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THE COLOURING EXISTENCE THEOREM REVISITED

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Abstract. We prove a better colouring theorem for $\aleph_4$ and even $\aleph_3$. This has a general topology consequence.

1. Introduction

1.1. Background. Our aim is to improve some colouring theorems of [10], [6, Ch. III, §4], they continue Todorćević [5] (introducing the walks) and [9], [8, §3] (and [11]), see history in [6], [7, §10]. After these works Moore [3] proved $\aleph_1 \mapsto [\aleph_1; \aleph_1]^2_{\aleph_0}$; Eisworth [1] and Rinot [4] proved equivalence of some colouring theorems on successor of singular cardinals.

Our aim is to prove better colouring theorems on successor of regular cardinals (when not too small), e.g. $\mathrm{Pr}_1(\aleph_3, \aleph_3, \aleph_3, (\aleph_0, \aleph_1))$, see §1. We have looked at the matter again because Juhász–Shelah [2] needs such theorem in order to solve a problem in general topology, see 2.10(3).

1.2. Results. The paper is self contained.

Here we formulate $\mathrm{Pr}_\ell(\lambda, \mu, \sigma, \bar\theta)$ where $\bar\theta$ is a pair (θ_0, θ_1) of cardinals rather than a single cardinal θ and prove e.g. $\mathrm{Pr}_1(\lambda, \lambda, \lambda, (\theta, \theta^+))$ when $\lambda = \theta^{+3}$ and θ is regular.

That is, we shall prove (see Definition 2.1 and Conclusion 2.10(1)):

THEOREM 1.1. 1) *For any regular* κ *we have* $\mathrm{Pr}_1(\kappa^{+4}, \kappa^{+4}, \kappa^{+4}, \kappa^+)$.

2) *For any regular* κ *we have* $(\mathrm{Pr}_1(\kappa^{+4}, \kappa^{+4}, \kappa^{+4}, (\kappa, \kappa^+))$ *and* $(\mathrm{Pr}_{0,0}(\kappa^{+4}, \kappa^{+4}, \kappa^{+4}, (\aleph_0, \kappa^+))$.

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REMARK 1.2. Note that the statement $\mathrm{Pr}_0(\kappa^{+4},\kappa^{+4},2,\kappa^+)$ is also called by Juhasz $\mathrm{Col}(\kappa^{+4},\kappa)$, see more in the end of §1.

Moreover by 2.11 in 1.1(2) we can replace κ^{+4} by κ^{+3}, (thus half solving Problem 1 of [2], i.e. for $\aleph_3$ though not for $\aleph_2$) so we naturally ask:

QUESTION 1.3. 1) *Do we have* $\mathrm{Pr}_1(\aleph_2,\aleph_2,\sigma,\aleph_1)$ *for* $\sigma=\aleph_2$*? For* $\sigma=2$*?*
2) *Do we have at least* $\mathrm{Pr}^{\mathrm{uf}}_{0,0}(\aleph_2,\aleph_2,2,(\aleph_0,\aleph_1))$*?*

Concerning the result of Juhász–Shelah [2] by using 2.8(1) instead of [6, Ch. III, §4] we can deduce $\mathrm{Pr}_0(\aleph_4,\aleph_4,2,(\aleph_0,\aleph_1))$ which is sufficient for the topological result there. Moreover by 3.5 + 2.5 even $\mathrm{Pr}_{0,0}(\aleph_3,\aleph_3,2,(\aleph_0,\aleph_1))$ holds, see 2.10 so there is a topological space as desired in [2] with weight $\aleph_3$, see 2.11(2).

We can also generalize the other conclusion of [6, Ch. III, §4] replacing θ by (θ_0,θ_1). This may be dealt with later. Also in [12] and better [13] we intend to improve 2.11 for most cardinals.

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2. Definitions and some connections

DEFINITION 2.1. Assume $\lambda\ge\mu\ge\sigma+\theta_0+\theta_1,\bar\theta=(\theta_0,\theta_1)$; if $\theta_0=\theta_1$ we may write θ_0 instead of $\bar\theta$.

1) Let $\mathrm{Pr}_0(\lambda,\mu,\sigma,\bar\theta)$ mean that there is $\mathbf{c}:[\lambda]^2\to\sigma$ witnessing it which means:

$(*)_{\mathbf{c}}$ if (a) then (b) where:

(a) (α) for $\iota=0,1,\bar\zeta^\iota=\langle\zeta^\iota_{\alpha,i}:\alpha<\mu,i<\mathbf{i}_\iota\rangle$ is a sequence without repetitions of ordinals $<\lambda$ and $\mathrm{Rang}(\bar\zeta^0),\mathrm{Rang}(\bar\zeta^1)$ are disjoint and $\mathbf{i}_0<\theta_0$, $\mathbf{i}_1<\theta_1$

(β) $h:\mathbf{i}_0\times\mathbf{i}_1\to\sigma$

(b) for some $\alpha_0<\alpha_1<\mu$ we have:

• if $i_0<\mathbf{i}_0$ and $i_1<\mathbf{i}_1$ then $\mathbf{c}\{\zeta^0_{\alpha_0,i_0},\zeta^1_{\alpha_1,i_1}\}=h(i_0,i_1)$.

2) For $\iota\in\{0,1\}$ let $\mathrm{Pr}_{0,\iota}(\lambda,\mu,\sigma,\bar\theta)$ be defined similarly but we replace $(a)(\beta)$ and (b) by $(a)(\beta)'$ and $(b)'$, where

(a) $(\beta)'$ $h:\mathbf{i}_\iota\to\sigma$

$(b)'$ for some $\alpha_0<\alpha_1<\mu$ we have

•$'$ if $i_0<\mathbf{i}_0$ and $i_1<\mathbf{i}_1$ then $\mathbf{c}\{\zeta^0_{\alpha_0,i_0},\zeta^1_{\alpha_1,i_1}\}=h(i_\iota)$.

3) Let $\mathrm{Pr}^{\mathrm{uf}}_{0,\iota}(\lambda,\mu,\sigma,\bar\theta)$ mean that some $\mathbf{c}:[\lambda]^2\to\sigma$ witnesses it which means:

$(*)^{\mathrm{uf}}_{\mathbf{c}}$ if (a) then (b) where

(a) (α) as above

(β) $h:\mathbf{i}_\iota\to\sigma$ and D is an ultrafilter on $\mathbf{i}_{1-\iota}$

(b) for some $\alpha_0<\alpha_1<\mu$ we have

- if $i < \mathbf{i}_\iota$ then $\{j < \mathbf{i}_{1-\iota} : \mathbf{c}\{\zeta^\iota_{\alpha_\iota,i}, \zeta^{1-\iota}_{\alpha_{1-\iota},j}\} = h(i)\}$ belongs to D.

DEFINITION 2.2. Assume $\lambda \geq \mu \geq \sigma + \theta_0 + \theta_1, \bar\theta = (\theta_0, \theta_1)$. Let $\mathrm{Pr}_1(\lambda, \mu, \sigma, \bar\theta)$ mean that there is $\mathbf{c} : [\lambda]^2 \to \sigma$ witnessing it, which means:

$(*)_{\mathbf{c}}$ if (a) then (b), where:

(a) for $\iota = 0, 1, \mathbf{i}_\iota < \theta_\iota$ and $\bar\zeta^\iota = \langle \zeta^\iota_{\alpha,i} : \alpha < \mu, i < \mathbf{i}_\iota \rangle$ are sequences of ordinals of λ without repetitions, $\mathrm{Rang}(\bar\zeta^\iota)$ are disjoint and $\gamma < \sigma$

(b) there are $\alpha_0 < \alpha_1 < \mu$ such that $\forall i_0 < \mathbf{i}_0$, $\forall i_1 < \mathbf{i}_1, \mathbf{c}\{\zeta^0_{\alpha_0,i_0}, \zeta^1_{\alpha_1,i_1}\} = \gamma$.

REMARK 2.3. 1) So if $\theta_0 = \theta = \theta_1$ and $\bar\theta = (\theta_0, \theta_1)$ then for $\ell \in \{0, 1\}$, $\mathrm{Pr}_\ell(\lambda, \mu, \sigma, \bar\theta)$ is $\mathrm{Pr}_\ell(\lambda, \mu, \sigma, \theta)$ from [6, Ch. III].

2) We do not write down the monotonicity and trivial implications concerning Definitions 2.1 and 2.5 below.

3) The disjointness of $\{\zeta^0_{\alpha,i} : \alpha < \mu, i < \mathbf{i}_0\}, \{\zeta^1_{\alpha,i} : \alpha < \mu, i < \mathbf{i}_1\}$ in Definition 2.1(1)(a)(α) and 2.1(2), 2.1(3) and 2.2(a) is not really necessary.

NOTATION 2.4. $\mathrm{pr} : \mathrm{Ord} \times \mathrm{Ord} \to \mathrm{Ord}$ is the standard pairing function.

Variants are

DEFINITION 2.5. Let $\lambda \geq \mu \geq \sigma + \theta_0 + \theta_1$ and $\bar\theta = (\theta_0, \theta_1)$.

1) Let $\mathrm{Qr}_0(\lambda, \mu, \sigma, \bar\theta)$ mean that there is $\mathbf{c} : [\lambda]^2 \to \sigma$ witnessing it which means:

$(*)_{\mathbf{c}}$ if (a) then (b) where

(a) (α) $u^\iota_\alpha \in [\lambda]^{<\theta_\iota}$ for $\iota < 2$ and $\alpha < \mu$

(β) $u_\alpha = u^0_\alpha \cup u^1_\alpha$ for every $\alpha < \mu$

(γ) $\langle u_\alpha : \alpha < \mu \rangle$ are pairwise disjoint

(δ) $h^\iota_\alpha : u^\iota_\alpha \to \sigma$ for $\iota < 2, \alpha < \mu$ and $\mathrm{pr} : \sigma \times \sigma \to \sigma$

(b) for some $\alpha_0 < \alpha_1 < \mu$ for every $(\zeta_0, \zeta_1) \in (u^0_{\alpha_0} \times u^1_{\alpha_1})$ we have $\zeta_0 < \zeta_1$ and $\mathbf{c}\{\zeta_0, \zeta_1\} = \mathrm{pr}(h^0_{\alpha_0}(\zeta_0), h^1_{\alpha_1}(\zeta_1))$.

2) Let $\mathrm{Qr}_{0,\iota}(\lambda, \mu, \sigma, \bar\theta)$ be defined similarly but each $h^{1-\iota}_\alpha$ is constant.

3) Let $\mathrm{Qr}_1(\lambda, \mu, \sigma, \bar\theta)$ be defined as above but each h^0_α and each h^1_α is a constant function.

4) Let $\mathrm{Qr}^{\mathrm{uf}}_{0,\iota}(\lambda, \mu, \sigma, \bar\theta)$ be defined parallely to Definition 2.1.

So, e.g.

OBSERVATION 2.6. 1) *If* $\mathrm{cf}(\mu) \geq \sigma^+$, *then* $\mathrm{Pr}_1(\lambda, \mu, \sigma, \bar\theta)$ *is equivalent to* $\mathrm{Qr}_1(\lambda, \mu, \sigma, \bar\theta)$.

2) *Recall that* $\mathrm{Pr}_\ell(\lambda, \mu, \sigma, \theta)$ *is* $\mathrm{Pr}_\ell(\lambda, \mu, \sigma, (\theta, \theta))$.

3) $\mathrm{Qr}_0(\lambda, \mu, \sigma, \bar\theta)$ *implies* $\mathrm{Pr}_0(\lambda, \mu, \sigma, \bar\theta)$; *similarly for the other variants*, $\mathrm{Qr}_{0,\iota}, \mathrm{Qr}^{\mathrm{uf}}_{0,\iota}$.

PROOF. Should be clear. $\square_{2.6}$

OBSERVATION 2.7. *Let* $\bar\theta = (\theta_0, \theta_1)$ *and* $\iota \in \{0, 1\}$.

1) *If* $\iota < 2, \partial < \theta_\iota \Rightarrow \sigma^\partial < \mathrm{cf}(\mu)$ *and* $\theta_0, \theta_1 < \mathrm{cf}(\mu)$, *then* $\mathrm{Pr}_{0,\iota}(\lambda, \mu, \sigma, \bar{\theta})$ *is equivalent to* $\mathrm{Qr}_{0,\iota}(\lambda, \mu, \sigma, \bar{\theta})$.

2) *If* $\partial < \theta_0 + \theta_1 \Rightarrow \sigma^\partial < \mathrm{cf}(\mu)$, *then* $\mathrm{Pr}_0(\lambda, \mu, \sigma, \bar{\theta}) \Leftrightarrow \mathrm{Qr}_0(\lambda, \mu, \sigma, \bar{\theta})$.

PROOF. Obvious but we elaborate.

1) By 2.6(3) we have one implication; so assume $\mathrm{Pr}_{0,\iota}(\lambda, \mu, \sigma, \bar{\theta})$ and we shall prove $\mathrm{Qr}_{0,\iota}(\lambda, \mu, \sigma, \bar{\theta})$, so let $u_\alpha = u^0_\alpha \cup u^1_\alpha$ for $\alpha < \mu$ and $h^\iota_\alpha : u^\iota_\alpha \to \sigma$ and $\mathrm{pr} : \sigma \times \sigma \to \sigma$ be as in Definition 2.5(1) and each $h^{1-\iota}_\alpha$ is constant.

We should prove that there are $\alpha_0 < \alpha_1 < \mu$ as promised in Definition 2.5(2). As $|u^{1-\iota}_\alpha| < \theta_{1-\iota}$ and $\theta_{1-\iota} < \mathrm{cf}(\mu)$, without loss of generality for some $\varepsilon_{1-\iota} < \theta_{1-\iota}$ we have $\alpha < \mu \Rightarrow \mathrm{otp}(u^{1-\iota}_\alpha) = \varepsilon_{1-\iota}$. As $\theta_\iota < \mathrm{cf}(\mu)$ hence without loss of generality for some $\varepsilon_\iota < \theta_\iota$ we have $\alpha < \mu \Rightarrow \mathrm{otp}(u^\iota_\alpha) = \varepsilon_\iota$. Moreover, noting $\sigma^{|\varepsilon_\iota|} < \mathrm{cf}(\mu)$, without loss of generality $\{(\mathrm{otp}(\zeta \cap u^\iota_\alpha), h^\iota_\alpha(\zeta)) : \zeta \in u^\iota_\alpha\}$ is the same for all $\alpha < \mu$. Now we can apply $\mathrm{Pr}_{0,\iota}(\lambda, \mu, \sigma, \bar{\theta})$.

2) Similarly. $\square_{2.7}$

CLAIM 2.8. 1) *Let* $\iota < 2$. *If* $\mathrm{Pr}_1(\lambda, \mu, \sigma_1, \bar{\theta})$ *and* $\lambda = \mu = \mathrm{cf}(\mu)$, $\bar{\theta} = (\theta_0, \theta_1)$, $\theta = \theta_0 + \theta_1 < \mu$ *and* $2^\chi \ge \lambda, \chi^{<\theta_\iota} + (\sigma_2)^{<\theta_\iota} \le \sigma_1$ *and* $\chi^{<\theta_\iota} < \mu$ *and* $(\sigma_2)^{<\theta_\iota} < \mu$ *then* $\mathrm{Pr}_{0,\iota}(\lambda, \mu, \sigma_2, \bar{\theta})$ *and* $\mathrm{Qr}_{0,\iota}(\lambda, \mu, \sigma_2, \theta)$.

1A) *If the assumptions of part* (1) *hold for both* $\iota = 0$ *and* $\iota = 1$, *then we can conclude* $\mathrm{Pr}_0(\lambda, \mu, \sigma_2, \bar{\theta})$ *and* $\mathrm{Qr}_0(\lambda, \mu, \sigma_2, \bar{\theta})$.

2) *If* $\lambda = \sigma^+$ *and* $\sigma = \sigma^{<\theta_\iota}$ *then* $\mathrm{Pr}_{0,\iota}(\lambda, \lambda, \sigma, \bar{\theta})$ *implies* $\mathrm{Pr}_{0,\iota}(\lambda, \lambda, \lambda, \bar{\theta})$.

3) *If* $\lambda = \sigma^+$ *and* $\sigma = \sigma^{<(\theta_0+\theta_1)}$ *then* $\mathrm{Pr}_0(\lambda, \lambda, \sigma, \bar{\theta})$ *implies* $\mathrm{Pr}_0(\lambda, \lambda, \lambda, \bar{\theta})$.

4) *If* $\mathrm{Pr}_1(\lambda, \mu, \sigma, \bar{\theta})$ *and* $\sigma \le \chi = \chi^{<(\theta_0+\theta_1)} < \lambda \le 2^\chi$ *then* $\mathrm{Pr}_0(\lambda, \mu, \sigma, \bar{\theta})$.

5) *If* $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \bar{\theta}), \lambda = \partial^+$ *and* $\partial = \partial^{<(\theta_0+\theta_1)}$ *then* $\mathrm{Pr}_0(\lambda, \lambda, \lambda, \bar{\theta})$.

REMARK 2.9. 1) Claim 2.8(1) is similar to [6, Ch. III, 4.5(3), pp. 169-170] but we shall elaborate.

2) The condition $\lambda = \mu$ can be omitted if we systematically use $\mathbf{c} : \lambda \times \lambda \to \sigma$.

PROOF. 1) Recalling $\lambda \le 2^\chi$ and $\chi^{<\theta_\iota} + (\sigma_2)^{<\theta_\iota} \le \sigma_1$ hence $\chi^{<\theta_\iota} + 2^{<\theta_\iota} \le \sigma_1$, choose

$(*)_1$ (a) $A_\alpha \subseteq \chi$ (for $\alpha < \lambda$) which are pairwise distinct.

