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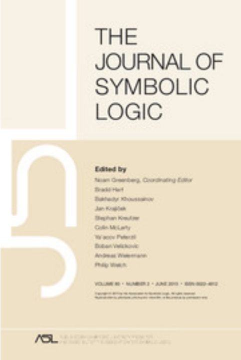

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The Journal of Symbolic Logic / Volume 59 / Issue 03 / September 1994, pp 984 - 996
DOI: 10.2307/2275922, Published online: 12 March 2014

Link to this article: http://journals.cambridge.org/abstract_S0022481200019654

How to cite this article:
Tapani Hyttinen and Saharon Shelah (1994). Constructing strongly equivalent nonisomorphic models for unsuperstable theories, Part A . The Journal of Symbolic Logic, 59, pp 984-996
doi:10.2307/2275922

THE JOURNAL OF SYMBOLIC LOGIC
Volume 59, Number 3, September 1994

CONSTRUCTING STRONGLY EQUIVALENT NONISOMORPHIC MODELS FOR UNSUPERSTABLE THEORIES, PART A

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Abstract. We study how equivalent nonisomorphic models an unsuperstable theory can have. We measure the equivalence by Ehrenfeucht-Fraisse games. This paper continues the work started in [HT].

§1. **Introduction.** In [HT] we looked at how equivalent nonisomorphic models first-order theories can have, i.e., we tried to strengthen S. Shelah's nonstructure theorems. We used Ehrenfeucht-Fraisse games to measure the equivalence (see Definition 2.2). If the theory is unstable, or has OTOP, or is superstable with DOP, then we were able to prove maximal results by assuming strong cardinal assumptions. We showed that if $\lambda^{<\lambda} = \lambda$, then there is a model $\mathscr{A}$ of the theory such that $|\mathscr{A}| = \lambda$ and for all λ^+, λ-trees t there is a model $\mathscr{B}$ such that $|\mathscr{B}| = \lambda$, $\mathscr{A} \not\cong \mathscr{B}$, and $\exists$ has a winning strategy in the Ehrenfeucht-Fraisse game $G_t^2(\mathscr{A}, \mathscr{B})$.

By assuming only that the theory is unsuperstable we were not able to say much if we tried to measure the equivalence by the length of Ehrenfeucht-Fraisse games in which $\exists$ has a winning strategy. But if instead we measured the equivalence by the length of Ehrenfeucht-Fraisse games in which $\forall$ does not have a winning strategy, then we were able to get rather strong results.

In this paper we look at the unsuperstable case again. We measure the equivalence by the length of Ehrenfeucht-Fraisse games in which $\exists$ has a winning strategy. We study λ^+, $\kappa + 1$-trees (see Definition 2.1) and give a rather complete answer to the question: how equivalent nonisomorphic λ^+, $\kappa + 1$-trees can there be? In §3 we show that if $\lambda = \mu^+$, $cf(\mu) = \mu$, $\kappa = cf(\kappa) \leq \mu$ and $\lambda^{<\kappa} = \lambda$, then there are λ^+, $\kappa + 1$-trees I_0 and I_1 such that $|I_0| \cup |I_1| \leq \lambda^\kappa$, $I_0 \not\cong I_1$, and

$$I_0 \equiv^{\lambda}_{\mu \times \kappa} I_1$$

(see Definition 2.2 and Definition 2.4(iii)). Instead of two trees it is possible to get 2^λ such trees.

In §4 we show that if in addition $\lambda \in I[\lambda]$, then the result of §3 is best possible. As in [HT], this implies that essentially the same is true also for the models of the canonical example of superstable theories.

In [HS] we will prove the results of §3 for unsuperstable theories in general.

Received June 30, 1993; revised October 28, 1993.

*Partially supported by the United States Israel binational science foundation, publ. 474.

This paper was born during the first author's visit to the second author at Rutgers University. The first author wishes to express his gratitude to Rutgers University for the hospitality shown to him during the visit.

§**2. Basic definitions.** In this section we define the basic concepts we shall use.

2.1. Definition. Let λ be a cardinal, and let α be an ordinal. Let t be a tree (i.e., for all $x \in t$, the set $\{y \in t | y < x\}$ is well-ordered by the ordering of t). If $x, y \in t$ and $\{z \in t | z < x\} = \{z \in t | z < y\}$, then we denote $x \sim y$, and the equivalence class of x for $\sim$ we denote by $[x]$. By a λ, α-tree t we mean a tree which satisfies:

(i) $|[x]| < \lambda$ for every $x \in t$;

(ii) there are no branches of length $\geq \alpha$ in t;

(iii) t has a unique root;

(iv) if $x, y \in t$, x and y have no immediate predecessors and $x \sim y$, then $x = y$.

If t satisfies only (i), (ii), and (iii) above, we say that t is a wide λ, α-tree.

Note that in a λ, α-tree each ascending sequence of a limit length has at most one supremum, but in a wide λ, α-tree an ascending sequence may have more than one supreumum.

2.2. Definition. Let t be a tree, and let κ be a cardinal. The Ehrenfeucht-Fraisse game of length t between models $\mathscr{A}$ and $\mathscr{B}$, $G_t^\kappa(\mathscr{A}, \mathscr{B})$, is the following. At each move α:

(i) player $\forall$ chooses $x_\alpha \in t$, $\kappa_\alpha < \kappa$, and either $a_\alpha^\beta \in \mathscr{A}$, $\beta < \kappa_\alpha$ or $b_\alpha^\beta \in \mathscr{B}$, $\beta < \kappa_\alpha$; we will denote this sequence of elements of $\mathscr{A}$ or $\mathscr{B}$ by X_α;

(ii) if $\forall$ chose from $\mathscr{A}$, then $\exists$ chooses $b_\alpha^\beta \in \mathscr{B}$, $\beta < \kappa_\alpha$, else $\exists$ chooses $a_\alpha^\beta \in \mathscr{A}$, $\beta < \kappa_\alpha$; we will denote this sequence by Y_α.

$\forall$ must move so that $(x_\beta)_{\beta \leq \alpha}$ form a strictly increasing sequence in t. $\exists$ must move so that $\{(a_\gamma^\beta, b_\gamma^\beta) | \gamma \leq \alpha, \beta < \kappa_\gamma\}$ is a partial isomorphism from $\mathscr{A}$ to $\mathscr{B}$. The player who first has to break the rules looses.

We write $\mathscr{A} \equiv_t^\kappa \mathscr{B}$ if $\exists$ has a winning strategy for $G_t^\kappa(\mathscr{A}, \mathscr{B})$.

2.3. Remark. Notice that the Ehrenfeucht-Fraisse game $G_t^\kappa(\mathscr{A}, \mathscr{B})$ need not be determined, i.e., it may happen that neither $\exists$ nor $\forall$ has a winning strategy for $G_t^\kappa(\mathscr{A}, \mathscr{B})$ (see [MSV]).

2.4. Definition. Let t and t' be trees.

(i) If $x \in t$, then pred(x) denotes the sequence $(x_\alpha)_{\alpha < \beta}$ of the predecessors of x, excluding x itself, ordered by $<$. Alternatively, we consider pred(x) as a set. The notation succ(x) denotes the set of immediate successors of x. If $x, y \in t$ and there is z, such that $x, y \in \text{succ}(z)$, then we say that x and y are brothers.

(ii) By $t^{<\alpha}$ we mean the set

$$\{x \in t | \text{ the order type of pred}(x)\text{is } < \alpha\}.$$

Similarly, we define $t^{\leq \alpha}$.

(iii) If α and β are ordinals, then by $\alpha + \beta$ and $\alpha \times \beta$ we mean the ordinal sum and product (see [Je]). Notice that ordinals are also trees.

§**3. On nonstructure of trees of fixed height.** In this chapter we will assume that $\lambda = \mu^+$, $cf(\mu) = \mu$, $\kappa = cf(\kappa) \leq \mu$, and $\lambda^{<\kappa} = \lambda$.

