_j, \beta_{j+1})\}$$

is of size κ.

Using $\mu < \mathfrak{b}_\kappa$ again, we may find an increasing sequence $\bar{\alpha}$ such that

$$(\forall \gamma < \mu)(\exists i_0 < \kappa)(\forall i > i_0)(Z_\gamma \cap [\alpha_i, \alpha_{i+1}) \neq \emptyset).$$

Then letting $\delta_i = \beta_{\alpha_i}$ (for $i < \kappa$) we will have $X_{\eta_\gamma,\bar{\beta}} \subseteq X_{\nu,\bar{\delta}}$ for each γ and the desired conclusion easily follows. □

Claim 5.4 *(1) If* $\kappa = \sup(S^\kappa_{\text{inac}})$ *then* $\text{cov}(\text{Cohen}_\kappa) \leq \text{non}(\mathbb{Q}_\kappa)$.
(2) If $\kappa = \sup(S^\kappa_{\text{inac}})$ *then* $\text{cov}(\mathbb{Q}_\kappa) \leq \text{non}(\text{Cohen}_\kappa)$.

Proof Both follow by 3.7 and 3.8.
(1) Let $A_0 \in \text{id}(\text{Cohen}_\kappa)$, $A_1 \in \text{id}(\mathbb{Q}_\kappa)$ be a partition of ${}^\kappa 2$ (see 3.8). There is $X = \{\eta_\varepsilon : \varepsilon < \mu\} \subseteq {}^\kappa 2$ where $\mu = \text{non}(\mathbb{Q}_\kappa)$ such that $X \notin \text{id}(\mathbb{Q}_\kappa)$. Now, ${}^\kappa 2$ with addition $\oplus$ modulo 2, coordinatewise, is an Abelian Group and both ideals $\text{id}(\text{Cohen}_\kappa)$ and $\text{id}(\mathbb{Q}_\kappa)$ are closed under translations (see 3.7). Thus $\{\eta_\varepsilon \oplus A_0 : \varepsilon < \mu\}$ is a family of $\leq \mu$ members of $\text{id}(\text{Cohen}_\kappa)$ and it suffices to prove that $\bigcup\{\eta_\varepsilon \oplus A_0 : \varepsilon < \mu\} = {}^\kappa 2$. So let $\nu \in {}^\kappa 2$. Since $\{\eta_\varepsilon : \varepsilon < \mu\} \notin \text{id}(\mathbb{Q}_\kappa)$, also $\{\eta_\varepsilon \oplus \nu : \varepsilon < \mu\} \notin \text{id}(\mathbb{Q}_\kappa)$ and

Sh:1004

hence it is not included in A_1. Thus for some $\varepsilon < \mu$, $\eta_\varepsilon \oplus \nu \in A_0$, hence $\nu \in \eta_\varepsilon \oplus A_0$ as required.
(2) Same proof, just interchanging A_0 and A_1. □

Claim 5.5 *If* $\mathfrak{b}_\kappa > \text{cov}(\text{Cohen}_\kappa)$, *then* $\text{cov}(\mathbb{Q}_\kappa) \leq \text{cov}(\text{Cohen}_\kappa)$.

Proof If $\kappa > \sup(S^\kappa_{\text{inac}})$, then $\text{cov}(\mathbb{Q}_\kappa) = \text{cov}(\text{Cohen}_\kappa)$.

So suppose κ is an inaccessible limit of inaccessibles and $\mathfrak{b}_\kappa > \text{cov}(\text{Cohen}_\kappa)$. Assume towards contradiction that $\text{cov}(\mathbb{Q}_\kappa) > \text{cov}(\text{Cohen}_\kappa) := \mu$.

Using the assumption $\mathfrak{b}_\kappa > \mu = \text{cov}(\text{Cohen}_\kappa)$ and Observation 5.1 we can easily find an increasing sequence $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \kappa \rangle$ and a family $\Upsilon \subseteq \prod_{\varepsilon<\kappa} \theta_\varepsilon$ such that

$(*)_1$ $0 < \theta_\varepsilon < \kappa$ for each $\varepsilon < \kappa$, $|\Upsilon| = \mu$ and
$(*)_2$ $(\forall \nu \in \prod_{\varepsilon<\kappa} \theta_\varepsilon)(\exists \rho \in \Upsilon)(\forall^\infty \varepsilon < \kappa)(\rho(\varepsilon) \neq \nu(\varepsilon))$.

Next, by induction on $\varepsilon < \kappa$, we choose inaccessible cardinals ∂_ε such that:

$(*)_3$ $\partial_\varepsilon > \theta_\varepsilon + \sum_{\zeta<\varepsilon} \partial_\zeta$ and $\partial_\varepsilon > \sup(\partial_\varepsilon \cap S^\kappa_{\text{inac}})$.

For each $\varepsilon < \kappa$ fix a partition $\langle S_{\varepsilon,i} : i < \theta_\varepsilon \rangle$ of ∂_ε into stationary sets and

- for $0 < i < \theta_\varepsilon$ define $A_{\varepsilon,i} = \big\{\eta \in {}^{\partial_\varepsilon}2$: the set $\{\alpha \in S_{\varepsilon,i} : \eta(\alpha) = 1\}$ is stationary but for each $j < i$ the set $\{\alpha \in S_{\varepsilon,j} : \eta(\alpha) = 1\}$ is not stationary$\big\}$, and
- let $A_{\varepsilon,0} = {}^{\partial_\varepsilon}2 \setminus \bigcup_{i\in[1,\theta_\varepsilon)} A_{\varepsilon,i}$.

Note that $\langle A_{\varepsilon,i} : i < \theta_\varepsilon \rangle$ is a partition of ${}^{\partial_\varepsilon}2$ such that

$(*)_4$ $\nu \in {}^{\partial_\varepsilon >}2 \Rightarrow \{\eta \in A_{\varepsilon,i} : \nu \lhd \eta\} \notin \text{id}(\text{Cohen}_{\partial_\varepsilon})$.

Now, for $\rho \in \Upsilon$ and $\alpha < \kappa$ let

$$\begin{aligned}\mathscr{I}_{\rho,\alpha} = \big\{p \in \mathbb{Q}_\kappa : \ell g(\text{tr}(p)) > \alpha \text{ and for some } \varepsilon < \kappa \\ \alpha < \partial_\varepsilon < \ell g(\text{tr}(p)) \wedge \text{tr}(p){\restriction}\partial_\varepsilon \in A_{\varepsilon,\rho(\varepsilon)}\big\}.\end{aligned}$$

It should be clear that each $\mathscr{I}_{\rho,\alpha}$ is an open dense subset of $\mathbb{Q}_\kappa$ (remember that $\partial_\varepsilon > \sup(\partial_\varepsilon \cap S^\kappa_{\text{inac}})$ and use $(*)_4$).

As we are assuming towards contradiction that $\text{cov}(\mathbb{Q}_\kappa) > \mu$, the set $\bigcap_{\rho\in\Upsilon} \bigcap_{\alpha<\kappa} \text{set}(\mathscr{I}_{\rho,\alpha})$ is not empty. Let $\eta \in \bigcap_{\rho\in\Upsilon} \bigcap_{\alpha<\kappa} \text{set}(\mathscr{I}_{\rho,\alpha})$ and let $\nu \in \prod_{\varepsilon<\kappa} \theta_\varepsilon$ be such that

$$\varepsilon < \kappa \Rightarrow \eta{\restriction}\partial_\varepsilon \in A_{\varepsilon,\nu(\varepsilon)}.$$

By the choice of η, for every $\rho \in \Upsilon$ we have $\sup(\{\varepsilon < \kappa : \eta{\restriction}\partial_\varepsilon \in A_{\varepsilon,\rho(\varepsilon)}\}) = \kappa$. Hence

$$(\forall \rho \in \Upsilon)(\exists^\infty \varepsilon < \kappa)(\nu(\varepsilon) = \rho(\varepsilon)),$$

a clear contradiction with $(*)_2$. □

Sh:1004

Conclusion 5.6 *Assume that either*

(a) $\kappa > \sup(S^\kappa_{\mathrm{inac}})$*, or*
(b) $\mathfrak{b}_\kappa > \mathrm{cov}(\mathrm{Cohen}_\kappa)$*, or*
(c) there is a stationary non-reflecting set $S \subseteq S^\kappa_{\mathrm{pr}}$.

Then $\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathrm{add}(\mathrm{Cohen}_\kappa)$.

Proof If $\kappa > \sup(\kappa \cap S^\kappa_{\mathrm{inac}})$ then $\mathbb{Q}_\kappa$ is equivalent to Cohen_κ, and moreover $\mathrm{id}(\mathbb{Q}_\kappa) = \mathrm{id}(\mathrm{Cohen}_\kappa)$ (see 3.5(1)) and so $\mathrm{add}(\mathbb{Q}_\kappa) = \mathrm{add}(\mathrm{Cohen}_\kappa)$.

Let us assume $\mathfrak{b}_\kappa > \mathrm{cov}(\mathrm{Cohen}_\kappa)$. Then, by 5.3(4),

$(\bullet)_1$ $\mathrm{add}(\mathrm{Cohen}_\kappa) = \mathrm{cov}(\mathrm{Cohen}_\kappa)$

and by the Claim 5.5

$(\bullet)_2$ $\mathrm{cov}(\mathbb{Q}_\kappa) \leq \mathrm{cov}(\mathrm{Cohen}_\kappa)$.

Hence (first inequality trivial, holds for any ideal, e.g. see 0.9, the other two by $(\bullet)_2$ and $(\bullet)_1$)

$(\bullet)$ $\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathrm{cov}(\mathbb{Q}_\kappa) \leq \mathrm{cov}(\mathrm{Cohen}_\kappa) = \mathrm{add}(\mathrm{Cohen}_\kappa)$.

Finally, if $\mathfrak{b}_\kappa \leq \mathrm{cov}(\mathrm{Cohen}_\kappa)$ but there is a stationary non-reflecting set $S \subseteq S^\kappa_{\mathrm{pr}}$, then by 5.3(4) we have $\mathrm{add}(\mathrm{Cohen}_\kappa) = \mathfrak{b}_\kappa$ and by 4.12(1) + 4.11(1) we get

$$\mathrm{add}(\mathbb{Q}_\kappa) \leq \mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa) \leq \mathfrak{b}_\kappa = \mathrm{add}(\mathrm{Cohen}_\kappa).$$

So we are done. □

The following result is dual to 5.5.

Claim 5.7 *If* $\mathfrak{d}_\kappa < \mathrm{non}(\mathrm{Cohen}_\kappa)$*, then* $\mathrm{non}(\mathrm{Cohen}_\kappa) \leq \mathrm{non}(\mathbb{Q}_\kappa)$.

Proof If $\kappa > \sup(S^\kappa_{\mathrm{inac}} \cap \kappa)$ this holds trivially as in the proof of 5.6, so from now on assume $\kappa = \sup(S^\kappa_{\mathrm{inac}} \cap \kappa)$. For every $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \kappa \rangle$ with $1 < \theta_\varepsilon < \kappa$ we choose $\bar{\partial}_{\bar{\theta}} = \langle \partial_{\bar{\theta},\varepsilon} : \varepsilon < \kappa \rangle$, $\bar{S}_{\bar{\theta},\varepsilon} = \langle S_{\bar{\theta},\varepsilon,i} : i < \theta_\varepsilon \rangle$, $\bar{A}_{\bar{\theta},\varepsilon} = \langle A_{\bar{\theta},\varepsilon,i} : i < \theta_\varepsilon \rangle$ as in the proof of Claim 5.5. That is, $\bar{\partial}_{\bar{\theta}}$, $\bar{S}_{\bar{\theta},\varepsilon}$, $\bar{A}_{\bar{\theta},\varepsilon}$ satisfy for $\varepsilon < \kappa$:

$(\oplus)_1$ $\partial_{\bar{\theta},\varepsilon} < \kappa$ is an inaccessible cardinal such that $\partial_{\bar{\theta},\varepsilon} > \theta_\varepsilon + \sum_{\zeta<\varepsilon} \partial_{\bar{\theta},\zeta}$ and $\partial_{\bar{\theta},\varepsilon} > \sup(\partial_{\bar{\theta},\varepsilon} \cap S^\kappa_{\mathrm{inac}})$,
$(\oplus)_2$ $\langle S_{\bar{\theta},\varepsilon,i} : i < \theta_\varepsilon \rangle$ is a partition of ∂_ε into stationary sets, and
$(\oplus)_3$ for $0 < i < \theta_\varepsilon$, $A_{\bar{\theta},\varepsilon,i} = \big\{\eta \in {}^{\partial_\varepsilon}2 :$ the set $\{\alpha \in S_{\bar{\theta},\varepsilon,i} : \eta(\alpha) = 1\}$ is stationary but for each $j < i$ the set $\{\alpha \in S_{\bar{\theta},\varepsilon,j} : \eta(\alpha) = 1\}$ is not stationary$\big\}$, and
$(\oplus)_4$ $A_{\bar{\theta},\varepsilon,0} = {}^{\partial_\varepsilon}2 \setminus \bigcup_{i\in[1,\theta_\varepsilon)} A_{\bar{\theta},\varepsilon,i}$.

A mapping ${}^\kappa 2 \ni \eta \mapsto \nu_{\bar{\theta},\eta} \in \prod_{\varepsilon<\kappa} \theta_\varepsilon$ is defined by the condition $\eta{\restriction}\partial_{\bar{\theta},\varepsilon} \in A_{\bar{\theta},\varepsilon,\nu_{\bar{\theta},\eta}(\varepsilon)}$ for each $\varepsilon < \kappa$.

Choose $\Upsilon \subseteq {}^\kappa 2$, $\Upsilon \notin \mathrm{id}(\mathbb{Q}_\kappa)$, of cardinality $\mathrm{non}(\mathbb{Q}_\kappa)$. For any $\bar{\theta}$ as above let $\Upsilon_{\bar{\theta}} = \{\nu_{\bar{\theta},\eta} : \eta \in \Upsilon\}$. Then clearly

$(\oplus)_5$ $\Upsilon_{\bar{\theta}} \subseteq \prod_{\varepsilon<\kappa} \theta_\varepsilon$ and $\Upsilon_{\bar{\theta}}$ has cardinality $\leq \mathrm{non}(\mathbb{Q}_\kappa)$.

