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\aleph_n -free abelian group with no non-zero homomorphism to \mathbb{Z}

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ABSTRACT

We, for any natural n , construct an \aleph_n -free abelian groups which have few homomorphisms to \mathbb{Z} . For this we use " \aleph_n -free $(n+1)$ -dimensional black boxes". The method is hopefully relevant to other constructions of \aleph_n -free abelian groups.

RESUMEN

Para cualquier natural n , contruimos un grupo abeliano libre \aleph_n el cual tiene pocos homomorfismos hacia \mathbb{Z} . Para esto usamos \aleph_n cajas negras libres $(n+i)$ -dimensionales. El método es relevante para otras construcciones de grupos abelianos \aleph_n -libres.

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Annotated Content

§1 Constructing $\aleph_{k(*)+1}$ -free Abelian group

[We introduce “ \mathbf{x} is a combinatorial $k(*)$ -parameter”. We also give a short cut for getting only “there is a non-Whitehead $\aleph_{k(*)+1}$ -free non-free abelian group” (this is from 1.6 on). This is similar to [5, §5], so proofs are put in an appendix, except 1.14, note that 1.14(3) really belongs to §3.]

§2 Black boxes

[We prove that we have black boxes in this context, see 2.1; it is based on the simple black box. Now 2.3 belongs to the short cut.]

§3 Constructing abelian groups from combinatorial parameter

[For $\mathbf{x} \in \mathbf{K}_{k(*)+1}^{\text{cb}}$ we define a class $\mathcal{G}_{\mathbf{x}}$ of abelian groups constructed from it and a black box. We prove they are all $\aleph_{k(*)+1}$ -free of cardinality $|\Gamma|^{\mathbf{x}} + \aleph_0$ and some $G \in \mathcal{G}_{\mathbf{x}}$ satisfies $\text{Hom}(G, \mathbb{Z}) = \{0\}$.]

§4 Appendix 1

[We give adaptation of the proofs from [5] with the relevant changes.]

0 Introduction

For regular $\theta = \aleph_n$ we look for a θ -free abelian group G with $\text{Hom}(G, \mathbb{Z}) = \{0\}$. We first construct G and a pure subgroup $\mathbb{Z}_Z \subseteq G$ which is not a direct summand. If instead “not direct product” we ask “not free” so naturally of cardinality θ , we know much: see [1].

We can ask further questions on abelian groups, their endomorphism rings, similarly on modules; naturally questions whose answer is known when we demand \aleph_1 -free instead \aleph_n -free; see [2]. But we feel those two cases can serve as a base for significant number of such problems (or we can immitate the proofs). Also this concentration is reasonable for sorting out the set theoretical situation. Why not $\theta = \aleph_\omega$ and higher cardinals? (there are more reasonable cardinals for which such results are not excluded), we do not fully know: note that also in previous questions historically this was harder.

Note that there is such an abelian group of cardinality \aleph_1 , by [7, §4] and see more in Göbel-Shelah-Struüingman [3]. However, if MA then $\aleph_2 < 2^{\aleph_0} \Rightarrow$ any \aleph_2 -free abelian group of cardinality $< 2^{\aleph_0}$ fail the question.

The groups we construct are in a sense complete, like ${}^\omega\mathbb{Z}$. They are close to the ones from [5, §5] but there $S = \{0, 1\}$ as there we are interested in Borel abelian groups. See earlier [8], see representations of [8] in [10, §3], [1].

However we still like to have $\theta = \aleph_\omega$, i.e. \aleph_ω -free abelian groups. Concerning this we continue in [11].

We shall use freely the well known theorem saying

Theorem 0.1 A subgroup of a free abelian group is a free abelian group.

Definition 0.2 1) $\text{Pr}(\lambda, \kappa)$: means that for some \bar{G} we have:

- (a) $\bar{G} = \langle G_\alpha : \alpha \leq \kappa + 1 \rangle$
- (b) \bar{G} is an increasing continuous sequence of free abelian groups
- (c) $|G_{\kappa+1}| \leq \lambda$,
- (d) $G_{\kappa+1}/G_\alpha$ is free for $\alpha < \kappa$,
- (e) $G_0 = \{0\}$
- (f) G_β/G_α is free if $\alpha \leq \beta \leq \kappa$
- (g) some $h \in \text{Hom}(G_\kappa; \mathbb{Z})$ cannot be extended to $\hat{h} \in \text{Hom}(G_{\kappa+1}, \mathbb{Z})$.

2) We let $\text{Pr}^-(\lambda, \theta, \kappa)$ be defined as above, only replacing “ $G_{\kappa+1}/G_\alpha$ is free for $\alpha < \kappa$ ” by “ $G_{\kappa+1}/G_\kappa$ is θ -free.”

1 Constructing $\aleph_{k(*)+1}$ -free abelian groups

Definition 1.1 1) We say \mathbf{x} is a combinatorial parameter if $\mathbf{x} = (k, S, \Lambda) = (k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}})$ and they satisfy clauses (a)-(c)

(a) $k < \omega$

(b) S is a set (in [5], $S = \{0, 1\}$),

(c) $\Lambda \subseteq {}^{k+1}(\omega S)$ and for simplicity $|\Lambda| \geq \aleph_0$ if not said otherwise.

1A) We say \mathbf{x} is an abelian group k -parameter when $\mathbf{x} = (k, S, \Lambda, \mathbf{a})$ such that (a),(b),(c) from part (1) and:

(d) \mathbf{a} is a function from $\Lambda \times \omega$ to \mathbb{Z} .

2) Let $\mathbf{x} = (k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}})$ or $\mathbf{x} = (k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}}, \mathbf{a}^{\mathbf{x}})$. A parameter is a k -parameter for some k and $K_{k(*)}^{\text{cb}}/K_{k(*)}^{\text{gr}}$ is the class of combinatorial/abelian group $k(*)$ -parameters.

3) We may write $\mathbf{a}_{\bar{\eta}, n}^{\mathbf{x}}$ instead $\mathbf{a}^{\mathbf{x}}(\eta, n)$. Let $w_{k, m} = w(k, m) = \{\ell \leq k : \ell \neq m\}$.

4) We say \mathbf{x} is full when $\Lambda^{\mathbf{x}} = {}^{k(*)}(\omega S)$.

5) If $\Lambda \subseteq \Lambda^{\mathbf{x}}$ let $\mathbf{x} \upharpoonright \Lambda$ be $(k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda)$ or $(k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda, \mathbf{a} \upharpoonright (\Lambda \times \omega))$ as suitable. We may write $\mathbf{x} = (\mathbf{y}, \mathbf{a})$ if $\mathbf{a} = \mathbf{a}^{\mathbf{x}}, \mathbf{y} = (k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}})$.

Convention 1.2 If \mathbf{x} is clear from the context we may write k or $k(*)$, S , Λ , \mathbf{a} instead of $k^{\mathbf{x}}, S^{\mathbf{x}}, \Lambda^{\mathbf{x}}, \mathbf{a}^{\mathbf{x}}$.

A variant of the above is

Definition 1.3 1) For $\bar{S} = \langle S_n : m \leq k \rangle$ we define when \mathbf{x} is a \bar{S} -parameter: $\bar{\eta} \in \Lambda^{\mathbf{x}} \wedge m \leq k^{\mathbf{x}} \Rightarrow \eta_m \in {}^\omega(S_m)$.

2) We say $\bar{\alpha}$ is a $(\mathbf{x}, \bar{\chi})$ -black box or $\text{Qr}(\mathbf{x}, \bar{\chi})$ when:

(a) $\bar{\chi} = \langle \chi_m : m \leq k^{\mathbf{x}} \rangle$

(b) $\bar{\alpha} = \langle \bar{\alpha}_{\bar{\eta}} : \bar{\eta} \in \Lambda^{\mathbf{x}} \rangle$

(c) $\bar{\alpha}_{\bar{\eta}} = \langle \alpha_{\bar{\eta}, m, n} : m \leq k^{\mathbf{x}}, n < \omega \rangle$ and $\alpha_{\bar{\eta}, m, n} < \chi_m$

(d) if $h_m : \Lambda_m^{\mathbf{x}} \rightarrow \chi_m$ for $m \leq k^{\mathbf{x}}$ then for some $\bar{\eta} \in \Lambda^{\mathbf{x}}$ we have: $m \leq k^{\mathbf{x}} \wedge n < \omega \Rightarrow h_m(\bar{\eta} \upharpoonright \langle m, n \rangle) = \alpha_{\bar{\eta}, m, n}$, see Definition 1.4(a) below on " $\bar{\eta} \upharpoonright \langle m, n \rangle$ " and $\Lambda_m^{\mathbf{x}}$.

2A) We may replace $\bar{\chi}$ by χ if $\bar{\chi} = \langle \chi_\ell : \ell \leq k^{\mathbf{x}} \rangle$. We may replace \mathbf{x} by $\Lambda^{\mathbf{x}}$ (so say $\text{Qr}(\Lambda^{\mathbf{x}}, \bar{\chi})$ or say $\bar{\alpha}$ is a $(\Lambda, \bar{\chi})$ -black box).

3) We say a parameter \mathbf{x} is \bar{S} -full when $\Lambda^{\mathbf{x}} = \prod_{m \leq k} {}^\omega(S_m)$.

Definition 1.4 For an $k(*)$ -parameter \mathbf{x} and for $m \leq k(*)$ let

- (a) $\Lambda_m^{\mathbf{x}} = \Lambda_{\mathbf{x},m} = \{\bar{\eta} : \bar{\eta} = \langle \eta_\ell : \ell \leq k(*) \rangle \text{ and } \eta_m \in {}^\omega S \text{ and } \ell \leq k(*) \wedge \ell \neq m \Rightarrow \eta_\ell \in {}^\omega S \text{ and for some } \bar{\eta}' \in \Lambda \text{ we have } n < \omega, \bar{\eta} = \bar{\eta}' \upharpoonright \langle m, n \rangle\}$ where $\bar{\eta} = \bar{\eta}' \upharpoonright \langle m, n \rangle$ means $\eta_m = \eta'_m \upharpoonright n$ and $\ell \leq k(*) \wedge \ell \neq m \Rightarrow \eta_\ell = \eta'_\ell$
- (b) $\Lambda_{\leq k(*)}^{\mathbf{x}}$ is $\cup\{\Lambda_m^{\mathbf{x}} : m \leq k(*)\}$
- (c) $m(\bar{\eta}) = m$ if $\bar{\eta} \in \Lambda_m^{\mathbf{x}}$.

Definition 1.5 1) We say a combinatorial $k(*)$ -parameter \mathbf{x} is free when there is a list $\langle \bar{\eta}^\alpha : \alpha < \alpha(*) \rangle$ of $\Lambda^{\mathbf{x}}$ such that for every α for some $m \leq k(*)$ and some $n < \omega$ we have

$$(*) \quad \bar{\eta}_m^\alpha \upharpoonright \langle m, n \rangle \notin \{\eta_m^\beta \upharpoonright \langle m, n \rangle : \beta < \alpha\}.$$

2) We say a combinatorial k -parameter \mathbf{x} is θ -free when $\mathbf{x} \upharpoonright \Lambda = (k, S^{\mathbf{x}}, \Lambda)$ is free for every $\Lambda \subseteq \Lambda^{\mathbf{x}}$ of cardinality $< \theta$.

Remark 1) We can require in (*) even $(\exists^\infty n)[\eta_m^\alpha(n) \notin \cup\{\eta_\ell^\beta(n') : \ell \leq k, \beta < \alpha, n' < \omega\}]$.

At present this seems an immaterial change.