(b) Let $\{(a_i, d_i) : i < \sigma_1\}$ be a list (maybe with repetitions) of the pairs (a, d) satisfying $a \subseteq \chi, |a| < \theta_\iota$ and d a function from $\mathscr{P}(\mathrm{a})$ to σ_2 such that

$$\big|\{b : b \subseteq a \text{ and } d(b) \neq 0\}\big| < \theta_\iota.$$

Choose

$(*)_2$ $\mathbf{c}$ to be a symmetric two-place function from λ to σ_1 exemplifying

$$\mathrm{Pr}_1(\lambda, \mu, \sigma_1, \bar{\theta}).$$

Now we define the two place function $\mathbf{d}$ from λ to σ_2 as follows: for $\alpha_0 < \alpha_1$:

$$\mathbf{d}(\alpha_0, \alpha_1) = \mathbf{d}(\alpha_1, \alpha_0) := d_{\mathbf{c}(\alpha_0,\alpha_1)}(A_{\alpha_\iota} \cap a_{\mathbf{c}(\alpha_0,\alpha_1)}).$$

We shall show that $\mathbf{d}$ witnesses $\mathrm{Qr}_{0,\iota}(\lambda, \mu, \sigma_2, \bar{\theta})$ thus finishing upon using Observation 2.7(1) which yields the parallel assertion about $\mathrm{Pr}_{0,\iota}(\lambda, \mu, \sigma_2, \bar{\theta})$ because its assumption on the cardinals follows from those of 2.8(1), i.e. recall $\lambda = \mu = \mathrm{cf}(\mu)$ and $\theta_0 + \theta_1 < \lambda$ so $\theta_\iota < \mathrm{cf}(\mu)$ and $\sigma_2^{<\theta_\iota} < \mu$. So let $\langle t_\alpha : \alpha < \mu\rangle$ be pairwise disjoint subsets of $\lambda, t_\alpha = t^0_\alpha \cup t^1_\alpha$ and $h^\iota_\alpha : t^\iota_\alpha \to \sigma_2$ such that $h^{1-\iota}_\alpha$ is constant, $|t^0_\alpha| < \theta_0, |t^1_\alpha| < \theta_1$ and $\mathrm{pr} : \sigma_2 \times \sigma_2 \to \sigma_2$. As $\lambda = \mu = \mathrm{cf}(\mu)$ without loss of generality $\alpha < \beta < \mu \Rightarrow \sup(t_\alpha) < \min(t_\beta)$. We have to find $\alpha_0 < \alpha_1$ as in the definition of $\mathrm{Qr}_{0,\iota}(\lambda, \mu, \sigma_\iota, \bar{\theta})$ see Definition 2.5. As by assumption $\mu = \mathrm{cf}(\mu) > \theta$ and, of course, $\alpha < \mu \wedge \ell < 2 \Rightarrow \mathrm{otp}(t^\ell_\alpha) < \theta_\ell \le \theta$ without loss of generality there are $\varepsilon^*_0 < \theta_0$, $\varepsilon^*_1 < \theta_1$ such that $\bigwedge\limits_\alpha \mathrm{otp}(t^\ell_\alpha) = \varepsilon^*_\ell$ for $\ell = 0, 1$.

For each $\alpha < \mu$ and $\ell < 2$ let $t^\ell_\alpha = \{\zeta^\ell_{\alpha,\varepsilon} : \varepsilon < \varepsilon^*_\ell\}$ with $\zeta^\ell_{\alpha,\varepsilon}$ increasing with ε. As $|\{\langle h^\iota_\alpha(\zeta^\iota_{\alpha,\varepsilon}) : \varepsilon < \varepsilon^*_\iota\rangle : \alpha < \mu\}| \le \sigma_2^{|\varepsilon^*_\iota|} \le \sigma_2^{<\theta_\iota} < \mu = \mathrm{cf}(\mu)$, without loss of generality $h^\iota_\alpha(\zeta^\iota_{\alpha,\varepsilon}) = \xi^\iota_\varepsilon < \sigma_2$ for all $\varepsilon < \varepsilon^*_\iota$ and $h^{1-\iota}_\alpha(\zeta^{1-\iota}_{\alpha,\varepsilon}) = \xi^{1-\iota}_\varepsilon$ which does not depend on α. Renaming without loss of generality $\mathrm{pr}(\xi^0_{\varepsilon(0)}, \xi^1_{\varepsilon(1)}) = \xi_{\varepsilon(\iota)}$, so rename it $\xi_{\varepsilon(\iota)}$ for $\varepsilon(0) < \varepsilon^*_0, \varepsilon(1) < \varepsilon^*_1$.

We should find $\alpha_0 < \alpha_1 < \mu$ such that for $\varepsilon_0 < \varepsilon^*_0$, $\varepsilon_1 < \varepsilon^*_1$ we have $\zeta_{\alpha_0,\varepsilon_0} < \zeta_{\alpha_1,\varepsilon_1}$ (which follows) and $\mathbf{d}(\zeta^0_{\alpha_0,\varepsilon_0}, \zeta^1_{\alpha_1,\varepsilon_1}) = \mathrm{pr}(h^0_{\alpha_0}(\zeta^0_{\alpha_0,\varepsilon_0}), h^1_{\alpha_1}(\zeta^1_{\alpha_1,\varepsilon_1}))$ which is equal to $\mathrm{pr}(\xi_{\varepsilon_0}, \xi_{\varepsilon_1})$. Choose $a_\alpha \subseteq \chi$, $|a_\alpha| = |\varepsilon^*_\iota| < \theta_\iota$ such that $\langle A_{\zeta^\iota_{\alpha,\varepsilon}} \cap a_\alpha : \varepsilon < \varepsilon^*_\iota\rangle$ is a sequence of pairwise distinct subsets of a_α. As $\mathrm{cf}(\mu) = \mu > \chi^{<\theta_\iota}$ without loss of generality for every $\alpha < \lambda = \mu$ we have $a_\alpha = a^*$ and $A_{\zeta^\iota_{\alpha,\varepsilon}} \cap a^* = a^*_\varepsilon$ for all $\varepsilon < \varepsilon^*_\iota$.

For some $i < \sigma_1$ we have $a_i = a^*$ and $d_i(a^*_\varepsilon) = \xi_\varepsilon$ for every $\varepsilon < \varepsilon^*_\iota$. By the choice of $\mathbf{c}$ for some $\alpha_0 < \alpha_1 < \mu$ the function $\mathbf{c}{\restriction}t_{\alpha_0} \times t_{\alpha_1}$ is constantly i, so $\varepsilon_0 < \varepsilon^*_0 \wedge \varepsilon_1 < \varepsilon^*_1 \Rightarrow \mathbf{c}(\zeta^0_{\alpha_0,\varepsilon_0}, \zeta^1_{\alpha_1,\varepsilon_1}) = i$, hence for every $(\varepsilon_0, \varepsilon_1) \in \varepsilon^*_0 \times \varepsilon^*_1$ we have

$$\mathbf{d}(\zeta^0_{\alpha_0,\varepsilon_0}, \zeta^1_{\alpha_1,\varepsilon_1}) = d_i(A_{\zeta_{\alpha_\iota,\varepsilon_\iota}} \cap a_i) = d_i(a^*_{\varepsilon_\iota}) = \xi_{\varepsilon_\iota} = \mathrm{pr}(h^0_{\alpha_0}(\zeta^0_{\alpha_0,\varepsilon_0}), h^1_{\alpha_1}(\zeta^1_{\alpha_1,\varepsilon_1}))$$

as required.

1A) Similarly.

2) Similar to part (3), see remarks inside its proof.

3) Let $\theta = \theta_0 + \theta_1$ but for part (2) we let $\theta = \theta_\ell$ and let $\mathbf{c}_1 : [\lambda]^2 \to \sigma$ witness $\mathrm{Pr}_0(\lambda, \lambda, \sigma, \bar{\theta})$ and let $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$ be such that f_α is a one-to-one function from σ onto $\sigma + \alpha$. Let $\langle A_\alpha : \alpha < \lambda\rangle$ be a sequence of pairwise distinct subsets of σ and let $\langle (a_i, d_i) : i < \sigma\rangle$ list the pairs (a, d) such that $a \in [\sigma]^{<\theta}, d : \mathscr{P}(a) \times \mathscr{P}(a) \to \sigma$ and $|\{(b_1, b_2) : b_1 \subseteq a, b_2 \subseteq a$ and $\mathbf{c}_1(b_1, b_2) \neq 0\}| < \theta$; for part (2) we use $d : \mathscr{P}(a) \to \sigma$.

Now we define $\mathbf{c}_2 : [\lambda]^2 \to \lambda$ as follows: for $\alpha < \beta < \lambda$ let $\mathbf{c}_2(\{\alpha,\beta\}) = f_\beta((d_{\mathbf{c}_1(\{\alpha,\beta\})}(A_\alpha \cap a_{\mathbf{c}_1(\{\alpha,\beta\})}, A_\beta \cap a_{\mathbf{c}_1(\{\alpha,\beta\})})))$.

So let $\bar\zeta^\iota = \langle \zeta^\iota_{\alpha,i} : \alpha < \lambda, i < \mathbf{i}_\iota\rangle$ for $\iota < 2$ and $h : \mathbf{i}_0 \times \mathbf{i}_1 \to \lambda$ be as in Definition 2.1(1) but for part (2), $h : \mathbf{i}_\ell \to \lambda$, see 2.1(2). For $\iota = 0,1$ for each $\alpha < \lambda$ and $i < \mathbf{i}_\iota$ we can find $a_{\alpha,\iota} \in [\sigma]^{<\theta_\iota}$ such that $\bar b_{\alpha,\iota} := \langle A_{\zeta^\iota_{\alpha,i}} \cap a_{\alpha,\iota} : i < \mathbf{i}_\iota\rangle$ is a sequence of pairwise distinct sets.

Without loss of generality $\alpha < \lambda \wedge \iota < 2 \Rightarrow a_{\alpha,\iota} = a_\iota, \bar b^\iota_\alpha = \bar b_\iota$; also without loss of generality $\sup(\mathrm{Rang}(h)) \le \min\{\zeta^\iota_{\alpha,i} : \alpha < \lambda, i < \mathbf{i}_\iota$ and $\iota < 2\}$.

Next let $\bar\beta^\iota_\alpha = \langle \beta^\iota_{\alpha,i_0,i_1} : i_0 < \mathbf{i}_0$ and $i_1 < \mathbf{i}_1\rangle$ be a sequence of ordinals $< \sigma$ such that $f_{\zeta^\iota_{\alpha,i_1}}(\beta^\iota_{\alpha,i_0,i_1}) = h(i_0,i_1)$ and without loss of generality $\bar\beta^\iota_\alpha = \bar\beta^\iota$; actually for part (3) we use only $f_{\zeta^\iota_{\alpha,i_1}}$ but for part (2) we use $f_{\zeta^\iota_{\alpha,i_\iota}}$ for the ι from there.

Let $a = a_0 \cup a_1$ so $a \in [\sigma]^{<(\theta_0+\theta_1)}$ and let $d : \mathscr{P}(a) \times \mathscr{P}(a) \to \sigma$ be such that $d(b^0_{i_0}, b^1_{i_1}) = \beta^1_{i_0,i_1}$ and $d(b_0,b_1) = 0$ if $b_0, b_1 \subseteq a$ and $(b_0,b_1) \notin \{(b^0_{i_0}, b^1_{i_1}) : i_0 < \mathbf{i}_0, i_1 < \mathbf{i}_1\}$. Let $j < \sigma$ be such that $(a_j,d_j) = (a,d)$.

Lastly, by the choice of $\mathbf{c}_1$ we can find $\alpha < \beta$ such that $i_0 < \mathbf{i}_0 \wedge i_1 < \mathbf{i}_1 \Rightarrow \mathbf{c}_1(\{\zeta^0_{\alpha,i_0}, \zeta^1_{\alpha,i_1}\}) = j$; and now check.

4) Similarly to the proof of part (3).

5) As $\mathrm{Pr}_1(\lambda,\lambda,\lambda,\bar\theta)$ by monotonicity we have $\mathrm{Pr}_1(\lambda,\lambda,\partial,\bar\theta)$ hence by part (4) we have $\mathrm{Pr}_0(\lambda,\lambda,\partial,\bar\theta)$ and now by part (3) we can deduce $\mathrm{Pr}_0(\lambda,\lambda,\lambda,\bar\theta)$ as promised. $\square_{2.8}$

In Juhász–Shelah [2] we use $\mathrm{Col}(\lambda,\kappa)$, i.e. $\mathrm{Pr}_0(\lambda,\lambda,2,\kappa^+)$ quoting [6, Ch. III, §4] that e.g. $(\lambda,\kappa) = ((2^{\aleph_0})^{++} + \aleph_4, \aleph_0)$ is O.K. But in fact less suffices (see Definition 2.1).

Conclusion 2.10. 1) *For* $\lambda = \kappa^{+4}$ *we have* $\mathrm{Pr}_1(\lambda,\lambda,\lambda,\kappa^+)$ *which implies* $\mathrm{Pr}_{0,0}(\lambda,\lambda,\lambda,(\aleph_0,\kappa^+))$ *and hence trivially* $\mathrm{Pr}_{0,0}(\lambda,\lambda,2,(\aleph_0,\kappa^+))$ *holds.*

2) *If* $\mathrm{Pr}_{0,0}(\lambda,\lambda,\aleph_0,(\aleph_0,\kappa^+))$ *or just* $\mathrm{Pr}^{\mathrm{uf}}_{0,0}(\lambda,\lambda,\aleph_0,(\aleph_0,\kappa^+))$, *e.g.* $\lambda = \aleph_4$, $\kappa = \aleph_0$ *then we have*:

$(*)_{\lambda,\kappa}$ *there is a topological space* X *such that*

- (a) X *is* T_3, *even has a clopen basis and has weight* $\le \lambda$
- (b) *the closure of any set of* $\le \kappa$ *points is compact*
- (c) *any infinite discrete set has an accumulation point*
- (d) *the space is not compact*
- (e) *some non-isolated point is not the accumulation point of any discrete set.*

Proof. 1) First we apply Theorem 3.2 (or [6, Ch. III, §4]) with $(\kappa^{+4},\kappa^{+3},\kappa^+)$ here standing for $(\lambda,\partial,\theta)$ there. Clearly the assumptions there hold hence $\mathrm{Pr}_1(\kappa^{+4},\kappa^{+4},\kappa^{+4},\kappa^+)$ holds.

Second, we apply Claim 2.8(1) with 0, κ^{+4}, κ^{+4}, κ^{+3}, κ^{+3}, κ^+, $\aleph_0$, κ^+, κ^{+3} here standing for ι, λ, μ, σ_1, σ_2, θ, θ_0, θ_1, χ there. Clearly the assumptions there hold because:

$\bullet_1$ "$\mathrm{Pr}_1(\lambda,\mu,\sigma_1,\bar\theta)$" there means $\mathrm{Pr}_1(\kappa^{+4},\kappa^{+4},\kappa^{+3},(\aleph_0,\kappa^+))$ here which holds by the "first" above and monotonicity

$\bullet_2$ "$\chi^{<\theta_\iota}<\mu$" there means "$(\kappa^{+3})^{<\aleph_0}<\kappa^{+4}$"

$\bullet_3$ "$\chi^{<\theta_\iota}\le\sigma_1$" there means "$(\kappa^{+3})^{<\aleph_0}\le\kappa^{+3}$"

$\bullet_4$ "$2^\chi\ge\lambda$" there means "$2^{\kappa^{+3}}\ge\kappa^{+4}$"

$\bullet_5$ "$\sigma_2^{<\theta_\iota}\le\sigma_1$" there which means here "$(\kappa^{+3})^{<\aleph_0}\le\kappa^{+3}$"

$\bullet_6$ "$\sigma_2^{<\theta_\iota}<\mu$" there which means here $(\kappa^{+3})^{<\aleph_0}<\kappa^{+4}$

So all of them hold indeed.

Next, the conclusion of 2.8(1) is $\mathrm{Pr}_{0,\iota}(\lambda,\mu,\sigma_2,\bar\theta)$ which here means $\mathrm{Pr}_{0,0}(\kappa^{+4},\kappa^{+4},\kappa^{+3},(\aleph_0,\kappa^+))$.

Lastly, by 2.8(2) we get $\mathrm{Pr}_{0,0}(\kappa^{+4},\kappa^{+4},\kappa^{+4},(\aleph_0,\kappa^+))$.

2) By Claim 2.13 below, which generalize the proof of Juhász–Shelah [2], that is, let $\bar D=\langle D_i:i<\beth_2\rangle$ list the ultrafilters on $\sigma:=\aleph_0$ and let $\sigma_i=\sigma$ for $i<\beth_2$ and $\theta=\kappa^+$. So clause (A) of 2.13 below holds, hence we can apply 2.13 for $(\lambda,\theta)=(\lambda,\kappa^+)$ and $\bar D$. So clause (a) of 2.10(2) holds by (B)(a)(α) of 2.13, of course; clause (b) of 2.10(2) holds by (B)(a)(γ) recalling the choice of $\bar D$; clause (c) there holds by (B)(a)(ε); clause (d) there holds by (B)(a)(δ); and lastly, clause (e) there holds by (B)(b). So we are done. $\square_{2.10}$

Moreover

Claim 2.11. 1) *If κ is regular and $\lambda=\kappa^{+3}$ then* $\mathrm{Pr}_1(\lambda,\lambda,\lambda,(\aleph_0,\kappa^+))$ *hence* $\mathrm{Pr}_{0,0}(\lambda,\lambda,\lambda,(\aleph_0,\kappa^+))$.

2) $(*)_{\aleph_3,\aleph_0}$ *from* 2.10(2) *holds.*

3) $(*)_{\kappa^{+3},\kappa}$ *from* 2.10(2) *holds for κ regular.*

Proof. Like the proof of 2.10 using Theorem 3.5 instead of Theorem 3.2, that is, we apply 3.5 with $(\aleph_3,\aleph_2,\aleph_1,\aleph_0)$ standing for $(\lambda,\partial,\theta_1,\theta_0)$. $\square_{2.11}$

We conclude this section with an explicit proof of the topological statement in 2.10(2). We shall need the following:

Definition 2.12. Let X be a topological space, D an ultrafilter over σ.

1) An element $y\in X$ is the D-limit of a sequence of points $\langle x_j:j<\sigma\rangle$ in X *iff* $y\in u\Rightarrow\{j<\sigma:x_j\in u\}\in D$ whenever u is a open subset of X.

2) X is D-complete *iff* for every sequence of points $\langle x_j:j<\sigma\rangle$ in X there is $y\in X$ such that y is the D-limit of the sequence.

3) If $\bar D=\langle D_i:i<i_*\rangle$ is a sequence such that each D_i is an ultrafilter over $\sigma_i=\sigma(i)$ *then* X is $\bar D$-complete iff X is D_i-complete for every $i<i_*$.