Let $I_n^+ = \{\eta \in {}^{\leq\kappa}\lambda | \eta(0) = \eta\} - \{(\,)\}$ and $I_n^- = \{\eta \in {}^{<\kappa}\lambda | \eta(0) = n\} - \{(\,)\}$, $n = 0, 1$. We consider these as trees ordered by initial segment relation. Because for all $\delta \leq \kappa$, $(I_n^+)^{<\delta} = (I_n^-)^{<\delta}$ (see Definition 2.4), we denote this set by $I_n^{<\delta}$, and similarly, we define $I_n^{\leq\delta} = (I_n^+)^{\leq\delta}$ for all $\delta < \kappa$.

If $\eta \in I_0^+$ and $\xi \in I_1^+$, then we write $\eta R^- \xi$ and $\xi R^- \eta$ iff $\eta(j) = \xi(j)$ for all $0 < j < \min\{\text{length}(\eta), \text{length}(\xi)\}$ even. For all $i < \kappa$ odd, we define P_i to be the set of all $\eta \in I_0^-$ such that $\text{length}(\eta) = i$. Let $P = \bigcup\{P_i | i < \kappa,\ i \text{ odd}\}$.

3.1. Lemma. *There is a partition* $\{S_\eta | \eta \in P\}$ *of* λ *such that for all* $\eta \in P$

(i) $\{\delta \in S_\eta | cf(\delta) = \mu\}$ *is stationary*;

(ii) *if* $\delta \in S_\eta$ *and* $cf(\delta) = \mu$, *then* $\delta = \sup(\delta \cap S_\eta)$.

Proof. Because $|P| = \lambda$, we can find a partition of $\{\alpha < \lambda | cf(\alpha) = \mu\}$ which satisfies (i). Let this partition be $\{S'_{\eta_\gamma} | \gamma < \lambda\}$, where $\{\eta_\gamma | \gamma < \lambda\}$ is an enumeration of P. Let $\{\sigma_\gamma | \gamma < \lambda\}$ be an enumeration of $\{\alpha < \lambda | cf(\alpha) = \mu\}$ so that if $\sigma_\gamma > \sigma_{\gamma'}$, then $\gamma > \gamma'$. We may assume that if $\delta \in S'_{\eta_\gamma}$, $\gamma \neq 0$, then $\delta > \sigma_\gamma$. By induction on $\alpha \leq \lambda$ we define sets $S^\alpha_{\eta_\gamma}$. Let $S^0_{\eta_0} = S'_{\eta_0} \cup \sigma_0$ and for all $\gamma > 0$, $S^0_{\eta_\gamma} = S'_{\eta_\gamma}$. If α is a limit ordinal and $cf(\alpha) \geq \mu$, then we define $S^\alpha_{\eta_\gamma} = \bigcup_{\beta<\alpha} S^\beta_{\eta_\gamma}$ for all $\gamma < \lambda$. Assume α is a successor or limit ordinal with $cf(\alpha) < \mu$. Let $\sigma'_\alpha = \bigcup_{\delta<\alpha} \sigma_\delta$. Then we choose $S^\alpha_{\eta_\gamma}$ so that (a)–(f) below are satisfied:

(a) $\bigcup_{\delta<\alpha} S^\delta_{\eta_\gamma} \subseteq S^\alpha_{\eta_\gamma}$;

(b) $S^\alpha_{\eta_\gamma} \cap S^\alpha_{\eta_{\gamma'}} = \varnothing$ if $\gamma \neq \gamma'$;

(c) $\sigma_\alpha \subseteq \bigcup_{\gamma<\lambda} S^\alpha_{\eta_\gamma}$;

(d) $S^\alpha_{\eta_\gamma} - \sigma_\alpha = S^0_{\eta_\gamma} - \sigma_\alpha$ for all $\gamma < \lambda$;

(e) if $\sigma_\alpha \in S'_{\eta_\gamma}$, then $\sigma_\alpha = \sup(\sigma_\alpha \cap S^\alpha_{\eta_\gamma})$;

(f) if $\gamma \leq \alpha$, then $(\sigma_\alpha - \sigma'_\alpha) \cap S^\alpha_{\eta_\gamma} \neq \varnothing$.

Then clearly $S_{\eta_\gamma} = S^\lambda_{\eta_\gamma}$, $\gamma < \lambda$, is a partition of λ and (i) is satisfied. We show that (ii) is also satisfied: if $\sigma_\delta \in S_{\eta_\gamma}$ and δ is a successor or limit ordinal with $cf(\delta) < \mu$, then by (e) $\sigma_\delta = \sup(\sigma_\delta \cap S_{\eta_\gamma})$. Otherwise, we know that $\sigma_\delta > \sigma_\gamma$; i.e., $\delta > \gamma$ and $\sup\{\sigma_\beta | \beta < \delta\} = \sigma_\delta$. By (f) this implies that $\sigma_\delta = \sup(\sigma_\delta \cap S_{\eta_\gamma})$. $\square$

3.2. Definition. We define a relation $R \subseteq (I_0^+ - I_0^-) \times (I_1^+ - I_1^-)$. Let $\eta \in I_0^+ - I_0^-$ and $\xi \in I_1^+ - I_1^-$. Then $(\eta, \xi) \in R$ iff

(i) $\eta R^- \xi$;

(ii) for every $j < \kappa$ odd, η and ξ satisfy the following: for all $p \in P$, $\eta(j) \in S_p$ iff $\xi(j) \in S_p$ and if $\eta(j) \notin S_{\eta \restriction j}$, then $\eta(j) = \xi(j)$;

(iii) the set $W^\kappa_{\eta,\xi}$ is bounded in κ, where $W^\kappa_{\eta,\xi}$ is defined in the following way: Let $\delta \leq \kappa$, $\eta \in I_0^+ - I_0^{<\delta}$, and $\xi \in I_1^+ - I_1^{<\delta}$; then

$$\begin{aligned} W^\delta_{\eta,\xi} = \{j < \delta | j &\text{ odd and } \eta(j) \in S_{\eta \restriction j} \text{ and} \\ &cf(\eta(j)) = \mu \text{ and } \xi(j) \geq \eta(j)\}. \end{aligned}$$

In order to simplify the notation we write $\eta R \xi$ and $\xi R \eta$ for $(\eta, \xi) \in R$. Notice that by this we do not try to claim that the relation is symmetric—in fact, it is antisymmetric; if $(\eta, \xi) \in R$, then always $\eta \in I_0^+ - I_0^-$ and $\xi \in I_1^+ - I_1^-$. We also take liberty to write $W^\delta_{\xi,\eta}$ for $W^\delta_{\eta,\xi}$ when it is convenient.

Our first goal in this section is to prove the following theorem. We will prove it in a sequence of lemmas.

3.3. THEOREM. *If* I_0 *and* I_1 *are such that*

(i) $I_n^- \subseteq I_n \subseteq I_n^+$, $n = 0, 1$,

and

(ii) *if* $\eta R \xi$, $\eta \in I_0^+$, *and* $\xi \in I_1^+$, *then* $\eta \in I_0$ *iff* $\xi \in I_1$,

then $I_0 \equiv^{\lambda}_{\mu \times \kappa} I_1$.

From now on in this section we assume that I_0 and I_1 satisfy (i) and (ii) above.