Definition 1.6 For $k(*) < \omega$ and an abelian group $k(*)$ -parameter \mathbf{x} we define an abelian group $G = G_{\mathbf{x}}$ as follows: it is generated by $\{x_{\bar{\eta}} : m \leq k(*) \text{ and } \bar{\eta} \in \Lambda_m^{\mathbf{x}}\} \cup \{y_{\bar{\eta},n} : n < \omega \text{ and } \bar{\eta} \in \Lambda^{\mathbf{x}}\} \cup \{z\}$ freely except the equations:

$$\boxtimes_{\bar{\eta},n} \quad (n!)y_{\bar{\eta},n+1} = y_{\bar{\eta},n} + \mathbf{a}_{\bar{\eta},n}^{\mathbf{x}}z + \sum\{x_{\bar{\eta} \upharpoonright \langle m,n \rangle} : m \leq k(*)\}.$$

Explanation 1.7 A canonical example of a non-free group is $(\mathbb{Q}, +)$. Other examples are related to it after we divide by something. The y 's here play the role of provided (hidden) copies of \mathbb{Q} . What about x 's? For $\bar{\eta} \in \Lambda$ we consider $\langle y_{\bar{\eta},n} : n < \omega \rangle$, as a candidate to represent $(\mathbb{Q}, +), k(*) + 1$ "chances", "opportunities" to avoid having $(\mathbb{Q}, +)$ as a quotient, say by dividing K by a subgroup generated by some of the x 's. This is used to prove $G_{\mathbf{x}}$ is not free even not \aleph_{n+1} -free which is necessary. But for each $m \leq k(*)$ if $\langle x_{\bar{\eta} \upharpoonright \langle m,n \rangle} : n < \omega \rangle$ are not in K , or at least $x_{\bar{\eta} \upharpoonright \langle m,n \rangle}$ for n large enough then \mathbb{Q} is not represented using $\langle y_{\bar{\eta},n} : n < \omega \rangle$; so we have $k(*) + 1$ "ways", "chances", "opportunities" to avoid having $\langle y_{\bar{\eta},n} : n < \omega \rangle$ represents $(\mathbb{Q}, +)$ in the quotient, one for each infinite cardinal $\leq \aleph_{k(*)}$. This helps us prove $\aleph_{k(*)}$ -freeness. More specifically, for each $m(*) \leq k(*)$ if $H \subseteq G$ is the subgroup which is generated by $X = \{x_{\bar{\eta} \upharpoonright \langle m,n \rangle} : m \neq m(*) \text{ and } \bar{\eta} \in {}^{k(*)+1}({}^\omega S) \text{ and } m \leq k(*)\}$, still in G/H the set $\{y_{\bar{\eta},n} : n < \omega\}$ does not generate a copy of \mathbb{Q} , as witnessed by $\{x_{\bar{\eta} \upharpoonright \langle m(*),n \rangle} : n < \omega\}$.

As a warm up we note:

Claim 1.8 For $k(*) < \omega$ and $k(*)$ -parameter \mathbf{x} the abelian group $G_{\mathbf{x}}$ is an \aleph_1 -free abelian group.

Now systematically

Definition 1.9 Let \mathbf{x} be a $k(*)$ -parameter.

- 1) For $U \subseteq {}^\omega S$ let $G_U = G_U^{\mathbf{x}}$ be the subgroup of G generated by $Y_U = Y_U^{\mathbf{x}} = \{z\} \cup \{y_{\bar{\eta}, n} : \bar{\eta} \in \Lambda \cap {}^{k(*)+1}(U) \text{ and } n < \omega\} \cup \{x_{\bar{\eta} \langle m, n \rangle} : m \leq k(*) \text{ and } \bar{\eta} \in \Lambda \cap {}^{k(*)+1}(U) \text{ and } n < \omega\}$. Let $G_U^+ = G_U^{\mathbf{x},+}$ be the divisible hull of G_U and $G^+ = G^+({}^\omega S)$.
- 2) For $U \subseteq {}^\omega S$ and finite $u \subseteq {}^\omega S$ let $G_{U,u}$ be the subgroup ² of G generated by $\cup\{G_{U \cup (u \setminus \{\eta\})} : \eta \in u\}$; and for $\bar{\eta} \in {}^{k(*)} \geq U$ let $G_{U, \bar{\eta}}$ be the subgroup of G generated by $\cup\{G_{U \cup \{\eta_k : k < \ell g(\bar{\eta}) \text{ and } k \neq \ell\}} : \ell < \ell g(\bar{\eta})\}$.
- 3) For $U \subseteq {}^\omega S$ let $\Xi_U = \Xi_U^{\mathbf{x}} = \{\text{the equation } \boxtimes_{\bar{\eta}, n} : \bar{\eta} \in \Lambda \cap {}^{k(*)+1}U \text{ and } n < \omega\}$. Let $\Xi_{U,u} = \Xi_{U,u}^{\mathbf{x}} = \cup\{\Xi_{U \cup (u \setminus \{\beta\})} : \beta \in u\}$.

Claim 1.10 Let $\mathbf{x} \in K_{k(*)}$.

- 0) If $U_1 \subseteq U_2 \subseteq {}^\omega S$ then $G_{U_1}^+ \subseteq G_{U_2}^+ \subseteq G^+$.
- 1) For any $n(*) < \omega$, the abelian group G_U^+ (which is a vector space over \mathbb{Q}), has the basis $Y_U^{n(*)} := \{z\} \cup \{y_{\bar{\eta}, n(*)} : \bar{\eta} \in \Lambda \cap {}^{k(*)+1}(U)\} \cup \{x_{\bar{\eta} \langle m, n \rangle} : m \leq k(*), \bar{\eta} \in \Lambda \cap {}^{k(*)+1}(U) \text{ and } n < \omega\}$.
- 2) For $U \subseteq {}^\omega S$ the abelian group G_U is generated by Y_U freely (as an abelian group) except the set Ξ_U of equations.
- 3) If $m(*) < \omega$ and $U_m \subseteq {}^\omega S$ for $m < m(*)$ then the subgroup $G_{U_0} + \dots + G_{U_{m(*)-1}}$ of G is generated by $Y_{U_0} \cup Y_{U_1} \cup \dots \cup Y_{U_{m(*)-1}}$ freely (as an abelian group) except the equations in $\Xi_{U_0} \cup \Xi_{U_1} \cup \dots \cup \Xi_{U_{m(*)-1}}$.
- 3A) Moreover $G/(G_{U_0} + \dots + G_{U_{m(*)-1}})$ is \aleph_1 -free provided that
 - ⊗ if $\eta_0, \dots, \eta_{k(*)} \in \cup\{U_m : m < m(*)\}$ are such that

$$(\forall \ell \leq k(*))(\exists m < m(*))(\{\eta_0, \dots, \eta_{k(*)}\} \setminus \{\eta_\ell\} \subseteq U_m)$$
 then for some $m < m(*)$ we have $\{\eta_0, \dots, \eta_{k(*)}\} \subseteq U_m$.

4) If $m(*) \leq k(*)$ and $U_\ell = U \setminus U'_\ell$ for $\ell < m(*)$ and $\langle U'_\ell : \ell < m(*) \rangle$ are pairwise disjoint then ⊗ holds.

5) $G_{U,u} \subseteq G_{U \cup u}$ if $U \subseteq {}^\omega S$ and $u \subseteq {}^\omega S \setminus U$ is finite; moreover $G_{U,u} \subseteq_{pr} G_{U \cup u} \subseteq_{pr} G$.

6) If $\langle U_\alpha : \alpha < \alpha(*) \rangle$ is \subseteq -increasing continuous then also $\langle G_{U_\alpha} : \alpha < \alpha(*) \rangle$ is \subseteq -increasing continuous.

7) If $U_1 \subseteq U_2 \subseteq U \subseteq {}^\omega S$ and $u \subseteq {}^\omega S \setminus U$ is finite, $|u| < k(*)$ and $U_2 \setminus U_1 = \{\eta\}$ and $v = u \cup \{\eta\}$ then $(G_{U,u} + G_{U_2 \cup u}) / (G_{U,u} + G_{U_1 \cup u})$ is isomorphic to $G_{U_1 \cup v} / G_{U_1, v}$.

8) If $U \subseteq {}^\omega S$ and $u \subseteq {}^\omega S \setminus U$ has $\leq k(*)$ members then $(G_{U,u} + G_u) / G_{U,u}$ is isomorphic to $G_u / G_{\emptyset, u}$.

²note that if $u = \{\eta\}$ then $G_{U,u} = G_U$

Discussion 1.11 : For the reader we write what the group $G_{\mathbf{x}}$ is for the case $k(*) = 0$. So, omitting constant indexes and replacing sequences of length one by the unique entry we get that it is generated by $y_{\eta,n}$ (for $\eta \in {}^\omega S, n < \omega$) and x_ν (for $\nu \in {}^\omega > S$) freely as an abelian group except the equations $(n!)y_{\eta,n+1} = y_{\eta,n} + x_{\eta|n}$. Note that if K is the countable subgroup generated by $\{x_\nu : \nu \in {}^\omega > 2\}$ then G/K is a divisible group of cardinality continuum hence G is not free. So G is \aleph_1 -free but not free.

Now we have the abelian group version of freeness, see generally 1.13.

Claim 1.12 *The Freeness Claim* Let $\mathbf{x} \in K_{k(*)}$.

- 1) The abelian group $G_{U \cup u} / G_{U,u}$ is free if $U \subseteq {}^\omega S, u \subseteq {}^\omega S \setminus U$ and $|u| \leq k \leq k(*)$ and $|U| \leq \aleph_{k(*)-k}$.
- 2) If $U \subseteq {}^\omega S$ and $|U| \leq \aleph_{k(*)}$, then G_U is free.

Claim 1.13 1) If \mathbf{x} is a combinatorial $k(*)$ -parameter then \mathbf{x} is $\aleph_{k(*)+1}$ -free.
 2) If \mathbf{x} is an abelian group parameter and (k^*, S^*, Λ^*) is free, then $G_{\mathbf{x}}$ is free.

Proof. 1) Easily follows by (2).

2) Similar and follows from 3.2 + Def 3.3 as easily G belongs to $\mathcal{G}_{k(*)}$.

Claim 1.14 Assume $\mathbf{x} \in K_{k(*)}^{cb}$ is full (i.e. $\Lambda^* = k(*)+1({}^\omega(S^*))$).

- 1) If $U \subseteq {}^\omega S$ and $|U| \geq (|S| + \aleph_0)^{+(k(*)+1)}$, the $(k(*) + 1)$ -th successor of $|S| + \aleph_0$. Then $G_U^{\mathbf{x}}$ is not free.
- 2) If $|S^*| \geq \aleph_{k(*)+1}$ then $G_{\mathbf{x}}$ is not free.
- 3) Assume $\mathbf{x} \in K_{k(*)}^{cb}, |S_\ell^*| + \lambda_\ell < \lambda_{\ell+1}$ for $\ell < k(*)$ and $|\Lambda^*| \geq \lambda_{k(*)}$ and $G \in \mathcal{G}_{\mathbf{x}}$ (see §3) then G is not free.

Proof. 1) Assume toward contradiction that G_U is free and let χ be large enough; for notational simplicity assume $|U| = \aleph_{\alpha, k(*)+1}$, this is O.K. as a subgroup of a free abelian group is a free abelian group where $\aleph_\alpha = |S|$. We choose N_ℓ by downward induction on $\ell \leq k(*)$ such that

- (a) N_ℓ is an elementary submodel³ of $(\mathcal{H}(\chi), \in, <_\chi^*)$
- (b) $\|N_\ell\| = |N_\ell \cap \aleph_{\alpha+k(*)}| = \aleph_{\alpha+\ell}$ and $\{\zeta : \zeta \leq \aleph_{\alpha+\ell}\} \subseteq N_\ell$
- (c) $\langle x_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\leq k(*)}^* \rangle, \langle y_{\bar{\eta},n} : \bar{\eta} \in \Lambda^* \text{ and } n < \omega \rangle, U$ and G_U belong to N_ℓ and $N_{\ell+1}, \dots, N_{k(*)} \in N_\ell$.