Sh:1004

Dually to arguments in 5.5 we will argue now that

$(\oplus)_6$ for every $\rho \in \prod_{\varepsilon<\kappa} \theta_\varepsilon$, there is $\nu \in \Upsilon_{\bar\theta}$ such that $(\exists^\infty \varepsilon < \kappa)(\rho(\varepsilon) = \nu(\varepsilon))$.

Why? Suppose $\rho \in \prod_{\varepsilon<\kappa} \theta_\varepsilon$. For $\alpha < \kappa$ let

$$\mathscr{I}_\alpha = \big\{p \in \mathbb{Q}_\kappa : \ell g(\mathrm{tr}(p)) > \alpha \text{ and for some } \varepsilon < \kappa \\ \alpha < \partial_{\bar\theta,\varepsilon} < \ell g(\mathrm{tr}(p)) \ \wedge\ \mathrm{tr}(p)\restriction\partial_{\bar\theta,\varepsilon} \in A_{\bar\theta,\varepsilon,\rho(\varepsilon)}\big\}.$$

Clearly, each $\mathscr{I}_\alpha$ is an open dense subset of $\mathbb{Q}_\kappa$ (remember $\partial_{\bar\theta,\varepsilon} > \sup(\partial_{\bar\theta,\varepsilon} \cap S^\kappa_{\mathrm{inac}})$). Since $\Upsilon \notin \mathrm{id}(\mathbb{Q}_\kappa)$ we know that $\Upsilon \cap \bigcap_{\alpha<\kappa} \mathrm{set}(\mathscr{I}_\alpha) \neq \emptyset$. Let $\eta \in \Upsilon \cap \bigcap_{\alpha<\kappa} \mathrm{set}(\mathscr{I}_\alpha)$. Then $(\exists^\infty \varepsilon < \kappa)(\nu_{\bar\theta,\eta}(\varepsilon) = \rho(\varepsilon))$. Thus $(\oplus)_6$ is justified.

Easily by definition of $\mathfrak{d}_\kappa$ we may choose a family $\{\bar\alpha_\xi : \xi < \mathfrak{d}_\kappa\}$ such that

$(\oplus)_7$ (a) $\bar\alpha_\xi = \langle \alpha_{\xi,\varepsilon} : \varepsilon < \kappa\rangle$ is an increasing continuous sequence in κ (for each $\xi < \mathfrak{d}_\kappa$), and
(b) if $\langle \alpha_i : i < \kappa\rangle$ is an increasing sequence of ordinals below κ, then for some $\xi < \mathfrak{d}_\kappa$ we have

$$(\forall^\infty \varepsilon < \kappa)(\exists i < \kappa)(\alpha_{\xi,\varepsilon} < \alpha_i < \alpha_{i+1} < \alpha_{\xi,\varepsilon+1}).$$

Now, for each $\xi < \mathfrak{d}_\kappa$ let $\bar\theta_\xi = \langle \theta_{\xi,\varepsilon} : \varepsilon < \kappa\rangle$, where $\theta_{\xi,\varepsilon} = |^{[\alpha_{\xi,\varepsilon},\alpha_{\xi,\varepsilon+1})}2|$. Also, for each ξ,ε fix a bijection $\pi_{\xi,\varepsilon} : \theta_{\xi,\varepsilon} \longrightarrow {}^{[\alpha_{\xi,\varepsilon},\alpha_{\xi,\varepsilon+1})}2$ and for $\nu \in \prod_{\varepsilon<\kappa} \theta_{\xi,\varepsilon}$ (for $\xi < \mathfrak{d}_\kappa$) set $x_{\xi,\nu} = \bigcup_{\varepsilon<\kappa} \pi_{\xi,\varepsilon}(\nu(\varepsilon)) \in {}^\kappa 2$. Consider the set

$$\mathscr{X} = \big\{x_{\xi,\nu} : \xi < \mathfrak{d}_\kappa \ \wedge\ \nu \in \Upsilon_{\bar\theta_\xi}\big\}.$$

We claim that

$(\oplus)_8$ $\mathscr{X} \notin \mathrm{id}(\mathrm{Cohen}_\kappa)$.

If not, then for some $\eta \in {}^\kappa 2$ and an increasing continuous sequence $\bar\alpha = \langle \alpha_i : i < \kappa\rangle \subseteq \kappa$ we have $\mathscr{X} \subseteq X_{\eta,\bar\alpha}$. Let $\xi < \mathfrak{d}_\kappa$ be given by $(\oplus)_7$(b) for $\bar\alpha$ and let $\rho^* \in \prod_{\varepsilon<\kappa} \theta_{\xi,\varepsilon}$ be such that $\pi_{\xi,\varepsilon}(\rho^*(\varepsilon)) = \eta\restriction[\alpha_{\xi,\varepsilon}, \alpha_{\xi,\varepsilon+1})$ for each $\varepsilon < \kappa$. It follows from $(\oplus)_6$ that for some $\nu \in \Upsilon_{\bar\theta_\xi}$ we have $(\exists^\infty \varepsilon < \kappa)(\rho^*(\varepsilon) = \nu(\varepsilon))$. This implies that $\big(\exists^\infty \varepsilon < \kappa\big)\big(x_{\xi,\nu}\restriction[\alpha_{\xi,\varepsilon}, \alpha_{\xi,\varepsilon+1}) = \eta\restriction[\alpha_{\xi,\varepsilon}, \alpha_{\xi,\varepsilon+1})\big)$ and hence (remembering the choice of ξ) we get $\big(\exists^\infty i < \kappa\big)\big(x_{\xi,\nu}\restriction[\alpha_i, \alpha_{i+1}) = \eta\restriction[\alpha_i, \alpha_{i+1})\big)$. Consequently $x_{\xi,\nu} \notin X_{\eta,\bar\alpha}$, a contradiction.

It follows from $(\oplus)_8$ that $\mathfrak{d}_\kappa < \mathrm{non}(\mathrm{Cohen}_\kappa) \le |\mathscr{X}| \le \mathrm{non}(\mathbb{Q}_\kappa) + \mathfrak{d}_\kappa$ and therefore $\mathrm{non}(\mathrm{Cohen}_\kappa) \le \mathrm{non}(\mathbb{Q}_\kappa)$. □

Conclusion 5.8 *Assume that either*

(a) $\kappa > \sup(S^\kappa_{\mathrm{inac}})$*, or*
(b) $\mathfrak{d}_\kappa < \mathrm{non}(\mathrm{Cohen}_\kappa)$*, or*
(c) there is a stationary non-reflecting set $S \subseteq S^\kappa_{\mathrm{pr}}$.

Then $\mathrm{cf}(\mathrm{Cohen}_\kappa) \le \mathrm{cf}(\mathbb{Q}_\kappa)$.

Proof The proof is similar to the proof of 5.6.

If $\kappa > \sup(S^{\kappa}_{\text{inac}})$ then $\text{id}(\mathbb{Q}_\kappa) = \text{id}(\text{Cohen}_\kappa)$ and $\text{cf}(\mathbb{Q}_\kappa) = \text{cf}(\text{Cohen}_\kappa)$.

If $\mathfrak{d}_\kappa < \text{non}(\text{Cohen}_\kappa)$, then it follows from 5.3(3) that $\text{cf}(\text{Cohen}_\kappa) = \text{non}(\text{Cohen}_\kappa)$. Also, by 5.7 and 0.9(b), we have $\text{non}(\text{Cohen}_\kappa) \leq \text{non}(\mathbb{Q}_\kappa) \leq \text{cf}(\mathbb{Q}_\kappa)$. Together $\text{cf}(\text{Cohen}_\kappa) \leq \text{cf}(\mathbb{Q}_\kappa)$ (under present assumptions).

If $\mathfrak{d}_\kappa \geq \text{non}(\text{Cohen}_\kappa)$, but there is a non-reflecting stationary subset of S^{κ}_{pr}, then we use 4.12(3) to get $\text{cf}(\mathbf{nst}^{\text{pr}}_{\kappa}) \geq \mathfrak{d}_\kappa$. Now. 5.3(3) implies $\text{cf}(\text{Cohen}_\kappa) = \mathfrak{d}_\kappa$ and 4.11(2) gives $\text{cf}(\mathbb{Q}_\kappa) \geq \text{cf}(\mathbf{nst}^{\text{pr}}_{\mathbf{nst}})$. Together we conclude $\text{cf}(\mathbb{Q}_\kappa) \geq \text{cf}(\text{Cohen}_\kappa)$, as desired. □

Now we may summarize the results of this section in the form of diagrams.

Theorem 5.9 *Assume that κ is an inaccessible cardinal and $\kappa = \sup(S^{\kappa}_{\text{inac}})$. Then the inequalities represented by arrows in the following diagram hold true:*

$$
\begin{array}{ccccccccc}
 & \text{cov}(\mathbb{Q}_\kappa) & \to & \text{non}(\text{Cohen}_\kappa) & \to & \text{cf}(\text{Cohen}_\kappa) & & \text{cf}(\mathbb{Q}_\kappa) & \to 2^\kappa \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
 & | & & \mathfrak{b}_\kappa & \to & \mathfrak{d}_\kappa & & | & \\
 & | & & \uparrow & & \uparrow & & | & \\
\kappa^+ \to & \text{add}(\mathbb{Q}_\kappa) & & \text{add}(\text{Cohen}_\kappa) & \to & \text{cov}(\text{Cohen}_\kappa) & \to & \text{non}(\mathbb{Q}_\kappa) &
\end{array}
$$

plus the dependencies

- $\text{add}(\text{Cohen}_\kappa) = \min\{\text{cov}(\text{Cohen}_\kappa), \mathfrak{b}_\kappa\}$,
- $\text{cf}(\text{Cohen}_\kappa) = \max\{\text{non}(\text{Cohen}_\kappa), \mathfrak{d}_\kappa\}$,
- $\text{cov}(\mathbb{Q}_\kappa) \leq \text{non}(\mathbb{Q}_\kappa)$ *(see 6.6(3)).*

Moreover, we may add that one of the following four diagrams holds (where each arrow $\to$ represents the inequality $\leq$ and $\uparrow \neq$ represents the strict inequality $<$).

Case 1

$$
\begin{array}{cccccccc}
 & & & & & & \text{cf}(\mathbb{Q}_\kappa) & \to 2^\kappa \\
 & & & & & & \uparrow & \\
 & & \text{non}(\text{Cohen}_\kappa) & = & \text{cf}(\text{Cohen}_\kappa) & \to & \text{non}(\mathbb{Q}_\kappa) & \\
 & & \uparrow & & \uparrow \neq & & & \\
 & & \mathfrak{b}_\kappa & \to & \mathfrak{d}_\kappa & & & \\
 & & \uparrow \neq & & \uparrow & & & \\
 & \text{cov}(\mathbb{Q}_\kappa) \to & \text{add}(\text{Cohen}_\kappa) & = & \text{cov}(\text{Cohen}_\kappa) & & & \\
 & \uparrow & & & & & & \\
\kappa^+ \to & \text{add}(\mathbb{Q}_\kappa) & & & & & &
\end{array}
$$

Sh:1004

Case 2

$$\begin{array}{ccccccccccc}
 & & & & & & & & \mathrm{cf}(\mathbb{Q}_\kappa) & \to & 2^\kappa\\
 & & & & & & & & \uparrow & & \\
 & & \mathrm{cov}(\mathbb{Q}_\kappa) & \to & \mathrm{non}(\mathrm{Cohen}_\kappa) & = & \mathrm{cf}(\mathrm{Cohen}_\kappa) & \to & \mathrm{non}(\mathbb{Q}_\kappa) & & \\
 & & \uparrow & & \uparrow & & \uparrow\neq & & & & \\
 & & & & \mathfrak{b}_\kappa & \to & \mathfrak{d}_\kappa & & & & \\
 & & & & \| & & \uparrow & & & & \\
\kappa^+ & \to & \mathrm{add}(\mathbb{Q}_\kappa) & & \mathrm{add}(\mathrm{Cohen}_\kappa) & \to & \mathrm{cov}(\mathrm{Cohen}_\kappa) & & & &
\end{array}$$

Case 3

$$\begin{array}{ccccccccccc}
 & & & & \mathrm{non}(\mathrm{Cohen}_\kappa) & \to & \mathrm{cf}(\mathrm{Cohen}_\kappa) & & \mathrm{cf}(\mathbb{Q}_\kappa) & \to & 2^\kappa\\
 & & & & \uparrow & & \| & & \uparrow & & \\
 & & & & \mathfrak{b}_\kappa & \to & \mathfrak{d}_\kappa & & & & \\
 & & & & \uparrow\neq & & \uparrow & & & & \\
 & & \mathrm{cov}(\mathbb{Q}_\kappa) & \to & \mathrm{add}(\mathrm{Cohen}_\kappa) & = & \mathrm{cov}(\mathrm{Cohen}_\kappa) & \to & \mathrm{non}(\mathbb{Q}_\kappa) & & \\
 & & \uparrow & & & & & & & & \\
\kappa^+ & \to & \mathrm{add}(\mathbb{Q}_\kappa) & & & & & & & &
\end{array}$$

Case 4

$$\begin{array}{ccccccccccc}
 & & \mathrm{cov}(\mathbb{Q}_\kappa) & \to & \mathrm{non}(\mathrm{Cohen}_\kappa) & \to & \mathrm{cf}(\mathrm{Cohen}_\kappa) & & \mathrm{cf}(\mathbb{Q}_\kappa) & \to & 2^\kappa\\
 & & \uparrow & & \uparrow & & \| & & \uparrow & & \\
 & & & & \mathfrak{b}_\kappa & \to & \mathfrak{d}_\kappa & & & & \\
 & & & & \| & & \uparrow & & & & \\
\kappa^+ & \to & \mathrm{add}(\mathbb{Q}_\kappa) & & \mathrm{add}(\mathrm{Cohen}_\kappa) & \to & \mathrm{cov}(\mathrm{Cohen}_\kappa) & \to & \mathrm{non}(\mathbb{Q}_\kappa) & &
\end{array}$$

Remark 5.10 (1) In a later work we prove that $\mathrm{add}(\mathbf{nst}^{\mathrm{pr}}_\kappa) \le \mathfrak{d}_\kappa$ and $\mathfrak{b}_\kappa \le \mathrm{cf}(\mathbf{nst}^{\mathrm{pr}}_\kappa)$. Consequently, by 4.11, $\mathrm{add}(\mathbb{Q}_\kappa) \le \mathfrak{d}_\kappa$ and $\mathrm{cf}(\mathbb{Q}_\kappa) \ge \mathfrak{b}_\kappa$.

