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Martin's Maximum, Saturated Ideals and Non-Regular Ultrafilters. Part II
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Martin's Maximum, saturated ideals and non-regular ultrafilters. Part II

By M. Foreman†, M. Magidor*, and S. Shelah#

Abstract

We prove, assuming the existence of a huge cardinal, the consistency of fully non-regular ultrafilters on the successor of any regular cardinal. We also construct ultrafilters with ultraproducts of small cardinality. Part II is logically independent of Part I.

0. Introduction and notation

Non-regular ultrafilters arose in the study of ultraproducts. Early model theorists were interested in the cardinality of ultrapowers. They isolated a property of ultrafilters which they called "regularity" which implied that the cardinality of an ultraproduct was the expected one.

The question of whether all ultrafilters were regular naturally arose. In particular a question in [C–K] is:

Can there be an ultrafilter D on ω_1 such that $|\omega^{\omega_1}/D| = \aleph_1$?

Non-regular ultrafilters became interesting to set theorists because of their analogy to large cardinals. (See [Ka-M].) They were used as combinatorial devices in several papers, notably Magidor's, [M4].

In [M5], Magidor was able to get the consistency of non-regular ultrafilters on cardinals above $\aleph_1$ but these ultrafilters did not have the greatest degree of non-regularity.

Laver in [L2] showed that there is a non-regular ultrafilter on ω_1 in a model constructed by Woodin assuming the consistency of the theory "ZF + $AD_{\mathbf{R}}$ + θ-regular".

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After the authors obtained the results in this paper, Woodin showed how to get the consistency of an $\aleph_1$-dense ideal on $\aleph_1$ from an almost-huge cardinal. This allows one to use Laver's results directly to get a non-regular ultrafilter on $\aleph_1$. Woodin's results also get non-regular ultrafilters on other cardinals.

It was not known before the results of this paper were proved how to construct fully non-regular ultrafilters on any cardinal bigger than $\aleph_1$.

In Section 1 we introduce a refinement of the notion of a saturated ideal and show that under fairly common conditions the existence of an ideal with this property implies that we can force the existence of a fully non-regular ultrafilter.

In Section 2 we show that from much less than a huge cardinal we can construct models with these ideals on any successor of a regular cardinal.

In Section 3 we refine our construction to show that we can produce ultrafilters that yield ultrapowers of small cardinality.

Theorems 2, 3 and Corollary 4 were shown by the first author. Theorem 9 is due to the third author.

For notation, we advise the reader to consult Part I of this paper [F-M-S].

1. Layered ideals

We introduce the concept of a *layered* ideal. Layered ideals are saturated and can be made centered. Further, one can force over a layered ideal to get a fully non-regular ultrafilter. If one forces over a model with a layered ideal on ω_1, then one can get an ultrafilter D on ω_1 such that $|\omega^{\omega_1}/D| = \omega_1$. Finally one can force over a model with huge cardinal κ to get a layered ideal on κ, with $\kappa = \mu^+$ for a preselected $\mu < \kappa$.

Thus we will prove the consistency of a non-regular ultrafilter on μ^+ from a huge cardinal. All ideals will be normal and countably complete, and all ultrafilters will be uniform.

Definition. An ultrafilter U on κ is (μ, γ)-non-regular if and only if whenever $\langle X_\alpha: \alpha < \gamma \rangle \subseteq U$ there is an $S \subseteq \gamma$, $|S| = \mu$ and $\bigcap_{\alpha \in S} X_\alpha \neq \varnothing$.

There is an extensive literature on non-regular ultrafilters (see [T2], [Ka-T] or [Ka]).

If $\kappa = \lambda^+$ then the greatest degree of non-regularity an ultrafilter on κ can have is (λ, λ^+)-non-regularity. We will call such ultrafilters *fully* non-regular.

There is another kind of ultrafilter very similar to non-regular ultrafilters.

Definition. An ultrafilter U on κ is *weakly normal* if and only if whenever we have a regressive function $f: X \to \kappa$ for some $X \in U$ then there is a $\gamma < \kappa$ such that $\{\alpha: f(\alpha) < \gamma\} \in U$.

Notice that an ultrafilter U on κ is normal if and only if U is weakly normal and κ-complete.

Kanamori proved the following theorem:

THEOREM ([Ka]). *Suppose λ is regular. Then there is a weakly normal ultrafilter U on λ^+ such that $\{\alpha: \mathrm{cof}(\alpha) = \lambda\} \in U$ if and only if there is a fully non-regular ultrafilter on λ^+.*

For the readers' convenience we show the direction of this theorem that we will use:

PROPOSITION 1. *Let λ be regular. Suppose U is a weakly normal ultrafilter on λ^+ such that $\{\alpha < \lambda^+: \mathrm{cof}(\alpha) = \lambda\} \in U$. Then U is fully non-regular.*

Proof. Let $\langle X_\alpha: \alpha < \lambda^+\rangle \subseteq U$ be a counterexample to non-regularity. We may assume that $X_\alpha \cap \alpha = \varnothing$. Define a function f: $\{\alpha: \mathrm{cof}(\alpha) = \lambda\} \to \lambda^+$ by $f(\alpha) =$ least β for all $\gamma \geq \beta$, $\alpha \notin X_\gamma$. Then $f(\alpha) \leq \alpha$ and if f were not regressive at α_0 then $\{\beta: \alpha_0 \in X_\beta\}$ would show that $\langle X_\alpha: \alpha < \lambda^+\rangle$ is *not* a counterexample to non-regularity.

Hence f is regressive. By weak normality there is a $\gamma < \lambda^+$ such that $\{\alpha: f(\alpha) < \gamma\} \in U$. But then $X_\gamma \notin U$, a contradiction. □

We want to express a sufficient condition for weak normality in terms of ideals.

Suppose that $\mathscr{I}$ is a normal, κ-complete, κ^+-saturated ideal on κ. By a theorem of Shelah ([Sh3]), if $\lambda^+ = \kappa$ then $\{\alpha: \mathrm{cof}(\alpha) = \mathrm{cof}(\lambda)\} \in \breve{\mathscr{I}}$.

Suppose $\mathscr{F} \subseteq \mathscr{P}(\kappa)$ is an ultrafilter extending $\breve{\mathscr{I}}$ such that whenever $\langle x_\alpha: \alpha < \kappa\rangle$ is a maximal antichain in $\mathscr{P}(\kappa)/\mathscr{I}$ there is a $\gamma < \kappa$ such that $\bigvee_{\alpha < \gamma} x_\alpha \in \mathscr{F}$, then $\mathscr{F}$ is weakly normal.

To see this we consider any regressive function f: $X \to \kappa$ where $X \in \mathscr{F}$. Let $x_\alpha = \{\beta: f(\beta) = \alpha\}$. Then $\{\tilde{X}\} \cup \{\langle x_\alpha: \alpha < \kappa\rangle\}$ is a maximal antichain in $\mathscr{P}(\kappa)/\mathscr{I}$ since $\mathscr{I}$ is normal. But then $\bigvee_{\alpha < \gamma} x_\alpha \in \mathscr{F}$ for some $\gamma < \kappa$; hence $\{\alpha: f(\alpha) < \gamma\} \in \mathscr{F}$ as desired.

We want to find an ideal $\mathscr{I}$ that allows us to construct such an $\mathscr{F}$.

Definition. Suppose $\kappa = \lambda^+$ and $\mathscr{I}$ is a κ-complete, normal ideal on κ. Then $\mathscr{I}$ is *layered* if and only if $\mathscr{B} = \mathscr{P}(\kappa)/\mathscr{I} = \bigcup_{\alpha < \kappa^+} \mathscr{B}_\alpha$ where

i) $|\mathscr{B}_\alpha| = \kappa$ and $\langle \mathscr{B}_\alpha: \alpha < \kappa^+\rangle$ is a continuous, increasing chain.

ii) There is a stationary set $S \subseteq \kappa^+ \cap \mathrm{cof}(\kappa)$ such that for all $\alpha \in S$ and all $x \in \mathscr{B} \sim \{0\}$ there is a $y \in \mathscr{B}_\alpha \sim \{0\}$ such that for all $z \in \mathscr{B}_\alpha \sim \{0\}$, $z \leq y$ implies $z \wedge x \neq 0$.

iii) $\mathscr{B}_\alpha$ is $< \kappa$-complete for $\alpha \in S$.

We note that there are several equivalent versions of ii) which is the algebraic equivalent of $\mathscr{B}_\alpha$ being a *regular subalgebra* of $\mathscr{B}$ (or $\mathscr{B}_\alpha$ being *neatly* embedded in $\mathscr{B}$), i.e. a maximal antichain $A \subseteq \mathscr{B}_\alpha$ is also a maximal antichain in $\mathscr{B}$.

A layered ideal $\mathscr{I} \subseteq \mathscr{P}(\kappa)$ is κ^+-saturated for the following reason:

If $A \subseteq \mathscr{B}$ is a maximal antichain then $\{\alpha\colon A \cap \mathscr{B}_\alpha$ is maximal in $\mathscr{B}_\alpha\}$ is closed and unbounded in κ^+. Hence there is an $\alpha \in S$ such that $A \cap \mathscr{B}_\alpha$ is maximal in $\mathscr{B}_\alpha$. But then $A \cap \mathscr{B}_\alpha$ is maximal in $\mathscr{B}$. Hence $A \cap \mathscr{B}_\alpha = A$. But $|\mathscr{B}_\alpha| = \kappa$ so that $|A| \leq \kappa$.

If $\mathscr{C}$ is a regular subalgebra of $\mathscr{B}$ then there is a projection map

$$\pi\colon \mathscr{B} \to \mathrm{compl}(\mathscr{C})$$

where compl($\mathscr{C}$) is the completion of $\mathscr{C}$. Then for each $x \in \mathscr{B} \sim \{0\}$ and all $z \in \mathscr{C} \sim \{0\}$, $z \leq \pi(x)$ implies $z \wedge x \neq 0$. This map π does not, in general, take values in $\mathscr{C}$. Thus the projection of x may not exist; hence we define what a projection of x is:

Definition. Suppose $\mathscr{B}$ and $\mathscr{C}$ are Boolean algebras and $\mathscr{C} \subseteq \mathscr{B}$ and $b \in \mathscr{B}$. Then $c \in \mathscr{C} \sim \{0\}$ is *a projection* of b if and only if for all $d \in \mathscr{C} \sim \{0\}$, $d \leq c$ implies $d \wedge b \neq 0$. Note that $\{c \in \mathscr{C}\colon c$ is a projection of $b\}$ is closed under $\leq$ and joins that exist in $\mathscr{C}$.

THEOREM 2. *Let λ be regular. Suppose that there is a layered ideal on $\kappa = \lambda^+$ and $\diamondsuit\kappa$. Then there is a (κ^+, ∞)-distributive partial ordering* **P** *that adds a weakly normal ultrafilter on κ extending $\mathscr{I}$.*

(We note $\kappa^{\underline{\kappa}} = \kappa$ implies $\diamondsuit_\kappa$ for regular $\kappa > \omega_1$.)

Proof. Let $\mathscr{B} = \mathscr{P}(\kappa)/\mathscr{I} = \bigcup_{\alpha<\kappa^+}\mathscr{B}_\alpha$ and S be as in the definition of layered. We view the $\mathscr{B}_\alpha$'s as approximations to $\mathscr{B}$ and ultrafilters on $\mathscr{B}_\alpha$ as approximations to ultrafilters on $\mathscr{B}$.

We call an ultrafilter $U \subseteq \mathscr{B}_\alpha$ *semi-normal* if and only if whenever $\langle x_\beta\colon \beta < \kappa\rangle \subseteq \mathscr{B}_\alpha$ is a maximal antichain in $\mathscr{B}_\alpha$ then there is a $\gamma < \kappa$, $\bigvee_{\beta<\gamma} x_\beta \in U$.

If we have an ultrafilter $\mathscr{F} \subseteq \mathscr{B}$ such that for all $\alpha \in S$, $\mathscr{F} \restriction \mathscr{B}_\alpha$ is semi-normal, then $\mathscr{F}$ viewed as a ultrafilter on $\mathscr{P}(\kappa)$ is easily seen to be weakly normal. Thus we force with conditions that are semi-normal ultrafilters to build such an $\mathscr{F}$.

More precisely a condition U in our partial ordering **P** is a semi-normal ultrafilter on $\mathscr{B}_\alpha$ for some $\alpha \in S$.

If $U, V \in$ **P** then $U \Vdash V$ if and only if U is an ultrafilter on $\mathscr{B}_\alpha$, V is an ultrafilter on $\mathscr{B}_\beta$ for some $\alpha > \beta$ and $U \supseteq V$.

Main Claim. Suppose $0 \le j < \kappa$ and $\langle \psi_i: i < j \rangle \subseteq S$ is an increasing sequence of ordinals and $\langle U_i: i < j \rangle$ is an increasing sequence of semi-normal ultrafilters, $U_i \subseteq \mathscr{B}_{\psi_i}$. Then for any $\psi \in S$, $\psi > \sup_{i<j}\psi_i$, there is a semi-normal ultrafilter $V \subseteq \mathscr{B}_\psi$, $V \supseteq \bigcup_{i<j} U_{\psi_i}$.

Proof. We imitate Laver [L2]. Let $\langle S_\alpha: \alpha < \kappa \rangle$ be a $\diamondsuit$-sequence on κ. Since $|\mathscr{B}_\psi| = \kappa$ we can construe the $\diamondsuit$-sequence as a $\diamondsuit$-sequence on $\mathscr{B}_\psi$: let $\mathscr{B}_\psi = \langle b_\theta: \theta < \kappa \rangle$; then if $X \subseteq \mathscr{B}_\psi$, $\{\alpha: S_\alpha = \{b_\theta: \theta < \alpha\} \cap X\}$ is stationary in κ.