Let $G_\ell = G_U \cap N_\ell$, a subgroup of G_U . Now

³ $\mathcal{H}(\chi)$ is $\{x: \text{the transitive closure of } x \text{ has cardinality } < \chi\}$ and $<_\chi^*$ is a well ordering of $\mathcal{H}(\chi)$

- (*)₀ $G_U/(\Sigma\{G_\ell : \ell \leq k(*)\})$ is a free (abelian) group [easy or see [6], that is: as G_U is free we can prove by induction on $k \leq k(*) + 1$ then $G/(\Sigma\{G_{k(*)+1-\ell} : \ell < k\})$ is free, for $k = 0$ this is the assumption toward contradiction, the induction step is by Ax VI in [6] for abelian groups and for $k = k(*) + 1$ we get the desired conclusion.]

Now

- (*)₁ letting U_ℓ^1 be U for $\ell = k(*) + 1$ and $\bigcap_{m=\ell}^{k(*)} (N_m \cap U)$ for $\ell \leq k(*)$; we have: U_ℓ^1 has cardinality $\aleph_{\alpha+\ell}$ for $\ell \leq k(*) + 1$
 [Why? By downward induction on ℓ . For $\ell = k(*) + 1$ this holds by an assumption. For $\ell = k(*)$ this holds by clause (b). For $\ell < k(*)$ this holds by the choice of N_ℓ as the set $\bigcap_{m=\ell+1}^{k(*)} (N_m \cap U)$ has cardinality $\aleph_{\alpha+\ell+1} \geq \aleph_\ell$ and belong to N_ℓ and clause (b) above.]

- (*)₂ $U_\ell^2 =: U_{\ell+1}^1 \setminus (N_\ell \cap U)$ has cardinality $\aleph_{\alpha+1}$ for $\ell \leq k(*)$
 [Why? As $|U_{\ell+1}^1| = \aleph_{\ell+1} > \aleph_\ell = \|N_\ell\| \geq |N_\ell \cap U|$.]

- (*)₃ for $m < \ell \leq k(*)$ the set $U_{m,\ell}^3 =: U_\ell^2 \cap \bigcap_{r=m}^{\ell-1} N_r$ has cardinality $\aleph_{\alpha+m}$
 [Why? By downward induction on m . For $m = \ell - 1$ as $U_\ell^2 \in N_m$ and $|U_\ell^2| = \aleph_{\alpha+\ell+1}$ and clause (b). For $m < \ell$ similarly.]

Now for $\ell = 0$ choose $\eta_\ell^* \in U_\ell^2$, possible by (*)₂ above. Then for $\ell > 0, \ell \leq k(*)$ choose $\eta_\ell^* \in U_{0,\ell}^3$. This is possible by (*)₃. So clearly

- (*)₄ $\eta_\ell^* \in U$ and $\eta_\ell^* \in N_m \cap U \Leftrightarrow \ell \neq m$ for $\ell, m \leq k(*)$.
 [Why? If $\ell = 0$, then by its choice, $\eta_\ell^* \in U_\ell^2$, hence by the definition of U_ℓ^2 in (*)₂ we have $\eta_\ell^* \notin N_\ell$, and $\eta_\ell^* \in U_{\ell+1}^1$ hence $\eta_\ell^* \in N_{\ell+1} \cap \dots \cap N_{k(*)}$ by (*)₁ so (*)₄ holds for $\ell = 0$. If $\ell > 0$ then by its choice, $\eta_\ell^* \in U_{0,\ell}^3$ but $U_{m,\ell}^3 \subseteq U_\ell^2$ by (*)₃ so $\eta_\ell^* \in U_\ell^2$ hence as before $\eta_\ell^* \in N_{\ell+1} \cap \dots \cap N_{k(*)}$ and $\eta_\ell^* \notin N_\ell$. Also by (*)₃ we have $\eta_\ell^* \in \bigcap_{r=0}^{\ell-1} N_r$ so (*)₄ really holds.]

Let $\bar{\eta}^* = \langle \eta_\ell^* : \ell \leq k(*) \rangle$ and let G' be the subgroup of G_U generated by $\{x_{\bar{\eta}^* | \langle m, n \rangle} : m \leq k(*) \text{ and } \bar{\eta} \in {}^{k(*)+1}U \text{ and } n < \omega\} \cup \{y_{\bar{\eta}^*, n} : \bar{\eta} \in {}^{k(*)+1}U \text{ but } \bar{\eta} \neq \bar{\eta}^* \text{ and } n < \omega\}$. Easily $G_\ell \subseteq G'$ recalling $G_\ell = N_\ell \cap G_U$ hence $\Sigma\{G_\ell : \ell \leq k(*)\} \subseteq G'$, but $y_{\bar{\eta}^*, 0} \notin G'$ hence

- (*)₅ $y_{\bar{\eta}^*, 0} \notin \Sigma\{G_\ell : \ell \leq k(*)\}$.

But for every n

$$(*)_6 \quad \bar{n}! y_{\bar{n}^*, n+1} - y_{\bar{n}^*, n} = \Sigma \{x_{\bar{n}^* 1 < m, n \rangle} : m \leq k(*)\} \in \Sigma \{G_\ell : \ell \leq k(*)\}.$$

[Why? $x_{\bar{n}^* 1 < m, n \rangle} \in G_m$ as $\bar{n}^* \uparrow (k(*) + 1 \setminus \{m\}) \in N_m$ by $(*)_4$.]

We can conclude that in $G_U / \Sigma \{G_\ell : \ell \leq k(*)\}$, the element $y_{\bar{n}^*, 0} + \Sigma \{G_\ell : \ell \leq k(*)\}$ is not zero (by $(*)_5$) but is divisible by every natural number by $(*)_6$. This contradicts $(*)_0$ so we are done.
2),3) Left to the reader. ■

2 Black Boxes

Claim 2.1 1) Assume $k(*) < \omega$, $\chi = \chi^{\aleph_0}$ and $\lambda = \beth_{k(*)}(\chi)$, $S = \lambda$, $\Lambda_{k(*)} = {}^{k(*)+1}(\omega S)$ or just $S_\ell = \chi_\ell = \beth_\ell(\chi)$, $\lambda_\ell^{\aleph_0} = \chi_\ell$ for $\ell \leq k(*)$ and $\Lambda_{k(*)} = \prod_{\ell \leq k(*)} \omega(S_\ell)$ and

$\mathbf{x}^{k(*)} = (k(*), \lambda, \Lambda_{k(*)})$ so \mathbf{x} is a full combinatorial $\langle S_\ell : \ell \leq k(*) \rangle$ -parameter. Then Λ has a χ -black box, i.e. $\text{Qr}(\Lambda_{\mathbf{x}^{k(*)}}, \chi)$, see Definition 1.3.

2) Moreover, \mathbf{x} has the $\langle \chi_\ell : \ell \leq k(*) \rangle$ -black box, i.e. for every $\bar{B} = \langle B_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\leq k(*)}^{\aleph_0} \rangle$ satisfying clause (c) below we can find $\langle h_{\bar{\eta}} : \bar{\eta} \in \Lambda \rangle$ such that:

(a) $h_{\bar{\eta}}$ is a function with domain $\{\bar{\eta} \uparrow \langle m, n \rangle : m \leq k(*), 2 \leq n < \omega\}$

(b) $h_{\bar{\eta}}(\bar{\eta} \uparrow \langle m, n \rangle) \in B_{\bar{\eta} \uparrow \langle m, n \rangle}$

(c) $B_{\bar{\eta} \uparrow \langle m, n \rangle}$ is a set of cardinality $\beth_m(\chi)$

(d) if h is a function with domain $\Lambda_{\leq k(*)}^{\aleph_0}$ such that $h(\bar{\eta} \uparrow \langle m, n \rangle) \in B_{\bar{\eta} \uparrow \langle m, n \rangle}$ and $\alpha_\ell < \beth_\ell(\chi)$ for $\ell \leq k(*)$ then for some $\bar{\eta} \in \Lambda^{\aleph_0}$, $h_{\bar{\eta}} \subseteq h$ and $\eta_\ell(0) = \alpha_\ell$ for $\ell \leq k(*)$.

3) Assume $\chi_\ell = \lambda_\ell^{\aleph_0}$, $\chi_{\ell+1} = \chi_{\ell+1}^{\chi_\ell}$ for $\ell \leq k(*)$. If $S_\ell = \lambda_\ell$ for simplicity $\ell \leq k(*)$, \mathbf{x} is a full combinatorial $(\bar{S}, k(*))$ -parameter, and $|B_{\bar{\eta} \uparrow \langle m, n \rangle}| \leq \chi_{k(*)}$ for $\bar{\eta} \in \Lambda^{\aleph_0}$ then we can find $\langle h_{\bar{\eta}} : \bar{\eta} \in \Lambda^{\aleph_0} \rangle$ as in part (2) replacing $\beth_\ell(\chi)$ by λ_ℓ , moreover such that:

(e) if $\bar{\eta} \in \Lambda$ then η_ℓ is increasing

(f) if λ_ℓ is regular then we can in clause (d) above add: if E_ℓ is a club of λ_ℓ for $\ell \leq k(*)$ then we can demand: if $\bar{\eta} \in \Lambda^{\aleph_0}$ then for each ℓ for some $\alpha_\ell^* < \lambda_\ell$ we have $\eta_\ell \in {}^\omega(E_\ell \cup \{\alpha_\ell^*\})$

(g) if λ_ℓ is singular of uncountable cofinality, $\lambda_\ell = \Sigma \{\lambda_{\ell, i} : i < \text{cf}(\lambda_\ell)\}$, $\text{cf}(\lambda_{i, \ell}) = \lambda_{i, \ell}$ increasing with i we can add: if $u_\ell \subseteq \text{cf}(\lambda_\ell)$ is unbounded, $E_{\ell, i}$ a club of $\lambda_{\ell, i}$ then $\eta_\ell \in {}^\omega(E_{i, \ell} \cup \{\alpha_\ell^*\})$ for some $i \in u_\ell$.

Proof. Part (1) follows from part (2) which follows from part (3), so let us prove part (3). To uniformize the notation in 2.1(1), i.e. 1.3 and 2.1(2),(3) we shall denote:

$$\odot_1 h_{\bar{\eta}}(\bar{\eta} \upharpoonright \langle m, n \rangle) = \alpha_{\bar{\eta}, m, n}^{k(*)}$$

Note that without loss of generality $B_{\bar{\nu}} = |B_{\bar{\nu}}|$ and we use $\alpha_{k(*)}, m, n = h_{\bar{\eta}}(\bar{\eta} \upharpoonright \langle m, n \rangle)$ for $\bar{\eta} \in \Lambda_{\mathbf{x}}, m \leq k(*)$ and $n < \omega$. We prove part (3) by induction on $k(*)$. Let $\Lambda_k = \Lambda^{\mathbf{x}}$ and without loss of generality $S_\ell = \lambda_\ell$.

Case 1: $k(*) = 0$.

By the simple black box, see [9, III, §4], or better [4, VI, §2], see below for details on such a proof.

Case 2: $k(*) = k + 1$.