3.4. DEFINITION. Let $\alpha < \kappa$.

(i) G_α is the family of all partial functions f satisfying:

(a) f is a partial isomorphism from I_0 to I_1;

(b) $\operatorname{dom}(f)$ and $\operatorname{rng}(f)$ are closed under initial segments and for some $\beta < \lambda$ they are included in $\{\eta \in I_0^+ \mid \text{for all } j < \kappa,\ \eta(j) < \beta\}$ and $\{\xi \in I_1^+ \mid \text{for all } j < \kappa,\ \xi(j) < \beta\}$, respectively;

(c) if $f(\eta) = \xi$, then $\eta R^- \xi$;

(d) if $\eta \in I_0$, $\xi \in I_1$, $f(\eta) = \xi$, and $\operatorname{length}(\eta) = j + 1$, j odd, then η and ξ satisfy the following: for all $\rho \in P$, $\eta(i) \in S_\rho$ iff $\xi(i) \in S_\rho$ and if $\eta(j) \notin S_{\eta \restriction j}$, then $\eta(j) = \xi(j)$;

(e) assume $\eta \in I_0^+ - I_0^{<\delta}$ and $\{\eta \restriction \gamma \mid \gamma < \delta\} \subseteq \operatorname{dom}(f)$, and let $\xi = \bigcup_{\gamma<\delta} f(\eta \restriction \gamma)$, then $W^{\delta}_{\eta,\xi}$ has order type $\leq \alpha$;

(f) if $\eta \in \operatorname{dom}(f)$, then $\{\gamma < \lambda \mid \eta \frown (\gamma) \in \operatorname{dom}(f)\} = \{\gamma < \lambda \mid f(\eta) \frown (\gamma) \in \operatorname{rng}(f)\}$ is an ordinal.

(ii) We define $F_\alpha \subseteq G_\alpha$ by replacing (f) by

(f') if $\eta \in \operatorname{dom}(f)$, then $\{\gamma < \lambda \mid \eta \frown (\gamma) \in \operatorname{dom}(f)\} = \{\gamma < \lambda \mid f(\eta) \frown (\gamma) \in \operatorname{rng}(f)\}$ is an ordinal of cofinality $< \mu$.

3.5. DEFINITION. For $f, g \in G_\alpha$ we write $f \leq g$ if $f \subseteq g$ and if $\gamma < \delta \leq \kappa$, $\eta \in I_0^+ - I_0^{<\delta}$, $\eta \restriction \gamma \in \operatorname{dom}(f)$, $\eta \restriction (\gamma + 1) \notin \operatorname{dom}(f)$, $\eta \restriction j \in \operatorname{dom}(g)$ for all $j < \delta$, and $\xi = \bigcup_{j<\delta} g(\eta \restriction j)$, then $W^{\gamma}_{\eta,\xi} = W^{\delta}_{\eta,\xi}$.

Notice that $f \leq g$ is a transitive relation.

3.6. REMARK. Let $f \in G_\alpha$.

(i) We define $\overline{f}$ by

$$\operatorname{dom}(\overline{f}) = \operatorname{dom}(f) \cup \{\eta \in I_0 \mid \eta \restriction \gamma \in \operatorname{dom}(f) \text{ for all } \gamma < \operatorname{length}(\eta) \text{ and } \operatorname{length}(\eta) \text{ is a limit ordinal}\},$$

and if $\eta \in \operatorname{dom}(\overline{f}) - \operatorname{dom}(f)$, then

$$\overline{f}(\eta) = \bigcup_{\gamma < \operatorname{length}(\eta)} f(\eta \restriction \gamma).$$

(ii) If $f \in F_\alpha$, then $\overline{f} \in F_\alpha$; if $f \in G_\alpha$, then $\overline{f} \in G_\alpha$.

3.7. LEMMA. *Assume* $\alpha < \kappa$, $\delta \leq \mu$, $f_i \in F_\alpha$ *for all* $i < \delta$, *and* $f_i \leq f_j$ *for all* $i < j < \delta$.

(i) $\bigcup_{i<\delta} f_i \in G_\alpha$.

(ii) *If* $\delta < \mu$, *then* $\bigcup_{i<\delta} f_i \in F_\alpha$ *and* $f_j \leq \bigcup_{i<\delta} f_i$ *for all* $j < \delta$.

PROOF. The proof follows immediately from the definitions. ☐

3.8. LEMMA. *If* $\delta < \kappa$, $f_i \in G_i$ *for all* $i < \delta$, *and* $f_i \subseteq f_j$ *for all* $i < j < \delta$, *then*

$$\bigcup_{i<\delta} f_i \in G_\delta .$$

PROOF. The proof follows immediately from the definitions. □

3.9. LEMMA. *If* $f \in F_\alpha$ *and* $A \subseteq I_0 \cup I_1$, $|A| < \lambda$, *then there is* $g \in F_\alpha$ *such that* $f \leq g$ *and* $A \subseteq \mathrm{dom}(g) \cup \mathrm{rng}(g)$.

PROOF. Let $\eta \in \mathrm{dom}(f)$, and let

$$\{i < \lambda | \eta \frown (i) \in \mathrm{dom}(f)\} = \{i < \lambda | f(\eta) \frown (i) \in \mathrm{rng}(f)\} = \delta,$$

$cf(\delta) < \mu$, and let $\beta > \delta$. We show first that there are $f^{\eta\beta} \in F_\alpha$ and $\gamma \geq \beta$ such that $f^{\eta\beta} \geq f$, $cf(\gamma) < \mu$, and

$$\{i < \lambda | \eta \frown (i) \in \mathrm{dom}(f^{\eta\beta})\} = \{i < \lambda | f(\eta) \frown (i) \in \mathrm{rng}(f^{\eta\beta})\} = \gamma .$$

Let length$(\eta) = j$. If j is even, it is trivial to find $f^{\eta\beta}$ and γ. So we assume that j is odd. We choose $\gamma \geq \beta$ so that $cf(\gamma) < \mu$. For any $i \in \gamma - \delta$ satisfying:

(i) $cf(i) = \mu$

and

(ii) $i \in S_\eta$,

we choose $j_i \in i - \delta$ so that $j_i \in S_\eta$, $cf(j_i) < \mu$ and if $i \neq i'$, then $j_i \neq j_{i'}$. These j_i exist because $\sup i \cap S_\eta = i$ and $i \neq \delta$.

Then we define $f^{\eta\beta}(\eta \frown (i)) = f(\eta) \frown (j_i)$ and $f^{\eta\beta}(\eta \frown (j_i)) = f(\eta) \frown (i)$. For all other $i \in \gamma - \delta$ we let $f^{\eta\beta}(\eta \frown (i)) = f(\eta) \frown (i)$. It is easy to see that $f^{\eta\beta} \in F_\alpha$ and $f^{\eta\beta} \geq f$.

It is easy to see that we can choose $\eta_i \in I_0$ and $\beta_i < \lambda$, $i < \mu$, so that the following functions are well defined:

(i) $g_0 = f$;

(ii) $g_{i+1} = (g_i)^{\eta_i \beta_i}$;

(iii) $g_i = \overline{(\bigcup_{j<i} g_j)}$, if i is a limit ordinal;

and $A \subseteq \mathrm{dom}(\bigcup_{i<\mu} g_i) \cup \mathrm{rng}(\bigcup_{i<\mu} g_i)$. Furthermore we can choose η_i and β_i so that if $i \neq i'$, then $\eta_i \neq \eta_{i'}$. Then $g = \bigcup_{i<\mu} g_i$ is as wanted. □

3.10. LEMMA. *If* $f \in G_\alpha$, *then there is* $g \in F_{\alpha+1}$ *such that* $f \subseteq g$.

PROOF. The proof is essentially the same as the proof of Lemma 3.9. □

Theorem 3.3 now follows easily from the lemmas above.

In the rest of this section we prove that there are trees I_0 and I_1 which satisfy the assumptions of Theorem 3.3 and are not isomorphic. For this we use the following Black Box. We define $H_{<\kappa^+}(\lambda)$ to be the smallest set H such that

(i) $\lambda \subseteq H$

and

(ii) if $x \subseteq H$ and $|x| \leq \kappa$, then $x \in H$.