(2) Remember that by 5.6 and 5.8, if $\kappa > \sup(S^\kappa_{\mathrm{inac}})$ or there is a stationary non-reflecting set $S \subseteq S^\kappa_{\mathrm{pr}}$, then $\mathrm{add}(\mathbb{Q}_\kappa) \le \mathrm{add}(\mathrm{Cohen}_\kappa)$ and $\mathrm{cf}(\mathbb{Q}_\kappa) \ge \mathrm{cf}(\mathrm{Cohen}_\kappa)$.

6 $\mathbb{Q}_\kappa$ vs $\mathbf{Cohen}_\kappa$

6(A) Effect on the ground model

Claim 6.1 *If κ is an inaccessible limit of inaccessibles, then in $\mathbf{V}^{\mathbb{Q}_\kappa}$ the set $({}^\kappa 2)^{\mathbf{V}}$ is κ-meagre.*

Remark 6.2 (1) The dual is 6.3.
(2) The assumption is necessary by 3.5.

Sh:1004

Proof Let $\langle \partial_i : i < \kappa \rangle$ list in increasing order the (strongly) inaccessible cardinals below κ. We claim that

$$\Vdash_{\mathbb{Q}_\kappa} \text{"if } \nu \in ({}^\kappa 2)^{\mathbf{V}} \text{ \underline{then} for every } i < \kappa \text{ large enough } \underset{\sim}{\eta} \restriction (\partial_i + 1, \partial_{i+1}) \nsubseteq \nu, \\ \text{moreover} \quad \alpha < \partial_{i+1} \Rightarrow \underset{\sim}{\eta} \restriction (\alpha, \partial_{i+1}) \nsubseteq \nu\text{''}.$$

This clearly suffices by 5.1(2). Let $p \in \mathbb{Q}_\kappa$ and we shall fix $\nu \in ({}^\kappa 2)^{\mathbf{V}}$ and we shall find q and $i_* < \kappa$ such that $p \leq_{\mathbb{Q}_\kappa} q$ and $q \Vdash$ "if $i > i_*$ then $\underset{\sim}{\eta} \restriction (\partial_i + 1, \partial_{i+1}) \nsubseteq \nu$".

Let i_* be such that $\ell g(\mathrm{tr}(p)) < \partial_{i_*}$ and let $(\varrho, S_1, \bar{\Lambda})$ be a witness for $p \in \mathbb{Q}_\kappa$. Now let $S_2 = \{\partial_{i+1} : i > i_*\}$ and if $\partial = \partial_{i+1} \in S_2$ and $\alpha \in (\partial_i, \partial_{i+1})$ <u>then</u> we let

$$\mathscr{I}_{\partial,\alpha} = \big\{ r \in \mathbb{Q}_\partial : \ell g(\mathrm{tr}(r)) > \alpha \text{ and } \mathrm{tr}(r) \restriction [\alpha, \ell g(\mathrm{tr}(r)) \nsubseteq \nu \big\}.$$

Clearly, $\mathscr{I}_{\partial,\alpha}$ is a dense open subset of $\mathbb{Q}_\partial$. Now let $S' = S_1 \cup S_2$ and note that S_2 is nowhere stationary, so S' is too. Next, for $\partial \in S'$ put

$$\Lambda'_\partial = \begin{cases} \Lambda_\partial & \text{if } \partial \in S_1 \backslash S_2, \\ \Lambda_\partial \cup \{\mathscr{I}_{\partial,\alpha} : \alpha \in (\partial_i, \partial_{i+1})\} & \text{if } \partial = \partial_{i+1} \in S_1 \cap S_2, \\ \{\mathscr{I}_{\partial,\alpha} : \alpha \in (\partial_i, \partial_{i+1})\} & \text{if } \partial = \partial_{i+1} \in S_2 \backslash S_1, \end{cases}$$

and let $\bar{\Lambda}' = \langle \Lambda'_\partial : \partial \in S' \rangle$. Easily the triple $(\mathrm{tr}(p), S', \bar{\Lambda}')$ is a witness for some $q \in \mathbb{Q}_\kappa$ and this q is as required. $\square$

Claim 6.3 *If κ is inaccessible limit of inaccessibles and $\mathbf{V}_1$ is an extension of $\mathbf{V}$ (e.g. a forcing extension) <u>then</u> $\mathbf{V}_1 \models$ "$({}^\kappa 2)^{\mathbf{V}} \in \mathrm{id}(\mathbb{Q}_\kappa)$" provided that at least one of the following holds (each implying κ is still an inaccessible limit of inaccessibles in $\mathbf{V}_1$):*

- *(a) $\mathbf{V}_1 = \mathbf{V}^{\mathrm{Cohen}(\kappa)}$, see Definition 0.5(2).*
- *(b) In $\mathbf{V}_1$, κ is still inaccessible and there are sequences $\bar{\eta} = \langle \eta_\partial : \partial \in S \rangle$, $\bar{\alpha} = \langle \alpha_\partial : \partial \in S \rangle$ such that*
  - *(α) $S \subseteq \kappa$ is unbounded in κ,*
  - *(β) $\partial \in S \Rightarrow \alpha_\partial = \sup(S \cap \partial) < \partial$,*
  - *(γ) S is a set of inaccessibles (in $\mathbf{V}_1$ hence in $\mathbf{V}$),*
  - *(δ) $\eta_\partial \in {}^\partial 2$, really just $\eta_\partial \restriction (\alpha_\partial, \partial)$ matter,*
  - *(ε) if $\eta \in ({}^\kappa 2)^{\mathbf{V}}$ <u>then</u> for unboundedly many $\partial \in S$ we have $\eta \restriction (\alpha_\partial, \partial) \subseteq \eta_\partial$.*
- *(c) In $\mathbf{V}_1$, κ is still inaccessible limit of inaccessibles but $\mathscr{H}(\kappa)^{\mathbf{V}} \neq \mathscr{H}(\kappa)^{\mathbf{V}_1}$.*
- *(d) Like clause (b) but*
  - *$(\beta)'$ S is unbounded nowhere stationary in κ,*
  - *$(\delta)'$ $\bar{\Lambda} = \langle \Lambda_\partial : \partial \in S \rangle$, Λ_∂ a set $\leq \partial$ dense subset of $\mathbb{Q}_\partial$,*
  - *$(\varepsilon)'$ if $\eta \in ({}^\kappa 2)^{\mathbf{V}}$ <u>then</u> for unboundedly many $\partial \in S$, $\eta \restriction \partial$ does not fulfill Λ_∂.*

Remark 6.4 Of course, if κ is inaccessible not limit of inaccessibles then the conclusion of 6.3 fails because $\mathbb{Q}_\kappa$ is equivalent to Cohen_κ, see 3.5.

Proof <u>**Clause (a)**</u>: It suffices to prove that the assumptions of (b) holds.
Clearly the forcing preserves inaccessibility. Let $\underset{\sim}{\eta} \in {}^\kappa 2$ be the name of the κ-Cohen real and let:

Sh:1004

- $S_1 = \{\partial < \kappa : \partial$ inaccessible in $\mathbf{V}_1$ or $\mathbf{V}$, those are equivalent$\}$,
- $S = \{\partial \in S_1 : \partial > \sup(S_1 \cap \partial)\}$,
- $\underset{\sim}{\eta}_\partial = \underset{\sim}{\eta} \restriction \partial$,
- $\alpha_\partial = \sup(S_1 \cap \partial)$ for $\partial \in S$.

Clearly clauses (α), (γ) of (b) are satisfied by S_1 and by S and clause (β) is satisfied by the α_∂'s and S. Also recalling $\eta \in {}^\kappa 2$, it is the κ-Cohen real, the derived sequence $\langle \underset{\sim}{\eta}_\partial : \partial \in S\rangle$ satisfies clause (δ) by our choice above. Lastly, clause (ε) holds as $\text{Cohen}_\kappa = ({}^{\kappa>}2, \lhd)$, so all the assumptions of clause (b) hold indeed.

Clause (b): We work in $\mathbf{V}_1$.
For $\alpha < \partial \in S$ let

$$\mathscr{I}^*_{\partial,\alpha} = \big\{p \in \mathbb{Q}_\partial : \text{for some } \beta \text{ we have } \alpha < \beta < \partial,\ \beta < \ell g(\text{tr}(p)) \text{ and } \text{tr}(p)\restriction(\alpha,\beta) \nsubseteq \eta_\partial\big\}.$$

Easily $\mathscr{I}^*_{\partial,\alpha}$ is a dense open subset of $\mathbb{Q}_\partial$ and let

$$\mathscr{I} = \big\{p \in \mathbb{Q}_\kappa : \text{for some } \gamma < \kappa \text{ we have } S\backslash\gamma \subseteq S_p \text{ and } \partial \in S\backslash\gamma \Rightarrow \mathscr{I}^*_{\partial,\alpha_\partial} \in \Lambda_{p,\partial}\big\}.$$

Clearly $\mathscr{I}$ is a dense open subset of $\mathbb{Q}_\kappa$ and $p \in \mathscr{I} \Rightarrow \lim_\kappa(p) \cap ({}^\kappa 2)^{\mathbf{V}} = \emptyset$, so $\mathbf{V} \cap {}^\kappa 2 \in \text{id}_2(\mathbb{Q}_\kappa)$ and we are done (remember 3.2(5)).

Clause (c): Let S_1 be the set of inaccessibles in $\mathbf{V}_1$ which are $< \kappa$. Let $\alpha < \kappa$ and ν be such that $\nu \in ({}^\alpha 2)^{\mathbf{V}_1}$ but $\nu \notin ({}^\alpha 2)^{\mathbf{V}}$.

Now let

- $S = \{\partial \in S_1 : \partial > \alpha \text{ and } \partial > \sup(S_1 \cap \partial)\}$,
- $\mathscr{I}_\partial = \{p \in \mathbb{Q}_\partial$: for some β we have $\beta + \alpha \le \ell g(\text{tr}(p))$ and $\langle \text{tr}(p)(\beta + i) : i < \alpha\rangle = \nu\}$ for $\partial \in S$,
- $\Lambda_\partial = \{\mathscr{I}_\partial\}$ for $\partial \in S$.

Why is $\mathscr{I}_\partial$ a dense subset of $\mathbb{Q}_\partial$ for every $\partial \in S$? Let $p_1 \in \mathbb{Q}_\partial$ and we shall find p_2 such that $p_1 \le_{\mathbb{Q}_\partial} p_2 \in \mathscr{I}_\partial$. Let $p_2 \in \mathbb{Q}_\partial$ be such that $p_1 \le_{\mathbb{Q}_\partial} p_2$ and $\ell g(\text{tr}(p_2)) \ge \alpha + \sup\{\theta : \theta < \partial \text{ is inaccessible}\}$. (Why such p_2 exists? As $\partial \in S$ implies that ∂ is (strictly) above the ordinal on the right). But this implies $S_{p_2} = \emptyset$ hence there is p_3 such that $p_2 \le_{\mathbb{Q}_\partial} p_3$ and $(\text{tr}(p_3))(\alpha + \ell g(\text{tr}(p_2) + i) = \nu(i)$ for $i < \alpha$ hence $p_3 \in \mathscr{I}_\partial$. Hence the assumptions of clause (d) hold, so the result follows.

Clause (d): Like the proof of clause (b). □

Remark 6.5 If κ is inaccessible not limit of inaccessibles and $\mathbf{V}_1$ extends $\mathbf{V}$ and $\mathscr{H}(\kappa)^{\mathbf{V}_1} \ne \mathscr{H}(\kappa)^{\mathbf{V}}$ then $({}^\kappa 2)^{\mathbf{V}} \in \text{id}(\text{Cohen}_\kappa)^{\mathbf{V}_1}$ and $({}^\kappa 2)^{\mathbf{V}} \in \text{id}(\mathbb{Q}_\kappa)^{\mathbf{V}_1}$.

Claim 6.6 *Assume κ is inaccessible limit of inaccessibles. Then*

(1) $\Vdash_{\mathbb{Q}_\kappa} \mathbf{V} \cap {}^\kappa 2 \in \text{id}(\mathbb{Q}_\kappa)$.
(2) *$\mathbb{Q}_\kappa$ is asymmetric; that is, if $\mathbf{V}_1 \subseteq \mathbf{V}_2 \subseteq \mathbf{V}_3$, $\eta_\ell \in ({}^\kappa 2)^{\mathbf{V}_{\ell+1}}$ is $(\mathbb{Q}_\kappa, \underset{\sim}{\eta}_\kappa)$-generic over $\mathbf{V}_\ell$, for $\ell = 1, 2$, then η_1 is not $(\mathbb{Q}_\kappa, \underset{\sim}{\eta}_\kappa)$-generic over $\mathbf{V}_1[\eta_2]$.*

Sh:1004

(3) $\mathrm{cov}(\mathbb{Q}_\kappa) \le \mathrm{non}(\mathbb{Q}_\kappa)$.

Proof (1) Let $\langle \partial_\varepsilon : \varepsilon < \kappa \rangle$ list S^κ_{inac} in increasing order and let $S = \{\partial_{\varepsilon+1} : \varepsilon < \kappa\}$. For $\eta \in {}^\kappa 2$ and $\partial \in S$ let $\Lambda_{\eta,\partial}$ be a family of $\le \partial$ dense subsets of $\mathbb{Q}_\partial$ such that

$$\mathrm{set}(\Lambda_{\eta,\partial}) = \big\{\rho \in {}^\partial 2 : \text{ for arbitrarily large } \zeta < \partial \text{ we have } \rho(\zeta) \ne \eta(\partial + \zeta)\big\}.$$

Define

$$A_\eta = \big\{\nu \in {}^\kappa 2 : (\forall^\infty \partial \in S)(\nu \restriction \partial \in \mathrm{set}(\Lambda_{\eta,\partial}))\big\}.$$

Clearly, the set A_η is κ-Borel. Note that

$$\big\{p \in \mathbb{Q}_\kappa : (S \setminus \ell g(\mathrm{tr}(p))) \subseteq S_p \ \wedge\ (\forall \partial \in S)(\ell g(\mathrm{tr}(p) < \partial \ \Rightarrow\ \Lambda_{\eta,\partial} \subseteq \Lambda_{p,\partial})\big\}$$

is an open dense subset of $\mathbb{Q}_\kappa$. Hence,

$(*)_1$ for every $\eta \in {}^\kappa 2$ we have ${}^\kappa 2 \setminus A_\eta \in \mathrm{id}(\mathbb{Q}_\kappa)$.