Claim 2.13. *If (A) then (B) where*

(A) (a) $\lambda=\mathrm{cf}(\lambda)>\theta=\mathrm{cf}(\theta)>\aleph_0$

(b) $\bar D=\langle D_i:i<i_*\rangle$, *each D_i is a non-principal ultrafilter on σ_i and* $\sigma_i<\theta$

(c) $\mathrm{Pr}_{0,0}(\lambda,\lambda,2,(\aleph_0,\theta))$; *yes!* $\mathrm{Pr}_{0,0}$ *and not* Pr_0

(B) *there is a topological space X and a point $g\in X$ such that:*

(a) (α) *X is a subspace of ${}^\lambda 2$ hence has a clopen basis and is a T_3-space*

(β) *X is a dense subset of ${}^\lambda 2$ hence has no isolated point and its weight is λ*

(γ) *if every non-principal ultrafilter D on a cardinal $\sigma < \theta$ appears in $\bar{D}$ then for any set $Y \subseteq X$ of cardinality $< \theta$, the closure of Y is compact*

(δ) *X is not compact*

(ε) *any subset of X of cardinality $\geq \min\{\sigma_i : i < i_*\}$ has an accumulation point; so the cardinality can be $\aleph_0$*

(ζ) *X is $\bar{D}$-complete*

(b) (α) *$g \in X$ is not an accumulation point of any discrete set $Y \subseteq X \backslash \{g\}$*

(β) *moreover, g is not an accumulation point of any set $Y \subseteq X \backslash \{g\}$ of cardinality $< \lambda$*

(c) (α) *X has $\leq \lambda^{<\theta} + \sum_{\sigma < \theta} 2^{2^\sigma}$ points*

(β) *X has $\geq \lambda$ points*

(d) *if $i_* < \lambda$ and $\alpha < \lambda \Rightarrow |\alpha|^{<\theta} < \lambda$ then*

(α) *X has no discrete subset of cardinality $\geq \lambda$, moreover*

(β) *$hL^+(X) \leq \lambda$ so $\lambda = \mu^+ \Rightarrow hL(X) \leq \mu$.*

PROOF.

Stage A: We make some choices:

$(*)_1$ (a) let $\mathbf{c} : [\lambda]^2 \to \{0,1\}$ witness $\Pr_{0,0}(\lambda, \lambda, 2, (\aleph_0, \theta))$

(b) let $\bar{h}^* = \langle h^*_\alpha : \alpha < \lambda \rangle$ list the finite partial functions from λ to $\{0,1\}$; without loss of generality $\mathrm{dom}(h^*_\alpha) \subseteq \alpha$

(c) let $g \in {}^\lambda 2$ be constantly 1.

Further

$(*)_2$ for $\alpha < \lambda$ we define $f^*_\alpha \in {}^\lambda 2$ as follows:

- for $\beta < \lambda$ we let $f^*_\alpha(\beta)$ be

(a) $h^*_\alpha(\beta)$ if $\beta \in \mathrm{dom}(h^*_\alpha)$

(b) $\mathbf{c}\{\beta, \alpha\}$ if $\beta < \alpha \wedge \beta \notin \mathrm{dom}(h^*_\alpha)$

Our X will include each f^*_α for $\alpha < \lambda$ but more.

$(*)_3$ for $\beta \leq \lambda$ we let

(a) $\mathscr{F}_\beta = \{f^*_\alpha : \alpha < \beta\}$

(b) $\mathscr{F}^*_\beta = c\ell_{\bar{D}}(\mathscr{F}_\beta)$, i.e. $\mathscr{F}^*_\beta$ is the minimal subset of ${}^\lambda 2$ which includes $\mathscr{F}_\beta$ and is $\bar{D}$-closed

(c) $\mathscr{G}^*_\beta = \{f : f \in \mathscr{F}^*_\lambda$ and $f{\restriction}[\beta, \lambda)$ is constantly zero$\}$.

So

$(*)_4$ $\mathscr{F}^*_\lambda$ is the union of the $\subseteq$-increasing sequence $\langle \mathscr{F}^*_\beta : \beta < \lambda \rangle$.

[Why? Clearly $\langle \mathscr{F}_\beta : \beta < \lambda \rangle$ is $\subseteq$-increasing and as $\mathrm{cf}(\lambda) \geq \theta$ and D_i is an ultrafilter on $\sigma_i < \theta$ for $i < i_*$ clearly $(*)_4$ follows.]

Lastly, we choose X

$(*)_5$ X is the subspace of ${}^\lambda 2$ with set of elements $\mathscr{F}^*_\lambda \cup \{g\}$.

So it suffices to prove that X, g are as required in the claim.

$(*)_6$ if $f \in \mathscr{F}^*_\lambda$ *then* for some triple (u, v, D) we have:

(a) $u, v \in [\lambda]^{<\theta}$

(b) D an ultrafilter on u

(c) $f = \lim_D(\langle f^*_\alpha : \alpha \in u\rangle)$

(d) if $\beta \in \lambda\backslash v$, *then* $f(\beta) = 1 \Leftrightarrow \{\alpha \in u : \beta < \alpha \text{ and } \mathbf{c}\{\alpha, \beta\} = 1\} \in D$.

[Why? Recall $\mathscr{F}^*_\lambda$ is $c\ell_{\bar{D}}(\mathscr{F}_\lambda)$ and each D_i is an ultrafilter on some $\sigma_i < \theta$]. Hence we can find a sequence $\langle f^*_\alpha : \alpha \in [\lambda, \alpha_*)\rangle$ listing $\mathscr{F}^*_\lambda\backslash\mathscr{F}_\lambda$ and for each such $\alpha, i(\alpha) = i_\alpha < i_*$ and $\bar{\beta}_\alpha \in {}^{\sigma(i(\alpha))}\lambda$ are such that $f^*_\alpha = \lim_{D_{i(\alpha)}}(\langle f_{\beta_{\alpha,\varepsilon}} : \varepsilon < \sigma_{i(\alpha)}\rangle)$. As θ is regular, *clearly* there are $u \in [\lambda]^{<\theta}$ and an ultrafilter D on u such that clause (c) holds.

[Why? If $f = f^*_\alpha$, $\alpha < \lambda$ then $u = \{\alpha\}$ is as required and if $f = f^*_\alpha$, $\alpha \in [\lambda, \alpha_*)$ then we can prove this by induction on α.]

Now choose $v = \bigcup\{\mathrm{dom}(h^*_\alpha) : \alpha \in u\}$, clearly u, v are as required. E.g. if $f = f^*_\alpha, \alpha < \lambda$ the ultrafilter D is the unique principal ultrafilter on $\{\alpha\}$; for $(*)_6$(d) recall the choice of the f^*_α's for $\alpha < \lambda$.]

$(*)_7$ if $f \in \mathscr{F}^*_\lambda$ and $\delta < \lambda$ has cofinality $\geq \theta$, *then* for some $\gamma < \delta$, at least one of the following holds:

(a) if $\beta \in [\gamma, \lambda)$ then $f(\beta) = 0$

(b) for some $u = u_f \in [\lambda\backslash\delta]^{<\theta}$ and $v = v_f \in [\lambda\backslash\delta]^{<\theta}$ and ultrafilter D on u we have

- if $\beta \in [\gamma, \lambda)\backslash v_f$ then $f(\beta) = \lim_D(\langle \mathbf{c}\{\beta, \alpha\} : \alpha \in u\rangle)$.

[Why? Let u, v, D be as in $(*)_6$. If $u \cap \delta \in D$ then let γ be $\sup(u \cap \delta) < \delta$ and by $(*)_2(c) + (*)_6$(c) clearly clause (a) of $(*)_7$ holds. So we can assume $u \cap \delta \notin D$ and as D is an ultrafilter on u, necessarily $u\backslash\delta \in D$. Let $u' = u\backslash\delta$, $\gamma = \sup(\bigcup\{\mathrm{dom}(h^*_\alpha) \cap \delta : \alpha \in u\} \cup (v \cap \delta)) + 1$ and $D' = D \cap \mathscr{P}(u')$ and $v' = v\backslash\delta$, they clearly witness clause (b) of $(*)_7$. Together we are done.]

$(*)_8$ (a) if $f \in \mathscr{F}^*_\lambda$, *then* for some $\beta < \lambda$ we have $f \in \mathscr{F}^*_\beta$ which implies f is constantly zero on $[\beta, \lambda)$

(b) $\mathscr{F}^*_\beta \subseteq \mathscr{G}^*_\beta \subseteq \mathscr{F}^*_\lambda$

(c) $\mathscr{G}^*_\beta$ is $\subseteq$-increasing with β with union $\mathscr{F}^*_\lambda$.

[Why? Clause (a) holds by $(*)_3(b) + (*)_4$ above. Clauses (b), (c) are easy too recalling $(*)_3$(a).]

Stage B: Now we check the demands in (B) of the claim.

$\oplus_1$ X is a subspace of ${}^\lambda 2$ [so clause (B)(a)(α) holds] hence X is a T_3 topological space with a clopen base.

[Why? By its choice in $(*)_5$.]

$\oplus_2$ X is dense in ${}^\lambda 2$ hence clause (B)(a)(β) holds.

[Why? By the choice of $\bar{h}^*$ in $(*)_1$(b) because $h^*_\alpha \subseteq f^*_\alpha$ for $\alpha < \lambda$ by $(*)_2$(a).]

$\oplus_3$ X is D_i-complete for every $i < i_*$ hence clause (B)(a)(ζ) holds.

[Why? By the choice of $\mathscr{F}^*_\lambda$ in $(*)_3$(b) because $X\backslash\mathscr{F}^*_\lambda = \{g\}$ recalling $\lambda = \mathrm{cf}(\lambda) > \theta$.]

$\oplus_4$ $\lambda \le |X| \le \lambda^{<\theta} + \sum\limits_{\sigma<\theta} 2^{2^\sigma}$ and also $|X| \le \lambda^{<\theta} + 2^{\theta+|i_*|}$ hence clause (B)(c) holds.

[Why? Clearly $|\mathscr{F}_\lambda| = \lambda$ and $\mathscr{F}_\lambda \subseteq \mathscr{F}^*_\lambda \subseteq X$ hence $\lambda \le |X|$. As $|X \backslash \mathscr{F}^*_\lambda| = |\{g\}| = 1$ and by $(*)_6$ the other inequalities follow.]

$\oplus_5$ $g \notin c\ell(Y)$ *when* $Y \subseteq X \backslash \{g\}$ and at least one of the following holds:

(a) $|Y| < \lambda$

(b) for some $\beta < \lambda, Y \subseteq \mathscr{F}^*_\beta$

(c) for some $\beta < \lambda, Y \subseteq \mathscr{G}^*_\beta := \{f \in \mathscr{F}^*_\lambda : f{\restriction}[\beta, \lambda)$ is constantly zero$\}$.

[Why? If clause (a), i.e. $|Y| < \lambda = \mathrm{cf}(\lambda)$ as $\langle \mathscr{F}^*_\beta : \beta < \lambda\rangle$ is $\subseteq$-increasing with union $\mathscr{F}^*_\lambda$ by $(*)_4$, necessarily $Y \subseteq \mathscr{F}^*_\beta$ for some $\beta < \lambda$, i.e. clause (b); but this in turn implies clause (c) by $(*)_8$(b).

But if clause (c) holds for β, then $g \notin c\ell(Y)$ recalling that $g(\gamma) = 1$ for every $\gamma < \lambda$.]

Now comes a major point using the choice of $\mathbf{c}$, i.e. $\mathrm{Pr}_{0,0}(\lambda, \lambda, 2, (\aleph_0, \theta))$.

$\oplus_6$ if $Y \subseteq \mathscr{F}^*_\lambda$ and $\beta < \lambda \Rightarrow Y \nsubseteq \mathscr{G}^*_\beta$ *then* Y is not discrete and even not left separated (hence, together with $\oplus_5$, clause (B)(b) holds).

[Why? For $\alpha < \lambda$ choose $f_\alpha \in Y \backslash \mathscr{G}^*_\alpha \subseteq \mathscr{F}^*_\lambda \backslash \mathscr{F}_\alpha$ hence there is $\beta^1_\alpha \in [\alpha, \lambda)$ such that $f_\alpha(\beta^1_\alpha) = 1$ and there is $\beta^2_\alpha \in (\beta^1_\alpha, \lambda)$ such that $f_\alpha{\restriction}[\beta^2_\alpha, \lambda)$ is constantly zero.]

Recall that "Y is left separated (in the space X)" means that there is a well-ordering $<^*$ on Y such that for every $x \in Y$ the set $\{y \in Y : x <^* y\}$ is closed in the induced topology on Y.

Toward contradiction assume Y is discrete or just left separated. Fix a well-ordering $<^*$ on Y which witnesses this fact. Clearly we can find $\mathscr{U}_0 \in [\lambda]^\lambda$ such that $\langle \beta^1_\alpha : \alpha \in \mathscr{U}_0\rangle$ is an increasing sequence of ordinals and on $Y, <^*$ and the usual order agree.

Now by the choice of $<^*$ for some $\mathscr{U} \in [\mathscr{U}_0]^\lambda$ we can find a sequence $\bar{h} = \langle h_\alpha : \alpha \in \mathscr{U}\rangle, h_\alpha$ is a finite function from λ to $\{0, 1\}$ satisfying (the statements $\bullet_0 + \bullet_2$ by the definition of "$<^*$ witnesses Y is left separated"; the statement $\bullet_1$ holds as without loss of generality as increasing h_α makes no harm, and the statement $\bullet_3$ holds without loss of generality because we can replace $\mathscr{U}$ by any $\mathscr{U}' \in [\mathscr{U}]^\lambda$):

$\bullet_0$ $h_\alpha \subseteq f_\alpha$

$\bullet_1$ $\beta^1_\alpha, \beta^2_\alpha \in \mathrm{Dom}(h_\alpha)$

$\bullet_2$ if $\alpha_1 < \alpha_2$ then $h_{\alpha_1} \nsubseteq f_{\alpha_2}$. Also (not used)

$\bullet_3$ if $\alpha_1 < \alpha_2$ are from $\mathscr{U}$ then $\beta^2_{\alpha_1} < \beta^1_{\alpha_2}$ hence $h_{\alpha_2} \nsubseteq f_{\alpha_1}$.

Renaming without loss of generality

$\bullet_4$ $\mathscr{U} = \lambda$ and still $\beta^2_\alpha > \beta^1_\alpha \ge \alpha, f_\alpha(\beta^1_\alpha) = 1$ and $f_\alpha{\restriction}[\beta^2_\alpha, \lambda)$ is constantly zero.

For each $\delta \in S_1 := S^\lambda_\theta = \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ we consider $(*)_7$ with (f_δ, δ) here standing for (f, δ) there, now $\beta^1_\delta \ge \delta, f_\delta(\beta^1_\delta) = 1$ by $\bullet_4$ hence clause $(*)_7$(a) fails, so necessarily clause $(*)_7$(b) holds. So there is a quadruple

$(\gamma_\delta, u_\delta, v_\delta, D_\delta)$ as there[1] and let $\beta^3_\delta := \sup(\delta \cap (\mathrm{dom}(h_\delta)))$, as h_δ is a finite function, necessarily $\beta^3_\delta < \delta$. So by Fodor lemma for some $\gamma_* < \lambda$ the set $S_2 = \{\delta \in S_1 : \gamma_\delta, \beta^3_\delta \le \gamma_* < \delta\}$ is stationary hence so is $S_3 = \{\delta \in S_2$: if $\alpha < \delta$ then $u_\alpha, v_\alpha \subseteq \delta$, $\beta^1_\alpha < \delta$, $\beta^2_\alpha < \delta$ and $\mathrm{dom}(h_\alpha) \subseteq \delta\}$. As $\mathrm{dom}(h_\alpha)$ is finite and $\mathrm{range}(h_\alpha) \subseteq \{0,1\}$ clearly for some h_*, h_{**} the set $S_4 = \{\delta \in S_3 : h_\delta \restriction \delta = h_*$ and $h_{**} = \{(\mathrm{otp}(\mathrm{dom}(h_\delta) \cap \gamma), h_\delta(\gamma)) : \gamma \in \mathrm{dom}(h_\delta)\}\}$ is stationary.

For $\delta \in S_4$ let $u_{\delta,0} = \mathrm{Dom}(h_\delta) \backslash \mathrm{Dom}(h_*), h'_\delta = h_\delta \restriction u_{\delta,0}$ and $u_{\delta,1} = u_\delta$ and recall $u_\delta \cap \delta = \emptyset = v_\delta \cap \delta$, see $(*)_7$(b). Note that $\mathrm{Qr}_{0,0}(\lambda, \lambda, 2, (\aleph_0, \theta))$ holds, see Definition 2.5(1),(2) for $\iota = 0$, now it holds because we are assuming $\mathrm{Pr}_{0,0}(\lambda, \lambda, 2, (\aleph_0, \theta))$ by 2.7(1). So we can apply the definition of $\mathrm{Qr}_{0,0}(\lambda, \lambda, 2, (\aleph_0, \theta))$ and the choice of $\mathbf{c}$ to $\langle (u_{\delta,0}, u_{\delta,1} : \delta \in S_4\rangle$ and $\langle h'_\delta : \delta \in S_4\rangle$. So there are δ_1, δ_2 such that:

$\bullet_5$ $\delta_1 < \delta_2$ are from S_4

$\bullet_6$ if $\alpha \in u_{\delta_1,0}$ and $\beta \in u_{\delta_2,1}$ then $\mathbf{c}\{\alpha, \beta\} = h'_{\delta_1}(\alpha)$.

Next

$\bullet_7$ if $\alpha \in u_{\delta_1,0}$ then $f_{\delta_2}(\alpha) = \lim_{D_{\delta_2}}(\langle \mathbf{c}\{\alpha, \beta\} : \beta \in u_{\delta_2,1} = u_{\delta_2}\rangle)$.

[Why? By the choice of $(\gamma_{\delta_2}, u_{\delta_2}, D_{\delta_2}, h_*, h_{**})$ that is recalling $(*)_7$(b) because $\alpha \in u_{\delta_1,0} \Rightarrow \alpha \in \mathrm{dom}(h'_{\delta_1}) \Rightarrow \alpha \ge \delta_1 \Rightarrow \alpha \ge \gamma_* \ge \gamma_{\delta_2}$ and $\alpha \in u_{\delta_1,0} \cup v_{\delta_1}$ $\Rightarrow \alpha < \delta_2$.]

$\bullet_8$ if $\alpha \in \mathrm{dom}(h'_{\delta_1})$ then $f_{\delta_2}(\alpha) = h'_{\delta_2}(\alpha)$.