3.11. THEOREM ([Sh3, Lemma 6.5]). *There is* $W = \{(\overline{M}^\alpha, \eta^\alpha) | \alpha < \alpha(*)\}$ *such that*:

(i) $\overline{M}^\alpha = (M_i^\alpha | i \leq \kappa)$ *is an increasing continuous elementary chain of models belonging to* $H_{<\kappa^+}(\lambda)$ *and* $\eta^\alpha \in {}^\kappa\lambda$ *is increasing.*

(ii) *$M_i^\alpha \cap \kappa^+$ is an ordinal, $\kappa + 1 \subseteq M_i^\alpha$, $M_i^\alpha \in H_{<\kappa^+}(\eta^\alpha(i))$, $(M_j^\alpha | j \leq i) \in M_{i+1}^\alpha$ and $\eta^\alpha \restriction i \in M_{i+1}^\alpha$.*

(iii) *In the following game, $G(\kappa, \lambda, W)$, player $\forall$ does not have a winning strategy: The play lasts κ moves, in the ith move $\forall$ chooses a model $M_i \in H_{<\kappa^+}(\lambda)$ and then $\exists$ chooses $\gamma_i < \lambda$. $\forall$ must choose models M_i, $i < \kappa$, so that $(M_i | i \leq \kappa)$ is an increasing continuous elementary chain of models, $M_i \cap \kappa^+$ is an ordinal, $\kappa + 1 \subseteq M_i$, and $(M_j | j \leq i) \in M_{i+1}$. In the end $\exists$ wins the play if for some $\alpha < \alpha(*)$, $\eta^\alpha = (\gamma_i | i < \kappa)$ and $M_i = M_i^\alpha$ for all $i < \kappa$.*

(iv) *$\eta^\alpha \neq \eta^\beta$ for $\alpha \neq \beta$.*

Notice that in the game above $\forall$ can choose the similarity type of models freely as long as other requirements are satisfied.

We define I_0 and I_1 with the help of W. We do this by defining J_α, $\neg J_\alpha$, K_α, and $\neg K_\alpha$ by induction on $\alpha < \alpha(*)$ so that $J_\alpha \cap \neg J_\alpha = \varnothing$ and $K_\alpha \cap \neg K_\alpha = \varnothing$ and then letting $I_0 = I_0^- \cup \bigcup_{\alpha < \alpha(*)} J_\alpha$ and $I_1 = I_1^- \cup \bigcup_{\alpha < \alpha(*)} K_\alpha$. We assume that we have well-ordered $I_0^+ - I_0^-$.

We say that $\alpha < \alpha(*)$ is active, if there is an $\eta \in I_0^+ - I_0^-$ such that α and η satisfy (i)–(vii) or (i)–(v), (vi′) and (vii′) below.

(i) For all $i \leq \kappa$, the similarity type of M_i^α is $\{\in, I_0^-, I_1^-, g\}$, where $\in$ and g are two-ary relation symbols and I_0^- and I_1^- are unary relation symbols;

(ii) for all $i \leq \kappa$,

$$M_i^\alpha \restriction \{\in, I_0^-, I_1^-\} \prec (H_{<\kappa^+}(\lambda), \in, I_0^-, I_1^-);$$

(iii) for all $i < \kappa$, $\eta \restriction i \in M_{i+1}^\alpha$;

(iv) for all $i \leq \kappa$, $M_i^\alpha \models$ "g is an isomorphism from I_0^- to I_1^-";

(v) for all $\omega \leq i < \kappa$, if $i = \gamma + 2k$ for some γ limit and $k < \omega$, then $\eta(i) = \eta^\alpha(\gamma + k)$, and for all $i < \omega$, if $i = 2k + 2$, then $\eta(i) = \eta^\alpha(k)$;

let

$$\xi = \bigcup_{i<\kappa} g_\alpha(\eta \restriction i),$$

where g_α is the interpretation of g in M_κ^α,

(vi) $\eta R^- \xi$;

(vi′) $\eta \not{R}^- \xi$;

(vii) for all $i < \kappa$ odd, $\eta(i)$ satisfies:

(a) $cf(\eta(i)) = \mu$ and $\eta(i) \in S_{\eta \restriction i}$;

(b) $M_\kappa^\alpha \models$ "the set $\{\eta \restriction i \frown (j) | j < \eta(i)\} \cup \{g(\eta \restriction i) \frown (j) | j < \eta(i)\}$ is closed under g and g^{-1}";

(vii′) there is $j_\eta < \kappa$ such that for all $i > j_\eta$ odd the following holds:

(a) if $i = \gamma + 4n + 1$ for some limit ordinal γ and $n \in \omega$, then $\xi(i) \in S_{\eta \restriction i}$;

(b) if $i = \gamma + 4n + 3$ for some limit ordinal γ and $n \in \omega$, then $\eta(i) \in S_{\eta \restriction i}$, $cf(\eta(i)) = \mu$, and $\xi(i) \geq \eta(i)$.

If α is active and there exists such η that α and η satisfy (i)–(vii) above, then we define η_α to be the least such $\eta \in I_0^+ - I_0^-$ in the well-ordering of $I_0^+ - I_0^-$.

Otherwise, we let η_α be the least $\eta \in I_0^+ - I_0^-$ in the well-ordering of $I_0^+ - I_0^-$ such that α and η satisfy (i)–(v), (vi′) and (vii′) above. Let

$$\xi_\alpha = \bigcup_{i<\kappa} g_\alpha(\eta_\alpha \restriction i),$$

where g_α is the interpretation of g in M_κ^α. If α is active and $\eta_\alpha \not R^- \xi_\alpha$, then let $j_\alpha = j_{\eta_\alpha}$.

Let $\overline{R}$ be the transitive and reflexive closure of R.

3.12. Lemma. *If γ is active, then $\eta_\gamma \not\overline{R} \xi_\gamma$.*

Proof. Clearly, we may assume that $\eta_\gamma R^- \xi_\gamma$. For a contradiction, assume that there are $\rho_0, \dots, \rho_n$ such that $\rho_0 = \eta_\gamma$, $\rho_n = \xi_\gamma$, for all $m < n$, $\rho_m R \rho_{m+1}$ and for all $k < m \le n$, $\rho_k \neq \rho_m$. We choose $i < \kappa$ so that

(α) i is odd;

(β) for all $k < m \le n$, $\rho_k \restriction i \neq \rho_m \restriction i$;

(γ) for all $m < n$, $W^\kappa_{\rho_m, \rho_{m+1}} \subseteq i$.

Because $\eta_\gamma(i) \in S_{\eta_\gamma \restriction i}$ and $cf(\eta_\gamma(i)) = \mu$, $\rho_1(i) < \rho_0(i)$ and $\rho_1(i) \in S_{\eta_\gamma \restriction i}$. By the definition of R, $\rho_2(i) \in S_{\eta_\gamma \restriction i}$. By ($\beta$) above $\rho_2(i) = \rho_1(i)$ and $\rho_3(i) = \rho_2(i)$. We can continue this and get $\rho_n(i) = \cdots = \rho_1(i)$. So $\eta_\gamma(i) > g(\eta_\gamma \restriction (i+1))(i)$ which contradicts with (vii)(b) in the definition of active. □

3.13. Lemma. *Let α and β be active, $\alpha \neq \beta$, $\xi_\alpha \overline{R} \xi_\beta$, and $\eta_\alpha \not R^- \xi_\alpha$, then $\eta_\beta R^- \xi_\beta$.*

Proof. For a contradiction assume $\eta_\beta \not R^- \xi_\beta$. By (vii′)(a) in the definition of active we can find $i < \kappa$ odd such that $\xi_\alpha(i) \in S_{\eta_\alpha \restriction i}$ and $\xi_\beta(i) \in S_{\eta_\beta \restriction i}$. By Definition 3.2(ii) this implies $\xi_\alpha \not\overline{R} \xi_\beta$, a contradiction. □

3.14. Lemma. *Let α and β be active.*

(i) *If $\alpha \neq \beta$, then $\eta_\alpha \not\overline{R} \eta_\beta$.*

(ii) *If $\eta_\alpha R^- \xi_\alpha$, then for all active γ, $\eta_\alpha \not\overline{R} \xi_\gamma$.*

Proof. (i) By (vii)(a) and (or) (vii′)(b) in the definition of active there is $i < \kappa$ odd such that $\eta_\alpha(i) \in S_{\eta_\alpha \restriction i}$, $\eta_\beta(i) \in S_{\eta_\beta \restriction i}$ and $\eta_\alpha \restriction i \neq \eta_\beta \restriction i$. By Definition 3.2(ii) this implies $\eta_\alpha \not\overline{R} \eta_\beta$.