We are going to argue that

$(*)_2$ $\Vdash_{\mathbb{Q}_\kappa} \mathbf{V} \cap A_{\underset{\sim}{\eta}} = \emptyset$.

So let $\nu \in {}^\kappa 2$. Suppose that $p \in \mathbb{Q}_\kappa$ and $\xi < \kappa$. Choose $\partial \in S$ such that $\partial > \xi, \ell g(\mathrm{tr}(p))$ and then pick $\rho \in p \cap {}^\partial 2$. Now $\varrho = \rho^\frown(\nu \restriction \partial) \in p$ and

$$p^{[\varrho]} \Vdash_{\mathbb{Q}_\kappa} \nu \restriction \partial \notin \mathrm{set}(\Lambda_{\underset{\sim}{\eta},\partial}).$$

By standard density arguments we conclude that

$$\Vdash_{\mathbb{Q}_\kappa} \big(\exists^\infty \partial \in S\big)\big(\nu \restriction \partial \notin \mathrm{set}(\Lambda_{\underset{\sim}{\eta},\partial})\big)$$

and thus $\Vdash_{\mathbb{Q}_\kappa} \nu \notin A_{\underset{\sim}{\eta}}$.

(2) Assume that η_1 is $(\mathbb{Q}_\kappa, \underset{\sim}{\eta})$-generic over $\mathbf{V}$ and η_2 is $(\mathbb{Q}_\kappa, \underset{\sim}{\eta})$-generic over $\mathbf{V}[\eta_1]$. It follows from $(*)_2$ of part (1) that

$(*)_3$ $\mathbf{V}[\eta_1, \eta_2] \models \eta_1 \notin A_{\eta_2}$.

Therefore, by $(*)_1$, η_1 is not $(\mathbb{Q}_\kappa, \underset{\sim}{\eta})$-generic over $\mathbf{V}[\eta_2]$.

(3) Let $S, \Lambda_{\eta,\partial}$ and A_η for $\partial \in S$, $\eta \in {}^\kappa 2$ be defined as in 6.6(1). Then ${}^\kappa 2 \setminus A_\eta \in \mathrm{id}(\mathbb{Q}_\kappa)$. For $\nu \in {}^\kappa 2$ let $A^\nu = \{\eta \in {}^\kappa 2 : \nu \in A_\eta\}$. The argument in the end of part (1) shows that for each $\xi < \kappa$ the set

$$\big\{p \in \mathbb{Q}_\kappa : \big(\exists \partial \in S \setminus \xi\big)\big(\forall \eta \in \lim_\kappa(p)\big)\big(\nu \restriction \partial \notin \mathrm{set}(\Lambda_{\eta,\partial})\big)\big\}$$

is open dense in $\mathbb{Q}_\kappa$. Hence $A^\nu \in \mathrm{id}(\mathbb{Q}_\kappa)$.

Now suppose that $X \subseteq {}^\kappa 2$ is such that $X \notin \mathrm{id}(\mathbb{Q}_\kappa)$. We claim that then

$$\bigcup\{{}^\kappa 2 \setminus A_\eta : \eta \in X\} = {}^\kappa 2.$$

So suppose $\nu \in {}^{\kappa}2$. Let $\eta \in X \setminus A^{\nu} \neq \emptyset$. By the definition this implies $\nu \notin A_{\eta}$ and we are done.

In [28] we note that generally for a nice enough $\mathbf{i}$ asymmetry implies $\mathrm{cov}(\mathbf{i}) \leq \mathrm{non}(\mathbf{i})$. $\square$

6(B) When does $\mathbb{Q}_{\kappa}$ add a Cohen real?

Definition 6.7 Let S_{awc} be the class of inaccessible κ such that (awc stands for "anti weakly compact") in $\mathbf{V}^{\mathbb{Q}_{\kappa}}$ there is a Cohen κ-real over $\mathbf{V}$; equivalently:

(∗) there is a sequence $\langle \mathscr{I}_{\alpha} : \alpha < \kappa \rangle$, $\mathscr{I}_{\alpha} \subseteq \mathbb{Q}_{\kappa}$ such that[5] for every $p \in \mathbb{Q}_{\kappa}$ there is $\alpha < \kappa$ such that:
for every $\beta \in (\alpha, \kappa)$ and $\varrho \in {}^{[\alpha,\beta)}2$ there is q such that
- $p \leq_{\mathbb{Q}_{\kappa}} q$,
- if $\gamma \in [\alpha, \beta)$ and $\varrho(\gamma) = 1$ then q is above some member of $\mathscr{I}_{\gamma}$,
- if $\gamma \in [\alpha, \beta)$ and $\varrho(\gamma) = 0$ then q is incompatible with every member of $\mathscr{I}_{\gamma}$.

Claim 6.8 *If κ is (strongly inaccessible but) not Mahlo <u>then</u> $\kappa \in S_{\mathrm{awc}}$.*

Proof It is similar to 4.12(2), but let us elaborate. Choose a closed unbounded subset E of κ disjoint to $S^{\kappa}_{\mathrm{inac}}$. Let A be E or any unbounded subset of κ such that $\partial \in S^{\kappa}_{\mathrm{inac}} \Rightarrow \partial > \sup(A \cap \partial)$.

Define functions $F_0 : {}^{\kappa>}2 \longrightarrow {}^{\kappa>}2$ and $F_1 : \mathbb{Q}_{\kappa} \longrightarrow \mathbb{Q}_{\kappa}$ and $F_2 : \mathbb{Q}_{\kappa} \longrightarrow \mathrm{Cohen}_{\kappa}$ by

- $F_0(\eta)$ is the $\nu \in {}^{\kappa>}2$ of length $\mathrm{otp}(\ell g(\eta) \cap A)$ and

$$\alpha < \ell g(\eta) \wedge \alpha \in A \Rightarrow \nu(\mathrm{otp}(\alpha \cap A)) = \eta(\alpha)$$

  (for $\eta \in {}^{\kappa>}2$),
- $F_1(p) = \{F_0(\eta) : \eta \in p\}$ (for $p \in \mathbb{Q}_{\kappa}$),
- $F_2(p) = F_0(\mathrm{tr}(p)) = \mathrm{tr}(F_1(p))$ (for $p \in \mathbb{Q}_{\kappa}$).

Now,

$(*)_1$ if $p \in \mathbb{Q}_{\kappa}$ and $\mathrm{Cohen}_{\kappa} \models$ "$F_2(p) \trianglelefteq \nu$" then for some $q \in \mathbb{Q}_{\kappa}$ we have $\mathbb{Q}_{\kappa} \models$ "$p \leq q$" and $F_2(q) = \nu$.

[Why? By the choice of A and we prove this by induction on $\ell g(\nu)$ as in Sect. 1.]

$(*)_2$ If $p \in \mathbb{Q}_{\kappa}$ then $F_1(p) = \{\rho : \rho \trianglelefteq F_0(\mathrm{tr}(p))$ or $F_0(\mathrm{tr}(p)) \triangleleft \rho \in {}^{\kappa>}2\}$.

[Why? As in Sect. 1 or the proof of 6.9.]

$(*)_3$ if $\mathbb{Q}_{\kappa} \models$ "$p \leq q$" then $\mathrm{Cohen}_{\kappa} \models$ "$F_2(p) \trianglelefteq F_2(q)$".

[Why? Obvious.]

Together we are done $\square$

[5] So $\mathscr{I}_{\alpha}$ is not necessarily dense and not necessarily open; without loss of generality $\mathscr{I}_{\alpha}$ is an antichain (but not necessarily maximal). Of course the ϱ later is not necessarily constant.

Claim 6.9 *(1) Assume that $W \subseteq S^\kappa_{\rm pr}$ (see 4.3) is stationary but not reflecting. Then forcing with $\mathbb{Q}_\kappa$ adds a Cohen κ-real.*
(2) Above also $\Pr(\kappa)$ *holds.*

Remark 6.10 We can replace the assumption of 6.9(1) by

(∗) there is a sequence $\bar{\mathscr{I}} = \langle \mathscr{I}_i : i < \kappa \rangle$ of dense open sets such that for no $\partial \in S^\kappa_{\rm inac}$ and $p \in \mathbb{Q}_\partial$ do we have $\mathscr{I}_i \restriction \partial$ is predense in $\mathbb{Q}_\partial$ above p for every $i \in [\ell g(\mathrm{tr}(p)), \partial)$ where $\mathscr{I}_i \restriction \partial = \{p \cap {}^{\partial>}2 : p \in \mathscr{I}_i$ satisfies $\ell g(\mathrm{tr}(p)) < \partial\}$.

That is, if (∗) holds true, then $\mathbb{Q}_\kappa$ adds a κ-Cohen real. We intend to return to it in [29].

Proof (1) Let $W \subseteq S^\kappa_{\rm pr}$ be a non-reflecting stationary set. Choose a sequence $\bar{\rho} = \langle \rho_\partial : \partial \in W \rangle$ such that:

$(\bullet)_1$ $\partial \in W \Rightarrow \rho_\partial \in {}^{\kappa>}2$
$(\bullet)_2$ for each $\rho \in {}^{\kappa>}2$ the set $\{\partial \in W : \rho_\partial = \rho\}$ is stationary.

For every $\partial \in W$ we fix open dense sets $\mathscr{I}^\partial_\varepsilon \subseteq \mathbb{Q}_\partial$ (for $\varepsilon < \partial$) such that:

$(\bullet)_3$ if $p \in \mathbb{Q}_\partial$ then $\lim_\partial(p) \nsubseteq \bigcap_{\varepsilon<\partial} \mathrm{set}(\mathscr{I}^\partial_\varepsilon)$.

Then for $\partial \in W$ we define

$(\bullet)_4$ $A_\partial := {}^\partial 2 \setminus \bigcap_{\varepsilon<\partial} \mathrm{set}(\mathscr{I}^\partial_\varepsilon)$.

Clearly,

$(\bullet)_5$ $A_\partial \in \mathrm{id}(\mathbb{Q}_\partial)$ but $\lim_\partial(p) \cap A_\partial \neq \emptyset$ for every $p \in \mathbb{Q}_\partial$.

Now,

$(\bullet)_6$ for $\partial \in W$ we can find a partition $(A^1_\partial, A^2_\partial)$ of A_∂ such that: for every $p \in \mathbb{Q}_\partial$ we have $\lim_\partial(p) \cap A^\ell_\partial \neq \emptyset$ for $\ell = 1, 2$, equivalently for every $\mathscr{X} \in \mathrm{id}(\mathbb{Q}_\partial)$ and $p \in \mathbb{Q}_\partial$, $\lim_\partial(p) \cap A^\ell_\partial \neq \emptyset$ for $\ell = 1, 2$.

[Why? Since $\mathbb{Q}_\partial$ has cardinality 2^∂ and $\mathrm{id}(\mathbb{Q}_\partial)$ is generated by 2^∂ sets, it is enough to prove that for every $p \in \mathbb{Q}_\partial$ and $\mathscr{X} = {}^\partial 2 \setminus \mathrm{set}(\bar{\mathscr{I}}) \in \mathrm{id}(\mathbb{Q}_\partial)$, where $\bar{\mathscr{I}}$ is a sequence of ∂ maximal antichains of $\mathbb{Q}_\partial$, the set $\mathscr{X} \cap \lim_\partial(p) \cap A_\partial$ has cardinality 2^∂. Without loss of generality $(S_\partial, \bar{\Lambda}_\partial, \bar{p}_\partial, \bar{\mathscr{I}}_\partial)$ is as in 3.16. Given p and $\mathscr{X}$, i.e. $(S_\partial, \bar{\Lambda}_\partial, \bar{p}_\partial, \bar{\mathscr{I}}_\partial)$ we let E be a club of ∂ disjoint to S_p, S_∂ and W and to $[0, \ell g(\mathrm{tr}(p))$. So consider the tree $\mathscr{T} = (\bigcup_{\alpha\in E} {}^\alpha 2) \cup {}^\kappa 2$. Recall $p \cap \mathscr{T}$ is a really closed subtree and for each $\varepsilon < \partial$, $\langle p \cap \mathscr{T}_\partial : p \in \mathscr{I}_{\partial,\varepsilon} \rangle$ is a sequence of closed subtrees with no maximal nodes such that $\lim_\partial(p) = \lim(p \cap \mathscr{T}_\gamma)$ are pairwise disjoint. The rest should be clear.]

We let $\underset{\sim}{\ell}_\partial$ be a $\mathbb{Q}_\kappa$-name for an element of $\{0, 1, 2\}$ such that

$(\bullet)_7$ $\Vdash_{\mathbb{Q}_\kappa}$ "$\underset{\sim}{\ell}_\partial = \iota$ iff $\underset{\sim}{\eta} \restriction \partial \in A^\iota_\partial$" for $\iota = 1, 2$ and $\Vdash_{\mathbb{Q}_\kappa}$ "$\underset{\sim}{\ell}_\partial = 0$ iff $\underset{\sim}{\eta} \restriction \partial \notin A_\partial$".

Lastly, let $\underset{\sim}{\nu}$ be (the $\mathbb{Q}_\kappa$-name for) the concatenation of $\langle \rho_\partial : \partial \in W$ and $\underset{\sim}{\ell}_\partial = 2\rangle$. We will argue that $\Vdash_{\mathbb{Q}_\kappa}$ "$\underset{\sim}{\nu}$ is Cohen over $\mathbf{V}$". To this end we will prove that:

(⊞) if $p \in \mathbb{Q}_\kappa$, $\partial \in W$, $\partial > \ell g(\mathrm{tr}(p))$ then there is $\tau \in p \cap {}^\partial 2$ such that:
(a) $\tau \in A^2_\partial$, equivalently $p^{[\tau]} \Vdash$ "$\underset{\sim}{\ell}_\partial = 2$",

Sh:1004

(b) if $\theta \in W \cap \partial, \theta > \ell g(\mathrm{tr}(p))$ then $\tau \restriction \theta \notin A^2_\theta$, equivalently $p^{[\tau]} \Vdash$"$\underset{\sim}{\ell}_\theta$ is 0 or is 1".