[Why? By $\bullet_7$ because $u_{\delta_1,0} = \mathrm{dom}(h'_{\delta_1})$ and $\bullet_6$.]

$\bullet_9$ $h'_{\delta_1} \subseteq f_{\delta_2}$.

[Why? By $\bullet_8$.]

However, $h_{\delta_1} \subseteq f_{\delta_1}$ by $\bullet_0$ hence $h_* \subseteq h_{\delta_1} \subseteq f_{\delta_1}$ but $h_* \subseteq h_{\delta_2} \nsubseteq f_{\delta_1}$ by $\bullet_2$ and $h'_{\delta_2} = h_{\delta_2} \restriction (\mathrm{dom}(h_{\delta_2}) \backslash \mathrm{dom}(h_*)$ hence

$\bullet_{10}$ $h'_{\delta_2} \nsubseteq f_{\delta_1}$.

But $\bullet_{10}$ contradict $\bullet_9$, all this follows from the assumption toward contradiction in the beginning of the proof of $\oplus_6$, so $\oplus_6$ holds indeed.

Now we can check all the remaining demands in (B), e.g.

Clause (B)(d)(β): Assume toward contradiction that $hL^+(X) > \lambda$. This means that some $Y \subseteq X$ has cardinality λ and is right separated (by some well ordering). Now without loss of generality $g \notin Y$ and if $\beta < \lambda \Rightarrow Y \nsubseteq \mathscr{G}^*_\beta$ then we get a contradiction by $\oplus_6$. So we are left with the case $Y \subseteq \mathscr{G}^*_\beta$ for some $\beta < \lambda$. But by the clause assumption $|\mathscr{G}^*_\beta| \le |\beta|^{<\theta} + |i_*|$ which has cardinality $< \lambda$, so we are done proving (B)(d)(β).

We are done proving 2.13: most clauses of (B) were proved and we have to add that: clauses (B)(a)(γ) + (ε) hold by the choice of $\mathscr{F}^*_\lambda$ as $X \backslash \mathscr{F}^*_\lambda = \{g\}$. Clause (B)(a)($\delta$) is exemplified by any uniform ultrafilter D on λ such that $\{\alpha : f^*_\alpha(0) = r\} \in D$, exists by $(*)_3(c) + (*)_8$. $\square_{2.13}$

[1] They depend also on $f = f_\delta$, but δ determines f.

3. The colouring existence

We try to explain the proof of 3.1, 3.5; probably more of it will make sense after reading part of the proof.

Claim 3.1 should be understood as follows: given a set S and functions $F_\iota : S \to \kappa_\iota$ for $\iota = 0, 1$ and a sequence $\varrho \in {}^{\omega>}S, \mathbf{d}(\varrho)$ is a natural number which in the interesting case is a "place in the sequence", i.e. $\mathbf{d}(\varrho) < \ell g(\varrho)$.

In the interesting cases, $\varrho = \eta_0{}^\frown\nu_0{}^\frown\rho{}^\frown\nu_1{}^\frown\eta_1$ is as constructed during the proof of 3.5, and if (B)(a)-(d) of 3.1 holds, $\ell g(\eta_0) + \ell_4$ is a place in the sequence; so 3.1 tells us that it depends only on ϱ (and not on the representation $(\eta_0, \nu_0, \rho, \nu_1, \eta_1)$ of ϱ).

How does $\mathbf{d}$ help us in the proof of Theorem 3.5?

We shall describe it for the case of θ_1 colours, i.e. $\sigma = \theta_1$ and the colouring is called $\mathbf{c}_1$. Let $(\kappa_0, \kappa_1, \kappa_2) = (\theta_0, \theta_1, \lambda)$. We shall be given pairwise disjoint $t_\alpha = t^0_\alpha \cup t^1_\alpha$ for $\alpha < \lambda$ and a colour $j_* < \theta_1$ such that $|t^\iota_\alpha| < \theta_\iota$ for $\iota = 0, 1$ and $\alpha < \lambda$ and we shall carefully choose $\alpha_0 < \alpha_1$ exemplifying the desired conclusion.

Toward choosing the pair (α_0, α_1) we also choose $\delta_0 < \delta_1 < \delta_2 < \delta_3$ which will be from (α_0, α_1) such that $\sup(t_{\alpha_0}) < \delta_0$ and ℓ_4 such that:

(a) we let $\nu_0 = \rho_{\bar h}(\delta_3, \delta_2), \rho = \rho_{\bar h}(\delta_2, \delta_1), \nu_1 = \rho_{\bar h}(\delta_1, \delta_0)$ where $\rho_{\bar h}(\delta', \delta'')$ is derived from the sequence $\rho(\delta', \delta'')$, see before $\odot_2$ in the proof of 3.5

(b) $\ell_4 < \ell g(\nu_0)$ and $h'(F_1(\nu_0(\ell_4))) = j_*$ where $h' : \kappa_1 \to \kappa_2$ is chosen in $\odot_7$ in the proof 3.5

(c) let $\zeta_0 \in t^0_{\alpha_0}$ and $\zeta_1 \in t^1_{\alpha_1}$ and define $\eta_{1,\zeta_0} = \rho_{\bar h}(\delta_0, \zeta_0), \eta_{0,\zeta_1} = \rho_{\bar h}(\zeta_1, \delta_3)$

(d) continuing clause (c) by the construction $\varrho_{\zeta_1,\zeta_0} := \rho_{\bar h}(\zeta_1, \zeta_0)$ is equal to $\eta_{0,\zeta_1}{}^\frown\nu_0{}^\frown\rho{}^\frown\nu_1{}^\frown\eta_{1,\zeta_0}$.

So naturally we choose the colouring $\mathbf{c}_1$ such that

$$\mathbf{c}_1(\alpha_0, \alpha_1) = h'(F_1(\varrho(\ell g(\eta_0) + \ell_4)))$$

and 3.1 tells us that assuming (a)-(d) this will be j_*. Note it is desirable that in 3.1, the sequences η_0, η_1 in a sense have little influence on the result, as they vary, i.e. we like to get j_* for every $\zeta_0 \in t^0_{\alpha_0}$, $\zeta_1 \in t^1_{\alpha_1}$.

Why do we demand in clause (b), $h_2(F_1(\nu_0(\ell_4))) = j_*$ and not simply $F_1(\nu_0(\ell_4)) = j_*$ and similarly when defining $\mathbf{c}_1$ in $\odot_7$ in the proof? Because we do not succeed to fully control $F_1(\nu_0(\ell_4))$, but just to place it in some stationary $S \subseteq \theta_1$, however we can use θ_1 pairwise disjoint stationary set and h_1 tells us which one.

When we choose $\alpha_0 < \alpha_1$ (in stage C of the proof) we first choose a pair $\delta_1 < \delta_2$ hence ρ (in $\oplus_0$ of the proof), then we choose an ordinal $\delta_0 < \delta_1$ hence ν_1 (in $\oplus_{0.1}$ of the proof) then $\varepsilon_* \in s_{\delta_2} \subseteq \kappa_1$ after $\oplus_{0.2}$ of the proof, (see below) large enough. Only then using ε_* we choose δ_3 and then α_1 (also after $\oplus_{0.2}$) hence $\eta_{0,\zeta}$ for $\zeta \in t^1_{\alpha_1}$. Lastly, we choose $\alpha_0 < \delta_0$ hence η_{1,ζ_0} for $\zeta_0 \in t^0_{\alpha_0}$. Of course, those choices are under some restrictions. More specifically, (in stage B) though not determining any of η_{0,ζ_0}, ν_0, ρ, ν_1, η_{1,ζ_1} we restrict them in some ways.

Earlier, we first in $(*)_1$ choose $\mathscr{U}_1^{\rm up}$, α_1^*, $\varepsilon_{1,1}^{\rm up}$, $\varepsilon_{1,0}^{\rm up}$ with the intention that $\alpha_1 \in \mathscr{U}_1^{\rm up}$ "promising" that if $\alpha_1 \in \mathscr{U}_1^{\rm up}$ then $\mathrm{Rang}(F_1(\eta_0)) \subseteq \varepsilon_{1,1}^{\rm up} < \kappa_1$, i.e. $\zeta_1 \in t^1_{\alpha_1} \Rightarrow \mathrm{Rang}(F_1(\eta_{0,\zeta_1})) \subseteq \varepsilon_{1,1}^{\rm up}$, similarly in the further steps below. Second we do not "know" for which $\varepsilon < \kappa$ we shall use $S^{\kappa_1}_{\kappa_0,\varepsilon} \subseteq \kappa_1$, so we consider all of them, i.e. in $(*)_2$ we choose $\mathscr{U}_{2,\varepsilon}^{\rm up}$, $g_{2,\varepsilon}$, γ_ε^*, $\alpha_{2,\varepsilon}^*$ satisfying $g_{2,\varepsilon} : \mathscr{U}_{2,\varepsilon}^{\rm up} \to \mathscr{U}_1^{\rm up}$ such that later $\delta_3 \in \mathscr{U}_{2,\varepsilon}^{\rm up}$ and $\alpha_1 = g_{2,\varepsilon}(\delta_3)$. We still do not know what ν_2 will be hence how to compute ℓ_4, but $\rho_{\bar h}(\alpha_1, \delta_3)$ will be part of it and for each $\varepsilon < \kappa_1$ we can compute $\ell_{2,\varepsilon}$ which will be the first place ℓ in ν_0 in which $F_2(\nu_0(\ell)) = \varepsilon$, see $(*)_2$(f).)

In $(*)_3$ we choose $\mathscr{U}_4^{\rm up}$, $\mathscr{U}_3^{\rm up}$, $g^3_{3,\varepsilon}$, α_3^* and $\langle s_\delta : \delta \in \mathscr{U}_\ell^{\rm up}\rangle$ giving another part of ν_0. Then in $(*)_4$ we deal further with ν_0, in particular $s_\delta \subseteq \kappa_1$ is a stationary subset of $S^{\kappa_1}_{\kappa_0,j_*}$, promising $F_1(\nu_2(\ell_4)) \in s_{\delta_2}$.

Next we work on restricting the choices from below, choosing $\mathscr{U}_1^{\rm dn}$, $\varepsilon_{1,0}^{\rm dn}$, $\varepsilon_{1,1}^{\rm dn}$ in $(*)_5$ promising $\delta_0 \in \mathscr{U}_1^{\rm dn}$ so this restricts η_1.

Lastly, in $(*)_6$ we choose $\mathscr{U}_2^{\rm dn}$, $\varepsilon_{2,0}^{\rm dn}$, $\varepsilon_{2,1}^{\rm dn}$ promising $\delta_1 \in \mathscr{U}_2^{\rm dn}$ (recalling $\nu_1 = \rho_{\bar h}(\delta_1, \delta_2)$).

Claim 3.1. *Assume κ_1, κ_0 are cardinals and S is a set. There is a function $\mathbf{d} : {}^{\omega>}S \to \mathbb{N}$ such that (A) $\Rightarrow$ (B) where*

(A) (a) *$F_\iota : S \to \kappa_\iota$ for $\iota = 0, 1$*

(b) *for $\varrho \in {}^{\omega>}S$ and $\iota < 2$ we let $F_\iota(\varrho) = \langle F_\iota(\varrho(\ell)) : \ell < \ell g(\varrho)\rangle$*

(c) *we stipulate* $\max \mathrm{Rang}(F_\iota(\langle\rangle)) = -1$

(B) *$\mathbf{d}(\varrho) = \ell_4^\bullet$ when $\varrho = \eta_0\char`^\nu_0\char`^\rho\char`^\nu_1\char`^\eta_1$ satisfies (note that ℓ_1, $\ell_4^\bullet$–$\ell g(\eta_0)$ are places in ν_0, ℓ_3 is a place in ν_1, ℓ_2^* is a place in ρ and $\ell_2^\bullet$, $\ell_4^\bullet$ is a place in ϱ and $u \subseteq \{\ell g(\nu_0) + \ell : \ell < \ell g(\nu_0)\}$) the following:*

(a) (α)

$$\max \mathrm{Rang}(F_1(\varrho)) = \max(\mathrm{Rang}(F_1(\nu_0)) > \max(\mathrm{Rang}(F_1(\eta_0\char`^\rho\char`^\nu_1\char`^\eta_1))$$

(β) *let* $\ell_1 = \min\{\ell < \ell g(\nu_0) : F_1(\nu_0(\ell)) = \max \mathrm{Rang}(F_1(\varrho))\}$ *so* $\ell_1 < \ell g(\nu_0)$

(b) (α) $\max \mathrm{Rang}(F_0(\varrho\restriction(\ell g(\eta_0) + \ell_1, \ell g(\varrho)))) = \max \mathrm{Rang}(F_0(\rho)) > \max \mathrm{Rang}(F_0(\nu_0\restriction[\ell_1, \ell g(\nu_0))\char`^\nu_1\char`^\eta_1)$

(β) *let* $\ell_2^\bullet = \min\Big\{\ell < \ell g(\varrho) : \ell \geq \ell g(\eta_0) + \ell_1$ *and*

$$F_0(\varrho(\ell)) = \max \mathrm{Rang}(F_0(\varrho\restriction(\ell g(\eta_0) + \ell_1, \ell g(\varrho))))\Big\}$$

so $\ell_2^\bullet < \ell g(\varrho)$ *and* $\ell_2^* = \ell_2^\bullet - \ell g(\eta_0\char`^\nu_0)$

(γ) *hence* $\ell_2^\bullet \in [\ell g(\eta_0\char`^\nu_0), \ell g(\eta_0\char`^\nu_0\char`^\rho))$ *and* $\ell_2^* < \ell g(\rho)$

(c) (α)

$$\max \operatorname{Rang}(F_1(\nu_0)) > \max \operatorname{Rang}(F_1(\varrho{\restriction}[\ell_2^\bullet, \ell g(\varrho)))$$
$$= \max \operatorname{Rang}(F_1(\nu_1)) > \max\{F_1(\rho(\ell)) : \ell \in [\ell_2^*, \ell g(\rho))\}$$

(β) ℓ_3 *is such that*

$\bullet_1$ $\ell_3 < \ell g(\nu_1)$

$\bullet_2$ $F_1(\nu_1(\ell_3)) = \max\{F_1(\varrho)(\ell) : \ell \geq \ell_2^\bullet\}$

$\bullet_3$ ℓ_3 *is minimal under the above*

(d) (α) *let* $u := \{\ell : \ell \leq \ell_2^\bullet$ *and* $F_1(\varrho)(\ell) \geq F_1(\nu_1(\ell_3))\}$

(β) $\ell_4^\bullet \in u$ *is such that*

$\bullet_1$ $F_1(\varrho(\ell_4^\bullet)) = \min\{F_1(\varrho(\ell)) : \ell \in u\}$

$\bullet_2$ *under* $\bullet_1, \ell_4^\bullet$ *is minimal*

$\bullet_3$ *notation*: *if* $\ell_4^\bullet \in [\ell g(\eta_0), \ell g(\eta_0{}^\frown\nu_0))$ *then we let*

$$\ell_4^* = \ell_4^\bullet - \ell g(\eta_0).$$

PROOF. Assume $\varrho \in {}^{\omega>}S$. We have to show that $\mathbf{d}$ is well defined, i.e. $\mathbf{d}(\varrho) = \ell_4^\bullet$ does not depend on the specific representation of ϱ as $\eta_0{}^\frown\nu_0{}^\frown\rho{}^\frown\nu_1{}^\frown\eta_1$, i.e. we shall prove that $\ell_4^\bullet$ depends on ϱ only.

Toward this

(a) $\ell g(\eta_0) + \ell_1$ depends on ϱ only

[Why? Let $\ell_1^\bullet$ be the first natural number so that

$$F_1(\varrho(\ell_1^\bullet)) = \max \operatorname{Rang}(F_1(\varrho)).$$

By the strict $>$ in (B)(a)(α) we must have $\ell g(\eta_0) \leq \ell_1^\bullet$. Although one can decompose ϱ in different ways, yielding different values to $\ell g(\eta_0)$, the sum $\ell g(\eta_0) + \ell_1$ will be always $\ell_1^\bullet$, by the definition of ℓ_1. Now since only ϱ is mentioned in the definition of $\ell_1^\bullet$ we conclude that $\ell g(\eta_0) + \ell_1 = \ell_1^\bullet$ depends on ϱ only.]

(b) $\ell_2^\bullet$ depends on ϱ only by a similar argument, this time for the function F_0

(c) $\ell g(\eta_0{}^\frown\nu_0{}^\frown\rho) + \ell_3$ depends on ϱ only (for this statement notice that $\rho \neq \langle\rangle$, by $(b)(\alpha)$)

(d) $\{\ell g(\eta_0) + \ell : \ell \in u\}$ depends on ϱ only

(e) $\ell_4^\bullet$ depends on ϱ only.

By (e) clearly we are done. $\square_{3.1}$

THEOREM 3.2. *Assume* $\aleph_0 \leq \theta = \operatorname{cf}(\theta), \lambda \geq \theta^{+3}$ *and* λ *is a successor of a regular cardinal. Then* $\operatorname{Pr}_1(\lambda, \lambda, \lambda, \theta)$ *holds.*

PROOF. Firstly, let us spell out the definition of Pr_1.

Recall that $\lambda \geq \mu \geq \sigma, \theta_0, \theta_1$ and let $\bar\theta = (\theta_0, \theta_1)$. $\operatorname{Pr}_1(\lambda, \mu, \sigma, \bar\theta)$ means that there exists a function $\mathbf{c} : [\lambda]^2 \to \sigma$ such that for every two disjoint sequences $\langle \zeta_{\alpha,i}^0 : \alpha < \mu,\ i < \mathbf{i}_0 \rangle$, $\langle \zeta_{\alpha,i}^1 : \alpha < \mu,\ i < \mathbf{i}_1 \rangle$ of ordinals $< \lambda$ (without repetitions) such that $\mathbf{i}_0 < \theta_0$, $\mathbf{i}_1 < \theta_1$ and for every $\gamma < \sigma$, one can find $\alpha_0 < \alpha_1 < \mu$ so that:

(*) if $i_0 < \mathbf{i}_0$ and $i_1 < \mathbf{i}_1$ then $\mathbf{c}(\zeta^0_{\alpha_0,i_0}, \zeta^1_{\alpha_1,i_1}) = \gamma$.