(ii) If $\gamma = \alpha$, the claim follows immediately from Lemma 3.12. So assume $\gamma \neq \alpha$. We may also assume $\eta_\alpha R^- \xi_\gamma$, because otherwise we have proved the claim. Then $\eta_\gamma \not R - \xi_\gamma$. By (vii)(a) and (vii′)(a) in the definition of active we can find $i < \kappa$ odd such that $\eta_\alpha(i) \in S_{\eta_\alpha \restriction i}$, $\xi_\gamma(i) \in S_{\eta_\gamma \restriction i}$, and $\eta_\alpha \restriction i \neq \eta_\gamma \restriction i$. As above this implies $\eta_\alpha \not\overline{R} \xi_\gamma$. □

3.15. Lemma. *Let α and β be active. If $\eta_\alpha \overline{R} \xi_\beta$, then there is $l_{\alpha\beta} < \kappa$ such that for all $i > l_{\alpha\beta}$, $i = \gamma + 4k + 3$, γ limit, and $k \in \omega$, $\eta_\alpha(i) > \xi_\beta(i)$.*

Proof. by Lemma 3.14(ii) we may assume $\eta_\beta \not R^- \xi_\alpha$. For a contradiction, assume that there are $\rho_0, \dots, \rho_n$ such that $\rho_0 = \eta_\alpha$, $\rho_n = \xi_\beta$, for all $m < n$, $\rho_m R \rho_{m+1}$ and for all $k < m \le n$, $\rho_k \neq \rho_m$. We choose $l_{\alpha\beta} < \kappa$ so that

(α) $j_{\eta_\alpha} < l_{\alpha\beta}$;

(β) for all $k < m \le n$, $\rho_k \restriction i \neq \rho_m \restriction i$;

(γ) for all $m < n$, $W^\kappa_{\rho_m, \rho_{m+1}} \subseteq i$.

Let $i > l_{\alpha\beta}$, $i = \gamma + 4k + 3$, γ limit, and $k \in \omega$. Because $\eta_\alpha(i) \in S_{\eta_\alpha \restriction i}$ and $cf(\eta_\alpha(i)) = \mu$, $\rho_1(i) < \rho_0(i)$ and $\rho_1(i) \in S_{\eta_\alpha \restriction i}$. By the definition of R, $\rho_2(i) \in S_{\eta_\alpha \restriction i}$.

By (β) above $\rho_2(i) = \rho_1(i)$ and $\rho_3(i) = \rho_2(i)$. We can continue this and get $\rho_n(i) = \cdots = \rho_1(i)$. So $\eta_\alpha(i) > \xi_\beta(i)$. □

3.16. Lemma. *There does not exist a sequence* $(\tau_0, \ldots, \tau_n)$, $n \in \omega$, $n \geq 3$, *such that*

(1) *for all* $m \leq n$ *there is an active* α *such that* $\tau_m = \eta_\alpha$ *or* $\tau_m = \xi_\alpha$;

(ii) *for all* $m < n$ *either*

(a) $\tau_m \overline{R} \tau_{m+1}$

or

(b) *there is active* α *such that* $\tau_m = \eta_\alpha$ *and* $\tau_{m+1} = \xi_\alpha$ *or* $\tau_m = \xi_\alpha$ *and* $\tau_{m+1} = \eta_\alpha$ *and at least case* (b) *exists in the sequence*;

(iii) $\tau_0 = \tau_n$;

(iv) *for all* m, $m' < n$ *if* $m \neq m'$, *then* $\tau_m \neq \tau_{m'}$.

Proof. For a contradiction, assume that such a sequence exists. By (ii)(b) we may choose the sequence so that for some α, $\tau_0 = \xi_\alpha$ and $\tau_1 = \eta_\alpha$. Then by (iv) and because $n \geq 3$, $\tau_1 \overline{R} \tau_2$. By Lemma 3.12 $\eta_\alpha \overline{\not{R}} \xi_\alpha$, and so, we may drop elements from the sequence so that (i)–(iv) remain true, there are still at least four elements in the sequence and

(*) if $m < n - 1$ and $\tau_m \overline{R} \tau_{m+1}$, then $\tau_{m+1} \overline{\not{R}} \tau_{m+2}$.

By induction on $m < n$ we show that if $\tau_m \overline{\not{R}} \tau_{m+1}$,then $\tau_{m+1} \overline{R} \tau_{m+2}$ and if $\tau_m = \eta_\beta$ or $\tau_m = \xi_\beta$ for some β, then $\eta_\beta \not{R}^- \xi_\beta$. We showed above that $\eta_\alpha \overline{R} \tau_2$. By Lemma 3.14(i) $\tau_2 = \xi_\beta$ for some active β. By Lemma 3.14(ii) $\eta_\alpha \not{R}^- \xi_\alpha$. Then by (*) above $\tau_3 = \eta_\beta$. By (iv) and Lemma 3.14(i) $\tau_4 = \xi_\gamma$ for some active γ, $\gamma \neq \beta$, and $\eta_\beta \overline{R} \xi_\gamma$. By Lemma 3.14(ii) $\eta_\beta \not{R}^- \xi_\beta$. We can continue this and obtain the claim.

So there are active $\alpha_0, \ldots, \alpha_m$ such that the sequence is of the following form:

$$(\xi_{\alpha_0}, \eta_{\alpha_0}, \xi_{\alpha_1}, \eta_{\alpha_1}, \ldots, \eta_{\alpha_m}, \xi_{\alpha_0}).$$

We choose $i < \kappa$ so that for all $k \leq m$, $i > j_{\alpha_k}$, for all $k < m$, $i > l_{\alpha_k \alpha_{k+1}}$, $i > l_{\alpha_m \alpha_0}$ and $i = \gamma + 4p + 3$ for some limit γ and $p \in \omega$. By (vii′)(b) $\xi_{\alpha_0}(i) \geq \eta_{\alpha_0}(i)$. By Lemma 3.15 $\eta_{\alpha_0}(i) > \xi_{\alpha_1}(i)$. We can continue this and finally we get $\eta_{\alpha_m}(i) > \xi_{\alpha_0}(i)$. So $\xi_{\alpha_0}(i) > \xi_{\alpha_0}(i)$, a contradiction. □

We define now J_α, $\neg J_\alpha$, K_α, and $\neg K_\alpha$ by induction on $\alpha < \alpha(*)$. We say that $(J_\alpha, \neg J_\alpha, K_\alpha, \neg K_\alpha)$ is closed if

(i) $J_\alpha \cup K_\alpha$ and $\neg J_\alpha \cup \neg K_\alpha$ are closed under $\overline{R}$;

(ii) if β is active, then $\eta_\beta \in J_\alpha$ iff $\xi_\beta \in \neg K_\alpha$ and $\eta_\beta \in \neg J_\alpha$ iff $\xi_\beta \in K_\alpha$;

(iii) $J_\alpha \cap \neg J_\alpha = \varnothing$ and $K_\alpha \cap \neg K_\alpha = \varnothing$.

We assume that for all $\beta < \alpha$ we have defined J_β, $\neg J_\beta$, K_β, and $\neg K_\beta$ so that $(J_\beta, \neg J_\beta, K_\beta, \neg K_\alpha)$ is closed.

If α is not active or for some $\beta < \alpha$, $\eta_\alpha \in J_\beta \cup \neg J_\beta$, then we let $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$, $\neg J_\alpha = \bigcup_{\beta<\alpha} \neg J_\beta$, $K_\alpha = \bigcup_{\beta<\alpha} K_\beta$, and $\neg K_\alpha = \bigcup_{\beta<\alpha} \neg K_\beta$.