We define V be induction on α. At stage α we have defined V_α so that $\bigcup_{i<j} U_{\psi_i} \cup V_\alpha$ has the finite intersection property (f.i.p.).

If $V_\alpha \cup \bigcup_{i<j} U_{\psi_i} \cup \{\bigvee S_\alpha\}$ has the f.i.p. then let

$$V_{\alpha+1} = V_\alpha \cup \{\bigvee S_\alpha\}.$$

Otherwise, let $V_{\alpha+1} = V_\alpha$. Let V be the filter generated by $\bigcup_{\alpha<\kappa} V_\alpha$. We claim that V is a semi-normal ultrafilter.

Let $X = \langle x_\alpha: \alpha < \kappa \rangle$ be a maximal antichain in $\mathscr{B}_\psi$.

For each $d \in \mathscr{B}_\psi$, let $d \wedge X = \{d \wedge x_\alpha: \alpha < \kappa \text{ and } d \wedge x_\alpha \neq 0\}$. Then $d \wedge X$ is a maximal antichain below d.

Let $\langle c_\beta: \beta < \kappa \rangle$ be the elements of $\bigcup_{\alpha<\kappa} V_\alpha$ in the order that they were added. Let $\langle d_\beta: \beta < \kappa \rangle$ be all finite intersections of $\langle c_\beta: \beta < \kappa \rangle$. Then $\{\gamma: \langle d_\beta: \beta < \gamma \rangle = \text{all finite intersections of } \langle c_\beta: \beta < \gamma \rangle\}$ is closed and unbounded in κ.

Consider some d_β and $i < j$. Then there is a maximal antichain $A_i^\beta \subseteq \mathscr{B}_{\psi_i}$ such that for all $a \in A_i^\beta$, either $a \wedge d_\beta = 0$ or a is a projection of some element of $d_\beta \wedge X$.

Enumerate $A_i^\beta = \langle a_\xi: \xi < \kappa \rangle$. Since U_{ψ_i} is semi-normal there is a $\delta_i^\beta < \kappa$ such that $b_i^\beta = \bigvee \langle a_\xi: \xi < \delta_i^\beta \rangle \in U_{\psi_i}$.

Since $\{d_\beta\}$ has the finite intersection property with U_{ψ_i}, we may assume that for all $\xi < \delta_i^\beta$, $d_\beta \wedge a_\xi \neq \varnothing$.

Each a_ξ is a projection of $d_\beta \wedge x_\alpha$ for some α. Hence there is a $\gamma_i^\beta < \kappa$ such that for all $c \in \mathscr{B}_{\psi_i}$, $c \le b_i^\beta$,

$$(*) \qquad c \wedge \bigvee_{\alpha < \gamma_i^\beta} (d_\beta \wedge x_\alpha) \neq 0.$$

Choose a γ such that

1) $\{d_\beta: \beta < \gamma\} = \text{all finite intersections of } \langle c_\beta: \beta < \gamma \rangle$.
2) For all $\beta < \gamma$ and all $i < j$, $\gamma_i^\beta < \gamma$.
3) $S\gamma = \{x_\alpha: \alpha < \gamma\}$.

Claim. $V_\gamma \cup \bigcup_{i<j} U_{\psi_i} \cup \{\bigvee\langle x_\alpha : \alpha < \gamma\rangle\}$ has the f.i.p.

Otherwise there is a $\beta < \gamma$, $i < j$ and a $b \in U_{\psi_i}$ such that $d_\beta \wedge b \wedge \bigvee_{\alpha<\gamma} x_\alpha = 0$. By shrinking b we may assume that $b \leq b_i^\beta$. So

$$b \wedge \bigvee_{\alpha<\gamma}(d_\beta \wedge x_\alpha) = 0.$$

Since $\gamma_i^\beta < \gamma$ this contradicts $(*)$. Hence $\bigvee S_\gamma = \bigvee\langle x_\alpha : \alpha < \gamma\rangle \in V$. Hence V is weakly normal. We note that V is an ultrafilter because for each $b \in \mathscr{B}$ there is a stationary set of γ, $S_\gamma = \{b\}$. Thus we have proved the main claim. □

We remark that to build the ultrafilter V in the main claim we could have let $V_0 = \{b\}$ for an arbitrary $b \in \mathscr{B}_\psi$ such that $\{b\}$ has the f.i.p. with $\bigcup_{i<j} U_{\psi_i}$.

Two facts follow immediately from the main claim.

First, the main claim shows that our partial ordering is not the empty set and for any $\psi \in S$, any $U \in \mathbf{P}$ can be extended to a $V \in \mathbf{P}$ that is semi-normal on some $\mathscr{B}_\varphi$, $\varphi \geq \psi$. Hence our generic object is an ultrafilter on $\mathscr{B}$.

Secondly it shows that the forcing conditions are κ-closed. We will be done when we show that $\mathbf{P}$ is (κ^+, ∞)-distributive.

Let $\langle D_\alpha : \alpha < \kappa\rangle$ be a sequence of dense open sets in $\mathbf{P}$ and $U \in \mathbf{P}$. We want to show that there is a $V \Vdash U$ such that $V \in \bigcap_{\alpha<\kappa} D_\alpha$. Let $\mu \gg \kappa$ be regular. Let $M \prec H(\mu)$ be such that

a) $|M| = \kappa$, $\mathbf{P}$, $\langle D_\alpha : \alpha < \kappa\rangle$, $U \in M$

b) $\psi \in M \cap \kappa^+ \in S$ and $M^\kappa \subseteq M$.

We define a game. Players W and B take turns to build a sequence $\langle (a_\alpha, U_\alpha) : \alpha < \kappa\rangle$ (Player W plays the a_α's, B plays the U_α's) such that each $U_\alpha \in M$, $U_\alpha \supseteq \bigcup_{\beta<\alpha} U_\beta \cup \{a_\alpha\}$ and U_α is a semi-normal ultrafilter on some $\mathscr{B}_{\psi_\alpha}$. Further, each a_α has the f.i.p. with $\bigcup_{\beta<\alpha} U_\beta \cup U$. W goes first at limit ordinals:

W	a_0		a_1		$\cdots$	a_α		$a_{\alpha+1}$	
B		U_0		U_1	$\cdots$		U_α		$U_{\alpha+1}$

The game has length κ. Player W wins if and only if $\bigcup_{\alpha<\kappa} U_\alpha$ is a semi-normal ultrafilter on $\mathscr{B}_\psi$.

If player W has a winning strategy σ, then by playing this strategy against B while B plays elements of the sets $\langle D_\alpha : \alpha < \kappa\rangle$ gives a $V \in \bigcap_{\alpha<\kappa} D_\alpha$. (In fact it gives a master condition in $\mathbf{P}$ over M.)

Thus we must see that W has a winning strategy. Note that by the *Main Claim*, B is never stuck without a move at any stage α.

Let $\mathscr{B}_\psi = \langle b_\theta : \theta < \kappa\rangle$ and $\diamondsuit = \langle S_\alpha : \alpha < \kappa\rangle$ where $S_\alpha \subseteq \langle b_\theta : \theta < \alpha\rangle$ and for any $X \subseteq \mathscr{B}_\psi$, $\{\alpha : S_\alpha = X \cap \langle b_\theta : \theta < \alpha\rangle\}$ is stationary. At stage α, player W plays $a_\alpha = \bigvee S_\alpha$ if $\bigcup_{\beta<\alpha} U_\beta \cup \{\bigvee S_\alpha\}$ has the f.i.p. Otherwise he plays arbitrarily.

Using arguments similar to the proof of the main claim it is easy to check that this is a winning strategy for W.

Let $\mathscr{F} \subseteq \mathbf{P}$ be generic. Then $V[\mathscr{F}]$ has the same subsets of κ that V does. Further, if $\langle x_\beta \colon \beta < \kappa \rangle$ is a maximal antichain with respect to $\mathscr{P}(\kappa)/\mathscr{I}$, then for some $\alpha < \kappa^+$, $\alpha \in S$, $\langle x_\beta \colon \beta < \kappa \rangle$ is a maximal antichain in $\mathscr{B}_\alpha$. Hence for some $\gamma < \kappa$, $\bigvee_{\beta < \gamma} x_\beta \in \mathscr{F} \restriction \mathscr{B}_\alpha$; hence $\bigvee_{\beta < \gamma} x_\beta \in \mathscr{F}$. Thus $\mathscr{F}$ is a weakly normal ultrafilter. $\square$

Note that the weakly normal ultrafilter in Theorem 2 extends $\mathscr{I}$ and hence concentrates on points of cofinality κ. Thus it is (λ, λ^+)-non-regular.

2. The consistency of layered ideals

To show the consistency of non-regular ultrafilters from large cardinals we show that we can get a layered ideal by forcing over a model with a large cardinal.

We recall some definitions. Let $j \colon V \to M$ be an elementary embedding with critical point κ. Then j is a huge embedding if and only if $M^{j(\kappa)} \subseteq M$. Similarly j is an almost huge embedding if and only if $M^{\underset{\smile}{j(\kappa)}} \subseteq M$ (i.e. whenever $\langle x_\alpha \colon \alpha < \beta \rangle \subseteq M$ and $\beta < \kappa$, $\langle x_\alpha \colon \alpha < \beta \rangle \in M$). We will use a fair amount of technology involving huge cardinals and refer the reader to [S-R-K] as a primary source of information on these techniques.

THEOREM 3. *Suppose $j \colon V \to M$ is an almost huge embedding and κ is the critical point of j and $j(\kappa)$ is Mahlo. Let $\mu < \kappa$ be regular. Then there is a $< \mu$-closed partial ordering* $\mathbf{P}$ *such that*

$$V^{\mathbf{P}} \models \mu^+ \text{ carries a layered ideal and } \diamondsuit_{\mu^+}.$$

We note that if κ is a huge cardinal then there is such an embedding j with critical point κ. Thus we deduce:

COROLLARY 4. *If μ is regular and there is a huge cardinal $\kappa > \mu$ then there is a (μ, ∞)-distributive partial ordering Q such that $V^Q \models$ there is a (μ, μ^+)-non-regular ultrafilter on μ^+.*

To prove Theorem 3 we need to prove a lemma coming from [S-R-K]:

LEMMA 5. *Suppose $j \colon V \to M$ is an almost huge embedding with critical point κ and $j(\kappa)$ is Mahlo. Then there is a stationary set $S \subseteq j(\kappa)$ and almost huge embeddings $\langle j_\alpha \colon \alpha \in S \rangle$ and factor maps $\langle k_{\alpha,\beta} \colon \alpha < \beta \in S \rangle$ such that:*

a) *$j_\alpha \colon V \to M_\alpha$ has critical point κ and $j_\alpha(\kappa) = \alpha$.*
b) *$k_{\alpha,\beta} \colon M_\alpha \to M_\beta$ is elementary and* $\operatorname{crit}(k_\alpha) = \alpha$.
c) *If $\alpha < \beta \in S$ then $j_\beta = k_{\alpha,\beta} \circ j_\alpha$.*
d) *For all $\alpha < \beta < \gamma$, $k_{\alpha,\gamma} = k_{\beta,\gamma} \circ k_{\alpha,\beta}$.*

We will sketch a proof of this lemma. For details we refer the reader to [S-R-K].

Proof. For each λ, $\kappa < \lambda < j(\kappa)$ we let $f(\lambda) = \sup\{j(g)(j''\lambda)$: $g: [\lambda]^{<\kappa} \to \kappa\}$. There is a closed unbounded set $C \subseteq j(\kappa)$ such that for all $\gamma \in C$ and $\lambda < \gamma$, $f(\lambda) < \gamma$.

Let $S = \{\alpha \in C$: α is regular$\}$. Then for $\alpha \in S$ we can let N_α be the class of all sets of the form $j(g)(j''\lambda)$ where $\lambda < \alpha$ and $g: [\lambda]^{<\kappa} \to V$. Then $j: V \to N_\alpha$ is an elementary embedding. If $\alpha < \beta \in S$ then we can map $i_{\alpha,\beta}: N_\alpha \to N_\beta$ by the identity map, which is an elementary embedding. Letting M_α be the transitive collapse of N_α and $k_{\alpha,\beta}$ be the "transitive collapse" of $i_{\alpha,\beta}$ we get a commutative system of elementary embeddings.

We must check that $\alpha = N_\alpha \cap j(k)$ and $N_\alpha^\alpha \subseteq N_\alpha$.

If $\lambda < \alpha$, let $g: [\lambda]^{<\kappa} \to V$ be defined by $g(x) =$ o.t. x. Then $j(g)(j''\lambda) = \lambda$. Hence $\alpha \subseteq N_\alpha$.

Suppose $\gamma \in N_\alpha \cap j(k)$ and $\gamma \geq \alpha$. Then $\gamma = j(g^*)(j''\lambda^*)$ for some $\lambda^* < \alpha$ and $g^*: [\lambda^*]^{<\kappa} \to V$. We note that $\alpha = \sup\{j(g)(j''\lambda)$: $\lambda < \alpha$ and $g: [\lambda]^{<\kappa} \to \kappa\}$. Hence we may assume that $j''\lambda^* \in j(\{x \in [\lambda^*]^{<\kappa} | g^*(x) \geq \kappa\})$. But then $j(g^*)(j''\lambda^*) \geq j(\kappa)$, a contradiction.