It follows from the definition that if $\theta'_1 \le \theta_1$ and $\mathrm{Pr}_1(\lambda,\mu,\sigma,(\theta_0,\theta_1))$ then $\mathrm{Pr}_1(\lambda,\mu,\sigma,(\theta_0,\theta'_1))$. Let $\theta_0 = \theta$, $\theta_1 = \theta^+$ by Theorem 3.5 below we have $\mathrm{Pr}_1(\lambda,\lambda,\lambda,(\theta_0,\theta_1))$ and since $\theta_0 < \theta_1$ we have by the previous sentence $\mathrm{Pr}_1(\lambda,\lambda,\lambda,(\theta_0,\theta_0))$ which is also denoted $\mathrm{Pr}_1(\lambda,\lambda,\lambda,\theta_0)$, see Observation 2.6, so we are done by noticing that θ_0 of 3.5 is θ here. $\square_{3.2}$

Remark 3.3. 1) Can we replace θ by (θ^+,θ)?

2) Or, at least when $\theta = \aleph_0, \lambda = \aleph_2$ for (θ,θ^+) with an ultrafilter on the $< \theta^+$ sets? and 2 colours? may try to use the proof of the $\aleph_2$-c.c. not productive from [11].

3) For many purposes, $\mathrm{Pr}_1(\lambda,\lambda,2,(\theta,\theta^+))$ suffices and for this the proof (in 3.5) is somewhat simpler.

Conclusion 3.4. *Assume* $\lambda = \partial^+$, $\partial = \mathrm{cf}(\partial) > \theta^+$, $\theta = \mathrm{cf}(\theta) \ge \aleph_0$

(a) *if there is* $\chi = \chi^{<\theta} < \lambda \le 2^\chi$ *and* $\chi \ge \sigma$ (*so* $\sigma \le \partial$), *then* $\mathrm{Pr}_0(\lambda,\lambda,\sigma,\theta)$

(b) *if* $\chi = \partial$ *satisifes* $\chi = \chi^{<\theta}$ *then* $\mathrm{Pr}_0(\lambda,\lambda,\lambda,\theta)$.

Proof. *Clause* (a): We apply 2.8(4) with $(\lambda,\lambda,\chi,\sigma,\theta,\theta)$ here standing for $(\lambda,\mu,\chi,\sigma,\theta_0,\theta_1)$ there. We have to check the assumption of 2.8(4), the main point is "$\mathrm{Pr}_1(\lambda,\lambda,\sigma,(\theta,\theta))$" which holds by Theorem 3.2, the other assumptions are straightforward hence we get the conclusion, i.e. $\mathrm{Pr}_0(\lambda,\lambda,\sigma,\bar\theta)$.

Clause (b): First, $\mathrm{Pr}_0(\lambda,\lambda,\partial,\theta)$ holds as we can apply Clause (a) with $(\lambda,\partial,\partial,\partial,\theta)$ here standing for $(\lambda,\partial,\chi,\sigma,\bar\theta)$ there.

Second, we get $\mathrm{Pr}_0(\lambda,\lambda,\lambda,\theta)$ holds as we can apply 2.8(3) with $(\lambda,\partial,\theta)$ here standing for (λ,σ,θ) there. $\square_{3.4}$

Theorem 3.5. *If* λ *is a successor of a regular cardinal,* $\lambda \ge \theta_1^+$ *and* $\theta_1 > \theta_0 \ge \aleph_0$ *are regular cardinals, then* $\mathrm{Pr}_1(\lambda,\lambda,\lambda,(\theta_0,\theta_1))$.

Proof. *Stage* A: Let ∂ be the regular cardinal such that $\lambda = \partial^+$, so $\partial \ge \theta_1$.

Below we shall choose σ and κ_ι (for $\iota = 0,1,2$) to help in using this proof for proving other theorems.

Let $\sigma = \lambda$. Let $S \subseteq S^\lambda_\partial$ be stationary and $h : \lambda \to \sigma$ be such that $\alpha < \lambda \Rightarrow h(\alpha) < 1 + \alpha, h{\restriction}(\lambda \backslash S)$ is constantly zero and $S^*_\gamma := \{\delta \in S : h(\delta) = \gamma\}$ is a stationary subset of λ for every $\gamma < \lambda$. Let $(\kappa_0,\kappa_1,\kappa_2) = (\theta_0,\theta_1,\sigma)$ and let $F_\iota : \lambda = \sigma \to \kappa_\iota$ for $\iota = 0,1,2$ be such that for every $(\varepsilon_0,\varepsilon_1,\varepsilon_2) \in (\kappa_0 \times \kappa_1 \times \kappa_2)$ the set $W_{\varepsilon_0,\varepsilon_1,\varepsilon_2}(\kappa) = \{\gamma \in S^\lambda_\kappa : F_\iota(\gamma) = \varepsilon_\iota$ for $\iota \le 2\}$ is a stationary subset of λ for every $\kappa = \mathrm{cf}(\kappa) < \lambda$.

Let $\bar e = \langle e_\alpha : \alpha < \lambda \rangle$ be such that

$\odot_1$ (a) if $\alpha = 0$ then $e_\alpha = \emptyset$

(b) if $\alpha = \beta + 1$ then $e_\alpha = \{\beta\}$

(c) if α is a limit ordinal then e_α is a club of α of order type $\mathrm{cf}(\alpha)$ disjoint to S^λ_∂ hence to S.

Let[2] $h_\alpha = h{\restriction}e_\alpha$ for $\alpha < \lambda$ and $\bar{h} = \langle h_\alpha : \alpha < \lambda\rangle$. Note that h_α is non-zero only for successor α. We shall mostly use the h_α's rather than h.

Now (using $\bar{e}$) for $0 < \alpha < \beta < \lambda$, let

$$\gamma(\beta, \alpha) := \min\{\gamma \in e_\beta : \gamma \geq \alpha\}.$$

Let us define $\gamma_\ell(\beta, \alpha)$:

$$\gamma_0(\beta, \alpha) = \beta, \quad \gamma_{\ell+1}(\beta, \alpha) = \gamma(\gamma_\ell(\beta, \alpha), \alpha) \text{ (if defined)}.$$

If $0 < \alpha < \beta < \lambda$, let $k(\beta, \alpha)$ be the maximal $k < \omega$ such that $\gamma_k(\beta, \alpha)$ is defined (equivalently is equal to α) and let $\rho_{\beta,\alpha} = \rho(\beta, \alpha)$ be the sequence

$$\big\langle \gamma_0(\beta, \alpha), \gamma_1(\beta, \alpha), \ldots, \gamma_{k(\beta,\alpha)-1}(\beta, \alpha)\big\rangle.$$

Let $\gamma_{\ell t}(\beta, \alpha) = \gamma_{k(\beta,\alpha)-1}(\beta, \alpha)$ where ℓt stands for last.

Let

$$\rho_{\bar{h}}(\beta, \alpha) = \big\langle h_{\gamma_\ell(\beta,\alpha)}(\gamma_{\ell+1}(\beta, \alpha)) : \ell < k(\beta, \alpha)\big\rangle$$

and we let $\rho(\alpha, \alpha)$ and $\rho_{\bar{h}}(\alpha, \alpha)$ be the empty sequence. Now clearly:

$\odot_2$ if $0 < \alpha < \beta < \lambda$ then $\alpha \leq \gamma(\beta, \alpha) < \beta$

hence

$\odot_3$ if $0 < \alpha < \beta < \lambda, 0 < \ell < \omega$, and $\gamma_\ell(\beta, \alpha)$ is well defined, then

$$\alpha \leq \gamma_\ell(\beta, \alpha) < \beta$$

and

$\odot_4$ if $0 < \alpha < \beta < \lambda$, then $k(\beta, \alpha)$ is well defined and letting $\gamma_\ell := \gamma_\ell(\beta, \alpha)$ for $\ell \leq k(\beta, \alpha)$ we have

$$\alpha = \gamma_{k(\beta,\alpha)} < \gamma_{\ell t}(\beta, \alpha) = \gamma_{k(\beta,\alpha)-1} < \cdots < \gamma_1 < \gamma_0 = \beta$$

and

$$\alpha \in e_{\gamma_{\mathrm{lt}}(\beta,\alpha)}$$

i.e. $\rho(\beta, \alpha)$ is a (strictly) decreasing finite sequence of ordinals, starting with β, ending with $\gamma_{\ell t}(\beta, \alpha)$ of length $k(\beta, \alpha)$.

Note that if $\alpha \in S, \alpha < \beta$ then $\gamma_{\mathrm{lt}}(\beta, \alpha) = \alpha + 1$.

[2] For successor of regular we can omit h_α and below replace $\bar{h}$ and h^- by h and even let $\rho_h(\beta, \alpha) = \langle h(\gamma_\ell(\beta, \alpha)) : \ell < k(\beta, \alpha)\rangle$; but for other cases the present version is better, see more [6, Ch. III, §4]. But in later stages we may use h directly, e.g. the proof of $(*)_1$.

Also

$\odot_5$ if δ is a limit ordinal and $\delta < \beta < \lambda$, *then* for some $\alpha_0 < \delta$ we have: $\alpha_0 \le \alpha < \delta$ *implies*:

(i) for $\ell < k(\beta,\delta)$ we have $\gamma_\ell(\beta,\delta) = \gamma_\ell(\beta,\alpha)$

(ii) $\delta \in \mathrm{nacc}(e_{\gamma_{\ell t}(\beta,\delta)}) \Leftrightarrow \delta = \gamma_{k(\beta,\delta)}(\beta,\delta) = \gamma_{k(\beta,\delta)}(\beta,\alpha) \Leftrightarrow \neg[\gamma_{k(\beta,\delta)}(\beta,\delta) = \delta > \gamma_{k(\beta,\delta)}(\beta,\alpha)]$

(iii) $\rho(\beta,\delta) \trianglelefteq \rho(\beta,\alpha)$; i.e. is an initial segment

(iv) $\delta \in \mathrm{nacc}(e_{\gamma_{\ell t}(\beta,\delta)})$ (here always holds if $\delta \in S$) implies:

- $\rho(\beta,\delta)\hat{\ }\langle\delta\rangle \trianglelefteq \rho(\beta,\alpha)$ hence
- $\rho_{\bar h}(\beta,\delta)\hat{\ }\langle h_{\gamma_{\ell t}(\beta,\delta)}(\delta)\rangle \trianglelefteq \rho_{\bar h}(\beta,\alpha)$.

(v) if $\mathrm{cf}(\delta) = \partial$ *then* we have $\gamma_{\ell t}(\beta,\delta) = \delta + 1$

(vi) if $\mathrm{cf}(\delta) = \partial$ and $\delta \in e_\alpha$, *then* necessarily $\alpha = \delta + 1$.

Why? Just let

$$\alpha_0 = \mathrm{Max}\big\{ \sup(e_{\gamma_\ell(\beta,\delta)} \cap \delta) + 1 : \ell < k(\beta,\delta) \text{ and } \delta \notin \mathrm{acc}(e_{\gamma_\ell(\beta,\delta)})\big\}.$$

Notice that if $\ell < k(\beta,\delta) - 1$ then $\delta \notin \mathrm{acc}(e_{\gamma_\ell(\beta,\delta)})$ is immediate.

Note that the outer maximum (in the choice of α_0) is well defined as it is over a finite non-empty set of ordinals. The inner sup is on the empty set (in which case we get zero) or is the maximum (which is well defined) as $e_{\gamma_\ell(\beta,\delta)}$ is a closed subset of $\gamma_\ell(\beta,\delta)$, $\delta < \gamma_\ell(\beta,\delta)$ and $\delta \notin \mathrm{acc}(e_{\gamma_\ell(\beta,\delta)})$ – as this is required. For clauses (v), (vi) recall $\delta \in S^\lambda_\partial$ and $e_\gamma \cap S^\lambda_\partial = \emptyset$ when γ is a limit ordinal and $e_\gamma = \{\gamma - 1\}$ when γ is a successor ordinal.

$\odot_6$ (a) if $0 < \alpha < \beta < \lambda, \ell < k(\beta,\alpha), \gamma = \gamma_\ell(\beta,\alpha)$ *then*

$$\rho(\beta,\alpha) = \rho(\beta,\gamma)\hat{\ }\rho(\gamma,\alpha) \quad \text{and} \quad \rho_{\bar h}(\beta,\alpha) = \rho_{\bar h}(\beta,\gamma)\hat{\ }\rho_{\bar h}(\gamma,\alpha)$$

(b) if $0 < \alpha_0 < \dots < \alpha_k$ and $\rho(\alpha_k,\alpha_0) = \rho(\alpha_k,\alpha_{k-1})\hat{\ }\dots\hat{\ }\rho(\alpha_1,\alpha_0)$ *then* this holds for any subsequence of $\langle\alpha_0,\dots,\alpha_k\rangle$.

Now apply Claim 3.1 with λ, κ_1, κ_0, F_1, F_0 here standing for S, κ_1, κ_0, F_1, F_0 there and get $\mathbf{d} : {}^{\omega>}\lambda \to \mathbb{N}$.

Lastly, we define the colouring; as the proof is somewhat simpler if we use only κ_1 colours (which suffice for many purposes) we define two colourings: $\mathbf{c}_1$ with κ_1 colours and $\mathbf{c}_2$ with $\kappa_2 = \lambda$ colours, as follows:

$\odot_7$ (a) choose a function $h' : \kappa_1 \to \kappa_1$ such that $S^{\kappa_1}_{\kappa_0,\varepsilon} := \{\delta \in S^{\kappa_1}_{\kappa_0} : h'(\delta) = \varepsilon\}$ is stationary in κ_1 for every $\varepsilon < \kappa_1$

(b) if $\eta = \langle\zeta_0,\dots,\zeta_{n-1}\rangle$ then we let $h'(\eta) = \langle h'(\zeta_0),\dots,h'(\zeta_{n-1})\rangle$

(c) $\mathbf{c}_1 : [\lambda]^2 \to \kappa_1$ is defined for $\alpha < \beta$ by

$$\mathbf{c}_1(\{\alpha,\beta\}) = h'(F_1(\rho_{\bar h}(\beta,\alpha)))(\ell^1_{\beta,\alpha})$$

where $\ell^1_{\beta,\alpha} = \mathbf{d}(\rho_{\bar h}(\beta,\alpha))$.

Clearly

$\odot_8$ we can demand on h'_1 that we can choose h'_2 such that:

(a) h'_1, h'_2 are functions with domain κ_1

(b) h'_1 is onto κ_1

(c) h'_2 is onto $\mathbb{N}$

(d) for every $\zeta < \kappa_1$ and $n < \omega$ the set $S_{\kappa_1,\zeta,n} = \{\varepsilon < \kappa_1 : h'_1(\varepsilon) = \zeta$ and $h'_2(\varepsilon) = n\}$ is stationary

$\odot_9$ the colouring $\mathbf{c}_2$ with λ colours is chosen as follows: for $\alpha < \beta < \lambda$, $\mathbf{c}_2(\{\alpha,\beta\}) = (F_2(\rho_{\bar{h}}(\beta,\alpha)))(\ell^2_{\beta,\alpha})$ where letting $\varepsilon_{\alpha,\beta} = \mathbf{c}_1(\{\alpha,\beta\})$ we have $\ell^2_{\beta,\alpha}$ is the $h'_2(\varepsilon_{\beta,\alpha})$-th member of the[3] set $\{\ell < \ell g(\rho_{\bar{h}}(\beta,\alpha)) : F_1(\rho_{\bar{h}}(\beta,\alpha))(\ell) = h'_1(\varepsilon_{\beta,\alpha})\}$ if this set has $> h'_2(\varepsilon_{\alpha,\beta})$ members and is zero otherwise.

Stage B: So we have to prove that the colouring $\mathbf{c} = \mathbf{c}_1$ (with κ_1 colours) and moreover $\mathbf{c} = \mathbf{c}_2$ (with λ colours) is as required.

Now for the rest of the proof assume:

$\boxplus$ (a) $t_\alpha \subseteq \lambda$ for every $\alpha < \lambda$

(b) $t_\alpha = t^0_\alpha \cup t^1_\alpha$ and $1 \le |t^\iota_\alpha| < \theta_\iota$ for $\iota < 2$

(c) $\alpha \ne \beta \Rightarrow t_\alpha \cap t_\beta = \emptyset$

(d) $j_* < \kappa_1$ (when dealing with $\mathbf{c}_1$) or $j_* < \sigma$ (when dealing with $\mathbf{c}_2$).

Clearly (by $\boxplus$(c)), we can choose β_α by induction on $\alpha < \lambda$ by $\beta_\alpha = \min\{\beta : \beta > \alpha$ and $\min(t_\beta) > \alpha + \sup(\bigcup\{t_{\beta_{\alpha(1)}} : \alpha(1) < \alpha\})\}$. Now can use $t'_\alpha = t_{\beta_\alpha}$ for $\alpha < \lambda$, hence:

$(*)_0$ without loss of generality $\alpha < \min(t_\alpha)$ and $\alpha < \beta \Rightarrow \sup(t_\alpha) < \min(t_\beta)$.

We have to prove that for some $\alpha_0 < \alpha_1 < \lambda$ for every $(\zeta_0,\zeta_1) \in t^0_{\alpha_0} \times t^1_{\alpha_1}$ we have $\mathbf{c}\{\zeta_0,\zeta_1\} = j_*$.

$(*)_1$ We can find $\mathscr{U}^{\rm up}_1, \alpha^*_1, \varepsilon^{\rm up}_{1,1}$ such that:

(a) $\mathscr{U}^{\rm up}_1 \subseteq S$ is stationary

(b) $h{\restriction}\mathscr{U}^{\rm up}_1$ is constantly 0 (so actually $\mathscr{U}^{\rm up}_1 \subseteq S^*_0$)

(c) $\alpha^*_1 < \min(\mathscr{U}^{\rm up}_1)$ and $\varepsilon^{\rm up}_{1,1} < \kappa_1$

(d) if $\delta \in \mathscr{U}^{\rm up}_1$ and $\alpha \in [\alpha^*_1, \delta), \beta \in t^1_\delta$ (treating t^0_δ is unreasonable because t^1_δ may be of cardinality $\ge \theta_0 = \kappa_1, \varepsilon_{1,0}$ is defined for notational simplicity) *then*:

- $\rho_{\beta,\delta}{}^\frown\langle\delta\rangle \trianglelefteq \rho_{\beta,\alpha}$
- $\mathrm{Rang}(F_1(\rho_{\bar{h}}(\beta,\delta))) \subseteq \varepsilon^{\rm up}_{1,1}$.