If α is active and for all $\beta < \alpha$, $\eta_\beta \notin J_\beta \cup \neg J_\beta$, then we let $(J_\alpha, \neg J_\alpha, K_\alpha, \neg K_\alpha)$ be such that it is closed and $J_\alpha \supseteq \{\eta_\alpha\} \cup \bigcup_{\beta<\alpha} J_\beta$, $\neg J_\alpha \supseteq \bigcup_{\beta<\alpha} \neg J_\beta$, $K_\alpha \supseteq \bigcup_{\beta<\alpha} K_\beta$ and $\neg K_\alpha \supseteq \bigcup_{\beta<\alpha} \neg K_\beta$. We prove the existence of these sets by defining sets J^i_α, $\neg J^i_\alpha$, K^i_α, and $\neg K^i_\alpha$ by induction on $i < |\alpha(*)|^+$.

We let $J^0_\alpha = \{\eta_\alpha\} \cup \bigcup_{\beta<\alpha} J_\beta$, $\neg J^0_\alpha = \bigcup_{\beta<\alpha} \neg J_\beta$, $K^0_\alpha = \bigcup_{\beta<\alpha} K_\beta$, and $\neg K^0_\alpha = \bigcup_{\beta<\alpha} \neg K_\beta$. If $i < |\alpha(*)|^+$ is a limit, we let $J^i_\alpha = \bigcup_{j<i} J^j_\alpha$ and similarly for the

other sets. If $i = j + 1$ and is odd, then we let the sets J^i_β, $\neg J^i_\beta$, K^i_β, and $\neg K^i_\beta$ be the least sets so that $J^i_\alpha \supseteq J^j_\alpha$, $\neg J^i_\alpha \supseteq \neg J^j_\alpha$, $K^i_\alpha \supseteq K^j_\alpha$, $\neg K^i_\alpha \supseteq \neg K^j_\alpha$, and $J^i_\alpha \cup K^i_\alpha$ and $\neg J^i_\alpha \cup \neg K^i_\alpha$ are closed under $\overline{R}$. If $i = j + 1$ and is even, then if there is not an active γ such that

(1) $\eta_\gamma \in J^j_\alpha$ and $\xi_\gamma \notin \neg K^j_\alpha$,

or

(2) $\eta_\gamma \in \neg J^j_\alpha$ and $\xi_\gamma \notin K^j_\alpha$,

or

(3) $\xi_\gamma \in K^j_\alpha$ and $\eta_\gamma \notin \neg J^j_\alpha$,

or

(4) $\xi_\gamma \in \neg K^j_\alpha$ and $\eta_\gamma \notin J^j_\alpha$,

then we let $J^i_\alpha = J^j_\alpha$ and similarly for the other sets. Otherwise, we let γ be the least such ordinal and define

Case 1. $J^i_\alpha = J^j_\alpha$, $\neg J^i_\alpha = \neg J^j_\alpha$, $K^i_\alpha = K^j_\alpha$ and $\neg K^i_\alpha = \neg K^j_\alpha \cup \{\xi_\gamma\}$;

Case 2. $J^i_\alpha = J^j_\alpha$, $\neg J^i_\alpha = \neg J^j_\alpha$, $K^i_\alpha = K^j_\alpha \cup \{\xi_\gamma\}$ and $\neg K^i_\alpha = \neg K^j_\alpha$;

Case 3. $J^i_\alpha = J^j_\alpha$, $\neg J^i_\alpha = \neg J^j_\alpha \cup \{\eta_\gamma\}$, $K^i_\alpha = K^j_\alpha$ and $\neg K^i_\alpha = \neg K^j_\alpha$;

Case 4. $J^i_\alpha = J^j_\alpha \cup \{\eta_\gamma\}$, $\neg J^i_\alpha = \neg J^j_\alpha$, $K^i_\alpha = K^j_\alpha$ and $\neg K^i_\alpha = \neg K^j_\alpha$.

Finally, we define $J_\alpha = \bigcup_{i<|\alpha(*)|^+} J^i_\alpha$ and similarly for the other sets. If these sets are not as required, then for some $i = j + 1 < |\alpha(*)|^+$ even we have defined f.ex. $\neg K^i_\alpha = \neg K^j_\alpha \cup \{\xi_j\}$, while ξ_γ belongs already to K^j_α. If i is the least such ordinal, then we can easily find a circle such that it contradicts Lemma 3.16.

So the sets J_α, $\neg J_\alpha$, K_α, and $\neg K_\alpha$ exist.

We define $I_0 = I_0^- \cup \bigcup_{\alpha<\alpha(*)} J_\alpha$ and $I_1 = I_1^- \cup \bigcup_{\alpha<\alpha(*)} K_\alpha$.

3.17. LEMMA. $I_0 \not\cong I_1$.

PROOF. For a contradiction assume $g\colon I_0 \to I_1$ is an isomorphism. By Theorem 3.11(iii) there exists an active $\alpha < \alpha(*)$ such that for all $i \leq \kappa$,

$$M^\alpha_i \prec (H_{<\kappa^+}(\lambda), \in, I_0^-, I_1^-, g).$$

But then $\eta_\alpha \in I_0$ iff $\xi_\alpha \notin I_1$ and $g(\eta_\alpha) = \xi_\alpha$, which contradicts the assumption that g is an isomorphism. □

Conclusion 3.18. Assume $\lambda = \mu^+$, $cf(\mu) = \mu$, $\kappa = cf(\kappa) \leq \mu$, and $\lambda^{<\kappa} = \lambda$. Then there are λ^+, $\kappa + 1$-trees I_0 and I_1 such that $I_0 \not\cong I_1$ and

$$I_0 \equiv^\lambda_{\mu\times\kappa} I_1.$$

If $\lambda^\kappa = \lambda$, then I_0 and I_1 are of cardinality λ.

Notice that if we replace Theorem 3.11 with a slightly stronger black box (see [Sh3]); we can, instead of two λ^+, κ-trees, get 2^λ λ^+, κ-trees such that any two of them satisfy Conclusion 3.18.

§**4. On structure of trees of fixed height.** In this section we will show that trees of fixed height are isomorphic if they are equivalent up to some relatively small tree. This implies that essentially the same is true for the models of the canonical example of unsuperstable theories (see [HT]).

4.1. DEFINITION ([Sh1]). Let λ be a regular cardinal. We define $I[\lambda]$ to be the set of $A \subseteq \lambda$ such that there exist a cub $E \subseteq \lambda$ and $\mathscr{P} = \{P_\alpha | \alpha < \lambda\}$ satisfying

(i) P_α is a set of subsets of α and $|P_\alpha| < \lambda$;

(ii) for all limits $\delta \in A \cap E$ such that $cf(\delta) < \delta$, there exists $C \subseteq \delta$ such that

(a) the order type of C is $< \delta$ and $\sup C = \delta$;

(b) $C \cap \alpha \in \bigcup_{\beta<\delta} P_\beta$ for all $\alpha < \delta$.

Notice that, for example, $\omega_1 \in I[\omega_1]$: Let $E \subset \omega_1$ be the set of all limit ordinals $< \omega_1$ and let $\mathscr{P} = \{P_\alpha | \alpha < \lambda\}$ be such that $P_\alpha = \{B \subseteq \alpha |\ |B| < \omega\}$. Then (i) and (ii) above are satisfied. For further properties of $I[\lambda]$ see [Sh1].

4.2. Definition. Let λ be a regular cardinal, and let t be a λ^+, λ-tree of cardinality λ. Let $\{x_i | i < \lambda\}$ be an enumeration of t, and let t' be a subtree of t. Then $S[t']$ is the set of those limit ordinals $\delta < \lambda$ which satisfy the following condition $(*)$:

$(*)$ $\{x_i \in t' | i < \delta\}$ contains a branch of length δ.