Let $\beta < \alpha$ and $\langle x_\gamma: \gamma < \beta\rangle \subseteq N_\alpha$. Choose a $\lambda \in [\beta, \alpha)$ so large that for all $\gamma < \beta$ there is a $\lambda_\gamma < \lambda$ and a $g_\gamma: [\lambda_\gamma]^{<\kappa} \to V$ such that $j(g_\gamma)(j''\lambda_\gamma) = x_\gamma$. Let $g: [\lambda]^{<\kappa} \to V$ be defined by $g(x) = \{g_\gamma(x \cap \lambda_\gamma): \gamma \in x \cap \beta\}$. Then it is easy to check that $j(g)(j''\lambda) = \langle x_\gamma: \gamma < \beta\rangle$. Hence $\langle x_\gamma: \gamma < \beta\rangle \in N_\alpha$. □

Our construction of the partial ordering will be a slight modification of the Kunen partial ordering for producing an $\aleph_2$-saturated ideal on $\aleph_1$. The Kunen construction was first done with infinite supports by Laver who proved the chain condition. The construction we give here is a variation on Laver's version of the Kunen construction. We refer the reader to [L3], [K1], [F1] and [F3] for a detailed exposition of the Kunen construction.

For regular δ and inaccessible γ we define the Silver collapse $S(\delta, \gamma)$ to be the partial ordering of partial functions $p: \gamma \times \delta \to \gamma$ such that:

a) $|p| \leq \delta$,

b) for all $(\alpha, \beta) \in \operatorname{dom} p$, $p(\alpha, \beta) < \alpha$, and

c) there is an $\varepsilon < \delta$ such that $\operatorname{dom} p \subseteq \gamma \times \varepsilon$.

The ordering on $S(\delta, \gamma)$ is reverse inclusion. Standard facts show that $S(\delta, \gamma)$ is δ-closed and γ-c.c.

Let $j: V \to M$ be an almost huge embedding with critical point κ, and let $\mu < \kappa$ be a regular cardinal. Let $\lambda = j(k)$. Our partial ordering $\mathbf{P}$ will be of the form $Q * S^Q(\kappa, j(k))$ where $S^Q(\kappa, j(k))$ is the Silver collapse of $j(k)$ to be κ^+ as defined in V^Q. We define Q as an iteration of length κ with $< \mu$ supports.

Let $Q_0 = S(\mu, \kappa)$. At stage α, if id: $Q_\alpha \cap V_\alpha \hookrightarrow Q_\alpha$ is a neat embedding then we let $Q_{\alpha+1} * S^{Q_\alpha \cap V_\alpha}(\alpha, \kappa)$. Otherwise we let $Q_{\alpha+1} = Q_\alpha * 1$. Having described the supports of Q and what happens at successor stages, we completely specify Q by letting $Q = Q_\kappa$.

Standard arguments show that Q is $< \mu$-closed and κ-c.c. and makes $\kappa = \mu^+$.

Since Q is κ-c.c., id: $Q \hookrightarrow j(Q)$. Further, $j(Q) \cap V_\kappa = Q$. Thus $j(Q)_{\kappa+1} = j(Q)_\kappa * S^Q(\kappa, \lambda)$. Hence there is a neat embedding i: $Q * S^Q(\kappa, \lambda) \hookrightarrow j(Q)$.

Inspecting the arguments showing the κ-c.c. of Q in V we see that if j: $V \to M$ then $j(Q)$ is λ-c.c. in V. Hence in $V^{j(Q)}$, any sequence of conditions $\langle p_\alpha$: $\alpha < \beta < \lambda \rangle \subseteq j\,(S^Q(\kappa, \lambda))$ lies in $M^{j(Q)}$. (Here we are using the fact that j is an almost huge embedding.)

Let $\kappa < \alpha < \lambda$ and α be inaccessible. Let $G * H \subseteq Q * S^Q(\kappa, \lambda)$ be generic and $\hat{G} \subseteq j(Q)$ be generic for $j(Q)/i''G^*H$.

In $S^{j(Q)}(\lambda, j(\lambda))$ there is a condition $m_\alpha = \bigcup j''H \restriction \alpha$. (We will call this a master condition.) Further $m_\alpha \in S^{j(Q)}(\lambda, j(\alpha))$ and if $\alpha < \alpha'$ and we construct $m_{\alpha'}$, then $m_{\alpha'} \Vdash m_\alpha$.

Let $\hat{H} \subseteq S^{j(Q)}(\lambda, j(\alpha))$ be generic with $m_\alpha \in \hat{H}$. It is easy to check that j can be extended to $\hat{j}$: $V[G * H \restriction \alpha] \to M[G * \hat{H}]$ (see [B1]). The map $\hat{j}$ takes the realization of the $Q * S^Q(\kappa, \alpha)$-term τ to the realization of $j(\tau)$ in $M^{j(Q) * S^{j(Q)}(\lambda, j(\alpha))}$.

Working in $V[\hat{G}]$ we claim that in $S(\lambda, j(\alpha))$ if $s_\alpha \Vdash m_\alpha$ then there is an $r_\alpha \Vdash s_\alpha$, $r_\alpha \in S(\lambda, j(\alpha))$ for all $x \subseteq \kappa$, $x \in \mathscr{P}(\kappa)^{V[G * H \restriction \alpha]}$ either

$$M[\hat{G}] \models r_\alpha \Vdash \kappa \in j(x) \text{ or } M[\hat{G}] \models r_\alpha \Vdash \kappa \notin j(x).$$

To see this we note that in $M[\hat{G}]$, $|\mathscr{P}(\kappa)^{V[G * H \restriction \alpha]}| < \lambda$ and in $M[\hat{G}]$, $S(\lambda, j(\alpha))$ is λ-closed. Thus we can build r_α in a tower of length α.

Claim 6. Suppose $r_0 \in S(\lambda, j(\lambda))$ is compatible with each m_α and there is a $\delta < \lambda$ such that $\operatorname{supp} r_0 \subseteq j(\delta)$. Then there is a sequence of conditions $\langle r_\alpha$: $\delta < \alpha < \lambda$ and α is inaccessible$\rangle$ such that for all α, $r_\alpha \Vdash m_\alpha$ and for all $x \in \mathscr{P}(\kappa)^{V[G * H]}$, for some α,

$$M[\hat{G}] \models r_\alpha \Vdash \kappa \in \hat{j}(x) \text{ or } M[\hat{G}] \models r_\alpha \Vdash \kappa \notin \hat{j}(x).$$

Further, if $\alpha < \alpha'$, $r_{\alpha'} \Vdash r_\alpha$ and for $\alpha > \delta$, $\operatorname{supp} r_\alpha \subseteq j(\alpha)$.

(Here we should remark what exactly we mean by "$M[\hat{G}] \models r_\alpha \Vdash \kappa \in \hat{j}(x)$". The map $\hat{j}$ is determined by j' and the generic object $\hat{H} \restriction \alpha$. If $\alpha < \alpha'$ and $\hat{H} \restriction \hat{j}(\alpha')$ extends $\hat{H} \restriction j(\alpha)$ then $\hat{j}$: $V[G * H \restriction \alpha'] \to M[\hat{G} * \hat{H} \restriction j(\alpha')]$

extends $\hat{j}$: $V[G * H \restriction \alpha] \to M[\hat{G} * \hat{H} \restriction j(\alpha)]$. Thus the relation

$$\text{“}M[\hat{G}] \models r_\alpha \Vdash \kappa \in \hat{j}(x)\text{”}$$

is well defined without specifying exactly what the domain of $\hat{j}$ is.)

Proof of claim. We repeatedly apply the remarks previous to the claim to build the tower of conditions $\langle r_\alpha$: $\delta < \alpha < \lambda$ and α is inaccessible$\rangle$ by an induction of length λ. First, let $r_\delta = r_0 \cup m_\delta$.

At stage α, let $t_\alpha = \bigcup_{\beta < \alpha} r_\beta$. Let $\gamma = \sup\{\beta$: $\beta < \alpha$ and β is inaccessible$\}$. Then $\gamma \leq \alpha$ and $\operatorname{supp} t_\alpha \subseteq \sup j''\gamma$. Further, $t_\alpha \Vdash m_\alpha \restriction \operatorname{supp} j''\gamma$. Hence m_α and t_α are compatible. Let $s_\alpha = m_\alpha \cup t_\alpha$. Then $s_\alpha \in S^{V[\hat{G}]}(\lambda, j(\alpha))$ so that we can apply the previous remarks to find an r_α as desired. □

We will call a tower with the properties of Claim 6 a *strong tower*.

Let $G * H \subseteq Q * S^Q(\kappa, \lambda)$ be generic. Let $\hat{G} \subseteq j(Q)$ be generic extending $i''G * H$. Let $T = \langle r_\alpha$: $\delta < \alpha < \lambda$ and α inaccessible$\rangle$ be a strong tower. Then in $V[\hat{G}]$ we get an ultrafilter on $\mathscr{P}(\kappa)^{V[G * H]}$ by letting $x \in \mathscr{F}$ if and only if for some α, $M[\hat{G}] \models r_\alpha \Vdash \kappa \in \hat{j}(x)$.

It is easy to check that this defines an ultrafilter on $\mathscr{P}(\kappa)^{V[G * H]}$ that is closed under $< \kappa$-intersections of sequences that lie in $V[G * H]$. We define an ideal $\mathscr{I} \subseteq \mathscr{P}(\kappa)^{V[G * H]}$ in $V[G * H]$, by:

$$x \in \mathscr{I} \text{ if and only if } \|x \in \mathscr{F}\|_{j(Q)/i''G * H} = 0.$$

We claim that $\mathscr{I}$ is κ-complete, normal and κ^+-saturated in $V[G * H]$.

Since $\mathscr{F}$ is closed under $< \kappa$-intersections that lie in $V[G * H]$, $\mathscr{I}$ is κ-complete. To see that $\mathscr{I}$ is normal we consider $\langle x_\gamma$: $\gamma < \kappa\rangle \subseteq \mathscr{I}$. Then there is an inaccessible $\alpha < \lambda$ such that $\langle x_\gamma$: $\gamma < \kappa\rangle \in V[G * H \restriction \alpha]$. Choose an arbitrary $\hat{G} \supset i''G * H$ and an $\alpha' \geq \alpha$ such that for all γ, $M[\hat{G}] \models r_{\alpha'} \Vdash \kappa \notin x_\gamma$. Let $\hat{H} \subseteq S^{M[\hat{G}]}(\lambda, j(\alpha))$ be an arbitrary generic object containing $r_{\alpha'} \restriction j(\alpha)$.

Then we can extend j to $\hat{j}$: $V[G * H \restriction \alpha] \to M[\hat{G} * \hat{H}]$. Then for all $\gamma < \kappa$, $\kappa \notin j(x_\gamma)$. Hence $\kappa \notin \hat{j}(\nabla_{\gamma < \kappa} x_\gamma)$. Thus $M[\hat{G}] \models r_{\alpha'} \Vdash \kappa \notin \hat{j}(\nabla_{\gamma < \kappa} x_\gamma)$ and hence $M[\hat{G}] \models \nabla_{\gamma < \kappa} x_\gamma \notin \mathscr{F}$. Since $\hat{G}$ was arbitrary, $\nabla_{\gamma < \kappa} x_\gamma \in \mathscr{I}$.

Since $j(Q)/i''G * H$ is λ-c.c. and $\lambda = \kappa^+$ in $V[G * H]$, $\mathscr{P}(\kappa)/\mathscr{I}$ is κ^+-c.c. (If $\langle x_\alpha$: $\alpha < \lambda\rangle$ is an antichain in $\mathscr{P}(\kappa)/\mathscr{I}$ then $\langle \|x_\alpha \in \mathscr{F}\|$: $\alpha < \lambda\rangle$ is an antichain in $\mathscr{B}(j(Q)/i''G * H)$.)

Finally we want to argue that $\mathscr{F}$ viewed as an ultrafilter on $\mathscr{P}(\kappa)/\mathscr{I}$ is generic over $V[G * H]$. To see this, we examine a maximal antichain $A \subseteq \mathscr{P}(\kappa)/\mathscr{I}$. We must find an $x \in A$ such that $x \in \mathscr{F}$.

Since $\mathscr{I}$ has the κ^+-c.c., $|A| = \kappa$. We can enumerate A, $A = \langle [x_\gamma]_{\mathscr{I}}$: $\gamma < \kappa\rangle$ where $x_\gamma \subseteq \kappa$. By choosing representatives carefully we may assume that for all $\gamma > 0$, $x_\gamma \cap (\gamma + 1) = \emptyset$ and $\bigcup_{\gamma<\kappa} x_\gamma = \kappa$. Choose an inaccessible α such that $\langle x_\gamma : \gamma < \kappa \rangle \in V[G * H \restriction \alpha]$.

Let $\hat{G} \subseteq j'(Q)$ be an arbitrary generic ultrafilter extending $i''G * H$. Let $\mathscr{F}$ be the corresponding ultrafilter on κ. Choose $\alpha' > \alpha$ so that for all $x \in \mathscr{P}(\kappa)^{V[G*H\restriction\alpha]}$ either

$$M[\hat{G}] \models r_{\alpha'} \Vdash \kappa \in \hat{j}(x) \quad \text{or}$$

$$M[\hat{G}] \models r_{\alpha'} \Vdash \kappa \notin \hat{j}(x).$$

Let $\hat{H} \subseteq S^{M[\hat{G}]}(\lambda, j(\alpha))$ be generic with $r_{\alpha'} \restriction j(\alpha) \in \hat{H}$. Consider $\hat{j}: V[G * H \restriction \alpha] \to M[\hat{G} * \hat{H}]$. Then $M[\hat{G} * \hat{H}] \models \bigcup \hat{j}(\langle x_\gamma : \gamma < \kappa \rangle) = \hat{j}(\kappa)$ and hence, if we call $\hat{j}(\langle x_\gamma : \gamma < \kappa \rangle)$ by $\langle x_\gamma^j : \gamma < j(\kappa) \rangle$ there is a γ, such that $\kappa \in x_\gamma^j$. But since $x_\gamma^j \cap \gamma + 1 = \emptyset$ for $\gamma > 0$, $\kappa \in x_\gamma^j$ for some $\gamma < \kappa$. But then $x_\gamma^j = \hat{j}(x_\gamma)$. Thus $\kappa \in \hat{j}(x_\gamma)$ so that $x_\gamma \in \mathscr{F}$. But $[x_\gamma] \in A$, hence $\mathscr{F}$ meets the maximal antichain A. Further, by the definition of $\mathscr{I}$, if $x \in \mathscr{P}(\kappa)^{V[G*H]}$ and $x \notin \mathscr{I}$ then there is a $\hat{G}$ such that $x \in \mathscr{F}$.