[Why? For every $\delta \in S^*_0 \subseteq S$ and $\zeta \in t_\delta$ let $\alpha^*_{1,\delta,\zeta} < \delta$ be such that

$$(\forall\alpha)\big(\alpha \in [\alpha^*_{1,\delta,\zeta},\delta\big) \Rightarrow \rho_{\zeta,\delta}{}^\frown\langle\delta\rangle \trianglelefteq \rho_{\zeta,\alpha}),$$

it exists by $\odot_5$ of Stage A.

Let $\alpha^*_{1,\delta} = \sup\{\alpha^*_{1,\delta,\zeta} : \zeta \in t_\delta\}$ and for $\iota = 1$ let

$$\varepsilon^{\rm up}_{1,1,\delta} = \sup\big\{F_1(h(\gamma_\ell(\zeta,\delta))) + 1 : \zeta \in t^1_\delta \text{ and } \ell < k(\zeta,\delta)\big\}$$

[3] So $\mathbf{d}$ is used only via the definition of $\ell^2_{\beta,\alpha}$.

$$= \sup \bigcup \left\{ \mathrm{Rang}(F_1(\rho_{\bar{h}}(\beta,\delta))+1) : \beta \in t^1_\delta \right\};$$

as $\mathrm{cf}(\delta) = \partial = \mathrm{cf}(\partial) > |t^1_\delta|$ and $\kappa_1 = \mathrm{cf}(\kappa_1) \geq \theta_1 > |t^1_\delta|$, necessarily $\alpha^*_{1,\delta} < \delta$ and $\varepsilon^{\rm up}_{1,1,\delta} < \kappa_\iota$.

Lastly, there are $\alpha^*_1 < \lambda$ and $\varepsilon^{\rm up}_{1,0} < \kappa_0, \varepsilon^{\rm up}_{1,1} < \kappa_1$ and $\mathscr{U}^{\rm up}_1 \subseteq S^*_0$ as required in $(*)_1$ by Fodor lemma.]

$(*)_2$ for each $\varepsilon \in S^{\kappa_1}_{\kappa_0} \backslash \varepsilon^{\rm up}_{1,1}$ we can find $g_{2,\varepsilon}, \mathscr{U}^{\rm up}_{2,\varepsilon}, \gamma^*_\varepsilon, \alpha^*_{2,\varepsilon}, \ell_{2,\varepsilon}$ such that:

(a) $\gamma^*_\varepsilon < \lambda$ satisfies $F_2(\gamma^*_\varepsilon) = j_*, F_1(\gamma^*_\varepsilon) = \varepsilon, F_0(\gamma^*_\varepsilon) = 0$

(b) $\mathscr{U}^{\rm up}_{2,\varepsilon} \subseteq S^*_{\gamma^*_\varepsilon}$ is stationary

(c) $\alpha^*_1 < \alpha^*_{2,\varepsilon} < \min(\mathscr{U}^{\rm up}_{2,\varepsilon})$

(d) $g_{2,\varepsilon}$ is a function with domain $\mathscr{U}^{\rm up}_{2,\varepsilon}$ such that $\delta \in \mathscr{U}^{\rm up}_{2,\varepsilon} \Rightarrow \delta < g_{2,\varepsilon}(\delta) \in \mathscr{U}^{\rm up}_1$

(e) if $\delta \in \mathscr{U}^{\rm up}_{2,\varepsilon}$ and $\alpha \in [\alpha^*_{2,\varepsilon}, \delta)$ and $\beta \in t_{g_{2,\varepsilon}(\delta)}$ then $\rho_{g_{2,\varepsilon}(\delta),\delta}{}^\frown\langle\delta\rangle \trianglelefteq \rho_{g_{2,\varepsilon}(\delta),\alpha}$ hence (recalling $\odot_6, (*)_1$(d))

- if $\beta \in t_{g_{2,\varepsilon}(\delta)}$ then $\rho_{\beta,\delta}{}^\frown\langle\delta\rangle \trianglelefteq \rho_{\beta,\alpha}$

(f) $\ell^*_{2,\varepsilon}$ is well defined where for any $\delta \in \mathscr{U}^{\rm up}_{2,\varepsilon}$ we have $\ell^*_{2,\varepsilon} = \ell g(\rho_{g_{2,\varepsilon}(\delta),\delta})$ hence if $\alpha \in (\alpha^*_{2,\varepsilon}, \delta)$ then $\rho_{g_{2,\varepsilon}(\delta),\alpha}(\ell^*_{2,\varepsilon}) = \delta$.

(g) Lastly, if $\alpha \in (\alpha^*_{2,\varepsilon}, \delta)$ then $\ell^\bullet_{2,\varepsilon} = \min\{\ell : \ell < \ell g(\rho_{g_{2,\varepsilon}(\delta),\alpha})$ and $F_1(\rho_{\bar{h}}(g_{2,\varepsilon}(\delta),\alpha))(\ell) = \varepsilon\}$ so $\ell^\bullet_{2,\varepsilon} \leq \ell^*_{2,\varepsilon}$; recall that $\varepsilon > \varepsilon^{\rm up}_{1,1}$ hence necessarily $\beta \in t_{g_{2,\varepsilon}(\delta)} \Rightarrow \varepsilon > \sup \mathrm{Rang}(F_1(\rho_{\bar{h}}(\beta, g_{2,\varepsilon}(\delta))))$.

[Why? First, choose γ^*_ε as in clause (a) of $(*)_2$, (possible by the choice of F_0, F_1, F_2 in the beginning of Stage A). Second, define $g'_\varepsilon : S^*_{\gamma^*_\varepsilon} \to \mathscr{U}^{\rm up}_1$ such that $\delta \in S^*_{\gamma^*_\varepsilon} \Rightarrow \delta < g'_\varepsilon(\delta) \in \mathscr{U}^{\rm up}_1$. Third, do as in the proof of $(*)_1$ above for each $\delta \in S^*_{\gamma^*_\varepsilon}$ separately, i.e. find $\alpha'_{2,\varepsilon,\delta} < \delta$ above α^*_1 and $\ell^*_{2,\varepsilon,\delta}, \ell^\bullet_{2,\varepsilon,\delta}$ such that the parallel of clauses (c), (e), (f), (g) of $(*)_2$ holds. Fourth, use Fodor lemma to get a stationary $\mathscr{U}^{\rm up}_{2,\varepsilon} \subseteq S^*_{\gamma^*_\varepsilon}$ such that $\langle(\alpha'_{2,\varepsilon,\delta}, \ell^*_{2,\varepsilon,\delta}, \ell^\bullet_{2,\varepsilon,\delta}) : \delta \in \mathscr{U}^{\rm up}_{2,\varepsilon}\rangle$ is constantly $(\alpha^*_{2,\varepsilon}, \ell^*_{2,\varepsilon}, \ell^\bullet_{2,\varepsilon})$ and lastly let $g_{2,\varepsilon} = g'_\varepsilon \restriction \mathscr{U}^{\rm up}_{2,\varepsilon}$.]

$(*)_3$ we can find $\mathscr{U}^{\rm up}_3, \bar{g}^3, \alpha^*_3$ such that:

(a) $\mathscr{U}^{\rm up}_3 \subseteq S$ is stationary

(b) $\min(\mathscr{U}^{\rm up}_3) > \alpha^*_3 > \sup\{\alpha^*_{2,\varepsilon} : \varepsilon \in S^{\kappa_1}_{\kappa_0} \backslash \varepsilon^{\rm up}_{1,1}\}$

(c) $\bar{g}^3 = \langle g_{3,\varepsilon} : \varepsilon \in S^{\kappa_1}_{\kappa_0} \backslash \varepsilon^{\rm up}_{1,1}\rangle$

(d) $g_{3,\varepsilon}$ is a function with domain $\mathscr{U}^{\rm up}_3$

(e) if $\delta \in \mathscr{U}^{\rm up}_3$ and $\varepsilon \in S^{\kappa_1}_{\kappa_0} \backslash \varepsilon^{\rm up}_{1,1}$ then $\delta < g_{3,\varepsilon}(\delta) \in \mathscr{U}^{\rm up}_{2,\varepsilon}$

(f) if $\alpha \in [\alpha^*_3, \delta)$, $\delta \in \mathscr{U}^{\rm up}_3$ and $\varepsilon \in S^{\kappa_1}_{\kappa_0} \backslash \varepsilon^{\rm up}_{1,1}$ then $\rho_{g_{3,\varepsilon}(\delta),\delta}{}^\frown\langle\delta\rangle \trianglelefteq \rho_{g_{3,\varepsilon}(\delta),\alpha}$ hence

$(f)'$ if in addition $\beta \in t^1_{g_{2,\varepsilon}(g_{3,\varepsilon}(\delta))}$ then $\rho_{\beta,\delta}{}^\frown\langle\delta\rangle \trianglelefteq \rho_{\beta,\alpha}$ this follows.

[Why? First, let $\alpha^*_2 = \sup\{\alpha^*_{2,\varepsilon} + 1 : \varepsilon \in S^{\kappa_1}_{\kappa_0} \backslash \varepsilon^{\rm up}_{1,1}\} < \lambda$ and choose $g''_\varepsilon : S \backslash \alpha^*_2 \to \mathscr{U}^{\rm up}_{2,\varepsilon}$ such that $g''_\varepsilon(\delta) > \delta$ for every $\delta \in S \backslash \alpha^*_2$ and second for each $\delta \in S \backslash \alpha^*_2$ choose $\alpha^*_{3,\delta} < \delta$ as in clauses (f), (f′) of $(*)_3$, i.e. such that $\alpha \in [\alpha^*_{3,\delta}, \delta)$

$\Rightarrow \rho_{g''_\varepsilon(\delta),\delta}{}^\frown\langle\delta\rangle \trianglelefteq \rho_{g''_\varepsilon(\delta),\alpha}$ for every $\varepsilon \in S^{\kappa_1}_{\kappa_0}\backslash\varepsilon^{\rm up}_{1,1}$ and such that the relevant part of clause (b) of $(*)_3$, holds, that is, $\alpha^*_{3,\delta} > \alpha^*_2 = \sup\{\alpha^*_{2,\varepsilon} : \varepsilon < S^{\kappa_1}_{\kappa_0}\backslash\varepsilon^{\rm up}_{1,1}\}$, possible as $\kappa_1 < \partial$. Third, use Fodor lemma to find $\alpha^*_3 < \lambda$ such that $\mathscr{U}^{\rm up}_3 = \{\delta \in S : \alpha^*_{3,\delta} = \alpha^*_3\}$ is a stationary subset of λ. Fourth, let $g_{3,\varepsilon} = g''_\varepsilon{\restriction}\mathscr{U}^{\rm up}_3$.]

$(*)_4$ recalling[4] $j_* < \kappa_1$, there are $\mathscr{U}^{\rm up}_4, \varepsilon^*_{4,1}, \varepsilon^*_{4,0}$ and $\langle s_\delta : \delta \in \mathscr{U}^{\rm up}_4\rangle$ such that:

(a) $\mathscr{U}^{\rm up}_4 \subseteq \mathscr{U}^{\rm up}_3$ is a stationary subset of λ

(b) $\varepsilon^{\rm up}_{1,1} < \varepsilon^{\rm up}_{4,1} < \kappa_1$ and $\varepsilon^{\rm up}_{4,0} < \kappa_0$

(c) if $\delta \in \mathscr{U}^{\rm up}_4$ then s_δ is a stationary (in κ_1) subset of $S^{\kappa_1}_{\kappa_0,j_*}\backslash\varepsilon^{\rm up}_{4,1}$

(d) if $\delta \in \mathscr{U}^{\rm up}_4, \varepsilon \in s_\delta$ *then*

(α) $\mathrm{Rang}(F_1(\rho_{\bar h}(g_{2,\varepsilon}(g_{3,\varepsilon}(\delta)),\delta))) \cap \varepsilon \subseteq \varepsilon^{\rm up}_{4,1}$ hence by clause (b)

(β) if $\beta \in t_{g_{2,\varepsilon}(g_{3,\varepsilon}(\delta))}$ then $\mathrm{Rang}(F_1(\rho_{\bar h}(\beta,\delta)) \cap \varepsilon \subseteq \varepsilon^{\rm up}_{4,1}$

(γ) also $\mathrm{Rang}(F_0(\rho_{\bar h}(g_{2,\varepsilon}(g_{3,\varepsilon}(\delta)),\delta))) \subseteq \varepsilon^{\rm up}_{4,0}$.

[Why? Recall that κ_1 is regular uncountable (being θ_1) and $\kappa_0 < \kappa_1$ is regular (being θ_0). First, for each $\delta \in \mathscr{U}^{\rm up}_3$ we use Fodor lemma on $S^{\kappa_1}_{\kappa_0,j_*}\backslash\varepsilon^{\rm up}_{1,1}$ to choose s_δ, $\varepsilon^{\rm up}_{4,1,\delta}$, $\varepsilon^{\rm up}_{4,0,\delta}$ as in clauses (c) + (d); second use the Fodor Lemma on $\mathscr{U}^{\rm up}_3$ to get $\mathscr{U}^{\rm up}_4$, $\varepsilon^{\rm up}_{4,1}$, $\varepsilon^{\rm up}_{4,0}$; we cannot do it for s_δ as maybe $2^{\kappa_1} \geq \lambda$.]

Let us verify (d)(β) and (d)(γ). For (d)(β) notice that $\mathrm{Rang}(F_1(\rho_{\bar h}(\beta,\delta))) \subseteq \varepsilon^{\rm up}_{1,1} < \varepsilon^{\rm up}_{4,1}$ for every $\beta \in t_{g_{2,\varepsilon}(g_{3,\varepsilon}(\delta))}$ by $(*)_1$(d). This requirement is easy since $|t_{g_{2,\varepsilon}(g_{3,\varepsilon}(\delta))}| < \kappa_1$ and $\rho_{\bar h}(\beta,\delta)$ is finite for every $\beta \in t_{g_{2,\varepsilon}(g_{3,\varepsilon}(\delta))}$.

For (d)(γ) we apply Fodor's lemma twice.

First, fix an ordinal $\delta \in \mathscr{U}^{\rm up}_4$. For every $\varepsilon \in s_\delta$, the sequence

$$F_0(\rho_{\bar h}(g_{2,\varepsilon}(g_{3,\varepsilon}(\delta)))$$

is finite and hence bounded in κ_0. But $\kappa_0 < \kappa_1 = \mathrm{cf}(\kappa_1)$ and hence by shrinking s_δ if needed we may assume that all the values are bounded by the same ordinal $\sigma_\delta < \kappa_0$.

Now for each $\delta \in \mathscr{U}^{\rm up}_4$ we choose $\sigma_\delta \in \kappa_0$ in this way, so by shrinking $\mathscr{U}^{\rm up}_4$ if needed we may assume that $\sigma_\delta = \sigma$ for some fixed $\sigma < \kappa_0$ and every $\delta \in \mathscr{U}^{\rm up}_4$. Now choose $\varepsilon^{\rm up}_{4,0} > \max\{\sigma, \varepsilon^{\rm up}_{1,0}\}$.

$(*)_5$ we can find $\mathscr{U}^{\rm dn}_1, \varepsilon^{\rm dn}_{1,0}, \varepsilon^{\rm dn}_{1,1}$ such that:

(a) $\mathscr{U}^{\rm dn}_1 \subseteq S^*_0$ is stationary in λ

(b) $\alpha < \delta \in \mathscr{U}^{\rm dn}_1 \Rightarrow t_\alpha \subseteq \delta$

(c) $\varepsilon^{\rm dn}_{1,\iota} < \kappa_\iota$ for $\iota = 0, 1$

(d) if $\delta \in \mathscr{U}^{\rm dn}_1$ *then* for arbitrarily large $\alpha < \delta$ we have $\beta \in t^\iota_\alpha \wedge \iota \in \{0,1\} \Rightarrow \mathrm{Rang}(F_\iota(\rho_{\bar h}(\delta,\beta))) \subseteq \varepsilon^{\rm dn}_{1,\iota} < \kappa_\iota$.

[Why? Clearly $E = \{\delta < \lambda : \delta$ a limit ordinal such that $\alpha < \delta \Rightarrow t_\alpha \subseteq \delta\}$ is a club of λ. For every $\delta \in S^*_0 \cap E$ and $\alpha < \delta$ we can find $(\varepsilon^{\rm dn}_{1,0,\delta,\alpha}, \varepsilon^{\rm dn}_{1,1,\delta,\alpha})$

[4] Recall that in this stage we are dealing with $\mathbf{c} = \mathbf{c}_1$ hence $j_* < \kappa_1$.

as in clauses (c),(d) above because $|t^\iota_\alpha| < \kappa_\iota = \mathrm{cf}(\kappa_\iota)$. So recalling that $\mathrm{cf}(\delta) = \partial > \theta_1 = \kappa_1 > \kappa_0 = \theta_0$ it follows that there is a pair $(\varepsilon^{\mathrm{dn}}_{1,0,\delta}, \varepsilon^{\mathrm{dn}}_{1,1,\delta})$ such that $\delta = \sup\{\alpha < \delta : (\varepsilon^{\mathrm{dn}}_{1,0,\delta,\alpha}, \varepsilon^{\mathrm{dn}}_{1,1,\delta,\alpha}) = (\varepsilon^{\mathrm{dn}}_{1,0,\delta}, \varepsilon^{\mathrm{dn}}_{1,1,\delta})\}$. Then recalling $\lambda = \mathrm{cf}(\lambda) > \kappa_1 + \kappa_0$ we can choose $(\varepsilon^{\mathrm{dn}}_{1,0}, \varepsilon^{\mathrm{dn}}_{1,1})$ such that the set $\mathscr{U}^{\mathrm{dn}}_1 = \{\delta \in S^*_0 : (\varepsilon^{\mathrm{dn}}_{1,0,\delta}, \varepsilon^{\mathrm{dn}}_{1,1,\delta}) = (\varepsilon^{\mathrm{dn}}_{1,0}, \varepsilon^{\mathrm{dn}}_{1,1})\}$ is stationary.]