From now on we assume that whenever we talk about a tree t, we have fixed an enumeration $\{x_i | i < |t|\}$ for it. We assume that the enumeration is such that if $x_i < x_j$, then $i < j$.

4.3. Definition. Let λ and κ be regular cardinals, let $\kappa < \lambda$, and let t be a λ^+, λ-tree of cardinality λ. Let $\{x_i | i < \lambda\}$ be the enumeration of t. We say that t is λ, κ-large if t satisfies the following condition: There are sets E_ξ, $\xi \le \kappa$, such that

(i) $E_\xi \subseteq t$ and if $\xi \ne \xi'$, then $E_\xi \cap E_{\xi'} = \varnothing$;

(ii) for $\xi < \delta$ and $x \in E_\delta$ there is a unique $y \in E_\xi$ such that $y < x$;

(iii) if $\delta \le \kappa$ is a limit ordinal, $x_\xi \in E_\xi$ for all $\xi < \delta$, and $(x_\xi)_{\xi<\delta}$ is increasing, then there is $y \in E_\delta$ such that $x_\xi < y$ for all $\xi < \delta$;

(iv) if $\xi < \kappa$, $x \in E_\xi$, then we write

$$t_x = \{y \in t | x \le y \text{ and there is } z \in E_{\xi+1} \text{ such that } y < z\}$$

and require that there exists a set Θ of regular cardinals $< \lambda$ such that

(a) $S[t_x] \cup \{\delta < \lambda | cf(\delta) < \delta,\ cf(\delta) \in \Theta\}$ contains a cub set (in λ);

(b) $\{\delta < \lambda | cf(\delta) < \delta,\ cf(\delta) \in \Theta,\ \delta \notin S[t_x]\} \in I[\lambda]$;

(c) for $\delta \in \Theta$ there is $y \in t_x$ such that the order type of $\{z | x \le z < y\}$ is δ;

(v) if $\gamma = \beta + 1 < \kappa$, $(x_\xi)_{\xi<\delta}$ is an increasing sequence in t, $x_0 \in E_\beta$, and for all $\xi < \delta$ there is $y_\xi \in E_\gamma$ such that $x_\xi < y_\xi$, then there is $y \in E_\gamma$ such that $x_\xi < y$ for all $\xi < \delta$.

Notice that if $\lambda = \mu^+$, $\lambda \in I[\lambda]$, and $\kappa < \lambda$ is regular, then $\mu \times \kappa + 1$ is a λ, κ-large λ^+, λ-tree. If λ is weakly compact, then there is no λ, κ-large λ^+, λ-trees.

The proof of the theorem below is a modification of the proof of a related result in [HT]. The most conspicuous difference is the use of elementary submodels of $H(\lambda^*)$. They are used only to make it easier to define the closures needed in the proof.

4.4. Theorem. *Let λ and κ be regular cardinals, $\kappa < \lambda$, and let I_0 and I_1 be λ^+, $\kappa + 1$-trees. Assume t is a λ, κ-large λ^+, λ-tree of cardinality λ. Then*

$$I_0 \equiv^\lambda_t I_1 \Leftrightarrow I_0 \cong I_1.$$

Proof. Without loss of generality we may assume that I_0 and I_1 are such that if $x, y \in I_0$ $(\in I_1)$, they have no immediate predecessors, $x \sim y$, and $\mathrm{pred}(x)$ is of power $< \kappa$, then $x = y$.

Let ρ be a winning strategy of $\exists$ in $G_t^\lambda(I_0, I_1)$. We define by induction on $\alpha \leq \kappa$ the following:

(i) an isomorphism f_α from $I_0^{\leq\alpha}$ onto $I_1^{\leq\alpha}$;

(ii) for each $x \in I_0^{\leq\alpha} \cup I_1^{\leq\alpha}$ we define an initial segment $R_x = ((a_i, X_i, Y_i))_{i\leq\beta}$ of a play in $G_t^\lambda(I_0, I_1)$, such that $x \in \bigcup_{i\leq\beta}(\mathrm{rng}(X_i) \cup \mathrm{rng}(Y_i))$, $\mathrm{rng}(X_i) \cup \mathrm{rng}(Y_i) \subseteq I_0^{\leq\alpha} \cup I_1^{\leq\alpha}$ for all $i < \beta$, $\exists$ has used ρ and if x is not a leaf, then for some $\delta < \kappa$ there is $a_x \in E_\delta$ such that $a_i \leq a_x$ for all $i < \beta$. Furthermore, we require that if $x < x'$, then R_x is an initial segment of $R_{x'}$ and for each $x \in I_0^{\leq\alpha}$ $f_\alpha(x)$ is the element $\exists$ has chosen to be the image of x in R_x.

If we can do this, we have clearly proved the theorem. The cases $\alpha = 0$ and α is a limit ordinal are trivial. So we assume that $\alpha = \gamma + 1$.

Let $z \in I_0^{\leq\gamma} - \bigcup_{\delta<\gamma} I_0^{\leq\delta}$. Clearly, it is enough to define $f_\alpha \restriction \mathrm{succ}(z)$ and R_x for all $x \in \mathrm{succ}(z)$ so that $f_\alpha \restriction \mathrm{succ}(z)$ is onto $\mathrm{succ}(f_\gamma(z))$. Let $y = f_\gamma(z)$, and let $n\colon \lambda \to t$ to be the function that gives the enumeration of t, $t = \{n(i) | i < \lambda\}$ (see the assumption after Definition 4.2). Let $R_z = ((a_i, X_i, Y_i))_{i\leq\beta}$. By the inductive assumption there is $a_z \in E_\delta$, $\delta < \kappa$, such that $a_i < a_z$ for all $i \leq \beta$. Let E and $\mathscr{P} = \{P_i | i < \lambda\}$ be the sets which show that

$$\{\delta < \lambda | cf(\delta) < \delta, cf(\delta) \in \Theta, \delta \notin S[t_{a_z}]\} \in I[\lambda].$$

Let λ^* be large enough, say $(\beth_{10}(\lambda))^+$. We choose $\mathscr{A}_i$, $i < \lambda$, so that

(a) $|\mathscr{A}_i| < \lambda$ and $\mathscr{A}_i \prec (H(\lambda^*), \in, I_0, I_1, t, <_0, <_1, <)$, where $<_0$ denotes the ordering of I_0, $<_1$ denotes the ordering of I_1, and $<$ denotes the ordering of t;

(b) ρ, n, $(E_\xi | \xi \leq \kappa)$, E, $(P_i | i < \lambda)$, R_z, λ, β, $a_z \in \mathscr{A}_0$, $\kappa + 1 \subseteq \mathscr{A}_0$, and $i \subseteq A_i$;

(c) $\mathscr{A}_i \prec \mathscr{A}_j$ if $i < j$ and $\mathscr{A}_i = \bigcup_{j<i} \mathscr{A}_j$ if i limit;

(d) for all $i \leq \beta$, $\mathrm{dom}(X_i) \in \mathscr{A}_0$ (see Definition 2.2);

(e) $\mathscr{A}_i \cap \lambda$ is ordinal, $\mathscr{A}_i \in \mathscr{A}_{i+1}$ and $\mathscr{A}_i \cap \lambda \in \mathscr{A}_{i+1}$;

(f) $\mathrm{succ}(z) \cup \mathrm{succ}(y) \subseteq \bigcup_{i<\lambda} \mathscr{A}_i$;

(g) if $x \in t \cap \mathscr{A}_i$, $y \in t$, and $y < x$, then $y \in \mathscr{A}_i$.

Let

$$C \subseteq S[t_{a_z}] \cup \{\delta < \lambda | cf(\delta) < \delta \text{ and } cf(\delta) \in \Theta\}$$

be cub. We may assume that for all $c \in C$, $\mathscr{A}_c \cap \lambda = c$ and $c \in E$.