If $\alpha < \lambda$ is inaccessible we can consider $\mathscr{F}_\alpha = \mathscr{F} \cap \mathscr{P}(\kappa)^{V[G*H\restriction\alpha]}$ and form the ultrapower $N_\alpha = V[G * H \restriction \alpha]^\kappa / \mathscr{F} \restriction \alpha$ of $V[G * H \restriction \alpha]$ with respect to functions $f: \kappa \to V[G * H \restriction \alpha]$ that lie in $V[G * H \restriction \alpha]$. Then we can define a map $k: N_\alpha \to M[\hat{G} * \hat{H}]$ by letting $k([f]) = \hat{j}(f)(\kappa)$. Standard arguments show that k is well defined and elementary and if $i: V[G * H \restriction \alpha] \to N_\alpha$ is the usual embedding of $V[G * H \restriction \alpha]$ into an ultrapower then $j = k \circ i$. Thus, N_α is well-founded and we identify N_α with its transitive collapse.

Since $\alpha = \kappa^+$ in $V[G * H \restriction \alpha]$ and $\mathscr{P}(\kappa)^{V[G*H\restriction\alpha]} \subseteq N_\alpha$, $i(\kappa) \geq \alpha$ and $N_\alpha \models \kappa$ is not a cardinal. Since $j = h \circ i$, $\operatorname{crit}(k) \geq \kappa$ and since $\operatorname{crit}(k)$ must be a cardinal in N_α, $\operatorname{crit}(k) \geq \alpha$.

We now have developed most of the tools we need to see that knowing $\mathscr{F}$ is equivalent to knowing $\hat{G}$.

Claim 7. Consider a generic object $G * H \subseteq Q * S^Q(\kappa, \lambda)$. Suppose $q \in j(Q)$ is compatible with $i''G * H$. Then there is a set $x \subseteq \kappa$, $x \in V[G * H]$ such that:

$$x \in \mathscr{F} \text{ if and only if } q \in \hat{G}.$$

Proof. We will first show that there is a term τ in the forcing language of $\mathbf{R} = \mathscr{P}(\kappa)/\mathscr{I}$ for a function from κ into $V[G * H]$ lying in $V[G * H]$ such that for some α, $\|[\tau]_{N_\alpha} = q\|_{\mathbf{R}} = 1$.

Since $q \in j(Q)$ there is a $\beta < \lambda$ such that $q \in j(Q) \cap V_\beta$. Let $\alpha > \beta$ be inaccessible. Let G be any generic object and $\hat{H} \subseteq S^{M[\hat{G}]}(\lambda, j(\alpha))$ be generic such that for some $r_{\alpha'}$ determining $\mathscr{F}_\alpha$, $r_{\alpha'} \restriction j(\alpha) \in \hat{H}$. Consider the commutative triangle

$$\begin{array}{ccc} \hat{j}\colon V[G*H\restriction\alpha] & \longrightarrow & M[\hat{G}*\hat{H}] \\ & \searrow^{i} \quad \nearrow_{k} & \\ & N_\alpha. & \end{array}$$

As we argued earlier, $\mathrm{crit}(k) \geq \alpha$. Since $\beta < \alpha$, $V[G*H\restriction\alpha] \models |V_\beta| = \kappa$ and thus in N_α we have an enumeration of $i(Q) \cap (V_\beta)^{N_\alpha} = \langle q_\gamma\colon \gamma < \mu\rangle$. By elementarity $k(\langle q_\gamma\colon \gamma < \mu\rangle)$ is an enumeration of $j(Q) \cap V_\beta$. Hence for some γ, $k(q_\gamma) = q$. But k is the identity on V_β and hence $q_\gamma = q$. Thus q is represented in N_α by some function $\tau\colon \kappa \to V[G*H\restriction\alpha]$. Further, $V[G*H][\mathscr{F}]$ can recognize τ. Thus in $V[G*H]$ we have the $\mathbf{R}$-term τ as desired.

By standard theory of saturated ideals (see [So 2]) in $V[G*H]$, there is a function $f\colon \kappa \to V[G*H]$ such that $\|\{\gamma\colon f(\gamma) = \tau(\gamma)\} \in \mathscr{F}\|_{\mathbf{R}} = 1$. Note that without loss of generality $f\colon \kappa \to Q$.

Let $x = \{\gamma < \kappa\colon f(\gamma) \in G\}$. Then $q \in \hat{G}$ if and only if $N_\alpha \models q \in i(G)$ if and only if $x \in \mathscr{F}$. □

We now return to the proof of Theorem 3.

Fix a $j\colon V \to M$ and a μ satisfying the hypothesis of Lemma 5. Let S, $\langle j_\alpha\colon \alpha \in S\rangle$, $\langle k_{\alpha,\beta}\colon \alpha < \beta, \alpha, \beta \in S\rangle$ be as in Lemma 5. Since $k_{\alpha,\alpha'} \restriction V_\alpha \cap M_\alpha = \mathrm{id}$ and $j_\alpha(Q)$ is (really) α-c.c. when we force with $\underrightarrow{\lim}\langle k_\alpha(Q)\colon \alpha \in S\rangle$ (taken over the maps $\langle k_{\alpha,\alpha'}\colon \alpha < \alpha' \in S\rangle$), we get a generic object G^* such that for each $\alpha < \alpha' \in S$, G^* induces a generic $G_\alpha \subseteq j_\alpha(Q)$ and $k''_{\alpha,\alpha'}G_\alpha \subseteq G_{\alpha'}$. Hence $k_{\alpha,\alpha'}$ induces an elementary embedding $k_{\alpha,\alpha'}\colon M_\alpha[G_\alpha] \to M_{\alpha'}[G_{\alpha'}]$.

Each G_α induces a generic object $G*H_\alpha \subseteq Q*S^Q(\kappa,\alpha)$. If $\alpha < \alpha' \in S$ then $G*H_\alpha \subseteq G*H_{\alpha'}$.

We want to build a sequence of strong towers $\langle T_\alpha\colon \alpha \in S\rangle$ by induction on α so that T_α is a strong tower for j_α and if $\alpha, \alpha' \in S$, $\alpha < \alpha'$ and $p \in T_\alpha$, then there is a $q \in T_{\alpha'}$, and $q \Vdash k_{\alpha,\alpha'}(p)$.

We construct T_α by induction on $\alpha \in S$. Our induction hypothesis is that for each $\beta < \alpha$, T_β is a strong tower and if $\alpha' < \beta < \alpha$, $\alpha', \beta \in S$ then for all $p \in T_{\alpha'}$ there is an $r \in T_\beta$ such that $r \Vdash k_{\alpha',\beta}(p)$.

Case 1. There is a $\beta \in S \cup \{0\}$ such that α is the least member of S above β.

Consider $k''_{\beta,\alpha}T_\beta$. Then in $M_\alpha[G_\alpha]$ this is a directed set of conditions in $S(\alpha, j_\alpha(\alpha))$ of cardinality $< \alpha$ and hence $r_0 = \bigcup k''_{\beta,\alpha}T_\beta \in S(\alpha, j_\alpha(\alpha))$. Consider the tower of master conditions $\langle m_\gamma\colon \kappa < \gamma < \alpha$ and γ is inaccessible$\rangle$, where each $m_\gamma = \bigcup j''_\gamma H_\gamma$.

We want to see that r_0 is compatible with each m_γ. If r_0 is not compatible with some m_γ then there is some $p \in T_\beta$ such that m_γ is incompatible with $k_{\beta,\alpha}(p)$. But then there is an inaccessible $\beta' < \beta$ such that $\operatorname{supp} p \subseteq j_\beta(\beta')$. Hence $k_{\beta,\alpha}(p)$ is incompatible with $m_\gamma \restriction (k_{\beta,\alpha} \circ j_\beta)(\beta')$.

Since T_β is a strong tower, p is compatible with $\bigcup j_\beta'' H_{\beta'}$. Since $k_{\beta,\alpha}$ is elementary and $j_\alpha = k_{\beta,\alpha} \circ j_\beta$, $k_{\beta,\alpha}(p)$ is compatible with $\bigcup j_\alpha'' H_{\beta'}$. But $m_\gamma \restriction j_\alpha(\beta') = \bigcup j_\alpha'' H_{\beta'}$ and $j_\alpha(\beta') = (k_{\beta,\alpha} \circ j_\beta)(\beta')$. Hence $k_{\beta,\alpha}(p)$ is compatible with $m_\gamma \restriction (k_{\beta,\alpha} \circ j_\beta)(\beta')$ as desired.

We similarly check that $\operatorname{supp} r_0 \subseteq \sup(k_{\beta,\alpha} \circ j_\beta)''\beta \leq j_\alpha(\beta)$. Thus by Claim 6 we can build a strong tower T_α starting with r_0. It is easy to check the induction hypothesis.

Case 2. $\alpha \notin \lim S$ but not where Case 1 holds.

Then there is a $\gamma < \alpha$, $\gamma \in \lim S$ and $S \cap [\gamma, \alpha) = \varnothing$. We let $r_0 = \bigcup_{\beta \in S \cap \gamma} \bigcup k_{\beta,\alpha}'' T_\beta$. Again this is a condition since $\bigcup k_{\beta,\alpha}'' T_\beta$ is a condition and by the induction hypothesis if $\beta' < \beta$, $\beta', \beta \in S$ then

$$\bigcup k_{\beta,\alpha}'' T_\beta \Vdash \bigcup k_{\beta',\alpha}'' T_\beta'.$$

Hence r_0 is a union of a chain of length less than α. Now we argue as in Case 1 to see that we can build a strong tower below r_0.

Case 3. $\alpha \in \lim(S) \cap S$.

Consider $\{p\colon$ there are a $\beta \in S \cap \alpha$ and a $q \in T_\beta$, $p = k_{\beta,\alpha}(q)\}$. We claim that this set can be regrouped to form a strong tower. For each $\beta \in S \cap \alpha$ let $s_\beta = \bigcup k_{\beta,\alpha}'' T_\beta$. Then, as in Cases 1 and 2, s_β is a condition and further, by our induction hypothesis if $\beta' < \beta$, $\beta, \beta' \in S$, then $s_\beta \Vdash s_{\beta'}$.

Let $\langle r_\beta\colon \kappa < \beta < \lambda$ and β is inaccessible$\rangle$ be defined as follows. For each inaccessible $\beta \in [\kappa, \lambda)$ let $\beta^* \geq \beta$ be the least element of S such that $s_{\beta^*} \Vdash m_\beta$. Let $r_\beta = s_{\beta^*} \restriction j_\alpha(\beta)$. Then for all β, $r_\beta \Vdash m_\beta$ and for each $\beta' \in S$ there is a β such that $r_\beta \Vdash s_{\beta'}$. To see that the sequence $\langle r_\beta\colon \beta$ is inaccessible between κ and $\lambda\rangle$ is a strong tower we must show that for all $x \in \mathscr{P}(\kappa)^{V[G * H_\alpha]}$ there is a β such that either

$$M[G_\alpha] \vDash r_\beta \Vdash \kappa \in j_\alpha(x) \text{ or}$$

$$M[G_\alpha] \vDash r_\beta \Vdash \kappa \notin j_\alpha(x).$$

If $x \in \mathscr{P}(\kappa)^{V[G * H_\alpha]}$ then, since α is a limit of elements of S and $S(\kappa, \alpha)$ has the α-c.c., there is a $\beta \in S$ such that $x \in \mathscr{P}(\kappa)^{V[G * H_\beta]}$. Hence there is a $q \in T_\beta$

such that either

$$M[G_\beta] \models q \Vdash \kappa \notin j_\beta(x) \text{ or}$$
$$M[G_\beta] \models q \Vdash \kappa \in j_\beta(x).$$

But then

$$M[G_\alpha] \models k_{\beta,\alpha}(q) \Vdash \kappa \notin j_\alpha(x) \text{ or}$$
$$M[G_\alpha] \models k_{\beta,\alpha}(q) \Vdash \kappa \in j_\alpha(x).$$

Since $s_\beta \Vdash k_{\beta,\alpha}(q)$ we get that there is an $r_{\beta'}$ such that

$$M[G_\alpha] \models r_{\beta'} \Vdash \kappa \notin j_\alpha(x) \text{ or}$$
$$M[G_\alpha] \models r_{\beta'} \Vdash \kappa \in j_\alpha(x)$$

as desired.

Thus we have defined the tower T_α to satisfy the induction hypothesis.

Recall that we formed the direct limit of the $\langle j_\alpha(Q)\alpha \in S\rangle$ by the maps $\langle k_{\alpha,\alpha'}\colon \alpha < \alpha'$ and $\alpha, \alpha' \in S\rangle$. Forcing with this partial ordering gave us a generic object G^* which induced generic $G_\alpha \subseteq j_\alpha(Q)$. Further each G_α gave us a generic object $G * H_\alpha \subseteq Q * S^Q(\kappa, \alpha)$ and the $G * H_\alpha$ cohere as α varies.

If we let $H = \bigcup_{\alpha < j(\kappa)} H_\alpha$, then $H \subseteq S^Q(\kappa, j(\kappa))$ and, since every initial segment of H is generic and $S(\kappa, j(\kappa))$ has the $j(\kappa)$-c.c., H is generic. Further, $H \cap V_\alpha = H_\alpha$.