Let

$$\odot_2 \alpha^k = \langle \alpha_{\bar{\eta}, m, n}^k : \bar{\eta} \in \Lambda_k, n < \omega, m \leq k \rangle \text{ witness parts (2), (3) for } k, \text{ i.e. for } \mathbf{x}^k, \text{ hence no need to assume } \mathbf{x}^k \text{ is full.}$$

So $\lambda = \lambda_{k(*)}, \chi = \chi_{k(*)}$ and let $\mathbf{H} = \{h : h \text{ is a function from } \Lambda_k \text{ to } \chi\}$. So $|\mathbf{H}| \leq (\lambda)^{\lambda^{\aleph_0}} = \chi$. By the simple black box, see below, we can find $\langle \bar{h}_\eta : \eta \in {}^\omega \lambda \rangle$ such that

$$\odot_3 (\alpha) \quad \bar{h}_\eta = \langle h_{\eta, n} : n < \omega \rangle \text{ and } h_{\eta, n} \in \mathbf{H} \text{ for } \eta \in {}^\omega \lambda$$

$$(\beta) \quad \text{if } \bar{f} = \langle f_\nu : \nu \in {}^\omega \lambda \rangle \text{ and } f_\nu \in \mathbf{H} \text{ for every such } \nu \text{ and } \alpha < \lambda \text{ and } \rho \in {}^\omega \lambda \text{ is increasing then for some increasing } \eta \in {}^\omega \lambda \text{ we have } \rho \triangleleft \eta \text{ and } n < \omega \Rightarrow h_{\eta, n} = f_{\eta \upharpoonright n}$$

$$(\gamma) \quad \text{if } \text{cf}(\lambda) > \aleph_0 \text{ and } E \text{ is a club of } \lambda \text{ then we can add } \cup \{\eta(n) : n < \omega\} \in E.$$

[Why? First assume $\chi = \lambda$. Let $\langle \bar{g}_\alpha = \langle g_{\alpha, \ell} : \ell < n_\alpha \rangle : \alpha < \lambda \rangle$ enumerate ${}^\omega \mathbf{H}$ such that for each $\bar{g} \in {}^\omega \mathbf{H}$ the set $\{\alpha < \lambda : \bar{g}_\alpha = \bar{g}\}$ is unbounded in λ . Now for $\eta \in {}^\omega \lambda$ and $n < \omega$ let $h_{\eta, n} = g_{\eta(k), n}$ for every k large enough if well defined and $g_{\eta \upharpoonright (n+1), n}$ otherwise. So clause (α) of \odot_3 holds and as for clause (β) of \odot_3 , let $\bar{f} = \langle f_\nu : \nu \in {}^\omega \lambda \rangle$ be given, $f_\nu \in \mathbf{H}$.

Assume $\rho \in {}^\omega \lambda$ is increasing. We choose α_n by induction on $n < \omega$ such that:

$$\odot_4 (\alpha) \quad \alpha_n = \rho(n) \text{ if } n < \ell g(\rho)$$

$$(\beta) \quad \alpha_n < \lambda \text{ and } \alpha_n > \alpha_m \text{ if } n = m + 1$$

$$(\gamma) \quad \text{if } n \geq \ell g(\rho) \text{ then } \alpha_n \text{ satisfies } \bar{g}_{\alpha_n} = \langle f_{\langle \alpha_\ell : \ell < m \rangle} : m \leq n \rangle.$$

Now $\eta =: \langle \alpha_n : n < \omega \rangle$ is as required in (β) of \odot_3 ; to get also (γ) of \odot_3 we should add in clause (β) of \odot_4 then $\alpha_n > \min(E \setminus \alpha_m)$.

Second, if $\chi > \lambda$ but still $\chi \leq \lambda^{\aleph_0}$, let $\langle \bar{g}_\alpha : \alpha < \chi^{\aleph_0} \rangle$ list ${}^\omega \mathbf{H}$, and let $h_n : \chi \rightarrow \lambda$ for $n < \omega$ be such ⁴ that $\alpha < \beta < \chi \Rightarrow (\forall^\infty n)(h_n(\alpha) \neq h_n(\beta))$ and let $\text{cd}: \lambda \rightarrow {}^\omega \lambda$ be one to one onto. Now for $\eta \in {}^\omega \lambda$ and $n < \omega$ let $h_{\eta, n}$ be g_α where α is the unique ordinal $\alpha < \chi$ such that for every $k < \omega$ large enough $(\text{cd}(\eta(k)))(n) = h_n(\alpha)$ so in

⁴recall $(\forall^\infty N)$ means "for every large enough $n < \omega$ "

particular ($\ell g(\text{cd}(\eta(k)) : k < \omega)$ is going to infinity or $h_{\eta,n}$ is not well defined; in fact, we can use only the case $\ell g(\text{cd}(\eta(k)) = k$; stipulating $h_{\eta,n} \in {}^\omega\{0\}$ when not defined. So we have defined $\langle h_{\eta,n} : \eta \in {}^\omega\lambda, n < \omega \rangle$. Now we immitate the previous argument: clause (β) of \oplus_2 holds.

Next we shall define $\bar{\alpha}^{k(*)} = \langle \alpha_{\bar{\eta},m,n}^{k(*)} : \bar{\eta} \in \Lambda_{k+1}, m \leq k(*), n < \omega \rangle$ as required; so let $\bar{\eta} = \langle \eta_\ell : \ell \leq k(*) \rangle \in \Lambda_{k(*)}$ we define $\bar{\alpha}_{\bar{\eta}}^{k(*)} = \langle \alpha_{\bar{\eta},m,n}^{k(*)} : m \leq k(*), n < \omega \rangle$ as follows:

\odot_5 if $\eta_{k(*)} \in {}^\omega\lambda$ and $\langle \eta_0, \dots, \eta_{k(*)-1} \rangle \in \Lambda_k$ then for $m \leq k(*)$ and $n < \omega$

(α) if $m = k(*)$ then $\alpha_{\bar{\eta},m,n}^{k(*)} = h_{\eta_{k(*)},n}(\langle \eta_0, \dots, \eta_{k(*)-1} \rangle) < \lambda_m$

(β) if $m < k(*)$, i.e. $m \leq k$ then $\alpha_{\bar{\eta},m,n}^{k(*)} = \alpha_{\bar{\eta}|k(*),m,n}^{k(*)} < \lambda_m$.

Clearly $\alpha_{\bar{\eta},m,n}^{k(*)} < \lambda_m$ in all cases, as required, (in clause (a),(b),(c) of 2.1(2) and (e) of 2.1(3). But we still have to prove that $\langle \bar{\alpha}_{\bar{\eta}}^{k(*)} : \bar{\eta} \in \Lambda^{k+1}, m \leq k(*), n < \omega \rangle$ witness $\text{Qr}(\mathbf{x}^{k(*)}, \chi)$, see Definition 1.3(2) this suffices for 2.1(2), little more is needed for 2.1(3); just using (γ) of \odot_3 and the induction hypothesis.

Why does this hold? Let h be a function with domain $\Lambda_{\leq k(*)}^{\mathbf{x}^{k(*)}}$ as in part (3) and $\alpha_\ell^* < \lambda_\ell$ for $\ell \leq k(*)$.

For $\nu \in {}^\omega\lambda$ let $f_\nu : \Lambda_k \rightarrow \lambda = \lambda_{k(*)}$ be defined by: $f_\nu(\langle \eta_\ell : \ell \leq k \rangle) =: h(\langle \eta_\ell : \ell \leq k \rangle \hat{\ } \nu)$. So by \odot_3 above for some increasing $\eta_{k(*)}^* \in {}^\omega\lambda$ we have $\eta_{k(*)}^*(0) = \alpha_{k(*)}^*$ and

\odot_6 $n < \omega \Rightarrow f_{\eta_{k(*)}^* \upharpoonright n} = h_{\eta_{k(*)}^*,n}$.

Now substituting the definition of \bar{f} we have

\odot_7 $\langle \eta_0, \dots, \eta_k \rangle \in \Lambda_k \wedge n < \omega \Rightarrow h_{\eta_{k(*)}^*,n}(\eta_0, \dots, \eta_k) = h(\langle \eta_0, \dots, \eta_k, \eta_{k(*)}^* \upharpoonright n \rangle)$.

Substituting the definition of $\bar{\alpha}^k$ we have

\odot_8 if $\langle \eta_0, \dots, \eta_k \rangle \in \Lambda_k$ and $n < \omega$ then $\alpha_{\langle \eta_0, \dots, \eta_k, \eta_{k(*)}^* \upharpoonright n \rangle}^{k(*)} = h(\langle \eta_0, \dots, \eta_k, \eta_{k(*)}^* \upharpoonright n \rangle)$.

Now we define a function h' with domain $\Lambda_{\leq k}^{\mathbf{x}^k}$ by: if $\bar{\eta} \in \Lambda_{\leq k}^{\mathbf{x}^k}$ then $h'(\bar{\eta}) = h(\bar{\eta} \hat{\ } \langle \eta_{k(*)}^* \rangle)$.

So by the choice of $\bar{\alpha}^k$ in \odot_2 we can find $\langle \eta_0^*, \dots, \eta_k^* \rangle \in \Lambda_k$ with no repetitions such that $\eta_\ell^*(0) = \alpha_\ell^*$ for $\ell \leq k$ and in \odot_2

\odot_9 $m \leq k \wedge n < \omega \Rightarrow \alpha_{\langle \eta_0^*, \dots, \eta_k^* \rangle, m, n}^k = h'(\langle \eta_0^*, \dots, \eta_k^* \rangle \upharpoonright (m, n))$.

Let $\bar{\eta}^* = \langle \eta_0^*, \dots, \eta_k^*, \eta_{k+1}^* \rangle$, $\bar{\eta}' = \langle \eta_0^*, \dots, \eta_i^* \rangle$.

Note that

\odot_{10} if $m \leq k, n < \omega$ then $h'(\bar{\eta}' \upharpoonright \langle k, m \rangle) = h((\bar{\eta}' \upharpoonright \langle k, m \rangle) \hat{\ } \langle \eta_{k(*)}^* \rangle) = h(\bar{\eta}^* \upharpoonright \langle k, m \rangle)$.

Now by $\odot_9 + \odot_{10}$ and $\odot_5(\beta)$ this means

\odot_{11} if $m \leq k$ and $n < \omega$ then $\alpha_{\bar{\eta}^*, m, n}^{k(*)} = h(\bar{\eta}^* \upharpoonright \langle k, m \rangle)$.

So by putting together $\odot_8 + \odot_{11}$ we are clearly done, i.e. we can check that $\langle \eta_0^*, \dots, \eta_k^*, \eta_{k(*)}^* \rangle$ is as required. ■

Conclusion 2.2 For every $k < \omega$ there is an \aleph_{k+1} -free abelian group G of cardinality \beth_{k+1} and pure (non-zero) subgroup $\mathbb{Z}z \subseteq G$ such that $\mathbb{Z}z$ is not a direct summand of G .

Proof. Let $\chi = 2^{\aleph_0}$ and \mathbf{x} be a combinatorial k -parameter as guaranteed by 2.1. Now by 2.3(2) below we can expand \mathbf{x} to an abelian group k -parameter, so $G_{\mathbf{x}}$ is as required.

Claim 2.3 1) If \mathbf{x} is a combinatorial k -parameter such that $\text{Qr}(\mathbf{x}, 2^{\aleph_0})$ then for some $\mathbf{a}, (\mathbf{x}, \mathbf{a})$ is an abelian group k -parameter such that $h \in \text{Hom}(G_{\mathbf{x}}, \mathbb{Z}) \Rightarrow h(z) = 0$.
2) For every k there is an \aleph_{k+1} -free abelian group G of cardinality \beth_{k+1} and $z \in G$ a pure $z \in G$ as above.

Proof. 1) Let $\bar{\alpha}$ witness $\text{Qr}(\mathbf{x}, 2^{\aleph_0})$. We define $\text{Ord} \rightarrow \mathbb{Z}$ by $:\iota(\alpha)$ is α if $\alpha < \omega$, is $-\eta$ if $\alpha = \omega + n < \omega + \omega$ and zero otherwise. For each $\bar{\eta} \in \Lambda^{\mathbf{x}}$ we shall choose a sequence $\langle \mathbf{a}_{\bar{\eta}, n} : n < \omega \rangle$ of integers such that for any $b \in \mathbb{Z} \setminus \{0\}$ for no $\bar{c} \in {}^\omega \mathbb{Z}$ do we have

$\boxtimes_{\bar{\eta}}$ for each $n < \omega$ we have

$$n!c_{n+1} = c_n + \mathbf{a}_{\bar{\eta}, n}b + \Sigma\{\iota(\alpha_{\bar{\eta}, m, n}) : m \leq k(*)\}.$$

This is easy: for each pair $(b, c_0) \in \mathbb{Z} \times \mathbb{Z}$ the set of sequences $\langle \mathbf{a}_{\bar{\eta}, n} : n < \omega \rangle \in {}^\omega \mathbb{Z}$ there is a sequence $\langle c_0, c_1, c_2, \dots \rangle$ of integers such that $\boxtimes_{\bar{\eta}}$ holds for them, so the choice of $\langle \mathbf{a}_{\bar{\eta}, n} : n < \omega \rangle$ is possible.

