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A model of ZFA + PAC with no outer model of ZFAC with the same pure part

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Abstract We produce a model of ZFA + PAC such that no outer model of ZFAC has the same pure sets, answering a question asked privately by Eric Hall.

Keywords Set theory with atom · Axiom of choice · Forcing

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1 Models of ZFA

The axiom system ZFA is a natural modification of Zermelo–Fraenkel set theory (ZF) allowing for the existence of non-set elements, called *atoms*. We refer the reader to Chapter 4 of [7], pages 249–261 of [6] or Chapter 7 of [3] for a specific definition, and background for some of the techniques below. Sets in a model of ZFA whose

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transitive closures do not contain atoms are called *pure sets*. The pure sets form an inner model of ZF; the axiom PAC asserts that this inner model satisfies the Axiom of Choice. The theory ZFAC extends ZFA with the statement that Choice for all sets (given ZFA + PAC, this amounts to asserting that the set of atoms can be wellordered). In this paper we produce a model of ZFA + PAC such that no outer model of ZFAC has the same pure sets, answering a question asked privately by Eric Hall.

Given a nonempty set A disjoint from $\{\emptyset\}$, we define the following hierarchy over A, indexed by ordinals:

- $\mathcal{P}^{0,*}(A) = A$;
- $\mathcal{P}^{\alpha+1,*}(A) = (\mathcal{P}^{\alpha,*}(A) \cup \mathcal{P}(\mathcal{P}^{\alpha,*}(A)))\backslash\{\emptyset\}$;
- $\mathcal{P}^{\beta,*}(A) = \bigcup_{\alpha<\beta} \mathcal{P}^{\alpha,*}(A)$ when β is a limit ordinal;
- $\mathcal{P}^{\infty,*}(A) = \bigcup_{\alpha\in\mathrm{Ord}} \mathcal{P}^{\alpha,*}(A)$.

Let us say that an *atom set* is a nonempty set A such that no member of A is in the transitive closure of any other member. Letting any one element of an atom set A represent the emptyset, and the other members of A represent atoms, $\mathcal{P}^{\infty,*}(A)$ is the domain of a model of ZFA.

Remark 1.1 A bijection $\rho\colon A \to B$ between atom sets A and B naturally induces a class-sized isomorphism $\pi_\rho\colon \mathcal{P}^{\infty,*}(A) \to \mathcal{P}^{\infty,*}(B)$ which restricts, for each ordinal α, to a bijection from $\mathcal{P}^{\alpha,*}(A)$ to $\mathcal{P}^{\alpha,*}(B)$.

Our approach to models of ZFA differs from the traditional Fraenkel–Mostowski method (see [3,6,7]), and we do not know how to produce our result in their way. The models we consider will have as their domains subclasses of classes of the form $\mathcal{P}^{\infty,*}(A)$. We concentrate on subclasses of $\mathcal{P}^{\infty,*}(A)$ (for a given atom set A) which are constructed over A using certain elements of $\mathcal{P}^{\infty,*}(A)$ as predicates.

Given sets X and B in $\mathcal{P}^{\infty,*}(A)$, we let $\mathrm{Def}_B(X)$ denote the collection of nonempty subsets of X which are definable over X using parameters from X and predicates corresponding to the members of B. We then define:

- $U_0^{A,B} = A$;
- $U_{\alpha+1}^{A,B} = U_\alpha^{A,B} \cup \mathrm{Def}_B(U_\alpha^{A,B})$;
- $U_\beta^{A,B} = \bigcup_{\alpha<\beta} U_\alpha^{A,B}$ when β is a limit ordinal.
- $U_\infty^{A,B} = \bigcup_{\alpha\in\mathrm{Ord}} U_\alpha^{A,B}$.

Finally, given $a \in A$, we let $\mathbf{U}(a, A, B)$ be the model of ZFA with domain $U_\infty^{A,B}$, where a is interpreted as the emptyset. Then $\mathbf{U}(a, A, B)$ is (up to isomorphism) the smallest wellfounded proper class model of ZFA with $A\backslash\{a\}$ as its set of atoms and a as its emptyset which is closed under intersections with the members of B. A standard proof by induction shows that every element of $\mathbf{U}(a, A, B)$ is definable in $\mathbf{U}(a, A, B)$ from a finite set of its ordinals, a finite subset of A and finitely many predicates from B (i.e., restrictions of elements of B to $\mathbf{U}(a, A, B)$).

