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Non-existence of universal members in classes of abelian groups

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Abstract. We prove that if $\mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$, or if $\aleph_0 < \lambda < 2^{\aleph_0}$, then there is no universal reduced torsion-free abelian group of cardinality λ. We also prove that if $\beth_\omega < \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$, then there is no universal reduced separable abelian p-group in λ. We also deal with the class of $\aleph_1$-free abelian groups. Both results fail if (a) $\lambda = \lambda^{\aleph_0}$ or if (b) λ is a strong limit and $\mathrm{cf}(\mu) = \aleph_0 < \mu$.

0 Introduction

We deal with the problem of the existence of a universal member in $\mathfrak{K}_\lambda$ for a class $\mathfrak{K}$ of abelian groups, where $\mathfrak{K}_\lambda$ is the class of groups in $\mathfrak{K}$ of cardinality λ; universal means that every other member can be embedded into it. We are concerned mainly with the class of reduced torsion-free groups. Generally, for the history of the existence of universal members, see Kojman–Shelah [1]. From previous work, a natural division of the possible cardinals for such problems is as follows:

Case 0. $\lambda = \aleph_0$.
Case 1. $\lambda = \lambda^{\aleph_0}$.
Case 2. $\aleph_0 < \lambda < 2^{\aleph_0}$
Case 3. $2^{\aleph_0} + \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$.
Case 4. $2^{\aleph_0} + \mu^+ + \mathrm{cf}(\lambda) < \lambda < \mu^{\aleph_0}$.
Case 5. $\lambda = \mu^+$, $\mathrm{cf}(\mu) = \aleph_0, (\forall \chi < \mu)(\chi^{\aleph_0} < \mu)$.
Case 6. $\mathrm{cf}(\lambda) = \aleph_0, (\forall \chi < \lambda)(\chi^{\aleph_0} < \lambda)$.
Subcase 6a. λ is strong limit.
Subcase 6b. Case 6 but not 6a.

Our main interest was in Case 3, originally for $\mathfrak{K} = \mathfrak{K}^{\mathrm{rtf}}$, the class of torsion-free reduced abelian groups. Note that if we omit the condition ‘reduced’ then divisible torsion-free abelian groups of cardinality λ are universal. A second class is $\mathfrak{K}^{\mathrm{rs}(p)}$, the

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class of reduced separable p-groups (see Definition 2.3(4), and more in Fuchs [1]). However we are also interested in developing methods and in the class of $\aleph_1$-free abelian groups. Kojman–Shelah [3] show that for $\mathfrak{K} = \mathfrak{K}^{\mathrm{rtf}}, \mathfrak{K}^{\mathrm{rs}(p)}$ in Case 3 there is no universal member if we restrict the possible embeddings to pure embeddings. This underlines the point that universality depends not only on the class of structures but also on what embeddings are allowed. In [7] we allow any embeddings, but restrict the class of abelian groups to $(<\lambda)$-stable ones. In [8, Sections 1 and 5] we allow any embeddings and all $G \in \mathfrak{K}_\lambda$, but there is a further restriction on λ related to the pcf theory (see [6]). This restriction is weak in the sense that it is not clear if there is any cardinal (in any possible universe of set theory) not satisfying it. Here we prove the full theorem for $\lambda > \beth_\omega$ with no further restrictions:

$(*)$ for $\lambda > \beth_\omega$ in Case 3, and for $\mathfrak{K} = \mathfrak{K}^{\mathrm{rtf}}, \mathfrak{K}^{\mathrm{rs}(p)}$, there is no universal member in $\mathfrak{K}_\lambda$.

(Here we define inductively $\beth_0 = \aleph_0$, $\beth_{n+1} = 2^{\beth_n}$, $\beth_\omega = \sum_{n<\omega} 2^{\beth_n}$ and generally $\beth_\alpha = \aleph_0 + \sum_{\beta<\alpha} 2^{\beth_\beta}$.)

In Section 1 we deal with $\mathfrak{K}^{\mathrm{rtf}}$ using mainly type theory. In Section 2 we apply combinatorial ideals whose definition has some built-in algebra and purely combinatorial ones to obtain results on $\mathfrak{K}^{\mathrm{rs}(p)}$; there is more interaction between algebra and combinatorics than in [8]. Similarly in Section 3 we work on the class of $\aleph_1$-free abelian groups.