In each $V[G_\alpha]$ we get an ultrafilter $\mathscr{F}_\alpha$ on $\mathscr{P}(\kappa)^{V[G*H_\alpha]}$ as described earlier.

Since $k_{\alpha,\beta}$ is elementary and $k''_{\alpha,\beta}T_\alpha$ is majorized by T_β, $\mathscr{F}_\beta \cap \mathscr{P}(\kappa)^{V[G*H_\alpha]} = \mathscr{F}_\alpha$.

Similarly in $V[G * H_\alpha]$ we get an ideal $\mathscr{I}_\alpha$ from j_α and the tower T_α. We claim that if $\alpha < \beta$ and $\alpha, \beta \in S$ then $\mathscr{I}_\alpha = \mathscr{P}(\kappa)^{V[G*H_\alpha]} \cap \mathscr{I}_\beta$.

To see that $\mathscr{I}_\alpha \subseteq \mathscr{P}(\kappa)^{V[G*H_\alpha]} \cap \mathscr{I}_\beta$ we note that if $x \in \mathscr{I}_\alpha$ no matter what the choice of G_β is, $x \notin \mathscr{F}_\alpha$. Since $\mathscr{F}_\beta \cap \mathscr{P}(\kappa)^{V[G*H_\alpha]} = \mathscr{F}_\alpha$, no matter what the choice of G_β is, $x \notin \mathscr{F}_\beta$. Hence $x \in \mathscr{I}_\beta$. Similarly if x can never be in $\mathscr{F}_\beta \cap \mathscr{P}(\kappa)^{V[G*H,\alpha]}$ then x can never be in $\mathscr{F}_\alpha$. (Here we are using that any G_α can be extended to a G_β.)

Define an ideal $\mathscr{I}$ on $\mathscr{P}(\kappa)^{V[G*H]}$ by $\mathscr{I} = \bigcup_{\alpha < j(\kappa)} \mathscr{I}_\alpha$. Then $\mathscr{I}$ is a normal, κ-complete ideal and $\mathscr{I} \cap \mathscr{P}(\kappa)^{V[G*H_\alpha]} = \mathscr{I}_\alpha$ for $\alpha \in S$.

For $\alpha \in S$ we let $\mathscr{B}_\alpha = \mathscr{P}(\kappa)^{V[G*H_\alpha]}/\mathscr{I}_\alpha$ and we interpolate to make a continuous chain $\langle \mathscr{B}_\alpha\colon \alpha < \kappa^+ = j(\kappa)\rangle$. Then each $\mathscr{B}_\alpha$ is a subalgebra of $\mathscr{P}(\kappa)/\mathscr{I}$ and $\mathscr{B} = \mathscr{P}(\kappa)/\mathscr{I} = \bigcup_{\alpha \in \kappa^+} \mathscr{B}_\alpha$. Further, for $\alpha \in S$, $\mathscr{B}_\alpha$ is κ-complete and for all $\alpha \in \kappa^+$, $|\mathscr{B}_\alpha| = \kappa$.

We will have shown that $\mathscr{I}$ is a layered ideal on κ if we can show that each $\mathscr{B}_\alpha$ is neatly embedded in $\mathscr{B}$ for $\alpha \in S$.

Our algebraic equivalent of this is to show that no $x \in \mathscr{B}$ is disjoint from a dense subset of $\mathscr{B}_\alpha$.

Let $[x]_\mathscr{I} \in \mathscr{B}$. Then $[x]_\mathscr{I}$ is represented by some set $x \in \mathscr{P}(\kappa)^{V[G*H_\beta]}$ for some $\beta \in S$. Let $\alpha \in S$. We must show that there is no set $D \subseteq \mathscr{P}(\kappa)^{V[G*H_\alpha]}$ such that $\{[y]_{\mathscr{I}_\alpha}: y \in D\}$ is dense in $\mathscr{B}_\alpha$ and for all $y \in D$, $y \cap x \in \mathscr{I}_\beta$. Suppose that there is such a D.

Let $Q_\alpha = j_\alpha(Q)$ and $Q_\beta = j_\beta(Q)$. Since $x \notin \mathscr{I}_\beta$ there is a $q \in Q_\alpha/G * H_\beta$, $q \Vdash_{Q_\beta} x \in \mathscr{F}_\beta$. Let $q' = q \cap V_\alpha$. By the construction of Q_α and Q_β we see that $q' \in Q_\alpha/G * H_\alpha$ and for all $r \nparallel q'$, $r' \in Q_\alpha/G * H_\alpha$, $r = k_{\alpha,\beta}(r)$ is compatible with q. (Remember, $Q \subseteq V_\kappa$ so $j_\alpha(Q) \subseteq V_\alpha$ and $\mathrm{crit}(k_{\alpha,\beta}) = \alpha$. Hence $k_{\alpha,\beta} \restriction Q_\alpha = \text{identity}$.)

Let $y \in \mathscr{P}(\kappa)^{V[G*H_\alpha]}$ be the set guaranteed to exist by Claim 7 such that $y \in \mathscr{F}_\alpha$ if and only if $q' \in G_\alpha$. Let $z \in D$ with $[z] \leq [y]$ in $\mathscr{B}_\alpha \sim \{0\}$. Then there is a condition $r \in Q_\alpha/G * H_\alpha$ such that $r \Vdash z \in \mathscr{F}_\alpha$. Then $r \Vdash y \in \mathscr{F}_\alpha$ so that $r \Vdash q'$. But then $k_\alpha(r)$ and q are compatible. Let $G_\beta \subseteq j_\beta(Q)$ be generic containing both $k_\alpha(r)$ and q. Then $z \in \mathscr{F}_\beta$ and $x \in \mathscr{F}_\beta$. Hence $x \cap z \in \mathscr{F}_\beta$. But $x \cap z \in \mathscr{I}_\beta$ by the definition of D, a contradiction.

Hence we have shown that there is no such set D. Thus $\mathscr{B}_\alpha$ is a regular subalgebra of $\mathscr{B}$ and $\mathscr{I}$ is a layered ideal.

Further, $Q * S^Q(\kappa, \lambda)$ is $< \mu$-closed and $V^Q \models \kappa = \mu^+$ and $\lambda = \kappa^+$. Since $S^Q(\kappa, \lambda)$ is κ-closed in V^Q, $V^Q \models \diamondsuit_{\mu^+}$. This proves Theorem 3. □

3. Small ultraproducts

We now turn to the problem of the cardinality of ultrapowers. We say that a κ-complete, normal ideal $\mathscr{I} \subseteq \mathscr{P}(\kappa)$ is *strongly layered* if and only if we can write $\mathscr{P}(\kappa)/\mathscr{I} = \bigcup_{\alpha<\kappa^+} \mathscr{B}_\alpha$ where the sequence $\langle \mathscr{B}_\alpha: \alpha < \kappa^+ \rangle$ is increasing and continuous and for each $\alpha \in \mathrm{cof}(\kappa) \cap \kappa^+$, $\mathscr{B}_\alpha$ is $< \kappa$-complete, $|\mathscr{B}_\alpha| = \kappa$ and $\mathscr{B}_\alpha$ is a regular subalgebra of $\mathscr{P}(\kappa)/\mathscr{I}$.

Shelah has shown that strongly layered ideals are κ-centered. If $\mathscr{I}$ is a layered ideal on κ and $S \subseteq \kappa^+ \cap \mathrm{cof}(\kappa)$ is the stationary set witnessing layeredness then we can force to shoot a closed unbounded set through

$$S \cup \{\alpha < \kappa^+: \mathrm{cof}(\alpha) < \kappa\}$$

without adding any new subsets of κ (see [A]). In this forcing extension, $\mathscr{P}(\kappa)/\mathscr{I}$ has not changed and S is the intersection of a club set with $\{\alpha < \kappa^+: \mathrm{cof}(\alpha) = \kappa\}$. Thus by rearranging the sequence $\langle \mathscr{B}_\alpha: \alpha < \kappa^+ \rangle$ witnessing layering we get a sequence witnessing strong layering. Thus we have shown:

PROPOSITION 8. *If $\mathscr{I}$ is a layered ideal on κ then there is a (κ^+, ∞)-distributive partial ordering* **P** *such that in $V^{\mathbf{P}}$, $\mathscr{I}$ is a strongly layered ideal.*

Shelah has shown that a strongly layered ideal on κ is κ-centered. If $\mathscr{B} = \mathscr{P}(\omega_1)/\mathscr{I}$ is strongly layered then $|\mathscr{B}| = \omega_2$ and we can identify $\mathscr{B}$ with ω_2 by a function $H\colon \mathscr{B} \xrightarrow[\text{onto}]{1-1} \omega_2$ so that if $\operatorname{cof}(\alpha) = \omega_1$ then $f''\mathscr{B}_\alpha = \alpha$.

THEOREM 9. *Suppose* $\diamondsuit\omega_1$ *and there is a strongly layered ideal on* ω_1. *Then there is an* (ω_2, ∞)*-distributive forcing adding an ultrafilter* D *on* ω_1 *such that*

$$|\omega^{\omega_1}/D| = \omega_1.$$

Proof. Let $\mathscr{B} = \mathscr{P}(\omega_1)/\mathscr{I}$ and $\langle \mathscr{B}_\alpha\colon \alpha < \omega_2\rangle$ be a strong layering of $\mathscr{I}$. We assume that $|\mathscr{B}_0| = \omega_1$ and $\mathscr{B}_0$ is a regular subalgebra of $\mathscr{B}$. Our strategy will be as follows. We want to construct an ultrafilter $D \supseteq \mathscr{I}$ on ω_1 so that for all $f\colon \omega_1 \to \omega$ there is a $g\colon \omega_1 \to \omega$ such that for each n,

$$g^{-1}(n) \in \mathscr{B}_0 \text{ and } [f]_D = [g]_D.$$

If we succeed then $|\omega^{\omega_1}/D| = |\mathscr{B}_0|^\omega = |\omega_1^\omega| = \omega_1$.

We translate this into the language of Boolean algebra. To do this we consider D as an ultrafilter on $\mathscr{P}(\omega_1)/\mathscr{I}$. Every function $f\colon \omega_1 \to \omega$ induces a partition of $\mathscr{B}$, $\langle x_n\colon n \in \omega\rangle$ by letting $x_n = [f^{-1}\{n\}]_{\mathscr{I}}$. Similarly any partition $\langle x_n\colon n \in \omega\rangle$ gives rise to a function when we choose disjoint representatives X_n for x_n and let $f \restriction X_n = \{n\}$. Thus an equivalent statement to the property of functions mentioned in the previous paragraph is that for any partition $\langle x_n\colon n \in \omega\rangle$ of $\mathscr{B}$ there is a partition $\langle y_n\colon n \in \omega\rangle \subseteq \mathscr{B}_0$ of $\mathscr{B}_0$ such that $\bigvee_{n\in\omega}(x_n \wedge y_n) \in D$.

To motivate our construction we perform a sample computation. Let $\langle x_n\colon n \in \omega\rangle$ be a partition of $\mathscr{B}$ and $\lambda \gg \omega_1$. Let $M \prec \langle H(\lambda), \varepsilon, \mathscr{B}, \langle x_n\colon n \in \omega\rangle\rangle$ be countable and let D^* be an ultrafilter on $M \cap \mathscr{B}$ such that for all m, $\bigvee_{n<m} x_n \notin D^*$. We want to find $\langle y_n\colon n \in \omega\rangle \subseteq \mathscr{B}_0$ so that $\bigvee_{n\in\omega} x_n \wedge y_n$ has the finite intersection property with D^*. Then, if D' is the filter generated by $D^* \cup \{\bigvee_{n\in\omega} x_n \wedge y_n\}$ then $\langle x_n\colon n \in \omega\rangle$ and $\langle y_n\colon n \in \omega\rangle$ give rise to equivalent functions modulo D'.

Enumerate $D^* = \{b_j\colon j \in \omega\}$. We must find $\langle y_n\colon n \in \omega\rangle$ such that for all j, $b_j \wedge \bigvee_{n\in\omega}(y_n \wedge x_n) \neq 0$; i.e. for each j there is a y_{n_j} such that $b_j \wedge y_{nj} \wedge x_{nj} \neq 0$. We have countably many tasks corresponding to the b_j's and countably many opportunities corresponding to the y_n's.

Suppose we have inductively chosen $\langle n_j\colon j < j^*\rangle$ and $\langle y_{nj}\colon j < j^*\rangle$ such that $y_{nj} \in \mathscr{B}_0 \cap M$ and $\bigvee_{j<j^*} y_{nj} \notin D$. Then $b_{j^*} \sim (\bigvee_{j<j^*} y_{nj}) \in D^*$. Since $\bigvee_{n<\sup_{j<j^*} nj} x_n \notin D^*$, there is an $n_{j^*} > \sup_{j<j^*} n_j$ such that $(b_{j^*} \sim \bigvee_{j<j^*} y_{nj}) \wedge x_{nj^*} \neq 0$. Let y_{nj^*} be a projection of $(b_{j^*} \sim \bigvee_{j<j^*} y_{nj}) \wedge x_{nj^*}$ that lies in $(M \cap \mathscr{B}_0) \sim D^*$. The $\langle y_{nj}\colon j \in \omega\rangle$, suitably re-indexed with dummy indices give a

partition of $\mathscr{B}_0$ such that $\bigvee_{n\in\omega} x_n \wedge y_n$ has the finite intersection property with D^*.

Our approach will be to try to build our final ultrafilter D by building countable approximations to it. The main problem that arises is that D has cardinality $\aleph_2$ and there are $\aleph_2$ many partitions of $\mathscr{B}$. The construction just reviewed only works for countable D^*.