Now toward contradiction assume that h is a homomorphism from $G_{\mathbf{x}}$ to $z\mathbb{Z}$ such that $h(z) = bz, b \in \mathbb{Z} \setminus \{0\}$. We define $h' : \Lambda_{\leq k}^{\mathbf{x}} \rightarrow \chi$ by $h'(\bar{\eta}) = n$ if $n < \omega$ and $h(x_{\bar{\eta}}) = nz$ and $h'(\bar{\eta}) = \omega + n$ if $n < \omega$ and $h(x_{\bar{\eta}}) = (-n)z$.

By the choice of $\bar{\alpha}$, for some $\bar{\eta} \in \Lambda^{\mathbf{x}}$ we have: $m \leq k \wedge n < \omega \Rightarrow h'(\bar{\eta} \upharpoonright \langle m, n \rangle) = \alpha_{\bar{\eta}, m, n}$. Hence $h(x_{\bar{\eta} \upharpoonright \langle m, n \rangle}) = \iota(\alpha_{\bar{\eta}, m, n})z$ for $m \leq k, n < \omega$.

Let $c_n \in \mathbb{Z}$ be such that $h(y_{\bar{\eta}, n}) = c_n z$. Now the equation $\boxtimes_{\bar{\eta}, n}$ in Definition 1.6 is mapped to the n -th equation in $\boxtimes_{\bar{\eta}}$, so an obvious contradiction.

2) By part (1) and 2.2. ■

Remark 2.4 1) We can replace χ by a set of cardinality χ in Definition 1.3. Using $\mathbb{Z}z$ instead of χ simplify the notation in the proof of 2.3.

2) We have not tried to save in the cardinality of G in 2.3(2), using as basic of the induction the abelian group of cardinality \aleph_0 or \aleph_1 .

Claim 2.5 1) If $\chi_0 = \chi_0^{\aleph_0}, \chi_{m+1} = 2^{\chi_m}$ and $\lambda_m = \chi_m$ for $m \leq k$ there is a $\bar{\chi}$ -full \mathbf{x} such that $(\mathbf{x}, \bar{\chi})$ -black box exist.

Conclusion 2.6 Assume $\mu_0 < \dots < \mu_{k(*)}$ are strong limit of cofinality \aleph_0 (or $\mu_0 = \aleph_0$), $\lambda_\ell = \mu_\ell^+$, $\chi_\ell = 2^{\mu_\ell}$.

Then in 2.1 for $\bar{\eta} \in \Lambda^{\mathbf{x}}$ we can let $h_{\bar{\eta}, m}$ has domain $\{\bar{\nu} \in \Lambda_m^{\mathbf{x}} : [\nu_\ell = \eta_\ell \text{ for } \ell = m+1, \dots, k(*)]\}$.

3 Constructing abelian groups from combinatorial parameters

Definition 3.1 1) We say F is a μ -regressive function on a combinatorial parameter $\mathbf{x} \in K_{k(*)}^{cb}$ when: $S^{\mathbf{x}}$ is a set of ordinals and:

- (a) $\text{Dom}(F)$ is $\Lambda^{\mathbf{x}}$
- (b) $\text{Rang}(F) \subseteq [\Lambda^{\mathbf{x}} \cup \Lambda_{\leq k(*)}^{\mathbf{x}}]^{\leq \aleph_0}$
- (c) for every $\bar{\eta} \in \Lambda^{\mathbf{x}}$ and $m \leq k(*)$ we ⁵ have $\sup \text{Rang}(\eta_m) > \sup(\cup\{\text{Rang}(\nu_n) : \bar{\nu} \in F(\bar{\eta})\})$; note $\bar{\nu}_\ell \in \Lambda^{\mathbf{x}}$ or $\bar{\nu} \in \Lambda_{\leq k(*)}^{\mathbf{x}}$ as $F(\bar{\eta})$ is a set of such objects.

1A) We say F is finitary when $F(\bar{\eta})$ is finite for every $\bar{\eta}$.

1B) We say F is simple if $\eta_{k(*)}(0)$ determined $F(\bar{\eta})$ for $\bar{\eta} \in \Lambda^{\mathbf{x}}$.

2) For \mathbf{x}, F as above and $\Lambda \subseteq \Lambda^{\mathbf{x}}$ we say that Λ is free for (\mathbf{x}, F) when: $\Lambda \subseteq \Lambda^{\mathbf{x}}$ and there is a sequence $\langle \bar{\eta}^\alpha : \alpha < \alpha(*) \rangle$ listing $\Lambda' = \Lambda \cup \cup\{F(\bar{\eta}) : \bar{\eta} \in \Lambda\}$ and sequence $\langle \ell_\alpha : \alpha < \alpha(*) \rangle$ such that

- (a) $\ell_\alpha \leq k(*)$
- (b) if $\alpha < \alpha(*)$ and $\bar{\eta}^\alpha \in \Lambda$ then $F(\bar{\eta}^\alpha) \subseteq \{\bar{\eta}^\beta, \bar{\eta}^\beta \upharpoonright \langle m, n \rangle : \beta < \alpha, n < \omega, m \leq k(*)\}$
- (c) if $\alpha < \alpha(*)$ and $\bar{\eta}^\alpha \in \Lambda$ then for some $n < \omega$ we have $\bar{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\bar{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle : \beta < \alpha, \eta^\beta \in \Lambda\} \cup \{\bar{\eta}^\beta : \beta < \alpha\}$.

3) We say \mathbf{x} is θ -free for F is (\mathbf{x}, F) is μ -free when \mathbf{x}, F are as in part (1) and every $\Lambda \subseteq \Lambda^{\mathbf{x}}$ of cardinality $< \theta$ is free for (\mathbf{x}, F) .

Claim 3.2 1) If $\mathbf{x} \in K_{k(*)}^{cb}$ and F is a regressive function on \mathbf{x} then (\mathbf{x}, F) is $\aleph_{k(*)+1}$ -free provided that F is finitary or simple.

2) In addition: if $k \leq k(*)$, $\Lambda \subseteq \Lambda^{\mathbf{x}}$ has cardinality $\leq \aleph_k$ and $\bar{u} = \langle u_{\bar{\eta}} : \bar{\eta} \in \Lambda \rangle$ satisfies $u_{\bar{\eta}} \subseteq \{0, \dots, k(*)\}$, $|u_{\bar{\eta}}| > k$, then we can find $\langle \bar{\eta}^\alpha : \alpha < \aleph_k \rangle, \langle \ell_\alpha : \alpha < \aleph_k \rangle, \langle n_\alpha : \alpha < \aleph_k \rangle$ such that:

- (a) $\Lambda \subseteq \{\bar{\eta}^\alpha : \alpha < \aleph_k\}$
- (b) if $\bar{\eta}^\alpha \in \Lambda^{\mathbf{x}}$ then $\ell_\alpha \in u_{\bar{\eta}^\alpha}, n_\alpha < \omega$

⁵actually, suffice to have it for $\ell = k(*)$

(c) $\bar{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n_\alpha \rangle \notin \{\bar{\eta}^\beta \upharpoonright \langle \ell_\alpha, n_\alpha \rangle : \beta < \alpha\} \cup \{\bar{\eta}^\beta : \beta < \alpha\}$.

Proof. 1) Follows by part (2) for the case $k = k^*$, $u_{\bar{\eta}} = \{0, \dots, k^*\}$ for every $\bar{\eta} \in \Lambda$.

2) So we are assuming $\mathbf{x} \in K_{k^*}^{\text{cb}}$, F is a regressive function on \mathbf{x} , $k \leq k^*$, $\Lambda \subseteq \Lambda^{\mathbf{x}}$ has cardinality $\leq \aleph_k$ and without loss of generality Λ is closed under $\bar{\eta} \mapsto F(\bar{\eta}) \cap \Lambda^{\mathbf{x}}$. We prove this by induction on k .

Case 1: $k = 0$.

Subcase 1A: Ignoring F .

Let $\langle \bar{\eta}^\alpha : \alpha < |\Lambda| \rangle$ list Λ with no repetitions (so $\alpha < |\Lambda| \Rightarrow \alpha < \aleph_k = \aleph_0$). Now $\alpha < |\Lambda| \Rightarrow u_{\bar{\eta}^\alpha} \neq \emptyset$ and let $\ell_\alpha = \min(u_{\bar{\eta}^\alpha}) \leq k^*$. Hence for each $\alpha < |\Lambda|$ we know that $\beta < \alpha \Rightarrow \bar{\eta}^\beta \neq \bar{\eta}^\alpha$, hence for some $n = n_{\alpha, \beta} < \omega$ we have $\bar{\eta}^\beta \upharpoonright \langle \ell_\alpha, n_{\alpha, \beta} \rangle \neq \bar{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n_{\alpha, \beta} \rangle$.

Let $n_\alpha = \sup\{n_{\alpha, \beta} : \beta < \alpha\}$, it is $< \omega$ as $\alpha < \omega$. Now $\langle \langle \ell_\alpha, n_\alpha \rangle : \alpha < |\Lambda| \rangle$ is as required.

Subcase 1B: $\bar{\eta} \in \Lambda \Rightarrow F(\bar{\eta})$ is finite.

Let $\langle \bar{\eta}^\alpha : \alpha < |\Lambda| \rangle$ list Λ , we choose w_j by induction on $j \leq j^*$, $j^* \leq \omega$ such that:

(a) $w_j \subseteq |\Lambda|$ is finite

(b) $j \in w_{j+1}$

(c) if $\alpha \in w_j$ then $F(\bar{\eta}^\alpha) \cap \Lambda \subseteq \{\bar{\eta}^\alpha : \beta \in w_j\}$

(d) $w_{j^*} = |\Lambda|$ and $w_0 = \emptyset$

(e) $w_j \subseteq w_{j+1}$ and $j(x) = w \Rightarrow w_{j(x)} = \cup\{w_j : j < j(x)\}$.

No problem to do this (for clause (c) use “ F is regressive, the ordinals well ordered”).

Now let $\langle \beta(j, i) : i < i_j^* \rangle$ list $w_{j+1} \setminus w_j$ such that: if $i_1, i_2 < i_j^*$ and $\bar{\eta}^{\beta(j, i_1)} \in F(\bar{\eta}^{\beta(j, i_2)})$ then $i_1 < i_2$; we prove existence by F being regressive. Let $\langle \bar{\nu}_{j, i} : i < i_j^{**} \rangle$ list $\cup\{F(\bar{\eta}^\alpha) : \alpha \in w_{j+1} \setminus w_j\} \setminus \Lambda^{\mathbf{x}} \setminus \{F(\bar{\eta}^\alpha) : \alpha \in w_j\}$.

Let $\alpha_j^* = \Sigma\{i_j^{**}(1) + i_j^*(1) : j(1) < j\}$. Now we choose $\bar{\rho}_\varepsilon$ for $\varepsilon < \alpha_j^*$ for $j < j^*$ as follows:

(a) $\rho_{\alpha_j^* + i} = \nu_{j, i}$ if $i < i_j^{**}$

(b) $\bar{\rho}_{\alpha_j^* + i_j^{**} + i} = \bar{\eta}^{\beta(j, i)}$ if $i < i_j^*$.