Why is ($\boxplus$) enough? Recalling 5.1, let $(\eta, \bar{\alpha})$ be as there, and we shall show that $\Vdash_{\mathbb{Q}_\kappa}$ "$\underset{\sim}{\nu} \notin X_{\eta,\bar{\alpha}}$". Let $p \in \mathbb{Q}_\kappa$, $j < \kappa$ and let ν_* be the concatenation of

$$\left\{\rho_\partial : \partial \in W,\ \partial \leq \ell g(\mathrm{tr}(p)) \text{ and } \mathrm{tr}(p)\restriction\partial \in A^2_\partial\right\}.$$

Let $\rho_* \in {}^{\kappa>}2$ be such that for some $i \in [j, \kappa)$ we have

$(\bullet)_8$ $\nu_*{}^\frown\rho_*$ has length $\geq \alpha_{i+1}$ and it does include $\eta\restriction[\alpha_i, \alpha_{i+1})$".

Clearly it suffices to prove that for some q:

$(\bullet)_9$ $p \leq_{\mathbb{Q}_\kappa} q$ and $q \Vdash$ "$\nu_*{}^\frown\rho_* \trianglelefteq \underset{\sim}{\nu}$".

By the choice of $\bar{\rho}$, the set $W' = \{\partial \in W : \partial \notin S_p,\ \partial > \ell g(\mathrm{tr}(p))$ and $\rho_\partial = \rho_*\}$ is a stationary subset of κ. Pick $\partial_* \in W'$ and then choose $\tau \in p \cap {}^{\partial_*}2$ as in (a), (b) of ($\boxplus$). Let $q = p^{[\tau]}$.

So the conclusion of 6.9 follows and ($\boxplus$) is indeed enough, but we still owe: Why ($\boxplus$) is true? Let $p \in \mathbb{Q}_\kappa$ as witnessed by $(\mathrm{tr}(p), S_p, \bar{\Lambda}_p)$, and let $\partial \in W$, $\partial > \ell g(\mathrm{tr}(p))$. Put

- $\mathrm{tr}(q) = \mathrm{tr}(p)$,
- $S_q = S_p \cup (W \cap \partial)$, and
- if $\theta \in S_q \setminus S_p$, then $\Lambda_{q,\theta} = \{\mathscr{I}^\theta_\varepsilon : \varepsilon < \theta\}$, and
- if $\theta \in S_p \cap (W \cap \partial)$, then $\Lambda_{q,\theta} = \Lambda_{p,\theta} \cup \{\mathscr{I}^\theta_\varepsilon : \varepsilon < \theta\}$.

This determines a condition $q \in \mathbb{Q}_\kappa$ stronger than p. It follows from the definition of $\bar{\Lambda}_q$ and S_q that

$(\bullet)_{10}$ if $\ell g(\mathrm{tr}(q)) < \theta \in W \cap \partial$, then $q \cap {}^\theta 2 \subseteq \mathrm{set}(\Lambda_{q,\theta}) \subseteq {}^\theta 2 \setminus A_\theta$.

Anyhow by $(\bullet)_6$ we are done.

(2) Let A^ι_∂ for $\partial \in W$ be as in (1) above such that

$(\bullet)_{11}$ $\eta \in A^2_\partial$ implies that $\{\alpha < \partial : \eta(\alpha) = 1\}$ is stationary.

For $\alpha < \kappa$ define

$$\begin{aligned}\mathscr{I}_\alpha = \{p \in \mathbb{Q}_\kappa : \ell g(\mathrm{tr}(p)) > \alpha \text{ and for some } \partial \in (\alpha, \ell g(\mathrm{tr}(p)))\\ \cap W \text{ we have } \mathrm{tr}(p)\restriction\partial \in A^2_\partial\}.\end{aligned}$$

Clearly each $\mathscr{I}_\alpha$ is a dense open subset of $\mathbb{Q}_\kappa$. We will argue that $\langle \mathscr{I}_\alpha : \alpha < \kappa\rangle$ witnesses $\Pr(\kappa)$, that is we show that for each $p \in \mathbb{Q}_\kappa$ we have $\lim_\kappa(p) \nsubseteq \bigcap_{\alpha<\kappa} \mathrm{set}(\mathscr{I}_\alpha)$.

Let $p \in \mathbb{Q}_\kappa$ be witnessed by $(\eta, S, \bar{\Lambda})$ and let $\alpha = \ell g(\eta)$. We will show that $\lim_\kappa(p) \nsubseteq \mathrm{set}(\mathscr{I}_{\alpha+1})$. Towards this let E be a club of κ disjoint from S with $\min(E) = \alpha = \ell g(\mathrm{tr}(p))$ and

$$\min(E) < \alpha \in E \wedge \alpha > \sup(\alpha \cap E) \Rightarrow \alpha \text{ is singular.}$$

Let $\langle \alpha_i : i < \kappa\rangle$ be an increasing enumeration of E. By induction on $i < \kappa$ we choose η_i so that

$(*)_i$ (a) $\eta_i \in p \cap {}^{(\alpha_i)}2$,
(b) $j < i \Rightarrow \eta_j \triangleleft \eta_i \wedge \eta_i(\alpha_i) = 0$,
(c) if $\partial \in W \cap (\alpha_0, \alpha_i]$, then $\eta_i \restriction \partial \notin A^2_\partial$.

This is enough as letting $\eta = \bigcup_{i<\kappa} \eta_i$ we will have $\eta \in \lim_\kappa(p) \setminus \mathrm{set}(\mathscr{I}_{\alpha+1})$.
Why can we carry out the induction?

For $i = 0$ we put $\eta_0 = \mathrm{tr}(p)$,

for a limit i we put $\eta_i = \bigcup_{i<j} \eta_j$ noting that if $\alpha_i \in W$ then η_i is not in $A^2_{\alpha_i}$ by $(\bullet)_{11}$,

for a successor $i = j+1$ we proceed as in the proof of ($\boxplus$) of the first part recalling $\alpha_i \notin W$. □

Claim 6.11 *(1) The assumption of 6.9(1) holds when* $\mathbf{V} = \mathbf{L}$ *and* κ *is Mahlo not weakly compact.*

(2) When the assumption of 6.11(1) or of 6.9(1) hold for κ, *then*

$$\mathrm{cov}(\mathbb{Q}_\kappa) \le \mathrm{cov}(\mathrm{Cohen}_\kappa) \ \ and \ \ \mathrm{cov}(\mathbb{Q}_\kappa) \le \mathrm{non}(\mathrm{Cohen}_\kappa) \le \mathrm{non}(\mathbb{Q}_\kappa).$$

Remark 6.12 (1) So when 6.11(1) applies, the Cichoń diagram for $\mathrm{id}(\mathrm{Cohen}_\kappa)$ and $\mathrm{id}(\mathbb{Q}_\kappa)$ is very different than the $\kappa = \aleph_0$ case, i.e., we have additional inequalities.

(2) In 6.11(1), note that if κ is inaccessible not Mahlo then the conclusion of 6.9(1) holds by 6.8.

Proof (1) Since κ is Mahlo not weakly compact, by a result of Jensen we know that every stationary subset of κ contains a non-reflecting stationary subset. So we may use Observation 4.4(4) and argue that again we are in the case of 6.9(1).
(2) It follows from 6.9, that there is a $\mathbb{Q}_\kappa$-name $\underset{\sim}{\varrho}$ such that for some Borel function $\mathbf{B} : {}^\kappa 2 \longrightarrow {}^\kappa\kappa$ we have

$(*)_1$ $\Vdash_{\mathbb{Q}_\kappa}$ "$\underset{\sim}{\varrho}$ is a κ-Cohen real over $\mathbf{V}$ and $\underset{\sim}{\varrho} = \mathbf{B}(\underset{\sim}{\eta})$".

Hence

$(*)_2$ $\mathrm{cov}(\mathbb{Q}_\kappa) \le \mathrm{cov}(\mathrm{Cohen}_\kappa)$

Why? Let $\mu = \mathrm{cov}(\mathrm{Cohen}_\kappa)$ and let $\langle X_\zeta : \zeta < \mu\rangle$ be a sequence of κ-meagre κ-Borel sets with union ${}^\kappa 2$. Let $\mathbf{B}_\zeta \in \mathrm{id}(\mathbb{Q}_\kappa)$ be such that

$$\eta \in {}^\kappa 2 \setminus \mathbf{B}_\zeta \Rightarrow \mathbf{B}(\eta) \notin X_\zeta.$$

We claim that then $\bigcup_{\zeta<\mu} \mathbf{B}_\zeta = {}^\kappa 2$. If not, then we may pick $\eta \in {}^\kappa 2 \setminus \bigcup_{\zeta<\mu} \mathbf{B}_\zeta$. But now, for every $\zeta < \mu$, $\mathbf{B}(\eta) \notin X_\zeta$, so $\bigcup_{\zeta<\mu} X_\zeta \ne {}^\kappa 2$—a contradiction.

Similarly,

$(*)_3$ $\mathrm{non}(\mathrm{Cohen}_\kappa) \le \mathrm{non}(\mathbb{Q}_\kappa)$.

Why? Let $\{\eta_\zeta : \zeta < \mu\} \subseteq {}^\kappa 2$ be a set not belonging to $\mathrm{id}(\mathbb{Q}_\kappa)$. Then $\{\mathbf{B}(\eta_\zeta) : \zeta < \mu\}$ exemplifies $\mathrm{non}(\mathrm{Cohen}_\kappa) \le \mu$.

Also,

$(*)_4$ $\mathrm{cov}(\mathbb{Q}_\kappa) \le \mathrm{non}(\mathrm{Cohen}_\kappa)$.

Sh:1004

Why? By 5.4(1), noting that its assumption "$\kappa = \sup(S^\kappa_{\rm inac})$" follows by our present assumptions. □

Claim 6.13 *If* $\mathbf{V} = \mathbf{L}$, *then an inaccessible* κ *satisfies* $\Pr(\kappa)$ *iff* κ *is not weakly compact iff* $\mathbb{Q}_\kappa$ *adds a* κ*-Cohen.*

Proof We prove this by considering possible cases.

Case 1 κ *is not Mahlo.*
Then

(a) κ *is not weakly compact,*
(b) $\mathbb{Q}_\kappa$ *add a* κ*-Cohen real by 6.8,*
(c) $\Pr(\kappa)$ *holds by 4.4(1).*

Case 2 κ *is Mahlo not weakly compact.*
By 4.4(4), $S^\kappa_{\rm pr}$ *is a stationary subset of* κ*. By a result of Jensen there is a stationary* $W \subseteq S^\kappa_{\rm pr}$ *which does not reflect. Hence by 6.9 the forcing notion* $\mathbb{Q}_\kappa$ *adds a* κ*-Cohen real and* $\Pr(\kappa)$ *holds true.*

Case 3 κ *is weakly compact.*
Then $\mathbb{Q}_\kappa$ *is* κ*-bounding hence does not add a* κ*-Cohen by 1.9 and* $\Pr(\kappa)$ *fails by 4.4(2), i.e., 3.6(2).* □

7 What about the parallel to "amoeba forcing"?

Definition 7.1 (1) We say that $\mathscr{J} \subseteq \mathbb{Q}$ is nice if $\mathscr{J}^{[\alpha,\pi]} \subseteq \mathscr{J}$ for every $\alpha < \kappa$ and a permutation $\pi : {}^\alpha 2 \longrightarrow {}^\alpha 2$ (remember 4.1(2)).

(2) We say that a family Λ of subsets of $\mathbb{Q}_\kappa$ is nice <u>when</u>: $\Lambda^{[\alpha]} \subseteq \Lambda$ for every $\alpha < \kappa$ (remember 4.1(3)).
(Equivalently, if $\mathscr{I}_1 \in \Lambda$, $\mathscr{I}_2 \subseteq \mathbb{Q}_\kappa$, $\alpha < \kappa$ and $\mathscr{I}_1^{[\alpha,\pi]} = \mathscr{I}_2$ <u>then</u> $\mathscr{I}_2 \in \Lambda$).

(3) For $p \in \mathbb{Q}_\kappa$ let $\mathrm{nb}(p) = \{p^{[\eta,\nu]} : \eta \in p \cap {}^\alpha 2, \nu \in {}^\alpha 2$ for some $\alpha < \kappa\}$.

Claim 7.2 *If* $\Lambda \subseteq \{\mathscr{I} : \mathscr{I} \subseteq \mathbb{Q}_\kappa$ *is predense*$\}$ *has cardinality* $\leq \kappa$ <u>*then*</u> *so is* $\Lambda^{[<\kappa]}$ *and it is nice.*

Proof It follows from 4.2. □

Claim 7.3 *(1) If* $p \in \mathbb{Q}_\kappa$ <u>*then*</u> $\mathrm{nb}(p)$ *is a predense subset of* $\mathbb{Q}_\kappa$.

(2) If $p \in \mathbb{Q}_\kappa$ <u>*then*</u> $\mathrm{nb}(p)$ *is nice and*

$$\mathrm{set}(\mathrm{nb}(p)) = \big\{\eta \in {}^\kappa 2 : \textit{ there is } \nu \in \lim_\kappa(p) \textit{ such that } (\forall^\infty \alpha < \kappa)(\eta(\alpha) = \nu(\alpha))\big\}.$$

(3) [κ *weakly compact] If* $X \in \mathrm{id}(\mathbb{Q}_\kappa)$ <u>*then*</u> *for a dense set of* $p \in \mathbb{Q}_\kappa$ *we have* $\mathrm{set}(\mathrm{nb}(p)) \subseteq {}^\kappa 2 \backslash X$.

Proof (1) Clearly for every $p, q \in \mathbb{Q}_\kappa$ we can choose $\alpha \geq \max\{\ell g(\mathrm{tr}(p), \ell g(\mathrm{tr}(q))\}$ such that $\alpha < \kappa$ and then choose $\eta \in p \cap {}^\alpha 2$, $\nu \in q \cap {}^\alpha 2$ and $\pi \in \mathrm{Sym}({}^\alpha 2)$ such that $\pi(\eta) = \nu$, so $q_1 = p^{[\eta,\nu]} \in \mathrm{nb}(p)$ and q_1, q have a common member ν which is of length $\geq \ell g(\mathrm{tr}(q_1)), \ell g(\mathrm{tr}(q))$, hence q_1, q are compatible.