We comment briefly on the other cases. For Case 4 (which is like Case 3 but with λ singular), for $\mathfrak{K}^{\mathrm{rtf}}_\lambda$ and pure embedding, the non-existence of universals was shown in [3] subject to a weak pcf assumption, and in [8] this was done for embeddings under slightly stronger pcf assumptions. It is not clear whether either of these assumptions may fail. The results on consistency of existence of universals in this case cannot be attacked as long as more basic pcf problems remain open.

Concerning Case 5, if we want to prove the consistency of the existence of universals, it is natural first to prove the existence of the relevant club guessing; here we expect consistency results. (Of course, consistently there is club guessing (from $\bar{C} = \langle C_\delta : \delta \in S\rangle, S \subseteq \lambda$, $\mathrm{otp}(C_\delta) = \mu$) and then there is no universal.) We were interested first in the existence of universal reduced torsion-free groups under embeddings, but later we also considered some of the other cases here. See more in [12].

Case 1 ($\lambda = \lambda^{\aleph_0}$). By subsequent work there is a universal member of $\mathfrak{K}^{\mathrm{rtf}}_\lambda$, and (see Fuchs [1]) in $\mathfrak{K}^{\mathrm{rs}(p)}_\lambda$ there is a universal member, but in $\mathfrak{K}^{\aleph_1\text{-free}}_\lambda$ there is no universal member (see forthcoming work).

Case 0 ($\lambda = \aleph_0$). In $\mathfrak{K}^{\mathrm{rtf}}_\lambda$ there is no universal member (see above or 3.17) and in $\mathfrak{K}^{\mathrm{rs}(p)}_\lambda$ there is a universal member (see Fuchs [1]).

Case 2 ($\aleph_0 < \lambda < 2^{\aleph_0}$). For $\mathfrak{K}^{\mathrm{rtf}}_\lambda$ we prove here that there is no universal member (by 1.2), whereas for $\mathfrak{K}^{\mathrm{rs}(p)}_\lambda$ this is consistent with and independent of ZFC (see [5, Section 4]).

We have also dealt with Case 6 ($(\forall\chi < \lambda)\chi^{\aleph_0} < \lambda$, $\lambda > \mathrm{cf}(\lambda) = \aleph_0$). There is a universal member for $\mathfrak{K}^{\mathrm{trf}}_\lambda$ and also for $\mathfrak{K}^{\mathrm{rs}(p)}_\lambda$. See [12].

Notation. The cardinality of a set A is $|A|$, the cardinality of a structure G is $\|G\|$. $\mathscr{H}(\lambda^+)$ is the set of sets whose transitive closure has cardinality $\leqslant \lambda$, and $<^*_{\lambda^+}$ denotes a fixed well order of $\mathscr{H}(\lambda^+)$. For an ideal I, we denote by I^+ the family of subsets of $\mathrm{Dom}(I)$ which are not in I.

1 Non-existence of universals among reduced torsion-free abelian groups

The first result (1.2) deals with the case when λ satisfies $\aleph_0 < \lambda < 2^{\aleph_0}$ and it shows the non-existence of universal members in $\mathfrak{K}^{\mathrm{trf}}_\lambda$; this result improves [8]. The proof, by analysing subgroups and comparing Bauer's types, is straightforward.

Then we deal with the case when $2^{\aleph_0} + \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$. We add witnesses to bar the way to 'undesirable' extensions (see [12] on classes of modules). This is a critical new point compared with [8].

1.1 Definition. Let $\mathfrak{K}^{\mathrm{rtf}}$ denote the class of torsion-free reduced abelian groups G (where torsion-free means that $nx = 0,\ n \in \mathbb{Z},\ x \in G \Rightarrow n = 0 \vee x = 0$ and reduced means that $(\mathbb{Q}, +)$ cannot be embedded into G). The subclass of $G \in \mathfrak{K}^{\mathrm{rtf}}$ of cardinality λ is denoted by $\mathfrak{K}^{\mathrm{rtf}}_\lambda$. Moreover, $\mathfrak{K}^{\mathrm{tf}}$ is the class of torsion-free abelian groups.

1.2 Claim. (1) *If* $\aleph_0 < \lambda < 2^{\aleph_0}$ *then* $\mathfrak{K}^{\mathrm{rtf}}_\lambda$ *has no universal member.*

(2) *Moreover there is no member of* $\mathfrak{K}^{\mathrm{rtf}}_\lambda$ *universal for* $\mathfrak{K}^{\mathrm{rtf}}_{\aleph_1}$.