Remark 1.2 Let A be an atom set, let a be an element of A, let B be a set in $\mathcal{P}^{\infty,*}(A)$ and let $\rho\colon A \to A$ be a permutation. By Remark 1.1, ρ induces a class-sized automorphism π_ρ of $\mathcal{P}^{\infty,*}(A)$. If $\rho(a) = a$ and $\pi_\rho(b) = b$ for each $b \in B$, then we have the following standard facts.

- The restriction of π_ρ to $\mathbf{U}(a, A, B)$ is an automorphism of $\mathbf{U}(a, A, B)$.
- If X is a set in $\mathbf{U}(a, A, B)$ which is definable from sets which are fixed by π_ρ, then X is fixed by π_ρ.

The following is one version of our main theorem.

Theorem 1.3 *In a c.c.c. forcing extension* $\mathbf{L}[G]$ *of* $\mathbf{L}$ *there is a model* $\mathbf{U}$ *of* ZFA *of the form* $\mathbf{U}(a, A, B)$*, for some atom set* A *in* $\mathbf{L}$*, some element* a *of* A *and some* B *in* $\mathcal{P}^{\infty,*}(A)$*, such that the pure part of* $\mathbf{U}$ *is isomorphic to* $\mathbf{L}$ *and such that in no outer model of* $\mathbf{L}[G]$ *is there a model of* ZFAC *containing* $\mathbf{U}$ *whose pure part is isomorphic to* $\mathbf{L}$.

More specifically, the model $\mathbf{U}$ in the statement of Theorem 1.3 will contain a set such that any outer model of $\mathbf{U}$ wellordering this set will contain an injection from $(\omega_3^{\mathbf{L}})^{\mathbf{U}}$ to $\mathcal{P}(\omega_1^{\mathbf{L}})^{\mathbf{U}}$, and therefore will have a subset of $(\omega_3^{\mathbf{L}})^{\mathbf{U}}$ which is not in $\mathbf{L}^{\mathbf{U}}$.

In Sect. 2 we give a proof of Theorem 1.3. Our proof uses a model theoretic construction due to Hjorth which produces a sentence in $\mathcal{L}_{\aleph_1,\aleph_0}$ homogeneously characterizing $\aleph_1$ (in a sense which will be made precise). We briefly discuss this construction and related results in Sect. 3. Section 4 illustrates the need for such a construction. We refer the reader to pages 25–27 of [5] for a definition of $\mathcal{L}_{\aleph_1,\aleph_0}$.

2 The proof

Our proof requires sets (in a model of ZFA) which are not wellordered (and moreover admit sufficiently many automorphisms) and fixed upper or lower bounds for the cardinalities of these sets in outer models of ZFAC. In Sect. 4 we show that simply choosing a large or small set of atoms does not suffice for this. In our proof we use a partition into $\aleph_3$ many infinite sets to get a lower bound of $\aleph_3$ for one set (K), and Theorem 2.1 below to get an upper bound of $\aleph_1$ for another (Q).

Theorem 2.1 *There exist in* $\mathbf{L}$ *a definable countable relational vocabulary* τ *containing a unary predicate* Q *and a definable sentence* ϕ *in* $\mathcal{L}_{\aleph_1,\aleph_0}(\tau)$ *such that* ZF *proves the following:*

- ϕ *has a unique countable model, up to isomorphism;*
- ϕ *has no model of cardinality greater than* $\aleph_1$*;*
- *if* $\mathcal{M}$ *is a countable model of* ϕ *and* M *is the domain of* $\mathcal{M}$*, then* $\mathsf{Q}^{\mathcal{M}}$ *is infinite, and for each finite* $M' \subseteq M$ *there is a finite* $Q' \subseteq \mathsf{Q}^{\mathcal{M}}$ *such that every permutation of* $\mathsf{Q}^{\mathcal{M}}$ *fixing* Q' *pointwise extends to an automorphism of* $\mathcal{M}$ *fixing* M' *pointwise.*

Theorem 2.1 is an immediate consequence of arguments in each of [4] and [8] (in the latter case, as exposed in [1]). We discuss in Sect. 3 how to get Theorem 2.1 from the arguments in [4]. A result of Gao [2] shows that the cardinality bound of $\aleph_1$ for models of ϕ cannot be replaced with $\aleph_0$ in the statement of Theorem 2.1.