Sh:1004

(2) Should be clear.
(3) There is a family Λ of $\le \kappa$ maximal antichains of $\mathbb{Q}_\kappa$ such that $X \cap \mathrm{set}(\Lambda) = \emptyset$. Without loss of generality $\Lambda = \Lambda^{[<\kappa]}$ and hence the set $Y = {}^\kappa 2 \backslash \mathrm{set}(\Lambda) \in \mathrm{id}(\mathbb{Q}_\kappa)$ satisfies:

- if $\eta_1, \eta_2 \in {}^\kappa 2$ and $\kappa > \sup\{\alpha < \kappa : \eta_1(\alpha) \ne \eta_2(\alpha)\}$, then $\eta_1 \in Y \Leftrightarrow \eta_2 \in Y$.

Now, as $Y \in \mathrm{id}(\mathbb{Q}_\kappa)$ by 3.6(2) for a dense set of $p \in \mathbb{Q}_\kappa$, $\lim_\kappa(p)$ is disjoint to Y, but by the choice of Y this holds for any $p' \in \mathrm{nb}(p)$, so we are done. □

Definition 7.4 Let $\mathbb{Q}_\kappa^{\mathrm{am}}$ be the following forcing notion:

(A) a member of $\mathbb{Q}_\kappa^{\mathrm{am}}$ has the form (α, p, E) with $\alpha < \kappa$, $p \in \mathbb{Q}_\kappa$, E a club of κ disjoint to S_p and $\mathrm{tr}(p) = \langle\rangle$,
(B) the order on $\mathbb{Q}_\kappa^{\mathrm{am}}$ is: $(\alpha_1, p_1, E_1) \le (\alpha_2, p_2, E_2)$ <u>iff</u>
  (a) $\alpha_1 \le \alpha_2$,
  (b) $p_1 \le_{\mathbb{Q}_\kappa} p_2$,
  (c) $p_1 \cap {}^{(\alpha_1)}2 = p_2 \cap {}^{(\alpha_1)}2$,
  (d) $E_1 \supseteq E_2$ and $E_1 \cap \alpha_1 = E_2 \cap \alpha_1$.
(C) The generic of $\mathbb{Q}_\kappa^{\mathrm{am}}$ is $\underset{\sim}{p}_\kappa = \bigcup\{p \cap {}^{\alpha\ge}2 : (\alpha, p, E) \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_\kappa^{\mathrm{am}}}\}$.

Claim 7.5 *(1) $\mathbb{Q}_\kappa^{\mathrm{am}}$ is a κ-strategically complete κ^+-cc (nicely definable) forcing notion and $\underset{\sim}{p}_\kappa$ is indeed a generic for $\mathbb{Q}_\kappa^{\mathrm{am}}$.*
(2) $\Vdash_{\mathbb{Q}_\kappa^{\mathrm{am}}}$ "$\underset{\sim}{p}_\kappa \in \mathbb{Q}_\kappa$".
(3) Assume κ is weakly compact. If $\mathscr{I}$ is a predense subset of $\mathbb{Q}_\kappa$ (in $\mathbf{V}$) <u>then</u> $\Vdash_{\mathbb{Q}_\kappa^{\mathrm{am}}}$ "$\mathrm{set}(\mathscr{I}) \supseteq \mathrm{set}(\mathrm{nb}(\underset{\sim}{p}_\kappa))$".
(4) Assume κ is weakly compact. Then $\Vdash_{\mathbb{Q}_\kappa^{\mathrm{am}}}$ "${}^\kappa 2 \backslash \mathrm{set}(\mathrm{nb}(\underset{\sim}{p}_\kappa)) \subseteq {}^\kappa 2$ is a member of $\mathrm{id}(\mathbb{Q}_\kappa)$ including all the old κ-Borel sets from $\mathrm{id}(\mathbb{Q}_\kappa)$".

Proof (1) Easy.
(2) Recall that for every $p \in \mathbb{Q}_\kappa$ there is a canonical witness $(\mathrm{tr}(p), S_p, \bar{\Lambda}_p)$ (see 1.3(C)(a)). Let us define some $\mathbb{Q}_\kappa^{\mathrm{am}}$-names:

$(*)_1$ (a) $\underset{\sim}{E} = \bigcap\{E_p : p \in \underset{\sim}{\mathbf{G}}\}$,
  (b) $\underset{\sim}{S} = \bigcup\{S_p : p \in \underset{\sim}{\mathbf{G}}\}$,
  (c) for every $\partial \in \underset{\sim}{S}$, $\underset{\sim}{\Lambda}_\partial = \bigcup\{\Lambda_{p,\partial} : p \in \underset{\sim}{\mathbf{G}}$ satisfies $\partial \in S_p\}$,
  (d) $\underset{\sim}{\bar{\Lambda}} = \langle \underset{\sim}{\Lambda}_\partial : \partial \in \underset{\sim}{S}\rangle$,
  (e) $\underset{\sim}{\varrho}$ is $\langle\rangle$.

Now,

$(*)_2$ for every $\beta < \kappa$, the set

$$\mathscr{I}_\beta := \left\{(\alpha, p, E) \in \mathbb{Q}_\kappa^{\mathrm{am}} : \alpha \ge \beta\right\}$$

is a dense open subset of $\mathbb{Q}_\kappa^{\mathrm{am}}$.

[Why? If $\beta < \kappa$ and $(\alpha_1, p_1, E_1) \in \mathbb{Q}_\kappa^{\mathrm{am}}$ then $(\alpha_1 + \beta, p_1, E_1) \in \mathbb{Q}_\kappa^{\mathrm{am}}$ is above (α_1, p_1, E_1) and belongs to $\mathscr{I}_\beta$.]

$(*)_3$ $\Vdash$ "$\underset{\sim}{E}$ is a club of λ".

[Why? Unbounded as for every $\beta < \kappa$ and $(\alpha_0, p_0, E_0) \in \mathbb{Q}^{\rm am}_\kappa$, let $\alpha_1 = \min\{\delta \in E_0 : \delta > \alpha_0, \delta > \beta\}$ so $(\alpha_1 + 1, p_0, E_0)$ is above (α_0, p_0, E_0) and forces $\delta \in \underset{\sim}{E}$. Being closed is easy, too.]

$(*)_4$ $\Vdash$ "$\underset{\sim}{S}$ is a nowhere stationary subset of $S^\kappa_{\rm inac}$".

[Why? First, for every $\beta < \kappa$, by $(*)_2$ for a dense set of $(\alpha, p, E) \in \mathbb{Q}^{\rm am}_\kappa$ we have $\alpha > \beta$. Since $(\alpha, p, E) \Vdash$ "$\underset{\sim}{S} \cap \alpha = S_p \cap \alpha$", we get that $\underset{\sim}{S} \cap \alpha$ is nowhere stationary and hence $\underset{\sim}{S} \cap \beta$ is nowhere stationary. Second, $\Vdash$ "$\underset{\sim}{S}$ is not stationary" because $\Vdash$ "$\underset{\sim}{E}$ is a club of κ disjoint to $\underset{\sim}{S}$" by the definition of $\mathbb{Q}^{\rm am}_\kappa$. Together we are done.]

$(*)_5$ $\Vdash$ "$\underset{\sim}{\Lambda}_\partial$ is a set of $\leq \partial$ predense subsets of $\mathbb{Q}_\partial$ for $\partial \in \underset{\sim}{S}$".

[Why? Given $(\alpha_0, p_0, E_0) \in \mathbb{Q}^{\rm am}_\kappa$, without loss of generality $\alpha_0 > \partial$ and hence it forces $\underset{\sim}{\Lambda}_\partial$ is $\Lambda_{p_0,\partial}$ if $\partial \in S_{p_0}$, not defined (or $\emptyset$) otherwise; the rest is clear.]

$(*)_6$ $\Vdash$ "$(\underset{\sim}{\varrho}, \underset{\sim}{S}, \bar{\underset{\sim}{\Lambda}})$ witnesses $\underset{\sim}{p}_\kappa \in \mathbb{Q}_\kappa$.

[Why? Read 7.4(C) and $(*)_3$–$(*)_5$.]
(3) It suffices to prove the following:

$(*)_1$ if $\alpha < \kappa$ and $\eta \in {}^\alpha 2$, $\nu \in {}^\alpha 2$ then

$$\Vdash_{\mathbb{Q}^{\rm am}_\kappa} \text{“ if } \eta \in \underset{\sim}{p}_\kappa \cap {}^\alpha 2 \text{ then } \lim(\underset{\sim}{p}^{[\eta,\nu]}) \subseteq {\rm set}(\mathscr{I}) \text{ ”.}$$

Now,

$(*)_2$ fixing α, without loss of generality for every $\pi \in {\rm Sym}({}^\alpha 2)$ we have $\mathscr{I}^{[\alpha,\pi]} = \mathscr{I}$.

[Why? Let $\mathscr{I}_1 = \{p \in \mathbb{Q}_\kappa$: for every $\pi \in {\rm Sym}({}^\alpha 2)$, p is above some member of $\mathscr{I}^{[\alpha,\pi]}\}$. Clearly:

- $\mathscr{I}_1 \subseteq \mathbb{Q}_\kappa$ is predense,
- $\mathscr{I}_1^{[\alpha,\pi]} = \mathscr{I}_1$ for every $\pi \in {\rm Sym}({}^\alpha 2)$,
- ${\rm set}(\mathscr{I}_1) \subseteq {\rm set}(\mathscr{I})$.

Hence we can replace $\mathscr{I}$ by $\mathscr{I}_1$ so finishing the proof of $(*)_2$.]
So

$(*)_3$ in $(*)_1 + (*)_2$, without loss of generality $\nu = \eta$ so $\underset{\sim}{p}_\kappa^{[\nu,n]} = \underset{\sim}{p}_\kappa$.

Let

$(*)_4$ $(\alpha_0, p_0, E_0) \in \mathbb{Q}^{\rm am}_\kappa$ and $\eta \in {}^\alpha 2$.

We shall find $(\alpha_1, p_1, E_1) \in \mathbb{Q}^{\rm am}_\kappa$ above (α_0, p_0, E_0) and forcing that $\eta \notin \underset{\sim}{p}_\kappa$ or forcing the statement in $(*)_1$. First, by $(*)_2$ of the proof of part (2), without loss of generality $\ell g(\eta) < \alpha_0$; so if $\eta \notin p_0$ then $(\alpha_0, p_0, E_0) \Vdash$ "$\eta \notin \underset{\sim}{p}_\kappa$" and we are done. So we can assume $\eta \in p_0$.

As κ is weakly compact for some $\partial \in S^\kappa_{\rm inac}$ which is $> \alpha_0$ we have:

$(*)_5$ the set

$$\mathscr{I}_\partial = \left\{q \cap {}^{\partial>}2 : q \in \mathscr{I} \text{ and } \ell g(\mathrm{tr}(q)) < \partial\right\}$$

is predense in $\mathbb{Q}_\partial$.

Next,

$(*)_6$ for every $\nu \in \mathrm{set}(\mathscr{I}_\partial) \cap p_0$ choose $q_\nu \in \mathscr{I}$ such that $\ell g(\mathrm{tr}(q_\nu)) < \partial$ and $\nu \in \lim_\partial(q_\nu \cap {}^{\partial>}2)$ equivalently $\nu \in q_\nu \cap {}^\partial 2$.

Let

$(*)_7$ (a) $S' = \bigcup\{S_{q_\nu} \backslash \partial : \nu \in \mathrm{set}(\mathscr{I}_\partial) \cap p_0\} \cup S_{p_0} \cup \{\partial\}$,
(b) for $\theta \in S'$ let Λ'_θ be:
(α) $\bigcup\big\{\Lambda : \Lambda = \Lambda_{q_{\nu,\theta}}$ and $\nu \in \mathrm{set}(\mathscr{I}_\partial) \cap p_0$ and $\theta \in S_{q_\nu} \backslash \partial^+$ <u>or</u> $\Lambda = \Lambda_{p_0,\theta}$ and $\theta \in S_{p_0}\big\}$ if $\theta \in S' \backslash \partial^+$,
(β) $\Lambda_{p_0,\theta} \cup \{\mathscr{I}_\partial\}$ if $\theta = \partial \wedge \partial \in S_{p_0}$,
(γ) $\{\mathscr{I}_\partial\}$ if $\theta \in \partial \wedge \partial \notin S_{p_0}$,
(δ) $\Lambda_{p_0,\theta}$ if $\theta \in S_{p_0} \cap \partial$.

Let $p_1 \in \mathbb{Q}_\kappa$ be defined by

$(*)_8$ $(\langle\rangle, S', \bar{\Lambda}')$ will witness p_1, where
- S' is from $(*)_7$,
- $\bar{\Lambda}' = \langle \Lambda'_\theta : \theta \in S' \rangle$, see $(*)_7$,

and let $\alpha_1 = \alpha_0$ and $E_1 \subseteq E_0$ be a club disjoint from S' and such that $E_1 \cap \partial = E_0 \cap \partial$. Now one easily verifies that $(\alpha_1, p_1, E_1) \in \mathbb{Q}^{\mathrm{am}}_\kappa$ is a condition stronger than (α_0, p_0, E_0) and it forces that

$$\eta \lhd \nu \in \underset{\sim}{p}_\kappa \cap {}^\partial 2 \Rightarrow (\exists q \in \mathscr{I})\left(\mathrm{tr}(q) \lhd \nu \in q \wedge \underset{\sim}{p}_\kappa^{[\nu]} \subseteq q\right).$$

(4) Follows by part (3). □

8 Generics and absoluteness

Recall from Definition 0.1 that we say that a set $\mathbf{B} \subseteq {}^\kappa\mathscr{H}(\kappa)$ is

- a κ-stationary Borel if for some κ-Borel function $F : {}^\kappa\mathscr{H}(\kappa) \longrightarrow \mathscr{P}(\kappa)$ we have $\eta \in B \Leftrightarrow F(\eta)$ is stationary,
- κ-nowhere stationary Borel if there is a κ-Borel function $F : {}^\kappa\mathscr{H}(\kappa) \longrightarrow \mathscr{P}(\kappa)$ such that for every $\eta \in {}^\kappa\mathscr{H}(\kappa)$ we have: $\eta \in \mathbf{B}$ <u>iff</u> $F(\eta)$ is a nowhere stationary subset of κ.