Lastly, we choose $n_{\alpha_j + i} < \omega$ for $i < i_j^*$ as in case 1A.

Now check.

Subcase 1C: F is simple.

Note that $F(\bar{\eta})$ when defined is determined by $\eta_{k^*}(0)$ and is included in $\{\bar{\nu} \in \Lambda_{\leq k^*}^{\mathbf{x}} \cup \Lambda^{\mathbf{x}} : \sup \text{Rang}(\nu_{k^*}) < \eta_{k^*}(0)\}$. So let $u = \{\eta_{k^*}(0) : \bar{\eta} \in \Lambda\}$ and $u^* = u \cup \{\sup(u) + 1\}$ and for $\alpha \in u$ let $\Lambda_\alpha = \{\bar{\eta} \in \Lambda : \eta_{k^*}(0) = \alpha\}$ and for $\alpha \in U^+$

let $\Lambda_{<\alpha} = \cup\{\Lambda_\alpha : \alpha \in u\}$. Now by induction on $\beta \in u^*$ we choose $\langle(\bar{\eta}^\varepsilon, \ell_\varepsilon) : \varepsilon < \varepsilon_\beta\rangle$ such that it is a required for $\Lambda_{<\alpha}$. For $\beta = \min(u)$ this is trivial and if $\text{otp}(u \cap \beta)$ is a limit ordinal this is obvious. So assume $\alpha = \max(u \cap \beta)$, we use Subcase 1A on Λ_α , and combine them naturally promising $\ell_\alpha = k(*) \Rightarrow n_\alpha > 1$.

Case 2: $k = k_* + 1$ and $|\Lambda| = \aleph_k$.

Let $\langle\Lambda_\varepsilon : \varepsilon < \aleph_k\rangle$ be \subseteq -increasing continuous with union Λ , $|\Lambda_{1+\varepsilon}| = \aleph_{k_*}$, $\Lambda_0 = \emptyset$, each Λ_ε closed enough, mainly:

⊙₁ if $\bar{\eta}^i \in \Lambda_\varepsilon$ for $i < i(*) < \omega$, $\bar{\rho} \in \Lambda$ and $\{\rho_\ell : \ell \leq k(*)\} \subseteq \{\eta_\ell^i : \ell \leq k(*), i < i(*)\}$ then $\bar{\rho} \in \Lambda_\varepsilon$

⊙₂ Λ_ε is closed under $\bar{\eta} \mapsto F(\bar{\eta}) \cap \Lambda^*$.

Next

⊙ if $\varepsilon < \aleph_k$, $\bar{\eta} \in \Lambda_{\varepsilon+1} \setminus \Lambda_\varepsilon$ then $u'_\bar{\eta} = \{\ell \in u_\bar{\eta} : \text{for every or just some } n < \omega \text{ for some } \bar{\nu} \in \Lambda_\varepsilon \text{ we have } \bar{\eta} \upharpoonright \langle \ell, n \rangle = \bar{\nu} \upharpoonright \langle \ell, n \rangle\}$ has at most one member.

[Why? So assume toward contradiction that $\bar{\eta} \in \Lambda_{\varepsilon+1}$ and $\ell(1) \neq \ell(2)$ belong to $u'_\bar{\eta}$. Hence by the definition of $u'_\bar{\eta}$ there are $\bar{\nu}^1, \bar{\nu}^2 \in \Lambda_\varepsilon$ and $\eta_1, \eta_2 < \omega$ such that $\bar{\eta} \upharpoonright \langle \ell_1, \eta_1 \rangle \in \bar{\nu}^1 \upharpoonright \langle \ell_1, \eta_1 \rangle$ and $\bar{\eta} \upharpoonright \langle \ell_2, \eta_2 \rangle = \bar{\nu}^2 \upharpoonright \langle \ell_2, \eta_2 \rangle$. Now $m \leq k(*) \Rightarrow$ for some $i \in \{1, 2\}$, $m \leq \ell_i \Rightarrow \eta_m$ is $(\bar{\eta} \upharpoonright \langle \ell_i, n_i \rangle)_m \Rightarrow \eta_m \in \{\rho_\ell : \bar{\rho} \in \Lambda_\varepsilon\}$. Hence $\{\eta_\ell : \ell \leq k(*)\} \subseteq \{\rho_\ell : \ell \leq k(*)\}$ and $\bar{\rho} \in \Lambda_\varepsilon$. So by ⊙₁ we have $\bar{\eta} \in \Lambda_\varepsilon$, so we are done.]

Apply the induction hypothesis to $\Lambda_{\varepsilon+1} \setminus \Lambda_\varepsilon$ for each ε and get $\langle(\bar{\eta}^{\varepsilon, \alpha}, \ell_{\varepsilon, \alpha, n_{\varepsilon, \alpha}}) : \alpha < \alpha(\varepsilon)\rangle$ such that $\bar{\eta}^{\varepsilon, \alpha} \upharpoonright \langle \ell_{\varepsilon, \alpha, n_{\varepsilon, \alpha}} \rangle \notin \{\bar{\eta}^{\varepsilon, \beta} \upharpoonright \langle \ell_{\varepsilon, \beta, n_{\varepsilon, \beta}} \rangle : \beta < \alpha\}$.

Let $\alpha_* = \Sigma\{\alpha(\varepsilon) : \varepsilon < |\Lambda|\}$ and $\alpha = \Sigma\{\alpha(\zeta) : \zeta < \varepsilon\} + \beta$, $\alpha < \alpha(\varepsilon)$ let $\eta^\alpha = \eta^{\varepsilon, \beta}$, $\ell_\alpha = \ell_{\varepsilon, \beta}$, $\eta_\alpha = \eta_{\varepsilon, \beta}$. I.e. we combine but for $\Lambda_{\varepsilon+1} \setminus \Lambda_\varepsilon$ we use $\langle u_\bar{\eta} \setminus u'_\bar{\eta} : \bar{\eta} \in \Lambda_{\varepsilon+1} \setminus \Lambda_\varepsilon \rangle$, so $|u_\bar{\eta} \setminus u'_\bar{\eta}| \geq k - 1 = k_*$. ■

Definition 3.3 For a combinatorial parameter \mathbf{x} we define $\mathcal{G}_\mathbf{x}$, the class of abelian groups derived from \mathbf{x} as follows: $G \in \mathcal{G}_\mathbf{x}$ if there is a simple (or finitary) regressive F on Λ^* and G is generated by $\{y_{\bar{\eta}, n} : \bar{\eta} \in \Lambda^*, n < \omega\} \cup \{x_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\leq k(*)}^*\}$ freely except

$$\boxtimes_{\bar{\eta}, n} (n!)y_{\bar{\eta}, n+1} = y_{\bar{\eta}, n} + b_{\bar{\eta}, n}^{\mathbf{x}} z_{\bar{\eta}, n} + \sum \{x_{\bar{\eta} \upharpoonright \langle m, n \rangle} : m \leq k(*)\}$$

where

⊙ (a) $b_{\bar{\eta}, n} \in \mathbb{Z}$

(b) $z_{\bar{\eta}, n}$ is a linear combination of $\{x_{\bar{\nu}} : \bar{\nu} \in F(\bar{\eta}) \setminus \Lambda^*\} \cup \{y_{\bar{\eta}, n} : \bar{\eta} \in F(\bar{\eta}) \cap \Lambda^*\}$ and $(\forall m \leq k(*))(\bar{\eta} \upharpoonright \langle m, n \rangle) \in F(\bar{\eta})$.

Claim 3.4 If $\mathbf{x} \in K_{k(*)}^{cb}$ and $G \in \mathcal{G}_\mathbf{x}$ (i.e. G is an abelian group derived from \mathbf{x}), then G is $\aleph_{k(*)+1}$ -free.

Proof. We use claim 3.2. So let H be a subgroup of G of cardinality $\leq \aleph_{k(*)}$. We can find Λ such that

- (*) (a) $\Lambda \subseteq \Lambda^x$ has cardinality $\leq \aleph_{k(*)}$
- (b) every equation which $X_\Lambda = \{x_{\bar{\eta}1 < m, n >}, y_{\bar{\eta}, n} : m \leq k(*), n < \omega, \bar{\eta} \in \Lambda\}$ satisfies in G , is implied by the equations in $\Gamma_\Lambda = \cup\{\boxtimes_{\bar{\eta}, n} : \bar{\eta} \in \Lambda\}$
- (c) $H \subseteq G_\Lambda = \langle x_{\bar{\eta}1 < m, n >}, y_{\bar{\eta}, n} : \bar{\eta} \in \Lambda, m \leq k(*), n < \omega \rangle_G$.

So it suffices to prove that G_Λ is a free (abelian) group.

Let the sequence $\langle (\bar{\eta}^\alpha, \ell_\alpha) : \alpha < \alpha(*) \rangle$ be as proved to exist in 3.2. Let $\mathcal{U} = \{\alpha < \alpha(*) : \bar{\eta}^\alpha \in \Lambda\} \cup \{\alpha(*)\}$ and for $\alpha \in \mathcal{U}$ let $X_\alpha^0 = \{x_{\bar{\eta}^\alpha 1 < m, n >} : \beta \in \alpha \cap \mathcal{U}, m \leq k(*) \text{ and } n < \omega\}$ and $X_\alpha^1 = X_\alpha^0 \cup \{\bar{\eta}^\beta : \beta \in \alpha \setminus \mathcal{U}\}$. So for each $\alpha \in \mathcal{U}$ there is $\bar{n}_\alpha = \langle n_{\alpha, \ell} : \ell \in v_\alpha \rangle$ such that: $\ell_\alpha \in v_\alpha \subseteq \{0, \dots, k(*)\}$, $n_{\alpha, \ell} < \omega$ and $X_{\alpha+1}^1 \setminus X_\alpha^1 = \{x_{\bar{\eta}1 < \ell, n >} : \ell \in v_\alpha \text{ and } n \in [n_{\alpha, \ell}, \omega)\}$.

For $\alpha \leq \alpha(*)$ let $G_{\Lambda, \alpha} = \langle \{y_{\bar{\eta}^\beta, n}, x_{\bar{\nu}} : \beta \in \mathcal{U} \cap \alpha \text{ and } \bar{\nu} \in X_\beta^1\} \rangle_{G_\Lambda}$. Clearly $\langle G_{\Lambda, \alpha} : \alpha \leq \alpha(*) \rangle$ is purely increasing continuous with union G_Λ , and $G_{\Lambda, 0} = \{0\}$. So it suffices to prove that $G_{\Lambda, \alpha+1}/G_{\Lambda, \alpha}$ is free. If $\alpha \notin \mathcal{U}$ the quotient is trivial by a free group, and if $\alpha \in \mathcal{U}$ we can use $\ell_\alpha \in v_\alpha$ to prove that is free giving a basis. ■

Conclusion 3.5 For every $k(*) < \omega$ there is an $\aleph_{k(*)+1}$ -free abelian group G of cardinality $\lambda = \beth_{k(*)+1}$ such that $\text{Hom}(G, \mathbb{Z}) = \{0\}$.

Proof. We use \mathbf{x} and $\langle h_{\bar{\eta}} : \bar{\eta} \in \Lambda^x \rangle$ from 2.1(3), and we shall choose $G \in \mathcal{G}_x$. So G is $\aleph_{k(*)+1}$ -free by 3.4.

Let $\mathcal{S} = \{ \langle (a_i, \bar{\eta}_i) : i < i_1 \rangle \wedge \langle (b_j, \bar{\nu}_j, n_j) : j < j_1 \rangle : i_1 < \omega, a_i \in \mathbb{Z}, \bar{\eta}_i \in \Lambda_{\leq k(*)}^x \text{ and } j_1 < \omega, b_j \in \mathbb{Z}, \bar{\nu}_j \in \Lambda^x, n_j < \omega \}$ (actually $\mathcal{S} = \Lambda_{\leq k(*)}^x$ suffice noting $\bar{\nu}_j = \langle \nu_{j, \ell} : \ell \leq k(*) \rangle$).