Beginning the proof of Theorem 1.3, fix sets A, I and M in $\mathbf{L}$ such that

- M is a countably infinite subset of I disjoint from $\{\emptyset\} \cup (\omega_3^{\mathbf{L}} \times \omega)$;
- $I = \{\emptyset\} \cup (\omega_3^{L} \times \omega) \cup M$;

- $A = \{a_i : i \in I\}$ and, for all i, j in I, if $i \neq j$ then $a_i \neq a_j$;
- A is an atom set;
- $a_i = i$ for all $i \in M$.

Let $\rho_\emptyset$ be $\{(\emptyset, a_\emptyset)\}$; this is a bijection between atom sets. For each $x \in \mathbf{L}$, we let x^* denote $\pi_{\rho_\emptyset}(x)$ as defined in Remark 1.1 (so x^* is the copy of x in $\mathcal{P}^{\infty,*}(\{a_\emptyset\})$). We let $\mathbf{L}^*$ denote the class of sets of the form x^* for x in $\mathbf{L}$.

Fix a vocabulary τ and a sentence ϕ as in the statement of Theorem 2.1, and let $\mathcal{M}$ be a model of ϕ^* with domain M. Let C be the set of $\mathcal{M}$-interpretations of the relations in τ^*. Treating finite sequences as iterated ordered pairs, each element of C is in $\mathcal{P}^{\infty,*}(M)$.

Working in $\mathbf{L}$, let

- K denote the set of pairs $\{(\alpha^*, a_{\alpha,i}) : \alpha < \omega_3^{\mathbf{L}}, i \in \omega\}$,
- for each $\alpha < \omega_3^{\mathbf{L}}$, K_α denote the set $\{(\alpha^*, a_{\alpha,i}) : i \in \omega\}$ and
- for some enumeration $\langle \mathrm{T}_n : n \in \omega\rangle$ in $\mathbf{L}$ of the relation symbols in τ, T be the set of pairs (n^*, c) for which c is in the $\mathcal{M}$-interpretation of T_n^*.

Let $B_0 = \{K, M, T\}$. The model $\mathbf{U}_0 = \mathbf{U}(a_0, A, B_0)$ is definable in $\mathbf{L}$, so $\mathbf{L}^*$ is its class of pure sets. The model $\mathcal{M}$ is a member of $\mathbf{U}_0$.

We let $\mathbb{P}$ be the forcing whose conditions are finite partial functions

$$p\colon K \times \mathrm{Q}^{\mathcal{M}} \to 2^*,$$

ordered by containment. Then $\mathbb{P}$ is in $\mathbf{U}_0$.

Let $G \subseteq \mathbb{P}$ be a $\mathbf{U}_0$-generic filter. Let $F = \bigcup G$, and let $B = B_0 \cup \{F\}$. The model $\mathbf{U}(a_0, A, B)$ (which we will call $\mathbf{U}$) is equivalent to $\mathbf{U}_0[G]$.

The following lemma is the key step in the proof of our main theorem.

Lemma 2.2 *The pure sets of* $\mathbf{U}$ *are exactly the members of* $\mathbf{L}^*$.

Proof Suppose that τ is a $\mathbb{P}$-name in $\mathbf{U}_0$ for a set of ordinals, and that some condition $p_0 \in \mathbb{P}$ forces the realization of τ not to be an element of $\mathbf{L}^*$. Then for each condition p below p_0 there exist an ordinal γ and conditions q, q' below p such that $q \Vdash \check{\gamma} \in \tau$ and $q' \Vdash \check{\gamma} \notin \tau$. Using this one can find a sequence $\bar{Y} = \langle Y_i : i \in \omega\rangle$ in $\mathbf{U}_0$ such that each Y_i is a nonempty set of $\mathbb{P}$ conditions below p_0, closed under strengthenings, and such that members of distinct Y_i's are incompatible. We aim to show that such a sequence cannot exist.

The sequence $\bar{Y}$ is ordinal definable in $\mathbf{U}_0$ from a finite subset of $A \cup \{K, M, T\}$, which implies that it is definable from

- a finite set of $\mathbf{U}_0$-ordinals,
- K, M, T,
- a finite set $M' \subseteq M$ and
- a finite set $K' \subseteq K$.