For all $i < \lambda$ we define by induction $c_i \in C$ and $f_\alpha \restriction (\mathrm{succ}(z) \cap \mathscr{A}_{c_i})$. If i is limit, then $c_i = \bigcup_{j<i} c_j$ and $f_\alpha \restriction (\mathrm{succ}(z) \cap \mathscr{A}_{c_i})$ is already defined.

Assume that we have defined c_i and $f_\alpha \restriction (\mathrm{succ}(z) \cap \mathscr{A}_{c_i})$ as wanted and

$$\mathrm{rng}(f_\alpha \restriction (\mathrm{succ}(z) \cap \mathscr{A}_{c_i})) = \mathrm{succ}(y) \cap \mathscr{A}_{c_i}.$$

Let us define c_{i+1} and

$$f_\alpha \restriction (\mathrm{succ}(z) \cap (\mathscr{A}_{c_{i+1}} - \mathscr{A}_{c_i})).$$

Now either $c_i \in S[t_{a_z}]$ or $c_i \in \{\delta < \lambda | cf(\delta) < \delta \text{ and } cf(\delta) \in \Theta\}$.

(1) $c_i \in S[t_{a_z}]$: Let $B \in \mathscr{A}_{c_i+1}$ be a branch in

$$S[t_{a_z}] \cap \mathscr{A}_{c_i} = \{n(j) | j < c_i\}$$

of length c_i. Let $h \in \mathscr{A}_{c_i}$ be a one-to-one function from $(\mathrm{succ}(z) \cup \mathrm{succ}(y)) \cap \mathscr{A}_{c_i}$ to $\mathscr{A}_{c_i} \cap \lambda$. We let the players continue the play R_z so that in the next c_i moves $\forall$

chooses the sets $\{h^{-1}(\delta)\}$, $\delta < c_i$, from $I_0 \cup I_1$ and from t he chooses elements of B. We let $\exists$ follow ρ. If B' is an initial segment of B, then $B' = \{y \in t | a_z \leq y < x\}$ for some $x \in B$. So $B' \in \mathscr{A}_{c_i}$, which implies that every initial segment of the play belongs to $\mathscr{A}_{c_i}$. Because $\mathscr{A}_{c_i}$ is closed under ρ, all the elements $\exists$ chooses are from $\mathscr{A}_{c_i}$. It is also easy to see that this play belongs to $\mathscr{A}_\gamma$ for all $\gamma > c_i$.

By Definition 4.3(v) we can find $a \in E_{\delta+1} \cap \mathscr{A}_{c_i+1}$, such that a is larger than any element $b \in t$ chosen by $\forall$ in the play above. Let

$$C' \subseteq S[t_a] \cup \{\delta < \lambda | cf(\delta) < \delta \text{ and } cf(\delta) \in \Theta\}$$

be cub. Let $c_{i+1} \in C \cap C'$ be such that $c_{i+1} > c_i$. Then $a \in \mathscr{A}_{c_{i+1}}$. Now either $c_{i+1} \in S[t_a]$ or $c_{i+1} \in \{\delta < \lambda | cf(\delta) < \delta \text{ and } cf(\delta) \in \Theta\}$. In the first case we let $\forall$ play the elements $(\mathrm{succ}(z) \cup \mathrm{succ}(y)) \cap \mathscr{A}_{c_{i+1}}$ as above. So let us assume that $c_{i+1} \notin S[t_a]$ and $c_{i+1} \in \{\delta < \lambda | cf(\delta) < \delta \text{ and } cf(\delta) \in \Theta\}$. Especially then

$$(*) \qquad c_{i+1} \in E \cap \{\delta < \lambda | cf(\delta) < \delta, cf(\delta) \in \Theta, \delta \notin S[t_a]\}.$$

Let $h' \in \mathscr{A}_{c_{i+1}}$ be a one-to-one function from $(\mathrm{succ}(z) \cup \mathrm{succ}(y)) \cap \mathscr{A}_{c_{i+1}}$ to $c_{i+1} = \mathscr{A}_{c_{i+1}} \cap \lambda$. Let

$$D' \subseteq c_{i+1}$$

be a set such that for all $\xi < c_{i+1}$, $\xi \cap D' \in \bigcup_{j<c_{i+1}} P_j$, $\sup D' = c_{i+1}$, and the order type of D' is $cf(c_{i+1})$. The existence of this set follows from $(*)$ above. Let $D = \{d_j | j < cf(c_{i+1})\}$ be the closure of D' in c_{i+1}. Because $cf(c_{i+1}) \in \mathscr{A}_{c_{i+1}}$, it is easy to see that in $t_a \cap \mathscr{A}_{c_{i+1}}$ there is a branch B of length $cf(c_{i+1})$. We let the players continue the play above so that in the next $cf(c_{i+1})$ moves $\forall$ chooses the sets $\{h'^{-1}(k) | k < d_j\}$ from $I_0 \cup I_1$, $j < cf(c_{i+1})$, and from t he chooses elements of B. We let $\exists$ follow ρ.

Because $\bigcup_{i<c_{i+1}} P_i \subseteq \mathscr{A}_{c_{i+1}}$, every initial segment of this play is in $\mathscr{A}_{c_{i+1}}$ and so all elements chosen by $\exists$ from $I_0 \cup I_1$ are from $\mathscr{A}_{c_{i+1}}$. Then by using the moves of $\exists$ we can define

$$f_\alpha \restriction (\mathrm{succ}(z) \cap (\mathscr{A}_{c_{i+1}} - \mathscr{A}_{c_i})).$$

For each $x \in \mathrm{succ}(z) \cap (\mathscr{A}_{c_{i+1}} - \mathscr{A}_{c_i})$, R_x will be the play defined above.

(2) $c_i \notin S[t_{a_z}]$: Now we first let $\forall$ play the elements of $(\mathrm{succ}(z) \cup \mathrm{succ}(y)) \cap \mathscr{A}_{c_i}$ as in the second half of the case (1) and then continue as above. Notice that in this case (also) we have to define the first $cf(c_i)$ moves so that the play belongs to $\mathscr{A}_{c_{i+1}}$. We can guarantee this by choosing $D' \subseteq c_i$ so that $D' \in \mathscr{A}_{c_i+1}$. □

4.5. **Remark.** Let $\lambda = \mu^+$, and let $\kappa < \lambda$ be regular. Let I_0 and I_1 be λ^+, $\kappa + 1$-trees. Assume $\lambda \in I[\lambda]$. Above we proved that if $\alpha = \mu \times \kappa + 1$, then

$$(*) \qquad I_0 \equiv^\lambda_\alpha I_1 \Leftrightarrow I_0 \cong I_1.$$

In §3 we showed that if μ is regular, then this is best possible. But if μ is not regular, then we can get better results.

If $\kappa < cf(\mu) < \mu$, then $(*)$ is true if $\alpha = \mu$; if $\kappa = cf(\mu) < \mu$, then $(*)$ is true if $\alpha = \mu + 1$. This can be proved in the same way as in the proof of Theorem 4.4.

REFERENCES

[HS] T. Hyttinen and S. Shelah, *Constructing strongly equivalent nonisomorphic models for unsuperstable theories, part* B (to appear).

[HT] T. Hyttinen and H. Tuuri, *Constructing strongly equivalent nonisomorphic models for unstable theories*, ***Annals of Pure and Applied Logic***, vol. 52 (1991), pp. 203–248.

[Je] T. Jech, ***Set theory***, Academic Press, New York, 1978.

[MSV] A. Mekler, S. Shelah, and J Väänänen, *The Ehrenfeucht-Fraisse-game of length* ω_1, ***Transactions of the American Mathematical Society***, vol. 339 (1983), pp. 567–580.

[Sh1] S. Shelah, *Nonelementary classes: part II*, Classification theory (J. T. Baldwin, editor) Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, Berlin and New York, (1987).

[Sh2] ———, *Classification theory*, ***Studies in Logic and the Foundations of Mathematics (second edition)***, vol. 92, North-Holland, Amsterdam, 1990.

[Sh3] ———, *Non-structure theory* (to appear).

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