To overcome this problem, we view each $\mathscr{B}_\alpha$, $\alpha \in \mathrm{cof}(\omega_1) \cap \omega_2$, as an approximation to $\mathscr{B}$. Since $\mathscr{B}_\alpha$ has cardinality ω_1, we can enumerate all countable partitions of $\mathscr{B}_\alpha$ in order type ω_1 and build an ultrafilter $U_\alpha \subseteq \mathscr{B}_\alpha$ such that any partition $\langle x_n\colon n \in \omega\rangle \subseteq \mathscr{B}_\alpha$ is equivalent modulo U_α to a partition $\langle y_n\colon n \in \omega\rangle \subseteq \mathscr{B}_0$. If we succeed in building $U_\alpha \subseteq \mathscr{B}_\alpha$ for each $\alpha \in \mathrm{cof}(\omega_1) \cap \omega_2$ which has this property and also coheres (i.e. if $\alpha < \beta$ then $U_\alpha \subseteq U_\beta$) then $\bigcup_{\alpha<\omega_2} U_\alpha$ will be an ultrafilter with the desired property. We will force with conditions of the form U_α where U_α will have characteristics that allow us to perform this construction.

The problem with this strategy is extending a $U_\alpha \subseteq \mathscr{B}_\alpha$ to a $U_\beta \subseteq \mathscr{B}_\beta$ where $\alpha < \beta$.

At a countable stage in our construction of U_β any set we want to add to U must have the f.i.p. with all of U_α, i.e. be a filter of cardinality ω_1. This prevents a naive construction of this form.

We overcome this obstacle by requiring that a projection of the set we want to add to U_β lies in U_α. A priori this seems to add $\aleph_2$-requirements to the construction of U_α but we can use a $\diamondsuit$-sequence to thin this set of requirements down to a set of size ω_1.

To define what these requirements are we introduce the notion of the preprojection of an $x \in \mathscr{B}$ by a descending sequence of ordinals $\vec{\alpha} \in (\omega_2 \cap \mathrm{cof}(\omega_1))^{<\omega}$.

By induction on the length of $\vec{\alpha}$, for all $x \in \mathscr{B}$ we define $pp^{\vec{\alpha}}(x)$. If $\alpha \in \omega_2 \cap \mathrm{cof}(\omega_1)$ let $pp^{\alpha}(x) = \{y \in \mathscr{B}_\alpha\colon y \text{ is a projection of } x\}$. If $\vec{\alpha}$ is a descending sequence of elements of $\omega_2 \cap \mathrm{cof}(\omega_1)$ and $\beta < \min \vec{\alpha}$ then

$$pp^{\vec{\alpha}^\frown\beta}(x) = \left\{y \in \mathscr{B}_\beta\colon \text{there is a } z \in pp^{\vec{\alpha}}(x) \text{ such that } y \text{ is a projection of } z\right\}.$$

We list some properties of $pp^{\vec{\alpha}}(x)$ which we shall use:

1) If $\vec{\alpha}^\frown\beta \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$ is a descending sequence and $y \in pp^{\vec{\alpha}^\frown\beta}(x)$ and $z \le y$, $z \ne 0$ then $z \in pp^{\vec{\alpha}^\frown\beta}(x)$.

2) If $\vec{\alpha}$ is a subsequence of $\vec{\beta}$ and $\vec{\alpha}$ and $\vec{\beta}$ have the same last element then

$$pp^{\vec{\beta}}(x) \subseteq pp^{\vec{\alpha}}(x).$$

3) If $x \le y$ then $pp^{\vec{\alpha}}(x) \subseteq pp^{\vec{\alpha}}(y)$.

4) $\bigvee pp^{\vec{\alpha}}(x) \ge x$.

5) If $\vec{\alpha}^\frown\beta$ is a descending sequence of elements of $\mathrm{cof}(\omega_1) \cap \omega_2$ and $a \in \mathscr{B}_\beta$ then

$$pp^{\vec{\alpha}^\frown\beta}(x) \wedge a \subseteq pp^{\vec{\alpha}^\frown\beta}(x \wedge a).$$

(Here we are again using the convention that if $X \subseteq \mathscr{B}$ and $a \in \mathscr{B}$ then $X \wedge a = \{x \wedge a: x \in X\}$.)

6) If $x \in \mathscr{B}_\beta$ and $\vec{\alpha}^\frown\beta$ is a decreasing sequence of elements of $\mathrm{cof}(\omega_1) \cap \omega_2$ then $pp^{\vec{\alpha}^\frown\beta}(x) = \{y \in \mathscr{B}_\beta: y \leq x \text{ and } y \neq 0\}$.

7) If $\vec{\alpha}^\frown\beta$ is a decreasing sequence of ordinals in $\mathrm{cof}(\omega_1) \cap \omega_2$ and $b \in \mathscr{B}$ then $\bigcup_{z \in pp^{\vec{\alpha}}(b)} pp^\beta(b \wedge z) \supseteq pp^{\vec{\alpha}^\frown\beta}(b)$.

These properties are easy to verify and we leave this to the reader.

Let $\lambda \gg \omega_2$. Consider a countable well-founded structure $\mathscr{A} = (X, \varepsilon, \mathscr{C}, \langle x_n: n \in \omega\rangle)$ such that there is a partition $\langle y_n: n \in \omega\rangle$ of $\mathscr{B}$ with the property that $\mathscr{A} \equiv (H(\lambda), \varepsilon, \mathscr{B}, \langle y_n: n \in \omega\rangle)$. Let $\langle \mathscr{C}_\alpha: \alpha \in (\mathrm{cof}(\omega_1) \cap \omega_2)^{\mathscr{A}}\rangle$ be the strong layering of $\mathscr{C}$ and let $D^* \subseteq \mathscr{C}$ be an ultrafilter such that for no $m \in \omega$ is $\bigvee_{n<m} x_n \in D^*$. We want to construct a sequence $\langle a_{nj}: j \in \omega\rangle \subseteq \mathscr{C}_0$ so that for all $b \in D^*$ and all descending sequences $\vec{\alpha} \in (\mathrm{cof}(\omega_1) \cap \omega_2)^{\mathscr{A}}$, there are a j and a $c \in pp^{\vec{\alpha}}(a_{nj} \wedge x_{nj})$ such that $c \wedge b \neq 0$. We do this exactly as in the sample calculation; i.e. we enumerate $D^* = \{bj: j \in \omega\}$ and choose $\{a_{nj}: j \in \omega\} \subseteq \mathscr{C}_0$ by induction so that for all j^*, $\bigvee_{j<j^*} a_{nj} \notin D$ and $a_{nj^*} \in pp^0((b_{j^*} \sim \bigvee_{j<j^*} a_{nj}) \wedge x_{nj^*})$.

Let $\vec{\alpha}$ be any descending sequence of elements of $(\mathrm{cof}(\omega_1) \cap \omega_2)^{\mathscr{A}}$ and $j \in \omega$. Then by property 4, $\mathscr{A} \vDash \bigvee pp^{\vec{\alpha}}(a_{nj} \wedge x_{nj}) \geq a_{nj} \wedge x_{nj}$. Since $a_{nj} \wedge x_{nj} \wedge b_j \neq 0$,

$$\mathscr{A} \vDash \text{there is a } c \in pp^{\vec{\alpha}}(a_{nj} \wedge x_{nj}) \text{ such that } c \wedge b_j \neq 0.$$

For each $\mathscr{A}$ and D^* we fix such a choice $\langle a_{nj}: j \in \omega\rangle$. We are now ready to work towards the notation of an *obedient* ultrafilter.

For each $\beta \in \mathrm{cof}(\omega_1) \cap \omega_2$ let $\{\langle \mathscr{B}_\beta^\delta, s^\delta\rangle: \delta < \omega_1\}$ be a continuous approximation to $\mathscr{B}_\beta$ and β; i.e. the sequence $\langle \mathscr{B}_\beta^\delta: \delta < \omega_1\rangle$ is a continuous increasing chain of countable elementary substructures of $\mathscr{B}_\beta$ and $\bigcup_{\delta<\omega_1} \mathscr{B}_\beta^\delta = \mathscr{B}_\beta$. Further, $\{s^\delta: \delta < \omega_1\}$ is a continuous increasing chain of countable subsets of β, $\bigcup_{\delta<\omega_1} s^\delta = \beta$ and we have identified $\mathscr{B}_\beta^\delta$ with s^δ by the function H (see the remarks before Theorem 9). Note that any two such approximations agree on a closed unbounded set.

Let $\langle\langle \mathscr{A}_\delta, D_\delta\rangle: \delta < \omega_1\rangle$ be a $\diamondsuit$-sequence of structures such that each

$$\mathscr{A}_\delta = \langle X_\delta, \varepsilon, \mathscr{C}_\delta, \langle x_n: n \in \omega\rangle_\delta, H_\delta\rangle$$

where X_δ is a countable transitive set and $\mathscr{A}_\delta \equiv \langle H(\lambda), \varepsilon, \mathscr{B}, \langle y_n: n \in \omega\rangle, H\rangle$ for some partition $\langle y_n: n \in \omega\rangle \subseteq \mathscr{B}$. Further, D_δ is an ultrafilter on $\mathscr{C}_\delta$.

The $\diamondsuit$-property we want $\langle\langle \mathscr{A}_\delta, D_\delta\rangle: \delta < \omega_1\rangle$ to satisfy is that for each well-founded structure $M = \langle X, \varepsilon^M, \mathscr{C}^M, \langle x_n: n \in \omega\rangle, H^M\rangle$ of cardinality $\aleph_1$, that is elementarily equivalent to $\langle H(\lambda), \varepsilon, \mathscr{B}, \langle y_n: n \in \omega\rangle, H\rangle$ for some partition $\langle y_n: n \in \omega\rangle$ of $\mathscr{B}$ and for each ultrafilter $U \subseteq \mathscr{C}^M$ and all continuous approximations $\langle M_\delta: \delta < \omega_1\rangle$ and $\langle U_\delta: \delta < \omega_1\rangle$ to M and U, there are a δ and an isomorphism $\phi: M_\delta \to \mathscr{A}_\delta$ such that $\phi''U_\delta = D_\delta$. (Note that this is equivalent to there being a stationary set of such δ.) Such a $\diamondsuit$-sequence is gotten from an ordinary $\diamondsuit$-sequence by coding.

For $\beta \in \operatorname{cof}(\omega_1) \cap \omega_2$ and an ultrafilter $U \subseteq \mathscr{B}^\delta_\beta$ we say that a limit ordinal δ is a *risky ordinal* if:

a) o.t. $s^\delta <$ o.t. $\omega_2^{\mathscr{A}_\delta}$ and if $\pi: s^\delta \xrightarrow{\text{onto}} \gamma$ is the transitive collapse map, then for $\alpha \in s^\delta$, $\operatorname{cof}(\alpha) = \omega_1$ if and only if $\mathscr{A}_\delta \models \operatorname{cof} \pi(\alpha) = \omega_1$ and $\mathscr{A}_\delta \models \operatorname{cof}(\gamma) = \omega_1$.

Let $(\mathscr{C}_\delta)_\gamma$ be the γ^{th} element in $\mathscr{A}_\delta$'s layering of C_δ.

b) The map π induces an isomorphism $\mathscr{B}^\delta_\beta$ onto $(\mathscr{C}_\delta)_\gamma$. (Here we are using H to identify $\mathscr{B}^\delta_\beta$ with s^δ and H_δ to identify $(\mathscr{C}_\delta)_\gamma$ with γ.)

c) $\pi''U = D_\delta \cap (\mathscr{C}_\delta)_\gamma$.

d) If $b \in \mathscr{B}^\delta_\beta$ and $\alpha \in \operatorname{cof}(\omega_1) \cap s^\delta$ and $\mathscr{A}_\delta \Vdash c \in pp^{\pi(\alpha)}(\pi(b))$ then $\pi^{-1}(c) \in pp^\alpha(b)$.

If δ is a risky ordinal and $\pi: \mathscr{B}^\delta_\beta \to \mathscr{C}_\delta$ is the canonical monomorphism and $\psi \in \operatorname{cof}(\omega_1) \cap s^\delta$ and $\vec{\alpha}$ is a descending sequence of elements of $(\operatorname{cof}(\omega_1) \cap \omega_2)^{\mathscr{A}_\delta}$ and $b \in D_\delta$, we can form $pp^{\vec{\alpha}^\frown \pi(\psi)}(b \wedge a_{n_j} \wedge x_{n_j})$ inside $\mathscr{A}_\delta$. Then $\pi^{-1}(pp^{\vec{\alpha}^\frown \pi(\psi)}(b \wedge a_{n_j} \wedge x_{n_j})^{\mathscr{A}})$ is a countable subset of $\mathscr{B}_\psi$. Hence it has a join in $\mathscr{B}_\psi$. We let

$$z_{\vec{\alpha}^\frown \psi, b} = \bigvee_{j\in\omega} \bigvee \pi^{-1}\Big(pp^{\vec{\alpha}^\frown \pi(\psi)}\big(b \wedge a_{n_j} \wedge x_{n_j}\big)\Big)^{\mathscr{A}}.$$

This join is not zero since for some j, $c \in pp^{\vec{\alpha}^\frown \pi(\psi)}(a_{n_j} \wedge x_{n_j})$, $b \wedge c \neq 0$.

From properties 1–7, we see that $\{z_{\vec{\alpha}^\frown \psi, b}: \vec{\alpha}^\frown \pi(\psi)$ is a descending sequence of elements of cofinality ω_1 in $\mathscr{A}$ and $b \in D_\delta\} = T_{\delta,\psi}$ is a filter in $\mathscr{B}_\psi$ and $\bigcup_{\psi \in s^\delta \cap \operatorname{cof} \omega_1} T_{\delta,\psi}$ has the f.i.p. with $U \restriction \mathscr{B}^\delta_\psi$. Similarly we can form

$$z_{\vec{\alpha}^\frown \beta, b} = \bigvee_{j\in\omega} \bigvee \pi^{-1}\Big(pp^{\vec{\alpha}^\frown \gamma}\big(b \wedge a_{n_j} \wedge x_{n_j}\big)^{\mathscr{A}}\Big)$$

where $b \in D_\delta$. Then $T_{\delta,\gamma} = \{z_{\vec{\alpha}^\frown \beta, b}: \vec{\alpha}$ is a descending sequence in $(\operatorname{cof}(\omega_1) \cap (\omega_2 \sim \gamma))^{\mathscr{A}_\delta}$ and $b \in D_\delta\}$ is a filter in $\mathscr{B}_\beta$ and has the f.i.p. with $U \restriction \mathscr{B}^\delta_\beta$ and $\bigcup_{\psi \in s^\delta \cap \operatorname{cof}(\omega_1)} T_{\delta,\psi}$.