Let $Q' = M' \cap \mathrm{Q}^{\mathcal{M}}$. Expanding Q' if necessary, we may assume (using the fact that ϕ witnesses Theorem 2.1) that every permutation of $\mathrm{Q}^{\mathcal{M}}$ fixing Q' pointwise extends to an automorphism of $\mathcal{M}$ fixing M' pointwise. For each $i \in \omega$ let Y_i^* be the set of

$p \in Y_i$ whose domain contains $K' \times Q'$. Since each Y_i is closed under strengthenings, the sets Y_i^* are also nonempty.

Let Z be the set of permutations of A which

- fix the members of $\{a_\emptyset\} \cup K' \cup M'$ pointwise,
- fix K, M and the members of C setwise (i.e., restrict to automorphisms of $\mathcal{M}$) and
- for each $\alpha < \omega_3^{\mathbf{L}}$, fix K_α setwise.

Each member of Z induces an automorphism of the model $\mathbf{U}_0$ which maps the sequence $\langle Y_i^* : i \in \omega\rangle$ to itself. As no two members of different Y_i^*'s are compatible, it follows that no permutation in Z induces an automorphism which moves a member of one Y_i^* to a condition compatible with a member of another. We will derive a contradiction by finding an element of Z which does this.

Let us say that the *type* of a condition $p \in \mathbb{P}$ is its restriction to $K' \times Q'$. As there are only finitely many possible types, the following claim finishes the proof of the lemma.

Claim 2.3 *If $\mathbb{P}$-conditions p and q have the same type, then there is a permutation ρ in Z mapping p to a condition compatible with q.*

We fix p and q and prove the claim. We have that ρ must fix the members of $\{a_\emptyset\} \cup K' \cup M'$ pointwise and restrict to an automorphism of $\mathcal{M}$. The rest of $\rho \restriction K$ can be chosen so that each K_α ($\alpha < \omega_3^{\mathbf{L}}$) is fixed setwise and $(\rho(a), c) \notin \mathrm{dom}(q)$, for all $(a, b) \in \mathrm{dom}(p) \cap ((K \setminus K') \times \mathtt{Q}^{\mathcal{M}})$ and $c \in \mathtt{Q}^{\mathcal{M}}$. Now we can choose $\rho \restriction (\mathtt{Q}^{\mathcal{M}} \setminus Q')$ so that for all $(a, b) \in \mathrm{dom}(p) \cap (K \times (\mathtt{Q}^{\mathcal{M}} \setminus Q'))$ there is no $(a', b') \in \mathrm{dom}(q)$ with $\rho(b) = b'$. Finally, we can extend ρ to M to form an automorphism of $\mathcal{M}$. Any permutation ρ satisfying these conditions witnesses the claim. □

Now suppose that $\mathbf{U}^+$ is an outer model of $\mathbf{U}$ satisfying ZFAC. By Theorem 2.1, the set $\mathtt{Q}^{\mathcal{M}}$ has cardinality at most $\aleph_1$ in $\mathbf{U}^+$. Since K is partitioned into $\aleph_3^{\mathbf{L}^*}$ many nonempty disjoint sets in $\mathbf{U}_0$, K has cardinality at least $|\aleph_3^{\mathbf{L}^*}|$ in $\mathbf{U}^+$. For each pair of distinct elements a, a' of K, however, there exists by the genericity of G a $b \in \mathtt{Q}^M$ such that $F(a, b) \neq F(a', b)$. This gives $|\aleph_3^{\mathbf{L}^*}|$ many distinct functions from $\omega_1^{\mathbf{L}^*}$ to 2 in $\mathbf{U}^+$, and thereby a pure set not in $\mathbf{U}$ (either an injection from $\omega_3^{\mathbf{L}^*}$ to $\omega_2^{\mathbf{L}^*}$ or a new subset of $\omega_1^{\mathbf{L}^*}$, since $\mathbf{L}^* \models 2^{\aleph_1} = \aleph_2$).

3 Hjorth's construction

In this section we briefly discuss how to get a proof of Theorem 2.1 from Hjorth's [4] and its extension as exposed in [1]. The difference between the two presentations is that the relation $\mathtt{Q}$ does not appear in [4]. The addition of $\mathtt{Q}$ in [1] requires only routine modifications.