$(*)_6$ we can find $\mathscr{U}^{\mathrm{dn}}_2, \varepsilon^{\mathrm{dn}}_{2,0}, \varepsilon^{\mathrm{dn}}_{2,1}$ such that:

(a) $\mathscr{U}^{\mathrm{dn}}_2 \subseteq S^*_0 \backslash (\alpha^*_3 + 1)$ is stationary

(b) if $\delta \in \mathscr{U}^{\mathrm{dn}}_2$ and $\zeta < \kappa_1$ *then* $\delta = \sup(\mathscr{U}^{\mathrm{dn}}_1 \cap \delta)$ and for arbitrarily large $\delta_0 \in \mathscr{U}^{\mathrm{dn}}_1 \cap \delta$ we have $\zeta < \max \mathrm{Rang}(F_1(\rho_{\bar h}(\delta, \delta_0)))$ and

$$\mathrm{Rang}(F_0(\rho_{\bar h}(\delta, \delta_0))) \subseteq \varepsilon^{\mathrm{dn}}_{2,0} \quad \text{and} \quad \zeta \cap \mathrm{Rang}(F_1(\rho_{\bar h}(\delta, \delta_0))) \subseteq \varepsilon^{\mathrm{dn}}_{2,1}$$

(c) $\varepsilon^{\mathrm{dn}}_{2,0} \in (\varepsilon^{\mathrm{dn}}_{1,0}, \kappa_0)$ and $\varepsilon^{\mathrm{dn}}_{2,1} \in (\varepsilon^{\mathrm{dn}}_{1,1}, \kappa_1)$.

[Why? For every $\zeta < \kappa_1$ let $S'_\zeta = \{\alpha \in S : \alpha = \sup(\mathscr{U}^{\mathrm{dn}}_1 \cap \alpha)$ and $F_1(h(\alpha)) = \zeta\}$, clearly it is a stationary subset of λ.

Let $E = \{\delta < \lambda : \delta$ is a limit ordinal and $\zeta < \kappa_1 \Rightarrow \delta = \sup(\delta \cap S'_\zeta)\}$. Clearly it is a club of λ. If $\zeta \in S^{\kappa_1}_{\kappa_0} \backslash \varepsilon^{\mathrm{dn}}_{1,1}$ and $\delta \in E \cap S^*_0$ and $\alpha \in S'_\zeta \cap \delta$ let $\varepsilon^{\mathrm{dn}}_{2,0,\zeta,\delta,\alpha} = \sup \mathrm{Rang}(F_0(\rho_{\bar h}(\delta, \alpha))) + \varepsilon^{\mathrm{dn}}_{1,0} + 1$ and let

$$\varepsilon^{\mathrm{dn}}_{2,1,\zeta,\delta,\alpha} = \sup(\zeta \cap \mathrm{Rang}(F_1(\rho_{\bar h}(\delta, \alpha))) + 1 < \zeta.$$

Fixing δ and ζ, recalling $\mathrm{cf}(\delta) > \kappa_0 + \kappa_1$, for some pair $(\varepsilon^{\mathrm{dn}}_{2,0,\zeta,\delta}, \varepsilon^{\mathrm{dn}}_{2,1,\zeta,\delta}) \in \kappa_0 \times \kappa_1$ we have $\delta = \sup\{\alpha \in S'_\zeta \cap \delta : (\varepsilon^{\mathrm{dn}}_{2,0,\zeta,\delta,\alpha}, \varepsilon^{\mathrm{dn}}_{2,1,\zeta,\delta,\alpha}) = (\varepsilon^{\mathrm{dn}}_{2,0,\zeta,\delta}, \varepsilon^{\mathrm{dn}}_{2,1,\zeta,\delta})\}$.

Fixing δ apply Fodor lemma on $S^{\kappa_1}_{\kappa_0}$, for some pair $(\varepsilon^{\mathrm{dn}}_{2,0,\delta}, \varepsilon^{\mathrm{dn}}_{2,1,\delta})$ the set $b_\delta = \{\zeta \in S^{\kappa_1}_{\kappa_0} : (\varepsilon^{\mathrm{dn}}_{2,0,\zeta,\delta}, \varepsilon^{\mathrm{dn}}_{2,1,\zeta,\delta}) = (\varepsilon^{\mathrm{dn}}_{2,0,\delta}, \varepsilon^{\mathrm{dn}}_{2,1,\delta})\}$ is a stationary subset of κ_1.

Applying Fodor lemma on $\delta \in E \cap S^*_0$, there is a pair $(\varepsilon^{\mathrm{dn}}_{2,0}, \varepsilon^{\mathrm{dn}}_{2,1})$ such that $\mathscr{U}^{\mathrm{dn}}_2 := \{\delta \in S^*_0 : \delta \in E$ and $(\varepsilon^{\mathrm{dn}}_{2,0,\delta}, \varepsilon^{\mathrm{dn}}_{2,1,\delta}) = (\varepsilon^{\mathrm{dn}}_{2,0}, \varepsilon^{\mathrm{dn}}_{2,1})\}$ is stationary. Clearly we are done. We could have put b_ε in $(*)_6$(b) but it does not seem needed.]

Stage C: Now we shall find the required $\alpha_0 < \alpha_1$.

In this stage we deal with $\mathbf{c}_1$, so $j_* < \kappa_1$. First, there are δ_1, δ_2, $\varepsilon^{\mathrm{md}}_0$, $\varepsilon^{\mathrm{md}}_1$, α^*_4 such that:

$\oplus_0$ (a) $\delta_1 \in \mathscr{U}^{\mathrm{dn}}_2$ and $\delta_2 \in \mathscr{U}^{\mathrm{up}}_4$, see $(*)_6$ and $(*)_4$ respectively

(b) $\delta_1 < \delta_2$ and $\alpha^*_3 < \delta_1$

(c) $\varepsilon^{\mathrm{md}}_\iota := \max \mathrm{Rang}(F_\iota(\rho_{\bar h}(\delta_2, \delta_1))) > \varepsilon^{\mathrm{dn}}_{2,\iota} + \varepsilon^{\mathrm{up}}_{4,\iota} \geq \varepsilon^{\mathrm{dn}}_{1,\iota} + \varepsilon^{\mathrm{up}}_{1,\iota}$ for $\iota = 0,1$

(d) $\alpha^*_4 < \delta_1$ is $> \alpha^*_3$ and if $\alpha \in (\alpha^*_4, \delta_1)$ then $\rho_{\delta_2,\delta_1}{}^\frown\langle \delta_1 \rangle \trianglelefteq \rho_{\delta_2,\alpha}$.

[Why can we? Easy but we give details. First, let $\mathscr{W}_* = \{\delta \in S : \delta$ is a limit ordinal $> \alpha^*_3$ necessarily of cofinality ∂ such that $F_\iota(\delta) > \varepsilon^{\mathrm{dn}}_{2,\iota} + \varepsilon^{\mathrm{up}}_{4,\iota}$ for $\iota = 0,1$ and $\delta = \sup(\delta \cap \mathscr{U}^{\mathrm{dn}}_2)\}$, clearly it is a stationary subset of λ. Second, choose $\delta_2 \in \mathscr{U}^{\mathrm{up}}_4$ which is $> \alpha^*_3$ such that $\delta_2 = \sup(\mathscr{W}_* \cap \delta_2)$. Third, choose $\delta_* \in \mathscr{W}_* \cap \delta_2$ such that $\alpha^*_3 < \delta_*$. Fourth, let $\alpha_* < \delta_*$ be such that $\alpha_* > \alpha^*_3$

and $\alpha \in (\alpha_*, \delta_*) \Rightarrow \rho(\delta_2, \delta_*)\char`\^\langle \delta_* \rangle \trianglelefteq \rho(\delta_2, \alpha)$ (hence $\rho_{\bar h}(\delta_2, \delta_*)\char`\^\langle h_{\delta_*+1}(\delta_*)\rangle \trianglelefteq \rho_{\bar h}(\delta_2, \alpha)$). Fifth, choose $\delta_1 \in (\alpha_*, \delta_*) \cap \mathscr{U}_2^{\rm dn}$ hence $\delta_1 > \alpha_3^*$. Sixth, we choose $\varepsilon_\iota^{\rm md}$ for $\iota = 0, 1$ by clause (c), the inequality holds because $\delta_* \in \mathscr{W}_* \cap {\rm Rang}(\rho_{\bar h}(\delta_2, \delta_1))$.

Lastly, choose α_4^* as in $\oplus_0$(d). Easy to check that we are done proving $\oplus_0$.]

Let $\rho = \rho_{\bar h}(\delta_2, \delta_1)$.

Second, choose δ_0 such that

$\oplus_{0.1}$ (a) $\delta_0 \in \mathscr{U}_1^{\rm dn} \cap \delta_1$

(b) $(*)_6$(b) holds with $(\varepsilon_1^{\rm md}, \delta_1)$ here standing for (ζ, δ) there, that is, we have $\varepsilon_1^{\rm md} < \max {\rm Rang}(F_1(\rho_{\bar h}(\delta_1, \delta_0)))$ and ${\rm Rang}(F_0(\rho_{\bar h}(\delta_1, \delta_0))) \subseteq \varepsilon_{2,0}^{\rm dn}$ and $\varepsilon_1^{\rm md} \cap {\rm Rang}(F_1(\rho_{\bar h}(\delta_1, \delta_0))) \subseteq \varepsilon_{2,1}^{\rm dn}$

(c) $\delta_0 > \alpha_4^*$ recalling $\delta_1 > \alpha_4^* > \alpha_3^*$ by $\oplus_0(b)$,(d).

[Why can we choose δ_0? By $(*)_6$.]

Also choose α_5^* such that

$\oplus_{0.2}$ $\alpha_5^* < \delta_0$ is such that $\alpha \in (\alpha_5^*, \delta_0) \Rightarrow \rho_{\delta_1,\delta_0}\char`\^\langle \delta_0 \rangle \trianglelefteq \rho_{\delta_1,\alpha}$.

Third, choose $\varepsilon_* \in s_{\delta_2}$ (s_{δ_2} is from $(*)_4(c),(d)$) such that $\varepsilon_* > \varepsilon_{2,1}^{\rm md} := \max\big({\rm Rang}(F_1(\rho_{\bar h}(\delta_2, \delta_1) \cup {\rm Rang}(F_1(\rho_{\bar h}(\delta_1, \delta_0)))\big)$ which is $> \varepsilon_1^{\rm md}$, possible as s_{δ_2} is a stationary subset of κ_1.

Fourth, let $\delta_3 = g_{3,\varepsilon_*}(\delta_2)$.

Fifth, let $\alpha_1 = g_{2,\varepsilon_*}(\delta_3)$.

Lastly, choose $\alpha_0 < \delta_0$ large enough and as in $(*)_5$(d) such that $\alpha_0 > \alpha_5^* > \alpha_4^*$, that is, we have $\beta \in t_{\alpha_0}^1 \Rightarrow {\rm Rang}(F_1(\rho_{\bar h}(\delta_0, \beta))) \subseteq \varepsilon_{1,1}^{\rm dn} < \kappa_1$.

We shall prove below that the pair (α_0, α_1) is as promised.

So (finishing the case of κ_1 colours)

⊛ let $\zeta_0 \in t_{\alpha_0}^0, \zeta_1 \in t_{\alpha_1}^1$ and we should prove that $\mathbf{c}_1\{\zeta_0, \zeta_1\} = j_*$.

Note

$\oplus_1$ $\delta_2 \in \mathscr{U}_4^{\rm up} \subseteq \mathscr{U}_3^{\rm up}$ and $\alpha_0 < \delta_0 < \delta_1 < \delta_2$.

[Why? The first statement holds by the choice of δ_2, see $\oplus_0$(a) and $(*)_4$(a). The second statement holds by the choices of δ_1, i.e. $\oplus_0$(b), the choice of δ_0, i.e. $\oplus_{0.1}$(a) and the choice of α_0 (see "Lastly…" after $\oplus_{0.2}$).]

$\oplus_2$ $\delta_3 = g_{3,\varepsilon_*}(\delta_2) \in \mathscr{U}_{2,\varepsilon_*}^{\rm up}$ and $\delta_2 < \delta_3$.

[Why? By the choice of δ_3 (after $\oplus_{0.2}$ in "Fourth") and by $(*)_3$(d)+(e) (note that the assumption of $(*)_3$(e) in our case, which means $\delta_2 \in \mathscr{U}_3^{\rm up}$ and $\varepsilon_* \in S_{\kappa_0}^{\kappa_1} \backslash \varepsilon_{2,1}^{\rm md}$, holds by $\oplus_1$ and by the "Third" after $\oplus_{0.2}$ above (recalling $s_{\delta_2} \subseteq S_{\kappa_0}^{\kappa_1}$ and $\oplus_0$(c))).]

$\oplus_3$ $\alpha_1 = g_{2,\varepsilon_*}(\delta_3) \in \mathscr{U}_1^{\rm up}$ and $\delta_3 < \alpha_1$.

[Why? By the choice of α_1 in "Fifth" after $\oplus_{0,2}$ and $(*)_2$(d).]

$\oplus_4$ $\eta_0 := \rho_{\bar h}(\zeta_1, \alpha_1)$ satisfies ($\eta_0 \in {}^{\omega>}\lambda$ and):

- ${\rm Rang}(F_1(\eta_0)) \subseteq \varepsilon_{1,1}^{\rm up}$.

[Why? By $(*)_1$(d) recalling $\oplus_3$ of course, $\alpha_1 > \alpha_5^* > \alpha_1^*$.]

Recall that $(*)_1$(d) deals only with t_ε^1.

$\oplus_5$ $\nu_0 := \rho_{\bar{h}}(\alpha_1, \delta_2)$ satisfies ($\nu_0 \in {}^{\omega >}\lambda$ and)

(a) $\mathrm{Rang}(F_0(\nu_0)) \subseteq \varepsilon^{\mathrm{up}}_{4,0}$

(b) $\varepsilon_* \in \mathrm{Rang}(F_1(\nu_0))$

(c) $\mathrm{Rang}(F_1(\nu_0)) \cap \varepsilon_* \subseteq \varepsilon^{\mathrm{up}}_{4,1}$

(d) $\alpha_1 = g_{2,\varepsilon_*}(g_{3,\varepsilon_*}(\delta_2)) = g_{2,\varepsilon_*}(\delta_3)$

(e) $\rho(\alpha_1, \delta_2) = \rho(\alpha_1, g_{3,\varepsilon_*}(\delta_2))\hat{\ }\rho(g_{3,\varepsilon_*}(\delta_2), \delta_2)$.

[Why? Clause (d) of $\oplus_5$ holds by the choice of α_1 in "Fourth" and "Fifth" after $\oplus_{0.2}$ above (and see $\oplus_2$); similarly clause (e) holds. By $\oplus_1$ we have $\delta_2 \in \mathscr{U}^{\mathrm{up}}_4$ and by $(*)_4(d)(\gamma),(\alpha)$ and the choices of δ_3, α_1 we have clauses (a) + (c) of $\oplus_5$; that is, $(\alpha_1, \delta_{2,\varepsilon_*}, \varepsilon_*)$ here stand for $(g_{2,\varepsilon}(g_{3,\varepsilon}(\delta)), \delta, \varepsilon)$ in $(*)_4$(d). Now $\delta_3 \in \mathrm{Rang}(\rho(g_{3,\varepsilon_*}(\delta)), \delta_2)$ by $\oplus_2$ hence $\delta_3 \in \mathrm{Rang}(\rho(\alpha_1, \delta_2))$ by $\oplus_5$(e) hence $\delta_3 \in \mathrm{Rang}(\nu_0)$ by the choice of ν_0 (see the beginning of $\oplus_5$). This implies clause (b) of $\oplus_5$ because $F_1(\delta_3) = \varepsilon_*$ because $\delta_3 \in \mathrm{dom}(g_{2,\varepsilon_*}) \subseteq \mathscr{U}^{\mathrm{up}}_{2,\varepsilon_*}$ by $\oplus_2$ and $(\forall\delta)[\delta \in \mathscr{U}^{\mathrm{up}}_{2,\varepsilon_*} \Rightarrow \delta \in S^*_{\gamma^*_{\varepsilon_*}} \Rightarrow F_1(\delta) = \varepsilon_*]$ by $(*)_2(a)$,(b).]

$\oplus_6$ $\nu_1 := \rho_{\bar{h}}(\delta_1, \delta_0)$ satisfies:

(a) $\mathrm{Rang}(F_0(\nu_1)) \subseteq \varepsilon^{\mathrm{dn}}_{2,0}$

(b) $\varepsilon^{\mathrm{md}}_1 < \max \mathrm{Rang}(F_1(\nu_1))$

(c) $\mathrm{Rang}(F_1(\nu_1)) \subseteq \varepsilon_*$.

[Why? By $\oplus_0$(a) we have $\delta_1 \in \mathscr{U}^{\mathrm{dn}}_2$. So (a), (b) hold by $(*)_6$(b) and the choice of δ_0, i.e. $\oplus_{0.1}$(b); we use the first two conclusions of $(*)_6$(b) not the third. As for clause (c) it holds by the choice of ε_* in "Third" after $\oplus_{0.2}$.]

$\oplus_7$ (a) $\eta_1 := \rho_{\bar{h}}(\delta_0, \zeta_0)$ satisfies

- $\mathrm{Rang}(F_\iota(\eta_1)) \subseteq \varepsilon^{\mathrm{dn}}_{1,\iota}$ for $\iota = 0, 1$.

(b) $\rho = \rho_{\bar{h}}(\delta_2, \delta_1)$ satisfies

- $\max \mathrm{Rang}(F_\iota(\rho)) = \varepsilon^{\mathrm{md}}_\iota$ for $\iota = 0, 1$.

[Why? Clause (a) holds by $(*)_5$(d) and the choice of α_0 in "lastly" after $\oplus_{0.2}$ recalling $\zeta_0 \in t^0_{\alpha_0}$. Clause (b) holds by $\oplus_0$(c).]

$\oplus_8$ (a) $\rho_{\bar{h}}(\zeta_1, \zeta_0) = \rho_{\bar{h}}(\zeta_1, \alpha_1)\hat{\ }\rho_{\bar{h}}(\alpha_1, \delta_2)\hat{\ }\rho_{\bar{h}}(\delta_2, \delta_1)\hat{\ }\rho_{\bar{h}}(\delta_1, \delta_0)\hat{\ }\rho_{\bar{h}}(\delta_0, \zeta_0)$

(b) recalling $\rho = \rho_{\bar{h}}(\delta_2, \delta_1)$ and the choices of $\eta_0, \nu_0, \rho, \nu_1, \eta_1$ we have $\rho_{\bar{h}}(\zeta_1, \zeta_0) = \eta_0\hat{\ }\nu_0\hat{\ }\rho\hat{\ }\nu_1\hat{\ }\eta_1$.