An ultrafilter $U \subseteq \mathscr{B}_\beta$ is *obedient* if and only if there is a closed unbounded set $C \subseteq \omega_1$ such that for all $\delta \in C$ that are risky for $U \cap \mathscr{B}^\delta_\beta$ and all $\psi \in \operatorname{cof}(\omega_1) \cap (s^\delta \cup \{\beta\})$, $T_{\delta,\psi} \subseteq U$.

Note that obedience is independent of the representation of $\mathscr{B}_\beta$ and β.

Claim 10. a) For all $\beta \in \operatorname{cof}(\omega_1) \cap \omega_2$ there is an obedient ultrafilter on $\mathscr{B}_\beta$.

b) If $\alpha < \beta$, $\alpha, \beta \in \operatorname{cof}(\omega_1) \cap \omega_2$ then for any obedient ultrafilter $U_\alpha \subseteq \beta_\alpha$ there is an obedient ultrafilter $U_\beta \subseteq \mathscr{B}_\beta$ extending U_α.

Proof. a) Represent

$$\langle \mathscr{B}_\beta, \beta \rangle = \left\langle \bigcup_{\delta < \omega_1} \mathscr{B}^\delta_\beta, \bigcup_{\delta < \omega_1} s^\delta \right\rangle.$$

Build U_β in ω_1-stages, $\langle U^\delta_\beta \colon \delta < \omega_1 \rangle$. At a stage δ, if $U^\delta_\beta \subseteq \mathscr{B}^\delta_\beta$ and U^δ_β is an ultrafilter on $\mathscr{B}^\delta_\beta$, and δ is a risky ordinal for U^δ_β, let

$$U^{\delta+1}_\beta = U^\delta_\beta \cup \bigcup \left\{ T_{\delta,\psi} \colon \psi \in \left(s^\delta \cup \{\beta\} \right) \cap \operatorname{cof}(\omega_1) \right\}.$$

At non-risky δ, extend U^δ_β to a filter $U^{\delta+1}_\beta$ such that $U^{\delta+1}_\beta \cap \mathscr{B}^{\delta+1}_\beta$ is an ultrafilter on $\mathscr{B}^{\delta+1}_\beta$ and $|U^{\delta+1}_\beta \sim U^\delta_\beta| \leq \omega$.

Since $\langle \mathscr{A}_\delta \colon \delta < \omega_1 \rangle$ is a $\diamondsuit$-sequence, there is a stationary set of non-risky δ and hence $\bigcup_{\delta < \omega_1} U^\delta_\beta \subseteq \mathscr{B}_\beta$ is an ultrafilter. Further, it is obedient since at all risky δ we put

$$\bigcup \left\{ T_{\delta,\psi} \colon \psi \in \left(s^\delta \cup \{\beta\} \right) \cap \operatorname{cof}(\omega_1) \right\} \text{ in } U_\beta.$$

b) Let $U_\alpha \subseteq \mathscr{B}_\alpha$ be obedient and let $\langle (\mathscr{B}^\delta_\beta, s^\delta) \colon \delta < \omega_1 \rangle$ be a continuous increasing representation of $\mathscr{B}_\beta$ and β so that $\alpha \in s^0$. Let $\mathscr{B}^\delta_\alpha = \mathscr{B}^\delta_\beta \cap \mathscr{B}_\alpha$ and $t^\delta = s^\delta \cap \alpha$. Then without loss of generality, $\langle (\mathscr{B}^\delta_\alpha, t^\delta) \colon \delta < \omega_1 \rangle$ is a continuous increasing representation of $\mathscr{B}_\alpha$.

We build U_β extending U_α in a continuous increasing sequence $\langle U^\delta_\beta \colon \delta < \omega_1 \rangle$. Let $a \subseteq \mathscr{B}_\beta$ be an arbitrary countable set having the f.i.p. with U_α. Let $U^0_\beta = U_\alpha \cup a$. At non-risky stages δ, let $U^{\delta+1}_\beta$ extend U^δ_β so that $U^{\delta+1}_\beta \cap \mathscr{B}^{\delta+1}_\beta$ is an ultrafilter on $\mathscr{B}^{\delta+1}_\beta$ and $|U^{\delta+1}_\beta \sim U^\delta_\beta| \leq \omega$.

Claim. Suppose that δ is a risky ordinal for $U^\delta_\beta \cap \mathscr{B}^\delta_\beta$; then δ is a risky ordinal for $U_\alpha \cap \mathscr{B}^\delta_\alpha$.

Proof. t^δ is an initial segment of s^δ; so if $\pi \colon s^\delta \xrightarrow{\text{onto}} \gamma$ is the transitive collapse map and $\gamma <$ o.t. $\omega_2^{\mathscr{A}_\delta}$ then $\pi \restriction t^\delta$ is the transitive collapse map of t^δ and $\pi \restriction t^\delta \colon t^\delta \xrightarrow{\text{onto}} \gamma' < \gamma$. Hence a) in the definition of "risky" holds for t^δ.

We see that b) holds since $\mathscr{A} \models \pi(\alpha) \in \operatorname{cof}(\omega_1) \cap \omega_2$ and H_δ: $(\mathscr{C}_\delta)_{\pi(\alpha)} \xrightarrow[\text{onto}]{1-1} \pi(\alpha)$. Hence $\pi \restriction t^\delta$ induces an isomorphism from $\mathscr{B}^\delta_\alpha$ to $(\mathscr{C}_\delta)_{\pi(\alpha)}$.

Since $\pi'' U^\delta_\beta = D_\delta \cap (\mathscr{C}_\delta)_\gamma$, and $U^\delta_\beta \cap \mathscr{B}_\alpha = U_\alpha \cap \mathscr{B}^\delta_\alpha$, $\pi'' U^\delta_\alpha = D_\alpha \cap (\mathscr{C}_\delta)_{\pi(\alpha)}$. Thus c) holds. Clause d) is a local condition so it holds also. Hence δ is a risky ordinal for $U_\alpha \cap \mathscr{B}^\delta_\alpha$.

Since U_α is obedient there is a closed unbounded set $C \subseteq \omega_1$ such that for all risky $\delta \in C$ and all $\psi \in (t^\delta \cup \{\alpha\}) \cap \mathrm{cof}(\omega_1))$, $T_{\delta,\psi} \subseteq U_\alpha$.

We now show that for all risky $\delta \in C$ if $U_\beta^\delta \cap (\mathscr{B}_\beta \sim \mathscr{B}_\alpha) \subseteq \mathscr{B}_\beta^\delta$ then $\bigcup\{T_{\delta,\psi} | \psi \in \mathrm{cof}(\omega_1) \cap (s^\delta \cup \{\beta\})\}$ has the finite intersection property with U_β^δ. Otherwise there are a $d \in U_\alpha$ and a $b \in U_\beta^\delta \cap \mathscr{B}_\beta^\delta$ and $\psi_1 < \psi_2 < \cdots < \psi_k$, $z_1, \ldots, z_k$ with $z_i \in T_{\delta,\psi_i}$ such that $d \wedge b \wedge \bigwedge_{i=1}^k z_i = 0$.

Since $z_i \in T_{\delta,\psi_i}$, $z = z_{\vec{\eta}_i^\frown \psi_i, b_i}$ for some $\vec{\eta}_i \in [(\mathrm{cof}(\omega_1) \cap \omega_2)^{\mathscr{A}_\delta}]^{<\omega}$ and $b_i \in D_\delta$. By decreasing the b_i's we may assume that for some $b' \leq b$ and all i, $b_i = b'$. By property 2), adding more ordinals to $\vec{\eta}_i$ decreases z_i, so without loss of generality we may assume that if $i' < i$, $\vec{\eta}_i^\frown \pi(\psi_i)$ is an initial segment of $\vec{\eta}_{i'}$.

By the obedience of U_α, $z_{\vec{\eta}_1^\frown \pi(\psi_1)^\frown \alpha, b'} \in U_\alpha$. So $d \wedge z_{\vec{\eta}_1^\frown \pi(\psi_1)^\frown \alpha, b'} \neq 0$. Since

$$z_{\vec{\eta}_1^\frown \pi(\psi_1)^\frown \alpha, b'} = \bigvee_{j \in \omega} \bigvee \pi^{-1}\Big(\Big(pp^{\vec{\eta}_1^\frown \pi(\psi_1)^\frown \pi(\alpha)}\big(b' \wedge a_{n_j} \wedge x_{n_j}\big)^{\mathscr{A}_\delta}\Big),$$

there are a j and a $c \in pp^{\vec{\eta}_1^\frown \pi(\psi_1)^\frown \pi(\alpha)}(b' \wedge a_{n_j} \wedge x_{n_j})^{\mathscr{A}_\delta}$ such that $d \wedge \pi^{-1}(c) \neq 0$.

Subclaim. In $\mathscr{B}_\beta$, $\pi^{-1}(c) \in pp^\alpha(b \wedge \bigwedge_{i=1}^k z_i)$.

Proof. Let $c_0 = c$. Choose $c_1, \ldots, c_k$ such that $\mathscr{A}_\delta \models c_i \in pp^{\vec{\eta}_i^\frown \pi(\psi_i)}(b' \wedge a_{n_j} \wedge x_{n_j})$ and $\mathscr{A}_\delta \models c_{i-1} \in pp^{\pi(\psi_{i-1})}(c_i)(c_0 \in pp^{\pi(\alpha)}(c_1))$. (This is possible since $\vec{\eta}_i^\frown \pi(\psi_i)$ is a subsequence of $\vec{\eta}_{i-1}$ and property 2 of the preprojections.) Then $\pi^{-1}(c_i) \leq z_i$ and since $\mathscr{A}_\delta \models b' \leq b$, $\mathscr{A}_\delta \models c_i \in pp^{\pi(\psi_i)}(b)$. Thus, by clause d) of the definition of riskyness we have that in $\mathscr{B}$, $\pi^{-1}(c_{i-1}) \in pp^{\psi_{i-1}}\pi^{-1}(c_1)$ and $\pi^{-1}(c_1) \in pp^{\psi_i}(b)$.

Let $c' \leq \pi^{-1}(c)$, $c' \in \mathscr{B}_\alpha$, then an easy induction shows that $c' \wedge \pi^{-1}(c_1) \wedge \cdots \wedge \pi^{-1}(c_k) \wedge b \neq 0$. Since $\pi^{-1}(c_i) \leq z_i$,

$$c' \wedge z_1 \wedge \cdots \wedge z_k \wedge b \neq 0.$$

This proves the subclaim.

Since $\pi^{-1}(c) \in pp^\alpha(b \wedge \bigwedge_{i=1}^k z_i)$ and $d \wedge c \neq 0$, $d \wedge b \wedge \bigwedge_{i=1}^k z_i \neq 0$. This proves that $\bigcup\{T_{\delta,\psi} | \psi \in \mathrm{cof}(\omega_1) \cap (s^\delta \cup \{\beta\})\}$ has the finite intersection property with U_β^δ. Let $U_\beta^{\delta+1} \supseteq U_\beta^\delta \cup \bigcup\{T_{\delta,\psi} | \psi \in \mathrm{cof}(\omega_1) \cap (s^\delta \cup \{\beta\})\}$ be a filter such that $U_\beta^{\delta+1} \cap \mathscr{B}_\beta^{\delta+1}$ is an ultrafilter and $|U_\beta^{\delta+1} \sim U_\beta^\delta| \leq \omega$.

Let $U_\beta = \bigcup_{\delta < \omega_1} U_\beta^\delta$.

Then for all $\delta \in C \cap \{\delta^*\colon U_\beta^{\delta^*} \cap (\mathscr{B}_\beta \sim \mathscr{B}_\alpha) \subseteq \mathscr{B}_\beta^{\delta^*}\}$ that are risky for $U_\beta^\delta \cap \mathscr{B}_\beta^\delta$ and for all $\psi \in (\{\beta\} \cup s^\delta) \cap \mathrm{cof}(\omega_1) T_{\delta,\psi} \subseteq U_\beta$. Hence U is obedient. □

Note that we have shown that we can extend U_α to U_β with one arbitrary choice of countably many elements of $U_\beta \sim U_\alpha$.

We now show a lemma that justifies our work.