Proof. Let $\mathbf{P}^*$ be the set of all primes and let $\{\mathbf{Q}_i : i < 2^{\aleph_0}\}$ be a family of infinite subsets of $\mathbf{P}^*$ with pairwise finite intersection. Let $\rho_\alpha \in {}^\omega 2$ for $\alpha < \omega_1$ be pairwise distinct. Let H^* be the divisible torsion-free abelian group with $\{x_\alpha : \alpha < \omega_1\}$ a maximal independent subset. For $i < 2^{\aleph_0}$ let H^*_i be the subgroup of H^* generated by

$$\{x_\alpha : \alpha < \omega_1\} \cup \{p^{-n} x_\alpha : p \in \mathbf{P}^* \setminus \mathbf{Q}_i, \alpha < \omega_1 \text{ and } n < \omega\}$$
$$\cup \{p^{-n}(x_\alpha - x_\beta) : \alpha, \beta < \omega_1 \text{ and } p \in \mathbf{P}^* \text{ and } \rho_\alpha \restriction p = \rho_\beta \restriction p \text{ and } n < \omega\}.$$

Clearly $H^*_i \in \mathfrak{K}^{\mathrm{rtf}}$ and $\|H^*_i\| = \aleph_1 \leqslant \lambda$. Let $G \in \mathfrak{K}^{\mathrm{rtf}}_\lambda$. We shall prove that at most λ of the groups H^*_i are embeddable into G.

So assume that $Y \subseteq 2^{\aleph_0}$, $|Y| > \lambda$ and that for each $i \in Y$ we have an embedding h_i of H^*_i into G. We shall derive that G is not reduced, which is a contradiction. We choose by induction on n a set $\Gamma_n \subseteq {}^n\lambda$ and pure abelian subgroups G_η of G for $\eta \in \Gamma_n$, as follows. For $n = 0$ we let $\Gamma_0 = \{\langle\rangle\}$ and $G_{\langle\rangle} = G$. For $n+1$, for $\eta \in \Gamma_n$ such that $\|G_\eta\| > \aleph_0$ we let $\Gamma_{n,\eta} = \{\eta\hat{}\langle\zeta\rangle : \zeta < \|G_\eta\|\}$, and we let $\bar{G}_\eta = \langle G_{\eta\hat{}\langle\zeta\rangle} : \zeta < \|G_\eta\|\rangle$ be an increasing continuous sequence of subgroups of G_η of cardinality $< \|G_\eta\|$ with union G_η such that

$(*)$ for $\zeta < \|G_\eta\|$ we have
$G_{\eta\hat{}\langle\zeta\rangle} = G_\eta \cap (\text{Skolem hull of } G_{\eta\hat{}\langle\zeta\rangle} \text{ in } (\mathscr{H}(\lambda^+), \in, <^*_{\lambda^+}, G_\eta))$.

Let $\Gamma_{n+1} = \{\eta\hat{}\langle\zeta\rangle : \eta \in \Gamma_n, \|G_\eta\| > \aleph_0 \text{ with } \zeta < \|G_\eta\|\}$ and $\Gamma = \bigcup_{n<\omega} \Gamma_n$. For each $i \in Y$, let $\eta = \eta_i \in \Gamma$ be such that

(a) $\{\alpha < \omega_1 : h_i(x_\alpha) \in G_{\eta_i}\}$ is uncountable, and

(b) subject to (a), the cardinality of G_{η_i} is minimal.

Clearly η_i is well defined as (a) holds for $\eta = \langle\,\rangle$ and clearly G_{η_i} is uncountable. It is also clear that the cardinality $\|G_{\eta_i}\|$ has cofinality $\aleph_1$. Let

$$X_i = \{\alpha < \omega_1 : h_i(x_\alpha) \in G_{\eta_i}\},$$

and let $\beta_i < \omega_1$ be minimal such that $\{\rho_\alpha : \alpha \in \beta_i \cap X_i\}$ is a dense subset of $\{\rho_\alpha : \alpha \in X_i\}$. Let $\zeta_i < \|G_{\eta_i}\|$ be the minimal ζ such that

$$\{h_i(x_\alpha) : \alpha \in \beta_i \cap X_i\} \subseteq G_{\eta\hat{}\langle\zeta\rangle}$$

(ζ exists as $\mathrm{cf}(\|G_{\eta_i}\|) = \aleph_1$). Now by condition (b) the set

$$X_i' = \{\alpha < \omega_1 : h_i(x_\alpha) \in G_{\eta_i\hat{}\langle\zeta_i\rangle}\}$$

is countable, and hence we can find $\alpha_i \in X_i \backslash X_i'$.