The vocabulary τ consists of

- a unary relation symbol $\mathtt{Q}$,
- binary relation symbols $\mathtt{P}$ and $\mathtt{S}_n$ ($n \in \omega$),
- $(k+2)$-ary relation symbols $\mathtt{R}_k$ for each $k \in \omega$.

Modifying Hjorth's argument slightly, we define a preliminary sentence ϕ_0 consisting of the conjunction of the following assertions:

- $\forall x, y\,(\mathtt{P}(x, y) \rightarrow (\neg\mathtt{Q}(x) \wedge \mathtt{Q}(y)),$
- $\forall x\,(\neg\mathtt{Q}(x) \rightarrow \exists! y\,\mathtt{P}(x, y)),$
- (for each $n \in \omega$) $\forall x, y\,(\mathtt{S}_n(x, y) \rightarrow \neg\mathtt{Q}(x) \wedge \neg\mathtt{Q}(y) \wedge x \neq y),$
- for each $k \in \omega$, the assertion that for all $x_0, \ldots, x_{k+1}$

$$\mathtt{R}_k(x_0, \ldots, x_{k+1}) \rightarrow ((x_0 \neq x_1) \wedge \left(\bigwedge_{i<k+2} \neg\mathtt{Q}(x_i)\right),$$

- the sentence asserting that for all $x \neq y$ such that $\neg\mathtt{Q}(x)$ and $\neg\mathtt{Q}(y)$, there is a unique $n \in \omega$ such that $\mathtt{S}_n(x, y)$ holds,
- the sentence asserting that for each $k \in \omega$ and all $x_0, x_1, y_1, \ldots, y_{k-1}$, if $\mathtt{R}_k(x_0, x_1, y_0, \ldots, y_{k-1})$ holds, then $\{y_0, \ldots, y_{k-1}\}$ has size k and is the set of z such that for some $n \in \omega$, $\mathtt{S}_n(x_0, z) \wedge \mathtt{S}_n(x_1, z)$ holds,
- the sentence asserting that for all $x_0 \neq x_1$ such that $\neg\mathtt{Q}(x_0)$ and $\neg\mathtt{Q}(x_1)$, there exist $k \in \omega$ and $y_0, \ldots, y_{k-1}$ such that $\mathtt{R}_k(x_0, x_1, y_0, \ldots, y_{k-1})$ holds.

We list some examples of finite models of ϕ_0:

- the unique τ-structure with empty domain;
- for any finite set M, the τ-structure $\mathcal{M}$ with domain M such that $\mathtt{Q}^{\mathcal{M}} = M$ and all other relations in τ are interpreted as $\emptyset$;
- a τ-structure $\mathcal{M}$ with two elements a and b, with $\mathtt{P}^{\mathcal{M}} = \{(a, b)\}$, $\mathtt{Q}^{M} = \{b\}$ and all other relations in τ interpreted as $\emptyset$.

Lemma 3.1 below is essentially Lemma 3.1 of [4]. The only difference is that in in Lemma 3.1 of [4] there is no predicate $\mathtt{Q}$, so in effect the models there are simply the $\neg\mathtt{Q}$ part of the models here. Extending the argument there to accommodate the predicate $\mathtt{Q}$ causes no additional difficulties, and no additional work, as we can take $\mathtt{Q}^{\mathcal{M}_2}$ to be $\mathtt{Q}^{\mathcal{M}_0} \cup \mathtt{Q}^{\mathcal{M}_1}$ and π_1 and π_2 to be identity functions in the case where $M_0 \cap M_1 = M$. We note that the lemma holds even in the case where $M = \emptyset$. The lemma shows that we can build a countable limit model $\mathcal{M}^*$ (in the sense of Section 7.1 of [5]) with the following properties:

- $\mathcal{M}^* \models \forall y\,(\mathtt{Q}(y) \rightarrow \exists^{\infty} x\,\mathtt{P}(x, y));$
- every finite subset of the domain of $\mathcal{M}^*$ is contained in a finite substructure of $\mathcal{M}^*$ satisfying ϕ_0;
- every isomorphism between finite substructures satisfying ϕ_0 extends to an automorphism of $\mathcal{M}^*$.