So $|\mathcal{S}| = \aleph_{k(*)}$ and let \bar{p} be such that:

- (a) $\bar{p} = \langle p^\alpha : \alpha < \lambda \rangle$
- (b) \bar{p} lists \mathcal{S}
- (c) $p^\alpha = \langle (a_i^\alpha, \bar{\eta}_i^\alpha) : i < i_\alpha \rangle \wedge \langle (b_j^\alpha, \bar{\nu}_j^\alpha, n_j^\alpha) : j < j_\alpha \rangle$ so $\bar{\nu}_j^\alpha = \langle \nu_{j, \ell}^\alpha : \ell \leq k(*) \rangle$
- (d) $\sup \text{Rang}(\eta_{i, k(*)}^\alpha) < \alpha$, $\sup \text{Rang}(\nu_{j, k(*)}^\alpha) < \alpha$ if $i < i_\alpha, j < j_\alpha$.

Now to apply Definition 3.3 we have to choose z_α (for Definition 3.3) as $\Sigma\{a_i^\alpha x_{\bar{\eta}_i} : i < i_\alpha\} + \Sigma\{b_j^\alpha y_{\bar{\nu}_j^\alpha, n_j^\alpha} : j < j_\alpha\}$ and $z_{\bar{\eta}} = z_{\eta_{k(*)}(0)}$ for $\bar{\eta} \in \Lambda^x$ then for $\bar{\eta} \in \Lambda^x$ we choose $\langle b_{\bar{\eta}, n} : n < \omega \rangle \in {}^\omega \mathbb{Z}$ such that:

⊗ there is no function h from $\{z_{\bar{\eta}}\} \cup \{y_{\bar{\eta}, n} : n < \omega\} \cup \{x_{\bar{\eta}1 < m, n >} : m \leq k(*), n < \omega\}$ into \mathbb{Z} satisfying

- ⊗ (a) $h(z_{\bar{\eta}}) \neq 0$ and
- (b) $h(x_{\bar{\eta}1 < m, n >}) = h_{\bar{\eta}}(\bar{\eta} \upharpoonright \langle m, n \rangle)$ for $m \leq k(*), n < \omega$

(c) for every n

$$(*)_n \quad n!h(y_{\bar{\eta},n+1}) = h(y_{\bar{\eta},n}) + b_{\bar{\eta},n}h(z_{\bar{\eta}}) + \Sigma\{(x_{\bar{\eta}|<m,n>}) : m \leq k(*)\}.$$

E.g. for each $\rho \in {}^\omega 2$ we can try $b_n^\rho = \rho(n)$ and assume toward contradiction that for each $\rho \in {}^\omega 2$ there is h_ρ as above. Hence for some $c \in \mathbb{Z} \setminus \{0\}$ the set $\{\rho \in {}^\omega 2 : h_\rho(z_{\bar{\eta}}) = c\}$ is uncountable. So we can find $\rho_1 \neq \rho_2$ such that $h_{\rho_1} = c = h_{\rho_2}(x_\nu)$ and $\rho_1 \upharpoonright (|c| + 7) = \rho_2 \upharpoonright (|c| + 7)$. So for some $n \geq |c| + 7, \rho_1 \upharpoonright n = \rho_2 \upharpoonright n$ and $\rho_1(n) \neq \rho_2(n)$. Now consider the equation $(*)_n$ for h_{ρ_1} and h_{ρ_2} , subtract them and get $(\rho_1(n) - \rho_2(n))c$ is divisible by $n!$, clear contradiction.

So $G \in \mathcal{G}_x$ is well defined and is $\aleph_{k(*)+1}$ -free by 3.4. Suppose $h \in \text{Hom}(G, \mathbb{Z})$ is non-zero, so for some $\alpha < \lambda_{k(*)}, h(z_\alpha) \neq 0$ (actually as $G^1 = \langle \{x_{\bar{\nu}} : \bar{\nu} \in \Lambda_{\leq k(*)}^x\} \rangle_G$ is a subgroup such that G/G^1 is divisible necessarily $h \upharpoonright G^1$ is not zero hence in 2.1(2) for some $\bar{\nu} \in \Lambda_{\leq k(*)}^x$ we have $h(x_{\bar{\nu}}) \neq 0$. Let $y = \{\bar{\nu}\}$ and so by the choice of $\langle h_{\bar{\eta}} : \bar{\eta} \in \Lambda \rangle$ for some $\bar{\eta} \in \Lambda^x, \eta_{k(*)}(0) = \alpha$ and we have $h_{\bar{\eta}} = h \upharpoonright \{x_{\bar{\eta}|<m,n>} : m \leq k(*), n < \omega\}$. By \otimes we clearly get a contradiction. \blacksquare

Remark We can give more details as in the proof of 2.3.

Conclusion 3.6 *For every $n \leq m < \omega$ there is a purely increasing sequence $\langle G_\alpha : \alpha \leq \omega_n + 1 \rangle$ of abelian groups, $G_\alpha, G_\beta/G_\alpha$ are free for $\alpha < \beta \leq \omega_n$ and $G_{\omega_n+1}/G_{\omega_n}$ is \aleph_n -free and for some $h \in \text{Hom}(G_\kappa, \mathbb{Z})$ has no extension in $\text{Hom}(G_{\omega_n+1}, \mathbb{Z})$.*

Proof. Let G, z be as in 2.2. So also $G/\mathbb{Z}z$ is \aleph_n -free. Let $G_\alpha = \langle \{z\} \rangle_G$ for $\alpha \leq \omega_2, G_{\omega_n+1} = G$.

4 Appendix 1

Notation 4.1 If $\bar{\eta}^* \in \Lambda_m^x$ and $\bar{\eta} = \bar{\eta}^* \upharpoonright \{\ell \leq k(*) : \ell \neq m\}$ and $\nu = \eta_m^*$ then let $x_{m,\bar{\eta},\nu} := x_{\bar{\eta}^*}$. (See proof of 1.12).

Proof of 1.8. Let $U \subseteq {}^\omega S$ be countable (and infinite) and define G'_U like G restricting ourselves to $\eta_\ell \in U$; by the Löwenheim-Skolem argument it suffices to prove that G'_U is a free abelian group. List $\Lambda \cap {}^{k(*)+1}U$ without repetitions as $\langle \bar{\eta}_t : t < t^* \leq \omega \rangle$, and choose $s_t < \omega$ by induction on $t < \omega$ such that $\{r < t \ \& \ \bar{\eta}_r \upharpoonright k(*) = \bar{\eta}_t \upharpoonright k(*)\} \Rightarrow \emptyset = \{\eta_{t,k(*)} \upharpoonright \ell : \ell \in [s_t, \omega)\} \cap \{\eta_{r,k(*)} \upharpoonright \ell : \ell \in [s_r, \omega)\}$.

Let

$$Y_1 = \{x_{m,\bar{\eta},\nu} : m < k(*), \bar{\eta} \in {}^{k(*)+1} \setminus \{m\} U \text{ and } \nu \in {}^\omega > 2\}$$

$$Y_2 = \left\{ x_{m,\bar{\eta},\nu} : m = k(*), \bar{\eta} \in {}^{k(*)} U \text{ and for no } t < t^* \text{ do we have } \bar{\eta} = \bar{\eta}_t \upharpoonright k(*) \ \& \ \nu \in \{\eta_{t,k(*)} \upharpoonright \ell : s_t \leq \ell < \omega\} \right\}$$

$Y_3 = \{y_{\bar{\eta}_t, n} : t < t^* \text{ and } n \in [s_t, \omega)\}$. Now

(*)₁ $Y_1 \cup Y_2 \cup Y_3 \cup \{z\}$ generates G'_U .

[Why? Let G' be the subgroup of G'_U which $Y_1 \cup Y_2 \cup Y_3$ generates. First we prove by induction on $n < \omega$ that for $\bar{\eta} \in {}^{k(*)}U$ and $\nu \in {}^nS$ we have $x_{k(*), \bar{\eta}, \nu} \in G'$. If $x_{k(*), \bar{\eta}, \nu} \in Y_2$ this is clear; otherwise, by the definition of Y_2 for some $\ell < \omega$ (in fact $\ell = n$) and $t < \omega$ such that $\ell \geq s_t$ we have $\bar{\eta} = \bar{\eta}_t \upharpoonright k(*), \nu = \eta_{t, k(*)} \upharpoonright \ell$.

Now

(a) $y_{\bar{\eta}_t, \ell+1}, y_{\bar{\eta}_t, \ell}$ are in $Y_3 \subseteq G'$

(b) $x_{m, \bar{\eta}_t \upharpoonright \{i \leq k(*): i \neq m\}, \nu}$ belong to $Y_1 \subseteq G'$ if $m < k(*)$.

Hence by the equation $\boxtimes_{\bar{\eta}, n}$ in Definition 1.6, clearly $x_{k(*), \bar{\eta}, \nu} \in G'$. So as $Y_1 \subseteq G' \subseteq G'_U$, all the generators of the form $x_{m, \bar{\eta}, \nu}$ with each $\eta_\ell \in U$ are in G' .

Now for each $t < \omega$ we prove that all the generators $y_{\bar{\eta}_t, n}$ are in G' . If $n \geq s_t$ then clearly $y_{\bar{\eta}_t, n} \in Y_3 \subseteq G'$. So it suffices to prove this for $n \leq s_t$ by downward induction on n ; for $n = s_t$ by an earlier sentence, for $n < s_t$ by $\boxtimes_{\bar{\eta}_t, n}$. The other generators are in this subgroup so we are done.]

(*)₂ $Y_1 \cup Y_2 \cup Y_3 \cup \{z\}$ generates G'_U freely.

[Why? Translate the equations, see more in [5, §5].] ■

Proof of 1.10 0), 1) Obvious.

2), 3), 4) Follows.

5) Let $\langle \eta_\ell : \ell < m(*) \rangle$ list $u, U_\ell = U \cup (u \setminus \{\eta_\ell\})$ so $G_{U, u} = G_{U_0^+} \dots + G_{U_{m(*)-1}}$. First, $G_{U, u} \subseteq G_{U \cup u}$ follows by the definitions. Second, we deal with proving $G_{U, u} \subseteq_{\text{pr}} G_{U \cup u}$. So assume $z^* \in G, a^* \in \mathbb{Z}$ and $a^* z^*$ belongs to $G_{U_0} + \dots + G_{U_{m(*)}}$, so it has the form $\Sigma\{b_i x_{\bar{\eta}^i \upharpoonright < m_i, n_i} : i < i(*)\} + \Sigma\{c_j y_{\bar{\eta}_j, n_j} : j < j(*)\} + a z$ with $i(*) < \omega, j(*) < \omega$ and $a^*, b_i, c_j \in \mathbb{Z}$ and $\nu_i, \bar{\eta}^i, \bar{\eta}_j$ are suitable sequences of members of $U_{\ell(i)}, U_{\ell(i)}, U_{k(j)}$ respectively where $\ell(i), k(j) < m(*)$. We continue as in [5].

6) Easy.

7) Clearly $U_1 \cup v = U_2 \cup u$ hence $G_{U_1 \cup u} \subseteq G_{U_1 \cup v} = G_{U_2 \cup u}$ hence $G_{U, u} + G_{U_1 \cup u}$ is a subgroup of $G_{U, u} + G_{U_2 \cup u}$, so the first quotient makes sense.

Hence $(G_{U, u} + G_{U_2 \cup u}) / (G_{U, u} + G_{U_1 \cup u})$ is isomorphic to $G_{U_2 \cup u} / (G_{U_2 \cup u} \cap (G_{U, u} + G_{U_1 \cup u}))$. Now $G_{U_1, v} \subseteq G_{U_1 \cup v} = G_{U_2 \cup u} \subseteq G_{U, u} + G_{U_2, u}$ and $G_{U_1, v} \subseteq G_{U, v} = G_{U, v} \setminus U = G_{U, u} \subseteq G_{U, u} + G_{U_2, u}$. Together $G_{U_1, v}$ is included in their intersection, i.e.