Now the number of possible sequences $\langle \eta_i, \beta_i, \zeta_i, \alpha_i, h_i(x_{\alpha_i})\rangle$ is at most

$$|{}^{\omega>}\lambda| \times \aleph_1 \times \lambda \times \aleph_1 \times \lambda$$

(as $\Gamma \subseteq {}^{\omega>}\lambda$). So for some $\langle \eta, \beta, \zeta, \alpha, y\rangle$ and $i_0 < i_1$ from Y we have (for $l = 0, 1$)

$$\eta_{i_l} = \eta, \quad \beta_{i_l} = \beta, \quad \zeta_{i_l} = \zeta, \quad \alpha_{i_l} = \alpha, \quad h_{i_l}(x_{\alpha_l}) = y.$$

Now as h_{i_l} embeds $H_{i_l}^*$ into G and $h_{i_l}(x_\alpha) = y$ we must have

$(**)$ if $p \in \mathbf{P}^* \backslash \mathbf{Q}_{i_l}$ and $n < \omega$ then p^{-n} divides y in G.

So this holds for every $p \in (\mathbf{P}^* \backslash \mathbf{Q}_{i_0}) \cup (\mathbf{P}^* \backslash \mathbf{Q}_{i_1}) = \mathbf{P}^* \backslash (\mathbf{Q}_{i_0} \cap \mathbf{Q}_{i_1})$.

Now $\mathbf{Q}_{i_0} \cap \mathbf{Q}_{i_1}$ is finite; let $p^* \in \mathbf{P}^*$ be above its maximum. As $\{\rho_\gamma : \gamma \in X_{i_0}'\}$ is a dense subset of $\{\rho_\alpha : \alpha \in X_{i_0}\}$, there is $\gamma \in X_{i_0}'$ such that

$$\rho_\gamma \restriction p^* = \rho_\alpha \restriction p^* (= \rho_{\alpha_{i_0}} \restriction p^*).$$

Let $h_{i_0}(x_\gamma) = y^*$; thus $y^* \in G_{\eta\hat{}\langle\zeta\rangle}$.

So in $(\mathscr{H}(\lambda^+), \in, <^*_{\lambda^+}, G_\eta)$, the following formula is satisfied (recall that G_η is a pure subgroup of G):

$$\begin{aligned}\varphi(y, y^*) = \text{`in } G_\eta,\ & y \text{ is divisible by } p^n \text{ when } p \in \mathbf{P}^* \ \&\ p \geqslant p^* \ \&\ n < \omega \\ & \text{and } y - y^* \text{ is divisible by } p^n \text{ when } p \in \mathbf{P}^* \ \&\ p < p^* \ \&\ n < \omega\text{'}.\end{aligned}$$

Hence by $(*)$, i.e. by the choice of $\langle G_{\eta\hat{}\langle\xi\rangle} : \xi < \|G_\eta\|\rangle$, for some $y' \in G_{\eta\hat{}\langle\zeta\rangle}$ we must have $\varphi(y', y^*)$. Now $y \neq y'$ as $y' \in G_{\eta\hat{}\langle\zeta\rangle}$, $y \notin G_{\eta\hat{}\langle\zeta\rangle}$. Also $y - y'$ is divisible by p^n for $p \in \mathbf{P}^*, n < \omega$. (This is because if $p \geqslant p^*$ then both y and y' are divisible by p^n, and if $p < p^*$ then

$$y - y' = (y - y^*) - (y' - y^*)$$

and both $y - y^*$ and $y' - y^*$ are divisible by p^n.) As G is torsion-free, the pure closure in G of $\langle\{y - y'\}\rangle_G$ is isomorphic to $\mathbb{Q}$, and this is a contradiction since G is reduced. $\square_{1.2}$