The sentence ϕ from the statement of Theorem 2.1 is the Scott sentence of the limit model $\mathcal{M}^*$, which characterizes $\mathcal{M}^*$ up to isomorphism (see [5], for instance). Lemma 3.3 of [4] shows that ϕ has no model of cardinality $\aleph_2$ or greater (briefly, if $\mathcal{N} \prec \mathcal{N}'$ are models of ϕ, and $b \in (\neg\mathtt{Q})^{\mathcal{N}'} \setminus (\neg\mathtt{Q})^{\mathcal{N}}$, then the map that sends each $a \in (\neg\mathtt{Q})^{\mathcal{N}}$ to the unique n such that $(a, b) \in \mathtt{S}_n^{\mathcal{N}'}$ is injective). Theorem 2.1 then follows from Lemma 3.1, since for each finite M' we need only to find a finite substructure $\mathcal{M}'$ of $\mathcal{M}^*$ satisfying ϕ_0 with domain containing M', and let Q' be $\mathtt{Q}^{\mathcal{M}'}$.

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Lemma 3.1 *If $\mathcal{M}$ is a finite model of ϕ_0 with domain M and $\mathcal{M}_0$, $\mathcal{M}_1$ are finite models of ϕ extending $\mathcal{M}$ with domains M_0 and M_1 respectively, then there exist a finite model $\mathcal{M}_2$ of ϕ_0 and, letting M_2 be the domain of $\mathcal{M}_2$, τ-embeddings*

$$\pi_0 \colon M_0 \to M_2,\ \pi_1 \colon M_1 \to M_2$$

such that $\pi_0 \restriction M = \pi_1 \restriction M$. Moreover, if $M = M_0 \cap M_1$ then M_2 can be taken to be $M_0 \cup M_1$.

4 Cardinalities of atom sets

In this section we show that the cardinality of an atom set A in $\mathbf{L}$ has no effect, for any $a_0 \in A$, on the cardinality of A in outer models of $\mathbf{U}(a_0, A, \emptyset)$ satisfying Choice and having the same pure part. Note that a model of the form $\mathbf{U}(a_0, A, \{g_A\})$ satisfies Choice if g_A is a bijection between some ordinal of of $\mathbf{L}^{\mathbf{U}(a_0,A,\emptyset)}$ and A.

Theorem 4.1 *Let A be an infinite atom set in $\mathbf{L}$, let a_0 be an element of A and let κ_A be an infinite cardinal of $\mathbf{L}^{\mathbf{U}(a_0,A,\emptyset)}$. In any outer model of $\mathbf{L}$ in which $|A| = |\kappa_A|$, there is a bijection $g_A \colon \kappa_A \to A$ such that $\mathbf{U}(a_0, A, \{g_A\})$ has the same pure sets as $\mathbf{U}(a_0, A, \emptyset)$.*

Proof Let ρ_A be the set $\{(\emptyset, a_0)\}$. Then there is an infinite cardinal κ of $\mathbf{L}$ such that $\kappa_A = \pi_{\rho_A}(\kappa)$ (where π_{ρ_A} is as defined in Remark 1.1).

Let B be an atom set of cardinality κ in $\mathbf{L}$, and let b_0 be an element of B. Let ρ_B be the set $\{(\emptyset, b_0)\}$ and let κ_B be $\pi_{\rho_B}(\kappa)$. Let $g \colon \kappa \to B$ be a bijection, and let g_B be the set $\{(\pi_{\rho_B}(\alpha), g(\alpha)) : \alpha < \kappa\}$. Then g_B is a bijection from κ_B to B, and $\mathbf{U}(b_0, B, \{g_B\})$ has the same pure sets as $\mathbf{U}(b_0, B, \emptyset)$.

Now suppose that $\rho \colon B \to A$ is a bijection in some outer model of $\mathbf{L}$ sending b_0 to a_0. Let g_A be the set $\{(\pi_{\rho_A}(\alpha), \rho(g(\alpha))) : \alpha < \kappa\}$. Then the restriction of π_ρ to $\mathbf{U}(b_0, B, \{g_B\})$ maps it isomorphically to $\mathbf{U}(a_0, A, \{g_A\})$, and sends κ_B to κ_A. Then g_A is a bijection from κ_A to A, and $\mathbf{U}(a_0, A, \{g_A\})$ has the same pure sets as $\mathbf{U}(a_0, A, \emptyset)$. □

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