Claim 8.1 *1. "$p \in \mathbb{Q}_\kappa$" is*[6] *a κ-nowhere stationary Borel relation (see 0.1(5)), also it is $\Sigma^1_1(\kappa)$.*

[6] Using coding it does not matter whether we use ${}^\kappa 2$ or $\mathscr{P}(\kappa)$ or ${}^\kappa\mathscr{H}(\kappa)$ or $\mathscr{P}(\mathscr{H}(\kappa))$, etc.

Sh:1004

2. *Both "$p \leq_{\mathbb{Q}_\kappa} q$" and "$p, q \in \mathbb{Q}_\kappa$ are compatible" are κ-Borel relations (but pedantically there are κ-Borel relations whose restrictions to $\mathbb{Q}_\kappa$ are the above relations).*
3. *If κ is weakly compact, then "being κ-nowhere stationary Borel" is equivalent to "being κ-Borel".*
4. *If κ is weakly compact then "$\{p_i : i < \kappa\} \subseteq \mathbb{Q}_\kappa$ is predense" is κ-stationary Borel.*
5. *Changing the definition of $\mathbb{Q}_\kappa$, we may get that the relations "$p \in \mathbb{Q}_\kappa$", "$p \leq_{\mathbb{Q}_\kappa} q$" as well as "$p, q \in \mathbb{Q}_\kappa$ are compatible" are κ-Borel and for every limit $\delta < \kappa$ there is an δ-place κ-Borel function giving an increasing sequence of length δ an upper bound.*
   The change does not affect the generic and the derived ideal.

Proof (1, 2) Straightforward. Note that for "$p \in \mathbb{Q}_\kappa$" the main point is "there is a club E of κ disjoint to S_p", as for $S \subseteq \kappa$ statement "$(\forall \alpha < \kappa)(S \cap \alpha$ is not stationary)" is κ-Borel.
(3) Let $F : {}^\kappa\mathscr{H}(\kappa) \longrightarrow \mathscr{P}(\kappa)$ be κ-Borel and let $X = \{A \subseteq \mathscr{H}(\kappa) : F(A)$ is nowhere stationary$\}$. To show that X is κ-Borel it is enough to note that

$$A \subseteq \kappa \text{ is nowhere stationary if and only if } A \text{ does not reflect.}$$

So the assertion should be clear.
(4) We define $F : {}^\kappa(\mathbb{Q}_\kappa) \longrightarrow \mathscr{P}(\kappa)$ as follows. For $\bar{p} \in {}^\kappa(\mathbb{Q}_\kappa)$ let

$$F(\bar{p}) = \left\{\partial \in S^\kappa_{\text{inac}} : \{p_i \cap {}^{\partial>}2 : i < \partial \text{ and } \text{tr}(p) \in {}^{\partial>}2\} \text{ is predense in } \mathbb{Q}_\partial\right\}.$$

Clearly, F is a κ-Borel function (well, replacing ${}^\kappa 2$ by ${}^\kappa(\mathbb{Q}_\kappa)$) and we have:

(∗) $\{p_i : i < \kappa\} \subseteq \mathbb{Q}_\kappa$ is predense iff $F(\bar{p})$ is stationary in κ.

Why? First, if $\{p_i : i < \kappa\}$ is not predense let $q \in \mathbb{Q}_\kappa$ be incompatible with every p_i which means $(\text{tr}(q) \notin p_i) \vee (\text{tr}(p_i) \notin q)$, so easily for every $\partial \in (\ell g(\text{tr}(q)), \kappa)$, $q \cap {}^\partial 2$ witnesses $\partial \notin F(\bar{p})$. Second, if $\{p_i : i < \kappa\}$ is predense, use the proof of "$\mathbb{Q}_\kappa$ is κ-bounding". So we are done (replacing ${}^\kappa(\mathbb{Q}_\kappa)$ by ${}^\kappa 2$ via coding).
(5) We define $\mathbb{Q}'_\kappa$ as the set of all quadruples $q = (\varrho_q, S_q, \bar{\Lambda}_q, E_q)$ such that $(\varrho_q, S_q, \bar{\Lambda}_p)$ is as in Definition 1.3(A), for a unique $T_q = T[q]$ a subtree of ${}^{\kappa>}2$ and E_q is a club of κ disjoint to $S_p \backslash (\ell g(\varrho_q) + 1))$ such that $\ell g(\varrho_q) \in E_q$. We let $q_1 \leq q_2$ iff:

(a) $\varrho_{q_1} \trianglelefteq \varrho_{q_2}$, $S_{q_2} \supseteq S_{q_1} \backslash (\ell g(\varrho_2) + 1)$,
(b) $\partial \in S_{q_1} \backslash (\ell g(\varrho_2) + 1) \Rightarrow \Lambda_{q_1,\partial} \subseteq \Lambda_{q_2,\partial}$,
(c) $\mathbb{Q}_\kappa \models T[q_1] \leq T[q_2]$,
(d) $E_{q_1} \supseteq E_{q_2}$,
(e) if $q_1 \neq q_2$ then $\varrho_{q_1} \neq \varrho_{q_2}$.

[Why the choice of (e)? The motivation is that otherwise an increasing sequence $\bar{p} = \langle p_\alpha : \alpha < \delta < \kappa \rangle$ with $\text{tr}(p_\alpha)$ constant may have no upper bound because $\bigcup_{\alpha<\delta} S_{p_\alpha}$ may reflect in some $\partial > \ell g(\text{tr}(p_\alpha))$. But by the present definition: if $\bar{p}$

Sh:1004

is eventually constant this is trivial; if not then $\rho = \bigcup_{i<\delta} \mathrm{tr}(p_\alpha)$ has length which belongs to $\bigcap_{\alpha<\delta} E_{p_\alpha}$ and we can finish easily.] $\square$

Observation 8.2 *Assume κ is weakly compact. For a set $X \subseteq {}^\kappa\kappa$ we have (a) $\Leftrightarrow$ (c) and (b) $\Leftrightarrow$ (d), where*

(a) X is κ-stationary Borel,
(b) ${}^\kappa\kappa \setminus X$ is κ-stationary Borel,
(c) X is $\Sigma^1_1(\kappa)$,
(d) X is $\Pi^1_1(\kappa)$.

Remark 8.3 Note that the family $\{X \subseteq {}^\kappa\kappa : X \text{ is } \Sigma^1_1(\kappa)\}$ is closed under $(\exists Y \subseteq \kappa)$ and unions/intersections of $\leq \kappa$ elements.

Proof Clause (a) implies clause (c):

Let F_1 be a κ-Borel function from ${}^\kappa\kappa$ to $\mathscr{P}(\kappa)$ such that $X = \{\eta \in {}^\kappa\kappa : \mathbf{B}(\eta)$ is stationary$\}$. Without loss of generality

$(*)_1$ F_1 is defined by the sequence $\bar{\mathbf{B}}_1 = \langle \mathbf{B}_{1,\alpha} : \alpha < \lambda \rangle$, $\mathbf{B}_{1,\alpha}$ a Borel subset of ${}^\kappa\kappa$ such that $F_1(\eta) = \{\alpha : \eta \in \mathbf{B}_{1,\alpha}\}$.

Let $M_\kappa \prec (\mathscr{H}(2^\kappa)^+, \in)$ of cardinality κ be such that $[M_\kappa]^{<\kappa} \subseteq M_\kappa$, $F_1 \in M_\kappa$ (necessarily $\kappa + 1 \subseteq M_\kappa$). Let $\langle M_\alpha : \alpha < \kappa \rangle$ be $\prec$-increasing continuous with union M_κ such that $\|M_\alpha\| \leq |\alpha| + \aleph_0$ and $F_1 \in M_0$ (necessarily $\kappa \in M_0$).

Let $E = \{\mu : \mu < \kappa$ is strong limit cardinal such that $M_\mu \cap \kappa = \mu$ hence $M_\mu \cap \mathscr{H}(\kappa) = \mathscr{H}(\mu)$ and $\alpha < \mu \Rightarrow \|M_\alpha\| < \mu\}$. Clearly E is a club of κ. For $\mu \in E$ let N_μ be the Mostowski collapse of M_μ and let π_μ be the isomorphism from M_μ onto N_μ. Let $F^1_\mu = \pi_\mu(F_1)$ and $\bar{\mathbf{B}}_\mu = \langle \mathbf{B}_{\mu,\alpha} : \alpha < \mu \rangle = \pi_\mu(\bar{\mathbf{B}}_1)$. Now,

$(*)_2$ for $\mu \in E$ (only inaccessible interests us) we have $F^1_\mu : {}^\mu\mu \to \mathscr{P}(\mu)$,
$(*)_3$ for $\eta \in {}^\kappa\kappa$ the following conditions are equivalent:
- (α) $\eta \in X$,
- (β) $\mathscr{U}_\eta := \{\partial < \kappa : \eta{\restriction}\partial \in {}^\partial\partial$ and $F^1_\partial(\eta{\restriction}\partial)$ is a stationary subset of $\partial\}$ is stationary in κ,
- (γ) the tree $\mathscr{T}_\eta$ has no κ-branch, where $\mathscr{T}_\eta = \bigcup_{\alpha<\kappa} T_{\eta,\alpha}$ where $T_{\eta,\alpha}$ is the set of $\rho \in {}^\alpha\kappa$ such that:
  - $\bullet_1$ ρ is an increasing continuous sequence of cardinals from E,
  - $\bullet_2$ $\eta{\restriction}\rho(B) \in {}^{\rho(\beta)}\rho(\beta)$,
  - $\bullet_3$ $\langle F^1_{\rho(\beta)}(\eta{\restriction}\beta) : \beta < \ell g(\alpha)\rangle$ is increasing, i.e., if $\beta_1 < \beta_2 = \ell g(\rho)$ then $F^1_{\rho(\beta_1)}(\eta{\restriction}\beta_1) = F^1_{\rho(\beta_2)}(\eta{\restriction}\beta_2) \cap \beta_1$,
  - $\bullet_4$ $F^1_{\rho(\beta)}(\eta{\restriction}\beta)$ is a non-stationary subset of $\rho(\beta)$,
- (δ) for a stationary set of $\partial < \kappa$, the tree $\mathscr{T}_\eta \cap {}^{\partial>}\partial$ has no ∂-branch.

This suffices because by $(\alpha) \Leftrightarrow (\gamma)$ in $(*)_3$, clearly X is defined by (γ) and this can be expressed by a Π^1_1-formula.

Why does $(*)_3$ hold?

$(\alpha) \Rightarrow (\beta)$:

Let M'_λ be like M_λ but $\{M_\lambda, \bar{M}, \eta\} \in M'_\lambda$ and let $\bar{M}' = \langle M'_\alpha : \alpha < \kappa \rangle$ be like $\bar{M}$ for M'_λ and $\{\bar{M}_\gamma, \bar{M}, \eta\} \in M'_0$ and $E' \subseteq E$ is like E for $\bar{M}'$ and also $N'_\alpha, \pi_\alpha (\alpha \in E')$.

Sh:1004

Easily $\partial \in E' \Rightarrow \pi_\partial(\bar{\mathbf{B}}\restriction\partial) = \pi'_\partial(\bar{\mathbf{B}}_1\restriction\gamma)$, etc. So for a club of $\partial < \kappa$, $F_1(\eta) \cap \partial = F^1_\partial(\eta\restriction\partial)$ and we are easily done.

$(\beta) \Leftrightarrow (\gamma)$:

Easy, too.

$(\gamma) \Leftrightarrow (\delta)$:

Because κ is weakly compact.

Clause (c) implies (a):

Similarly.

Clause (b) iff clause (d):

Similarly. □

Claim 8.4 *Assume κ is weakly compact.*

(1) "$\{p_i : i < \kappa\} \subseteq \mathbb{Q}_\kappa$ is predense" is $\Pi^1_1(\kappa)$; this means $\{(i, \eta) : \eta \in p_i, i < \kappa\}$ is $\Pi^1_1(\kappa)$-set recalling 0.1(3).

(2) "$X = {}^\kappa 2 \setminus \bigcup\{\lim_\kappa(\mathscr{T}_\alpha) : \alpha < \kappa\}$ belongs to $\mathrm{id}(\mathbb{Q}_\kappa)$ each $\mathscr{T}_\alpha$ a subtree of ${}^{\kappa>}2$" is a κ-stationary-Borel relation.

Proof (1) By 8.1(4) and 8.2

(2) As κ is weakly compact, $X \in \mathrm{id}(\mathbb{Q}_\kappa)^+$ iff there is $p \in \mathbb{Q}_\kappa$ such that $\lim_\kappa(p) \subseteq X$ iff there are $\alpha < \kappa$ and q as in 8.1(5) above p such that $T[q] \subseteq \mathscr{T}_\alpha$. So $X \in \mathrm{id}(\mathbb{Q}_\kappa)^+$ is a $\Sigma^1_1(\kappa)$ condition hence "$X \in \mathrm{id}(\mathbb{Q}_\kappa)$" is a $\Pi^1_1(\kappa)$ condition and we finish by 8.2. □

Claim 8.5 *(1) Assume $\mathbb{P}$ is ($<\kappa$)-complete or just strategically κ-complete (i.e. for games with κ moves, COM winning if a play takes κ-moves).*

(a) Satisfying a κ-stationary-Borel is absolute between $\mathbf{V}$ and $\mathbf{V}^{\mathbb{P}}$.

(b) Satisfying a $\Sigma^1_1(\kappa)$ relation is absolute between $\mathbf{V}$ and $\mathbf{V}^{\mathbb{P}}$.

(2) If $\mathbb{P}$ is strategically θ-complete for every $\theta < \kappa$, then "$p \in \mathbb{Q}_\kappa$" is upward absolute from $\mathbf{V}$ to $\mathbf{V}^{\mathbb{P}}$.