LEMMA 11. *Let $\beta \in \operatorname{cof}(\omega_1) \cap \omega_2 \cap \lim(\operatorname{cof} \omega_1)$. Suppose that $\langle x_n : n \in \omega \rangle$ is a partition of $\mathscr{B}_\beta$ and U is an ultrafilter such that $U \,\hat{\wedge}\, \mathscr{B}_\beta = U_\beta$ is obedient. Then there is a partition $\langle y_n : n \in \omega \rangle$ of $\mathscr{B}_0$ such that $\bigvee_{n \in \omega} x_n \wedge y_n \in U_\beta$.*

Proof. Let $M \prec \langle H(\lambda), \varepsilon, \beta, \langle x_n : n \in \omega \rangle, H \rangle$ be an elementary substructure of cardinality ω_1 such that $\langle x_n : n \in \omega \rangle \in M$. Represent M by a continuous increasing sequence $\langle M_\delta : \delta < \omega_1 \rangle$ and U_β by $\langle U_\delta : \delta < \omega_1 \rangle$. Since $\langle \mathscr{A}_\delta : \delta < \omega_1 \rangle$ is a $\diamondsuit$-sequence there is a stationary set S of δ such that there is an isomorphism ϕ: $M_\delta \to \mathscr{A}_\delta$ and $\phi'' U_\delta \subseteq D_\delta$. Let $s^\delta = M_\delta \cap \beta$ and $\mathscr{B}_\beta^\delta = M_\delta \cap \mathscr{B}_\beta$. Then each $\delta \in S$ is risky for $\mathscr{B}_\beta^\delta$ and U_δ. Since U_β is obedient, there is a risky $\delta \in S$ such that $\bigcup_{\psi \in \operatorname{cof}(\omega_1) \cap s_\delta} T_{\delta, \psi} \subseteq U_\beta$. Since $\beta \in \lim \operatorname{cof}(\omega_1)$, there is a $\psi \in \operatorname{cof}(\omega_1) \cap s^\delta$ such that $\langle x_n : n \in \omega \rangle \in \mathscr{B}_\psi$.

Let π: $M_\delta \cong \mathscr{A}_\delta$ be the isomorphism. Since $pp^{\pi(\psi)}(a_{n_j} \wedge \pi(x_{n_j})) = \{ y \in \mathscr{C}_{\pi(\psi)} : y \le a_{n_j} \wedge \pi(x_{nj}) \}$; we get

$$\bigvee_{j \in \omega} \bigvee \pi^{-1}\Big(pp^{\pi(\psi)}\big(a_{n_j} \wedge \pi(x_{n_j})\big)\Big) \le \bigvee_{j \in \omega} \pi^{-1}(a_{n_j}) \wedge x_{n_j}.$$

Since $T_{\delta, \psi} \subseteq U_\beta$, $\bigvee_{j \in \omega} \bigvee \pi^{-1}(pp^{\pi(\psi)}(a_{n_j} \wedge \pi(x_{n_j}))) \in U_\beta$; hence $\bigvee_{j \in \omega}(\pi^{-1}(a_{n_j}) \wedge x_{n_j}) \in U_\beta$. Let $y_n = \pi^{-1}(a_{n_j})$ if there is a j, $n = n_j$ and $y_n = 0$ otherwise. $\square$

Claim 12. Suppose that $\langle \alpha_i : i \in \omega \rangle \subseteq \operatorname{cof}(\omega_1) \cap \omega_2$ is an increasing sequence of ordinals and $\langle U_i : i \in \omega \rangle$ is an increasing sequence of obedient ultrafilters with each $U_i \subseteq \mathscr{B}_{\alpha_i}$. Then for all $\beta > \sup_{i \in \omega} \alpha_i$, there is an obedient ultrafilter $V \subseteq \mathscr{B}_\beta$ such that for all i, $V \supseteq U_i$.

Proof. Let $\beta > \sup \alpha_i$, $\beta \in \operatorname{cof}(\omega_1) \cap \omega_2$. Let $\langle (\mathscr{B}_\beta^\delta, s^\delta) : \delta < \omega_1 \rangle$ be a continuous representation of $\mathscr{B}_\beta$ and β so that for all $i \in \omega$, $\alpha_i \in s^0$.

Then without loss of generality, $\langle (\mathscr{B}_\beta^\delta \cap \mathscr{B}_{\alpha_i}, s^\delta \cap \alpha_i) : \delta < \omega_1 \rangle$ is a continuous representation of $\mathscr{B}_{\alpha_i}$ and α_i for each i.

Since U_{α_i} is obedient there is a $C_i \subseteq \omega_1$ closed and unbounded such that for all risky $\delta \in C_i$ and all $\psi \in \operatorname{cof}(\omega_1) \cap (s^\delta \cap \alpha_i)$, $T_{\delta, \psi} \subseteq U_{\alpha_i}$.

Let $C = \bigcap_{i \in \omega} C_i$. We build the ultrafilter $U_\beta \subseteq \mathscr{B}_\beta$ in ω_1 stages $\langle U_\beta^\delta : \delta < \omega_1 \rangle$ such that $U_\beta^0 \supset \bigcup_{i \in \omega} U_{\alpha_i}$ and $U_\beta^\delta \cap \mathscr{B}_\beta^\delta$ is an ultrafilter.

As we argued in claim 10, if δ is a risky stage for $U_\beta^\delta \cap \mathscr{B}_\beta^\delta$ then δ is risky for each $U_\beta^\delta \cap \mathscr{B}_{\alpha_i}^\delta = U_{\alpha_i} \cap \mathscr{B}_{\alpha_i}^\delta$. Hence, if $\delta \in C$ and $U_\beta^\delta \cap (\mathscr{B}_\beta \sim \bigcup \mathscr{B}_{\alpha_i}) \subseteq \mathscr{B}_\beta^\delta$ then for all $\psi < \sup \alpha_i$, if $\psi \in \operatorname{cof}(\omega_1) \cap s^\delta$ then $T_{\delta, \psi} \subseteq \bigcup_{i \in \omega} U_{\alpha_i}$. As in claim 10 this implies that $\bigcup_{\psi \in \operatorname{cof}(\omega_1) \cap (s^\delta \cup \{\beta\})} T_{\delta, \psi}$ has the f.i.p. with U_β^δ. (Essentially, since for any $b \in U_\beta^\delta \cap \mathscr{B}_\beta^\delta$ and any $z \in T_{\delta, \psi}$ there is a projection of $b \wedge z$ in U_{α_i} and hence every element of U_{α_i} is compatible with $b \wedge z$.) For such δ, we let $U_\beta^{\delta+1}$ be any filter extending $U_\beta^\delta \cup \{T_{\delta,\psi}\colon \psi \in \mathrm{cof}(\omega_1) \cap (s^\delta \cup \{\beta\})\}$ such that $U_\beta^{\delta+1} \cap \mathscr{B}_\beta^{\delta+1}$ is an ultrafilter on $\mathscr{B}_\beta^\delta$ extending U_β^δ and $|U_\beta^{\delta+1} \sim U_\beta^\delta| \le \omega$.

At other δ we let $U_\beta^{\delta+1}$ extend U_β^δ arbitrarily so that $U_\beta^{\delta+1} \cap \mathscr{B}_\beta^{\delta+1}$ is an ultrafilter on $\mathscr{B}_\beta^{\delta+1}$ and $|U_\beta^{\delta+1} \sim U_\beta^\delta| \le \omega$.

By construction, at risky $\delta \in C \cap \{\delta^*\colon U_\beta^{\delta^*} \cap (\mathscr{B}_\beta \sim \bigcup_i \mathscr{B}_{\alpha_i}) \subseteq \mathscr{B}_\beta^{\delta^*}\}$ for all $\psi \in (\{\beta\} \cup s^\delta) \cap \mathrm{cof}(\omega_1)$, $T_{\delta,\psi} \subseteq U_\beta$. Hence U_β is obedient. This proves claim 12. As in claim 10 we could prove something stronger, namely, if $a \subseteq \mathscr{B}_\beta$ is countable and has the f.i.p. with $\bigcup_{i \in \omega} U_{\alpha_i}$ then there is an obedient ultrafilter $U_\beta \subseteq \mathscr{B}_\beta$ such that $\bigcup_{i \in \omega} U_{\alpha_i}$, $a \subseteq U_\beta$. □

We are now in a position to define our forcing conditions **P**: A condition $U \in \mathbf{P}$ is an obedient ultrafilter $U \subseteq \mathscr{B}_\alpha$ for some $\alpha \in \mathrm{cof}(\omega_1) \cap \omega_2$. If $U \subseteq \mathscr{B}_\alpha$ and $V \subseteq \mathscr{B}_\beta$ are obedient ultrafilters and $\alpha < \beta, \alpha, \beta \in \mathrm{cof}(\omega_1) \cap \omega_2$ then $V \Vdash U$ if and only if $V \supseteq U$.

Claims 10 and 12 show that if $G \subseteq \mathbf{P}$ is generic then $U = \bigcup G \subseteq \mathscr{B}$ is an ultrafilter such that for all $\alpha \in \mathrm{cof}(\omega_1) \cap \omega_2$, $U \cap \mathscr{B}_\alpha$ is obedient and **P** is countably closed forcing.

If we can show that **P** adds no new ω_1-sequences, then by Lemma 11 we will have shown that $|\omega^{\omega_1}/U| = \omega_1$ in $V^{\mathbf{P}}$. Thus we will have proved Theorem 9 if we can show:

Claim 13. **P** is (ω_2, ∞)-distributive.

Proof. Let $\langle D_\alpha\colon \alpha < \omega_1\rangle \subseteq \mathbf{P}$ be a collection of open dense sets and $U_0 \in \mathbf{P}$.

Let $M \prec (H(\lambda), \varepsilon, \mathscr{B}, H, \langle D_\alpha\colon \alpha < \omega_1\rangle, U_0, \langle \mathscr{A}_\delta\colon \delta < \omega_1\rangle, \Delta)$ be an elementary substructure of $H(\lambda)$ of cardinality ω_1 such that $M^{\omega_1} \subseteq M$. Let $\beta = M \cap \omega_2$. Let $\langle \gamma_i\colon i \in \omega_1\rangle$ be a continuous increasing sequence of ordinals cofinal in β such that for all $i \in \omega_1$, $\gamma_{i+1} \in \mathrm{cof}(\omega_1)$.

We construct a sequence of obedient ultrafilters $U_i \subseteq \mathscr{B}_{\alpha i}$, some α_i, and $U_{i+1} \in D_i$ such that $\bigcup_{i \in \omega_1} U_{\alpha_i}$ is an obedient ultrafilter on $\mathscr{B}_\beta$ and $\alpha_i \ge \gamma_i$.

Represent $\mathscr{B}_\beta$ and β by a continuous increasing sequence $\langle(\mathscr{B}_\beta^\delta, s^\delta)\colon \delta < \omega_1\rangle$. Then for $\delta^* < \omega_1$, $\langle(\mathscr{B}_\beta^\delta, s^\delta)\colon \delta < \delta^*\rangle \in M$. For each $\alpha < \beta$, the sequence $\langle(\mathscr{B}_\beta^\delta \cap \mathscr{B}_\alpha, s^\delta \cap \alpha)\colon \delta < \omega_1\rangle$ is a continuous representation of $\mathscr{B}_\alpha$ and α.

Suppose we have chosen $\langle U_i\colon i < \delta\rangle$. Then for each $i < \delta$ there is a closed unbounded set $C_i \subseteq \omega_1$ witnessing reliability of U_i for the representation $\langle(\mathscr{B}_\beta^\delta \cap \mathscr{B}_{\alpha_i}, s^\delta \cap \alpha_i)\colon \delta < \omega_1\rangle$. Let $U_\delta = \bigcup_{i<\delta} U_i$ and $\alpha_\delta = \sup_{i<\delta}\alpha_i$.

Case 1. $\delta \in \bigcap_{i<\delta} C_i$ and δ is risky for $\mathscr{B}_\beta^\delta$, s^δ and $\bigcup_{i<\delta} U_i \cap \mathscr{B}_\beta^\delta$ and $\{\alpha_i\colon i < \delta\}$ is cofinal in s^δ.

In this case we want to claim that $T_{\delta,\beta}$ has the f.i.p. with $\bigcup_{i<\delta} U_i$. Otherwise there is a $d \in U_i$ and a $z_{\vec{\eta}^\frown \beta, b} \in T_{\delta,\beta}$ such that $d \wedge z_{\vec{\eta}^\frown \beta, b} \neq 0$. But $d \wedge z_{\vec{\eta}^\frown \pi(b)^\frown \alpha_i, b} \neq 0$, a contradiction.

By claims 10 and 8 we can find an obedient $U_{\delta+1} \subseteq \mathscr{B}_{\alpha_{\delta+1}}$ such that $U_{\delta+1} \in D_\delta$, $\alpha_{\delta+1} \geq \gamma_{\delta+1}$ and $U_{\delta+1} \supseteq T_{\delta,\beta} \cup \bigcup_{i<\delta} U_i$.

Case 2. Otherwise. Let $U_{\delta+1} \in D_\delta$ be an arbitrary obedient ultrafilter on some $\mathscr{B}_{\alpha_{\delta+1}}$ with $U_{\delta+1} \supseteq \bigcup_{i<\delta} U_i$ and $\alpha_{\delta+1} \geq \gamma_{\delta+1}$.

We claim that $U = \bigcup_{i \in \omega_1} U_i$ is an obedient ultrafilter on $\mathscr{B}_\beta$. Since $\langle \alpha_i: i \in \omega_1 \rangle$ are cofinal in $\mathscr{B}$, $\bigcup_{i \in \omega_1} U_i$ is an ultrafilter on $\mathscr{B}_\beta$.

Let C_i witness the reliability of U_i for the sequence $\langle \mathscr{B}_\beta^\delta \cap \mathscr{B}_{\alpha_i}: \delta < \omega_1 \rangle$. Let

$$C = \mathop{\triangle}_{i<\omega_1} C_i \cap \left\{ \delta: \{\alpha_i: i < \delta\} \text{ is cofinal in } s^\delta \right\}.$$

Let $\delta \in C$ be risky for $\mathscr{B}_\beta^\delta$ and U. Then δ is risky for all U_{α_i}, $i < \delta$. Hence for all $\psi \in s^\delta$, $T_{\delta,\psi} \subseteq U$. Hence we are in case 1. Thus $T_{\delta,\beta} \subseteq U$.

But then for all risky $\delta \in C$ and all $\psi \in (s^\delta \cup \{\beta\}) \cap \omega_1$, $T_{\delta,\psi} \subseteq U$; hence U is obedient and $U \in \bigcap_{\delta \in \omega_1} D_\delta$. □

This completes the proof of Theorem 9.

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