$G_{U_2 \cup u} \cap (G_{U, u} + G_{U_1 \cup u})$ include $G_{U_1, v}$ and using part (1) both has the same divisible hull inside G^+ . But as $G_{U_1, v}$ is a pure subgroup of G by part (5) hence of $G_{U_1 \cup v}$. So necessarily $G_{U_1 \cup u} \cap (G_{U, u} + G_{U_1, u}) = G_{U_1, v}$, so as $G_{U_2 \cup u} = G_{U_1 \cup v}$ we are done. ■

8) See [5].

Proof of 1.12 1) We prove this by induction on $|U|$; without loss of generality $|u| = k$ as also $k' = |u|$ satisfies the requirements.

Case 1: U is countable.

So let $\{\nu_\ell^* : \ell < k\}$ list u be with no repetitions, now if $k = 0$, i.e. $u = \emptyset$ then $G_{U \cup u} = G_U = G_{U,u}$ so the conclusion is trivial. Hence we assume $u \neq \emptyset$, and let $u_\ell := u \setminus \{\nu_\ell^*\}$ for $\ell < k$.

Let $\{\bar{\eta}_t : t < t^* \leq \omega\}$ list with no repetitions the set $\Lambda_{U,u} := \{\bar{\eta} \in \Lambda^{\times} \cap^{k(*)+1}(U \cup u) : \text{for no } \ell < k \text{ does } \bar{\eta} \in {}^{k(*)+1}(U \cup u_\ell)\}$. Now comes a crucial point: let $t < t^*$, for each $\ell < k$ for some $r_{t,\ell} \leq k(*)$ we have $\eta_{t,r_{t,\ell}} = \nu_\ell^*$ by the definition of $\Lambda_{U,u}$, so $|\{r_{t,\ell} : \ell < k\}| = k < k(*) + 1$ hence for some $m_t \leq k(*)$ we have $\ell < k \Rightarrow r_{t,\ell} \neq m_t$ so for each $\ell < k$ the sequence $\bar{\eta}_t \upharpoonright (k(*) + 1 \setminus \{m_t\})$ is not from $\{\rho_s : s \leq k(*) \text{ and } s \neq m_t\} : \rho_s \in {}^\omega(U \cup u_\ell)$ for every $s \leq k(*)$ such that $s \neq m_t$.

For each $t < t^*$ we define $J(t) = \{m \leq k(*) : \{\eta_{t,s} : s \leq k(*) \text{ \& } s \neq m\}$ is included in $U \cup u_\ell$ for no $\ell \leq k\}$. So $m_t \in J(t) \subseteq \{0, \dots, k(*)\}$ and $m \in J(t) \Rightarrow \bar{\eta}_t \upharpoonright \{j \leq k(*) : j \neq m\} \notin {}^{k(*)+1} \setminus \{m\}(U \cup u_\ell)$ for every $\ell \leq k$. For $m \leq k(*)$ let $\bar{\eta}'_{t,m} := \bar{\eta}_t \upharpoonright \{j \leq k(*) : j \neq m\}$ and $\bar{\eta}'_t := \bar{\eta}'_{t,m_t}$. Now we can choose $s_t < \omega$ by induction on t such that

(*) if $t_1 < t, m \leq k(*)$ and $\bar{\eta}'_{t_1,m} = \bar{\eta}'_{t,m}$, then $\eta_{t,m} \upharpoonright s_t \notin \{\eta_{t_1,m} \upharpoonright \ell : \ell < \omega\}$.

Let $Y^* = \{x_{m,\bar{\eta}} \in G_{U \cup u} : x_{m,\bar{\eta}} \notin G_{U \cup u_\ell} \text{ for } \ell < k\} \cup \{y_{\bar{\eta},n} \in G_{U \cup u} : y_{\bar{\eta},n} \notin G_{U \cup u_\ell} \text{ for } \ell < k\}$.

Let

$$Y_1 = \{x_{m,\bar{\eta},\nu} \in Y^* : \text{for not } < t^* \text{ do we have } m = m_t \text{ \& } \bar{\eta} = \bar{\eta}'_t\}.$$

$$Y_2 = \{x_{m,\bar{\eta},\nu} \in Y^* : x_{m,\bar{\eta}} \notin Y_1 \text{ but for no } t < t^* \text{ do we have } m = m_t \text{ \& } \bar{\eta} = \bar{\eta}'_t \text{ \& } \eta_{t,m_t} \upharpoonright s_t \triangleleft \nu \triangleleft \eta_{t,m_t}\}$$

$$Y_3 = \{y_{\bar{\eta},n} : y_{\bar{\eta},n} \in Y^* \text{ and } n \in [s_t, \omega) \text{ for the } t < t^* \text{ such that } \bar{\eta} = \bar{\eta}'_t\}.$$

Now the desired conclusion follows from

(*)₁ $\{y + G_{U,u} : y \in Y_1 \cup Y_2 \cup Y_3\}$ generates $G_{U \cup u} / G_{U,u}$

(*)₂ $\{y + G_{U,u} : y \in Y_1 \cup Y_2 \cup Y_3\}$ generates $G_{U \cup u} / G_{U,u}$ freely.

Proof of (*)₁. It suffices to check that all the generators of $G_{U \cup u}$ belong to $G'_{U \cup u} =: \langle Y_1 \cup Y_2 \cup Y_3 \cup G_{U,u} \rangle_G$.

First consider $x = x_{m,\bar{\eta},\nu}$ where $\eta \in {}^{k(*)+1}(U \cup u), m < k(*)$ and $\nu \in {}^n S$ for some $n < \omega$. If $x \notin Y^*$ then $x \in G_{U,u_\ell}$ for some $\ell < k$ but $G_{U \cup u_\ell} \subseteq G_{U,u} \subseteq G'_{U \cup u}$ so we are done, hence assume $x \in Y^*$. If $x \in Y_1 \cup Y_2 \cup Y_3$ we are done so assume $x \notin Y_1 \cup Y_2 \cup Y_3$. As $x \notin Y_1$ for some $t < t^*$ we have $m = m_t$ \& $\bar{\eta} = \bar{\eta}'_t$. As $x \notin Y_2$, clearly for some t as above we have $\eta_{t,m_t} \upharpoonright s_t \triangleleft \nu \triangleleft \eta_{t,m_t}$. Hence by Definition 1.6 the equation $\boxtimes_{\bar{\eta},n}$ from Definition 1.6 holds, now $y_{\bar{\eta},n}, y_{\bar{\eta},n+1} \in G'_{U \cup u}$. So in order to deduce from the equation that $x = x_{\bar{\eta}'_t \upharpoonright < m_t, n}$ belongs to $G_{U \cup u}$, it suffices to show

that $x_{\bar{\eta}'_{t,j} \upharpoonright \langle j, n \rangle} \in G'_{U \cup u}$ for each $j \leq k(*), j \neq m_t$. But each such $x_{\bar{\eta}'_{t,j} \upharpoonright \langle j, n \rangle}$ belong to $G'_{U \cup u}$ as it belongs to $Y_1 \cup Y_2$.
 [Why? Otherwise necessarily for some $r < t^*$ we have $j = m_r, \bar{\eta}'_{t,j} = \bar{\eta}'_{r,m_r}$ and $\eta_{r,m_r} \upharpoonright s_r \trianglelefteq \eta_t \upharpoonright n \triangleleft \eta_{r,m_r}$ so $n \geq s_r$ and as said above $n \geq s_t$. Clearly $r \neq t$ as $m_r = j \neq m_t$, now as $\bar{\eta}'_{t,m_r} = \bar{\eta}'_{r,m_r}$ and $\bar{\eta}_t \neq \bar{\eta}_r$ (as $t \neq r$) clearly $\eta_{t,m_r} \neq \eta_{r,m_r}$. Also $\neg(r < t)$ by (*) above applied with r, t here standing for t_1, t there as $\eta_{r,m_r} \upharpoonright s_r \trianglelefteq \eta_{t,j} \upharpoonright n \triangleleft \eta_{r,m_r}$. Lastly for if $t < r$, again (*) applied with t, r here standing for t_1, t there as $n \geq m_t$ gives contradiction.]
 So indeed $x \in G'_{U \cup u}$.

Second consider $y = y_{\bar{\eta}, n} \in G_{U \cup u}$, if $y \notin Y^*$ then $y \in G_{U, u} \subseteq G'_{U \cup u}$, so assume $y \in Y^*$. If $y \in Y_3$ we are done, so assume $y \notin Y_3$, so for some $t, \bar{\eta} = \bar{\eta}_t$ and $n < s_t$. We prove by downward induction on $s \leq s_t$ that $y_{\bar{\eta}, s} \in G'_{U \cup u}$, this clearly suffices. For $s = s_t$ we have $y_{\bar{\eta}, s} \in Y_3 \subseteq G'_{U \cup u}$; and if $y_{\bar{\eta}, s+1} \in G'_{U \cup u}$ use the equation $\boxtimes_{\bar{\eta}_t, s}$ from 1.6, in the equation $y_{\bar{\eta}, s+1} \in G'_{U \cup u}$ and the x 's appearing in the equation belong to $G'_{U \cup u}$ by the earlier part of the proof (of (*)₁) so necessarily $y_{\bar{\eta}, s} \in G'_{U \cup u}$, so we are done.

Proof of (*)₂ We rewrite the equations in the new variables recalling that $G_{U \cup u}$ is generated by the relevant variables freely except the equations of $\boxtimes_{\bar{\eta}, n}$ from Definition 1.6. After rewriting, all the equations disappear.

Case 2: U is uncountable.

As $\aleph_1 \leq |U| \leq \aleph_{k(*)-k}$, necessarily $k < k(*)$.

Let $U = \{\rho_\alpha : \alpha < \mu\}$ where $\mu = |U|$, list U with no repetitions. Now for each $\alpha \leq |U|$ let $U_\alpha := \{\rho_\beta : \beta < \alpha\}$ and if $\alpha < |U|$ then $u_\alpha = u \cup \{\rho_\alpha\}$. Now

⊙₁ $\langle (G_{U, u} + G_{U_\alpha \cup u}) / G_{U, u} : \alpha < |U| \rangle$ is an increasing continuous sequence of subgroups of $G_{U \cup u} / G_{U, u}$.
 [Why? By 1.10(6).]

⊙₂ $G_{U, u} + G_{U_0 \cup u} / G_{U, u}$ is free.
 [Why? This is $(G_{U, u} + G_{\emptyset \cup u}) / G_{U, u} = (G_{U, u} + G_u) / G_{U, u}$ which by 1.10(8) is isomorphic to $G_u / G_{\emptyset, u}$ which is free by Case 1.]

Hence it suffices to prove that for each $\alpha < |U|$ the group $(G_{U, u} + G_{U_{\alpha+1} \cup u}) / (G_{U, u} + G_{U_\alpha \cup u})$ is free. But easily

⊙₃ this group is isomorphic to $G_{U_\alpha \cup u_\alpha} / G_{U_\alpha, u_\alpha}$.
 [Why? By 1.10(7) with $U_\alpha, U_{\alpha+1}, U, \rho_\alpha, u$ here standing for U_1, U_2, U, η, u there.]

⊙₄ $G_{U_\alpha \cup u_\alpha} / G_{U_\alpha, u_\alpha}$ is free.
 [Why? By the induction hypothesis, as $\aleph_0 + |U_\alpha| < |U| \leq \aleph_{k(*)-(k+1)}$ and $|u_\alpha| = k + 1 \leq k(*)$.]

2) If $k(*) = 0$ just use 1.8, so assume $k(*) \geq 1$. Now the proof is similar to (but easier than) the proof of case (2) inside the proof of part (1) above. ■

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