Proof Should be clear. □

Observation 8.6 *Being κ-stationary Borel is not equivalent to being κ-Borel.*

Proof Consider $\mathbf{A}_1 = \{S \subseteq \kappa : S$ is stationary$\}$ and $\mathbf{A}_0 = \mathscr{P}(\kappa) \setminus \mathbf{A}_1$. Clearly $\mathbf{A}_1$ is κ-stationary Borel and $\mathbf{A}_0$ is κ-non-stationary Borel (defined naturally). Assume towards contradiction that $\mathbf{A}_1$ is equal to a κ-Borel set $\mathbf{B}$. Let $\mathrm{Cohen}_\kappa = ({}^{\kappa>}2, \triangleleft)$, and let η be the κ-generic real. Then for some truth value $\mathbf{t}$ and $\nu \in {}^{\kappa>}2$ we have $\nu \Vdash_{\mathrm{Cohen}_\kappa}$ "$\underset{\sim}{\eta}^{-1}\{1\} \in \mathbf{B}$ iff $\mathbf{t} = 1$". Let $\iota < 2$, $M \prec (\mathscr{H}(\kappa^+), \in)$ be of cardinality κ, $[M]^{<\kappa} \subseteq M$ and $\mathbf{B}, \kappa \in M$. Now we can find $\nu_\iota \in {}^\kappa 2$ such that $\nu \triangleleft \nu_\iota$ and $\{\nu_\iota\restriction\alpha : \alpha < \kappa\}$ is a subset of Cohen_κ generic over M and $\nu_\iota(\alpha) = \iota$ for a club of $\alpha < \kappa$. By easy absoluteness we get $\nu_\iota \in \mathbf{B}$ iff $\mathbf{t} = 1$, easy contradiction. □

Claim 8.7 *(1) Consistently, κ is weakly compact but being predense in $\mathbb{Q}_\kappa$ is not absolute under κ-complete forcing and hence it is not κ-Borel.*

Sh:1004

(2) Assume κ is weakly compact and moreover (can be gotten by preliminary forcing) this is preserved by adding κ^+, κ-Cohen. Then adding a κ^+, κ-Cohens (i.e. forcing with $\mathrm{Cohen}_{\kappa,\kappa^+}$*) we get the above.*

(3) In part (2) also $\{S \subseteq \kappa : S$ stationary in $\kappa\}$ (is κ-stationary Borel but) its complement is not κ-stationary Borel.

Proof The counterexample will be gotten by forcing by $\mathrm{Cohen}_{\kappa,\kappa^+}$, e.g., when κ is Laver indestructible supercompact but similarly for κ weakly compact by a preliminary forcing and the set S_2 below being $\{\partial < \kappa : \partial$ not Mahlo$\}$.

Assume κ is Mahlo and let $S_1 \subseteq S^\kappa_{\mathrm{inac}}$ be nowhere stationary but unbounded. Let $S_2 \subseteq S^\kappa_{\mathrm{inac}}$ be a stationary subset of $\mathrm{acc}(S_1)$. We define a representation $\mathbb{Q}_1$ of Cohen_κ as follows:

$(*)_1$ (A) $p \in \mathbb{Q}_1$ iff:
 (a) $p = \langle \eta_\partial : \partial \in S_2 \cap \alpha\rangle = \langle \eta_{p,\partial} : \partial \in S_2 \cap \alpha_p\rangle$ for some $\alpha = \alpha_p < \kappa$,
 (b) for each $\partial \in S_2 \cap \alpha_p$, $\eta_\partial \in {}^\partial 2$.
 (B) $\mathbb{Q}_1$ is ordered by $\trianglelefteq$.
 (C) The generic of $\mathbb{Q}_1$ is $\bar{\eta} = \bigcup\{p : p \in \mathbf{G}_{\mathbb{Q}}\}$ and let $\underset{\sim}{Y} = \{\underset{\sim}{\eta}_\partial : \partial \in S_2\}$, where $p \Vdash$“ $\underset{\sim}{\eta}_\partial = \nu$” if $\partial \in S_2 \cap \alpha_p \wedge \eta_{p,\partial} = \nu$.
 (D) The length $\ell g(p)$ of p is the minimal $\alpha < \kappa$ such that $\mathrm{dom}(p) = S_2 \cap \alpha$.

Next we let $p_\eta = \{\rho \in {}^{\kappa>}2 : \rho \trianglelefteq \eta \vee \eta \trianglelefteq \rho\} \in \mathbb{Q}_\kappa$ for $\eta \in {}^{\kappa>}2$. Now

$(*)_2$ $\Vdash_{\mathbb{Q}_1}$ “$\{p_\eta : \eta \in \underset{\sim}{Y}\}$ is a predense subset of $\mathbb{Q}_\kappa$”.

[Why? If not, let $q \in \mathbb{Q}_1$, $q \Vdash_{\mathbb{Q}_1}$ “$\underset{\sim}{p} = (\nu, \underset{\sim}{S}, \langle \underset{\sim}{\Lambda}_\partial : \partial \in \underset{\sim}{S}\rangle) \in \mathbb{Q}_\kappa$ is incompatible with every p_η for $\eta \in \underset{\sim}{Y}$ and $\underset{\sim}{E}_1$ is a club of κ disjoint to $\underset{\sim}{S}$”.

Let $\langle q_i : i < \kappa\rangle$ be increasing continuous in $\mathbb{Q}_1$, $q_0 = q$ and q_{i+1} forces a value to $\underset{\sim}{S} \cap i$, $\langle \underset{\sim}{\Lambda}_\partial : \partial \in \underset{\sim}{S} \cap i\rangle$ and to $\min(\underset{\sim}{E}_1 \backslash i)$ called γ_i. Let

$$E = \left\{\delta < \kappa : \delta \text{ is a limit ordinal and } i < \delta \Rightarrow \ell g(q_i) < \delta \wedge \gamma_i < \delta\right\}.$$

Clearly E is a club of κ, so we can choose $\partial \in S_2 \cap E$. Then $q_\partial \in \mathbb{Q}_1$ is well defined and of length ∂ and it forces a value $(S', \langle \Lambda'_\theta : \theta \in S'\rangle)$ to $(\underset{\sim}{S} \cap \partial, \langle \underset{\sim}{\Lambda}_\theta : \theta \in \underset{\sim}{S} \cap \partial\rangle)$ and this value represents a condition $r \in \mathbb{Q}_\partial$. Moreover, q_∂ forces that $\partial = \sup\{\gamma_i : i < \partial\} = \sup(\underset{\sim}{E}_1 \cap \partial) \in \underset{\sim}{E}_1$ and hence it forces $\partial \notin \underset{\sim}{S}$. Choose $\nu \in \lim_\partial(r) \in {}^\partial 2$ and let $q'_{\partial+1}$ be above q_∂ such that $q'_{\partial+1}(\partial) = \nu$, i.e. $q'_{\delta+1} \Vdash$ “$\nu \in \underset{\sim}{Y}$” and we arrive to an easy contradiction.]

Next, in $\mathbf{V}^{\mathbb{Q}_1}$ we define $\mathbb{Q}_2 = \mathbb{Q}_2[\underset{\sim}{\eta}^1_\kappa]$, $\underset{\sim}{\eta}^1_\kappa$ the generic for $\mathbb{Q}_1$, by

$(*)_3$ (A) $p \in \mathbb{Q}_2$ iff
 (a) $p = (\alpha, \bar{\Lambda}) = (\alpha_p, \bar{\Lambda}_p)$,
 (b) $\alpha_p < \kappa$, $\bar{\Lambda}_p = \langle \Lambda_{p,\partial} : \partial \in S_1 \cap \alpha_p\rangle$,
 (c) each $\Lambda_{p,\partial}$ is a family of $\leq \partial$ dense subsets of $\mathbb{Q}_\partial$ (for $\partial \in S_1 \cap \alpha_p$),
 (d) if $\theta \in S_2 \cap (\alpha + 1)$, then $\theta = \sup\{\partial \in S_1 \cap \theta : \eta_\theta{\restriction}\partial \notin \mathrm{set}(\Lambda_{p,\partial})\}$ (recall $S_2 \subseteq \mathrm{acc}(S_1)$);
 (B) the order is being an initial segment.
 (C) The generic is $\underset{\sim}{\bar{\Lambda}} = \langle \underset{\sim}{\Lambda}_\partial : \partial \in S_1\rangle$.

Sh:1004

Now in $\mathbf{V}^{\mathbb{Q}_1}$ the forcing notion $\mathbb{Q}_2$ is not $(<\kappa)$-complete and even not strategically κ-complete but it is strategically $(<\kappa)$-complete. (It is not strategically κ-complete because given **st**, let $M \prec (\mathscr{H}(\chi), \in)$, $\chi = (2^\kappa)^+$, $M \cap \kappa = \partial \in S_2$, $\|M\| = \partial$, $[M]^{<\partial} \subseteq M$, $\mathbf{st} \in M$, $\underset{\sim}{Y} \in M$).

Now in $\mathbf{V}^{\mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}}_2}$ easily $p = (\langle\rangle, S_1, \bar{\underset{\sim}{\Lambda}})$ belongs to $\mathbb{Q}_\kappa$ and it exemplifies that $\langle p_\eta : \eta \in \underset{\sim}{Y}\rangle$ is not predense. Also $\mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}}_2$ has a dense set closed subset equivalent to κ-Cohen and similarly $\mathbb{Q}_1$, hence $\Vdash_{\mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}}_2}$"$\kappa$ is weakly compact" and $\Vdash_{\mathbb{Q}_1}$"κ is weakly compact". So there are κ-Borel functions $\mathbf{B}_1, \mathbf{B}_2$ with domain ${}^\kappa 2$ and such that

$$\Vdash_{\text{Cohen}_\kappa} \text{“ } \mathbf{B}_1(\underset{\sim}{\eta}_\kappa) \text{ is generic over } \mathbf{V} \text{ for } \mathbb{Q}_1 \text{ and } \mathbf{B}_2(\underset{\sim}{\eta}_\kappa) \text{ is generic over } \mathbf{V}[\mathbf{B}_1(\underset{\sim}{\eta}_\kappa)] \text{ for } \mathbb{Q}_2[\mathbf{B}_1(\underset{\sim}{\eta}_\kappa)] \text{ ”.}$$

Assume that in $\mathbf{V}^{\mathbb{Q}_1}$, $\mathbf{B}$ is a (definition of a) κ-Borel subset of $[\mathscr{H}(\kappa)]^\kappa$ which is the set of predense subsets of $\mathbb{Q}_\kappa$, so in $\mathbf{V}^{\mathbb{Q}_1 * \underset{\sim}{\mathbb{Q}}_2}$, $\mathbf{B}$ no longer satisfies this. This is somewhat weaker than the desired conclusion, but if $\bar{\eta} = \langle \eta_\gamma : \gamma < \kappa^+\rangle$ is generic for $\text{Cohen}_{\kappa,\kappa^+}$ and $\mathbf{B} \in \mathbf{V}[\bar{\eta}]$ is a (definition of a) κ-Borel subset of $[\mathscr{H}(\kappa)]^\kappa$, for some $\alpha < \kappa$, $\mathbf{B} \in \mathbf{V}[\bar{\eta}\restriction\alpha]$ and interpret η_α as the generic $\mathbb{Q}_1 * \mathbb{Q}_2$. Consider $\bar{p} = \mathbf{B}_1(\eta_\alpha)$.

Now we can compute $\mathbf{B}_1(\bar{p})$ in $\mathbf{V}[\eta\restriction\alpha, \bar{p}]$ and in $\mathbf{V}[\eta\restriction\alpha, \eta_\alpha]$. As $\mathbf{B}$ is κ-Borel, we should get the same result, but they are not the same. A contradiction. □

Definition 8.8 (1) We say M is a κ-model <u>when</u>:

- (a) $M \subseteq (\mathscr{H}(\kappa^+), \in)$ is transitive of cardinality κ, $[M]^{<\kappa} \subseteq M$ and M is a model of ZFC^- (i.e. power set axiom omitted);
- (b) similarly for $(\mathscr{H}_{<\kappa^+}(\mathbf{U}), \in)$, $\mathbf{U}$ a set of ure-elements.

(2) We say η is a $(M, \mathbb{Q}, \underset{\sim}{\eta})$-generic κ-real <u>when</u> (as in [38]):

- (a) $\mathbb{Q}$ is a forcing notion definable in M, (absolutely enough in the interesting cases),
- (b) $\underset{\sim}{\eta} \in M$ a $\mathbb{Q}$-name of κ-real, defined by a Borel function from a sequence of κ truth values of the form "$p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}}$",
- (c) there is $\mathbf{G} \subseteq \mathbb{Q}^M$ generic over M such that $\underset{\sim}{\eta}[\mathbf{G}] = \eta$.

Observation 8.9 *(1) A κ-Borel set $\mathbf{B}$ belongs to* $\text{id}(\mathbb{Q}_\kappa)$ *<u>iff</u> for some κ-real $\mathbf{c} = \mathbf{c}_\mathbf{B}$ for every κ-model M to which $\mathbf{c}$ belongs we have:*

- *if ν is $(M, \mathbb{Q}_\kappa, \underset{\sim}{\eta})$-generic real <u>then</u> $\nu \notin \mathbf{B}$.*

(2) If M is a κ-model, $M \models$ "$\mathbb{Q}$ is $(<\kappa)$-strategically complete forcing notion (set or class in M sense) (or a definition of $\mathbb{Q}$)" and $\mathbf{G} \subseteq \mathbb{Q}^M$ is generic over M <u>then</u> $M[\mathbf{G}]$ is a κ-model.

Definition 8.10 1. We say a set $X \subseteq {}^\kappa\mathscr{H}(\kappa)$ is $\kappa - \text{id}_\kappa$-Borel <u>when</u>:

- (a) id_κ is an ideal on $\mathscr{P}(\kappa)$,
- (b) for some κ-Borel function $F : {}^\kappa\mathscr{H}(\kappa) \longrightarrow \mathscr{P}(\kappa)$ for every $\eta \in {}^\kappa\mathscr{H}(\kappa)$ we have: $\eta \in X$ iff $F(\eta) \in \text{id}$.

Here (in (2), (3)) we may omit κ when clear from the context.

2. Similarly for id_κ^+.
3. Let $\mathrm{id}_{\mathrm{wc}}(\kappa)$ be the weakly compact ideal on κ.

So

Observation 8.11 *Letting* $\mathrm{id}_{\mathbf{nst}}(\kappa)$ *be the non-stationary ideal on* κ*,* κ–$\mathrm{id}^+_{\mathbf{nst}}(\kappa)$*-Borel means* κ*-stationary Borel.*

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