[Why? Clause (a) holds by the choices of α^*_0 in $(*)_1$(c)(d) and of α^*_3 in $(*)_3$(f),(f)$'$ and $\delta_1 > \alpha^*_3$ by $\oplus_0$(b) and as "$\delta_0 > \alpha^*_3$" recalling $\oplus_{0.1}$(c) and "$\alpha_0 > \alpha^*_5$", see "Lastly" after $\oplus_{0.2}$. Clause (b) holds by clause (a) and the definitions of η_0, ν_0, ρ, ν_1, η_1 above, that is, in $\oplus_4$, in $\oplus_3$, before $\oplus_{0.1}$, in $\oplus_6$, in $\oplus_7$ respectively.]

$\oplus_9$ $\ell^\bullet_4 := \mathbf{d}(\rho_{\bar{h}}(\zeta_1, \zeta_0))$ satisfies $F_1(\varrho(\ell^\bullet_4)) = \varepsilon_*$.

[Why? We shall use $\oplus_8$(a),(b) freely; now $\mathbf{d}$ was chosen by Claim 3.1 and letting $\varrho = \eta_0\hat{\ }\nu_0\hat{\ }\rho\hat{\ }\nu_1\hat{\ }\eta_1$ we apply the claim to $(\eta_0, \nu_0, \rho, \nu_1, \eta_1)$, so it suffices to show that clauses (B)(a)–(d) of 3.1 hold.

$\oplus_{9.1}$ clause (B)(a)(α) of 3.1 holds.

Why? First, $\varepsilon_* \leq \max \mathrm{Rang}(F_1(\nu_0))$ by $\oplus_5$(b).

Second, $\mathrm{Rang}(F_1(\eta_0)) \subseteq \varepsilon^{\mathrm{up}}_{1,1}$ by $\oplus_4$ and $\varepsilon^{\mathrm{up}}_{1,1} \le \varepsilon^{\mathrm{up}}_{4,1}$ by $(*)_4$(b) and $\varepsilon^{\mathrm{up}}_{4,1} \le \varepsilon^{\mathrm{md}}_1$ by $\oplus_0$(c) and $\varepsilon^{\mathrm{md}}_1 < \varepsilon_*$ by the choice of ε_* in "Third" after $\oplus_{0.2}$.

Third, $\mathrm{Rang}(F_1(\rho)) \subseteq \varepsilon_*$ as

$$\mathrm{Rang}(F_1(\rho)) = \mathrm{Rang}(F_1(\rho_{\bar h}(\delta_2, \delta_1))) \subseteq \varepsilon^{\mathrm{md}}_1 + 1$$

by $\oplus_0$(c) and $\varepsilon^{\mathrm{md}}_1 < \varepsilon_*$ by the choice of ε_*.

Fourth, $\mathrm{Rang}(F_1(\nu_1)) \subseteq \varepsilon_*$ by $\oplus_6$(c).

Fifth, $\mathrm{Rang}(F_1(\eta_1)) \subseteq \varepsilon_*$ as $\mathrm{Rang}(F_1(\eta_1)) \subseteq \varepsilon^{\mathrm{dn}}_{1,1}$ by $(*)_5$ and $\varepsilon^{\mathrm{dn}}_{1,1} \le \varepsilon^{\mathrm{dn}}_{2,1}$ by $(*)_6$(c) and $\varepsilon^{\mathrm{dn}}_{2,1} < \varepsilon^{\mathrm{md}}_1$ by $\oplus_0$(c) and $\varepsilon^{\mathrm{md}}_1 < \varepsilon_*$ by the choice of ε_*.

Together $\oplus_{9.1}$ holds.

$\oplus_{9.2}$ let $\ell_1 < \ell g(\nu_0)$ be as in clause (B)(a)(β) of 3.1

$\oplus_{9.3}$ clause $(B)(b)(\alpha)$ of 3.1 holds.

Why? First, $\max \mathrm{Rang}(F_0(\rho)) = \varepsilon^{\mathrm{md}}_0$ by $\oplus_0$(c).

Second, $\mathrm{Rang}(F_0(\eta_0)) \subseteq \varepsilon^{\mathrm{md}}_0$ is unreasonable see $\oplus_4$ and not necessary.

Third, $\mathrm{Rang}(F_0(\nu_0)) \subseteq \varepsilon^{\mathrm{md}}_0$ because $\mathrm{Rang}(F_0(\nu_0)) \subseteq \varepsilon^{\mathrm{up}}_{4,0}$ by $\oplus_5$(a) and $\varepsilon^{\mathrm{up}}_{4,0} \le \varepsilon^{\mathrm{md}}_0$ by $\oplus_0$(c).

Fourth, $\mathrm{Rang}(F_0(\nu_1)) \subseteq \varepsilon^{\mathrm{md}}_0$ because $\mathrm{Rang}(F_0(\nu_1)) \subseteq \varepsilon^{\mathrm{dn}}_{2,0}$ by $\oplus_6$(a) and $\varepsilon^{\mathrm{dn}}_{2,0} \le \varepsilon^{\mathrm{md}}_0$ by $\oplus_0$(c).

Fifth, $\mathrm{Rang}(F_0(\eta_1)) \subseteq \varepsilon^{\mathrm{md}}_0$ because $\mathrm{Rang}(F_0(\eta_1)) \subseteq \varepsilon^{\mathrm{dn}}_{1,0}$ by $\oplus_7$(a) and $\varepsilon^{\mathrm{dn}}_{1,0} < \varepsilon^{\mathrm{dn}}_{2,0}$ by $(*)_6$(c) and $\varepsilon^{\mathrm{dn}}_{2,0} \le \varepsilon^{\mathrm{md}}_0$ by $\oplus_0$(c).

Together $\oplus_{9.3}$ holds.

$\oplus_{9.4}$ (a) let $\ell^\bullet_2 < \ell g(\varrho)$ be as in clause (B)(b)(β) of 3.1
(b) let $\ell^*_2 = \ell^\bullet_2 - \ell g(\eta_0 \hat{}\, \nu_0)$

$\oplus_{9.5}$
(a) $\ell^\bullet_2 \in [\ell g(\eta_0 \hat{}\, \nu_0), \ell g(\eta_0 \hat{}\, \nu_0 \hat{}\, \rho))$
(b) clause $(B)(c)(\alpha)$ holds, i.e.
$\bullet_1$ $\max \mathrm{Rang}(F_1(\nu_0)) > \max \mathrm{Rang}(F_1(\varrho{\restriction}[\ell^\bullet_2, \ell g(\varrho)))$
$\bullet_2$

$$\max \mathrm{Rang}(F_1)(\varrho{\restriction}[\ell^\bullet_2, \ell g(\varrho))) = \max \mathrm{Rang}(F_1(\nu_1)) > \max \mathrm{Rang}(\rho \hat{}\, \eta_1)$$

(c) let $\ell_3 < \ell g(\nu_1)$ be as in clause (B)(c)(β) of 3.1
(d) $F_1(\nu_1(\ell_3)) \ge \varepsilon^{\mathrm{md}}_1$.

Why? Clause (a) follows by $(B)(b)(\alpha)$ proved in $\oplus_{9.3}$ above. Clause (b), $\bullet_1$ holds by $\oplus_{9.1}$. Clause (b),$\bullet_2$ follows because: first $\mathrm{Rang}(F_1(\rho)) \subseteq \varepsilon^1_{\mathrm{md}} + 1$ by $\oplus_0$(c) and $\varepsilon^1_{\mathrm{md}} + 1 < \varepsilon$ by second; $\mathrm{Rang}(F_1(\nu_1)) \nsubseteq \varepsilon^{\mathrm{md}}_1 + 1$ by $\oplus_6$(b) and third, $\mathrm{Rang}(F_1(\eta_1)) \subseteq \varepsilon^{\mathrm{dn}}_{1,1}$ by $\oplus_7$(a) and $\varepsilon^{\mathrm{dn}}_{1.1} < \varepsilon^{\mathrm{md}}_1$ by $\oplus_0$(d) by the choice of ε_*.

By clause (b), it follows that ℓ_3 from Clause (c) are well defined and Clause (d) holds

$\oplus_{9.6}$ (a) $\mathrm{Rang}(F_1(\eta_0 \hat{}\, (\rho{\restriction}\ell^*_2) \hat{}\, \nu_1 \hat{}\, \eta_1)) \subseteq \varepsilon^{\mathrm{md}}_1 + 1$

(b) $\varepsilon_* \in \mathrm{Rang}(F_1(\nu_0))$ is $> \varepsilon_1^{\mathrm{md}}$

(c) $\mathrm{Rang}(F_1(\nu_0)) \cap \varepsilon_* \subseteq \varepsilon_1^{\mathrm{md}}$

Why? First, $\mathrm{Rang}(F_1(\eta_0)) \subseteq \varepsilon_1^{\mathrm{md}}$ because $\mathrm{Rang}(F_1(\eta_0)) \subseteq \varepsilon_{1,1}^{\mathrm{up}}$ by $\oplus_4$ and $\varepsilon_{1,1}^{\mathrm{up}} \le \varepsilon_{4,1}^{\mathrm{up}}$ by $(*)_4$(b) and $\varepsilon_{4,1}^{\mathrm{up}} \le \varepsilon_1^{\mathrm{md}}$ by $\oplus_0$(c).

Second, $\mathrm{Rang}(F_1(\rho{\restriction}\ell_2^*)) \subseteq \mathrm{Rang}(\rho) \subseteq \varepsilon_1^{\mathrm{md}} + 1$ and $\mathrm{Rang}(F_1(\nu_1{}^\frown\eta_1)) \subseteq \varepsilon_1^{\mathrm{md}}$ by $\oplus_0$(c).

Third, $\mathrm{Rang}(F_1(\eta_0{}^\frown(\rho{\restriction}\ell_2^*)) \subseteq \mathrm{Rang}(F_1(\eta_0)) \cup \mathrm{Rang}(F_1(\rho{\restriction}\ell_2^*)) \subseteq \varepsilon_1^{\mathrm{md}} + 1$ by the last two sentences, so clause (a) of $\oplus_{9.6}$ holds.

Fourth, clause (b), i.e. $\varepsilon_* \in \mathrm{Rang}(F_1(\nu_0))$ holds by $\oplus_5$(b).

Fifth, $\mathrm{Rang}(F_1(\eta_0{}^\frown\nu_0)) \cap \varepsilon_* \subseteq \varepsilon_{4,1}^{\mathrm{up}}$ by $(*)_4$(d) with $(\delta, \beta, \varepsilon)$ there standing for $(\delta_2, \zeta_1, \varepsilon_*)$ here (recalling $\delta_2 \in \mathscr{U}_4^{\mathrm{up}}$ and $\zeta_1 \in t_{\alpha_1}^1 = t^1_{g_{2,\varepsilon_*}(g_{3,\varepsilon_*}(\delta_2))}$) and $\varepsilon_{4,1}^{\mathrm{up}} \le \varepsilon_1^{\mathrm{md}}$ by $\oplus_0$(c). Hence, $\mathrm{Rang}(F_1(\nu_0)) \cap \varepsilon_* \subseteq \mathrm{Rang}(F_1(\eta_0{}^\frown\nu_0)) \cap \varepsilon_* \subseteq \varepsilon_{4,1}^{\mathrm{up}} \subseteq \varepsilon_1^{\mathrm{md}}$, so also clause (c) of $\oplus_{9.6}$ holds.

$\oplus_{9.7}$ (a) let $\ell_4^\bullet$ from $\oplus_9$ be as in (B)(d)(β)

(b) $F_1(\varrho(\ell_4^\bullet)) = \varepsilon_*$

(c) (used in stage D) $\ell_4^\bullet \in [\ell g(\eta_0), \ell g(\eta_0{}^\frown\nu_1))$.

[Why? By $\oplus_{9.6}$, $\ell_4^\bullet$ is well defined and belongs to $[\ell g(\eta_0), \ell g(\eta_0{}^\frown\nu_0))$; moreover, $F_1(\varrho(\ell_4^\bullet)) = \varepsilon_*$.]

So indeed $\oplus_9$ holds.

$\oplus_{10}$ $\mathbf{c}_1\{\zeta_0, \zeta_1\} = j_*$.

[Why? Because $\mathbf{d}(\varrho) = \ell_4^\bullet$ and $(F_1(\varrho))(\ell_4^\bullet) = \varepsilon_*$ and so by $\odot_7(c)$, $h''(\varepsilon_*) = \ell_4^\bullet$ we have $\mathbf{c}_1\{\zeta_0, \zeta_1\} = h'(\varepsilon_*)$ and $h'(\varepsilon_*) = j_*$ because $\varepsilon_* \in s_{\delta_2}$ by the choice of ε_* and $h'(\varepsilon_*)$ is j_* by $(*)_4$(c) recalling the definition of $S^{\kappa_1}_{\kappa_0, j_*}$ in $\odot_7$(a).]

Stage D:

We would like to have λ colours (not just κ_1 colours), but (unlike earlier versions) we rely on what was proved (i.e. the properties of $\mathbf{c}_1$) instead of repeating it. So we shall assume $\boxplus$ from the beginning of Stage B and $j_* < \lambda$ in $\boxplus$(d).

Now

$\boxplus_1$ for some $\mathscr{W}_1, \varepsilon_{0,1}^{\mathrm{up}}, \alpha_{0,1}^*$

(a) $\alpha_{0,1}^* < \lambda, \varepsilon_{0,1}^{\mathrm{up}} < \kappa_1$

(b) $\mathscr{W}_1 \subseteq S$ is stationary and $\min(\mathscr{W}_1) > \alpha_{0,1}^*$

(c) if $\delta \in \mathscr{W}_1$ and $\beta \in t_\delta$ then $\mathrm{Rang}(F_1(\rho_{\bar{h}}(\beta, \delta))) \subseteq \varepsilon_{0,1}^{\mathrm{up}}$

(d) if $\delta \in \mathscr{W}_1$ and $\alpha \in [\alpha_{0,1}^*, \delta)$ and $\beta \in t_\delta$ then $\rho(\beta, \delta){}^\frown\langle\delta\rangle \trianglelefteq \rho(\beta, \alpha)$.

[Why? As in the proof of $(*)_1$ in Stage B.]

$\boxplus_2$ (a) let $\mathscr{W}_2 = \{\delta \in S : F_2(h(\delta)) = j_*, F_1(h(\delta)) = \varepsilon_{0,1}^{\mathrm{up}}$ and $\delta > \alpha_{0,1}^*\}$, so stationary

(b) let $g_1^* : \mathscr{W}_2 \to \mathscr{W}_1$ be such that $\delta < g_1^*(\delta) \in \mathscr{W}_1$

$\boxplus_3$ there are $\mathscr{W}_3, \alpha_{0,2}^*$ and n_* such that:

(a) $\mathscr{W}_3 \subseteq \mathscr{W}_2$ is stationary and $\min(\mathscr{W}_3) > \alpha_{0,2}^* > \alpha_{0,1}^*$

(b) if $\delta \in \mathscr{W}_3$ and $\alpha \in [\alpha^*_{0,2}, \delta)$ and $\beta \in t_{g^*_1(\delta)}$ then $\rho(\beta, g^*_1(\delta))\hat{}\langle g^*_1(\delta)\rangle \trianglelefteq \rho(\beta, g^*_1(\delta))\hat{}\rho(g^*_1(\delta), \delta)\hat{}\langle\delta\rangle \trianglelefteq \rho(\beta, \alpha)$

(c) if $\delta \in \mathscr{W}_3$ and $\beta \in t_{g^*_1(\delta)}$ *then*

(α) $\mathrm{Rang}(F_1(\rho_{\bar h}(\beta, g^*_1(\delta))) \subseteq \varepsilon^{\mathrm{up}}_{0,1}$

(β) $n_* = |\{\ell < k(\beta, \delta) : (F_1(\rho_{\bar h}(\beta, \delta))(\ell) = \varepsilon^{\mathrm{up}}_{0,1}\}|$

(γ) hence if $\alpha < \delta$ and $\rho(\beta, \delta)\hat{}\langle\delta\rangle \trianglelefteq \rho(\beta, \alpha)$ then the $(n_* + 1)$-th member of the set $\{\ell < k(\beta, \alpha) : F_1(\rho_{\bar h}(\beta, \alpha))(\ell) = \varepsilon^{\mathrm{up}}_{0,1}\}$ is $\ell g(\rho(\beta, \delta))$.

[Why? As usual, e.g. how do we justify n_* in clause $(c)(\beta)$ not depending on $\beta \in t_\delta$? First, find δ, then for any $\beta \in t_\delta$ we have

- $\rho(\beta, \delta) = \rho(\beta, g^*_1(\delta))\hat{}\rho(g^*_1(\delta), \delta)$.

[Why? Recall $\boxplus_1$(d).]

- $\mathrm{Rang}(F_1(\rho_{\bar h}(\beta, g^*_1(\delta))) \subseteq \varepsilon^{\mathrm{up}}_{0,1}$.

[Why? Recall $\boxplus_1$(c).]

Together, n_* depends just on $\rho_{\bar h}(g^*_1(\delta), \delta)$ which depends only on δ (not β). Second, as choosing $\mathscr{W}_3$ we can make n_* not depend on δ.]

Let $j_{**} < \kappa_1$ be such that $h'_1(j_{**}) = \varepsilon^{\mathrm{up}}_{0,1}$, $h'_2(j_{**}) = n_*$. Next let $g_* : \lambda \to \mathscr{W}_3$ be increasing and define $s_\alpha = t_{g_*(\alpha)}$, $s^\iota_\alpha = t^\iota_{g_*(\alpha)}$ for $\iota = 0, 1$. Now by what was proved in the earlier stages we can find $\alpha_0 < \alpha_1 < \lambda$ such that if $\zeta_0 \in s^0_{\alpha_0} \wedge \zeta_1 \in s^1_{\alpha_1}$ then $\mathbf{c}_1\{\zeta_0, \zeta_1\} = j_{**}$.

Let $(\zeta_0, \zeta_1) \in s^0_{\alpha_0} \times s^1_{\alpha_1}$. By the choice of $\mathbf{c}_1$, in $\odot_7$ we have $\mathbf{c}_2$ from $\odot_9$ and by $\boxplus_3(c)(\gamma)$ we have $\mathbf{c}_2(\{\zeta_0, \zeta_1\}) = j_*$. But $(s^0_{\alpha_0}, s^1_{\alpha_1}) = (t^0_{g_*(\alpha_0)}, t^1_{g_*(\alpha_1)})$ so $\alpha'_0 = g_*(\alpha_0)$, $\alpha'_1 = g_*(\alpha_1)$ are as required. $\square_{3.2}$

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