1.3 Definition. (1) Let $\mathbf{P}^*$ be the set of all primes.

(2) For $G \in \mathfrak{K}^{\mathrm{rtf}}$ and $x \in G\backslash\{0\}$ let

(a) $\mathbf{P}(x, G) = \{p \in \mathbf{P}^* : x \in \bigcap_{n<\omega} p^n G\}$; thus $p \in \mathbf{P}(x, G)$ if and only if x is divisible in G by p^n for every $n < \omega$;

(b) $\mathbf{P}^-(x, G) = \{p : p \in \mathbf{P}^*$, but $p \notin \mathbf{P}(x, G)$ and there is $y \in G\backslash\{0\}$ such that $\mathbf{P}^*\backslash\{p\} \subseteq \mathbf{P}(y, G)$ and $p \in \mathbf{P}(x - y, G)\}$.

(3) $G \in \mathfrak{K}^{\mathrm{rtf}}$ is called full if $\mathbf{P}^* = \mathbf{P}(x, G) \cup \mathbf{P}^-(x, G)$ for every $x \in G\backslash\{0\}$.

(4) The class of full groups G in $\mathfrak{K}^{\mathrm{rtf}}$ is denoted by $\mathfrak{K}^{\mathrm{stf}}$, and $\mathfrak{K}^{\mathrm{stf}}_\lambda = \mathfrak{K}^{\mathrm{stf}} \cap \mathfrak{K}^{\mathrm{rtf}}_\lambda$. (We use s as it is the successor to r in the alphabet.)

1.4 Fact. (1) If $G \in \mathfrak{K}^{\mathrm{rtf}}$ then for any $x \in G$ the sets $\mathbf{P}(x, G)$ and $\mathbf{P}^-(x, G)$ are disjoint subsets of $\mathbf{P}^*$.

(2) If G_2 is an extension of G_1, both in $\mathfrak{K}^{\mathrm{rtf}}$, and $x \in G_1\backslash\{0\}$, then

(a) $\mathbf{P}(x, G_1) \subseteq \mathbf{P}(x, G_2)$, with equality if G_1 is a pure subgroup of G_2, and

(b) $\mathbf{P}^-(x, G_1) \subseteq \mathbf{P}^-(x, G_2)$.

(3) For every $G \in \mathfrak{K}^{\mathrm{rtf}}$ there is a G' such that

(a) G' is full, $G' \in \mathfrak{K}^{\mathrm{rtf}}$, and

(b) G is a pure subgroup of G' and $\|G'\| = \|G\|$.

Proof. Assertions (1), (2) are trivial. To prove (3) it suffices to show the following:

($*$) if $G \in \mathfrak{K}^{\mathrm{rtf}}$ and $x \in G\backslash\{0\}$, and $p \in \mathbf{P}^*\backslash\mathbf{P}(x, G)$, then for some pure extension G' of G with $\mathrm{rk}(G/G') = 1$ we have $p \in \mathbf{P}^-(x, G')$ and $G' \in \mathfrak{K}^{\mathrm{rtf}}$.

Given G, x, let $\hat{G}$ be the divisible hull of G and let

$$G_0 = \{y \in \hat{G} : \text{for some } n > 0, p^n y \in G\},$$

$$G_1 = \{y \in \hat{G} : \text{for some } b \in \mathbb{Z}, b > 0 \text{ not divisible by } p \text{ we have } by \in G\}.$$

Clearly $G = G_0 \cap G_1$. We define the following subsets of $\hat{G} \times \mathbb{Q}$:

$$H_0 = \{(y, 0) : y \in G\} \quad (\text{so } G \text{ is isomorphic to } H_0);$$

$$H_1 = \{(p^n bx, p^n b) : b, n \in \mathbb{Z}\};$$

$$H_2 = \{(0, c_1/c_2) : c_1, c_2 \in \mathbb{Z} \text{ and } c_2 \text{ not divisible by } p\}.$$

All three subsets are additive subgroups of $\hat{G} \times \mathbb{Q}$, and $H_2 \cong \mathbb{Z}_{(p)}$. Let G' be the subgroup $H_0 + H_1 + H_2$ of $\hat{G} \times \mathbb{Q}$.

We claim that $G' \cap (\hat{G} \times \{0\}) = H_0$. The inclusion $\supseteq$ is clear. For the other inclusion, let $z \in G' \cap (\hat{G} \times \{0\})$; as $z \in G'$ there are $(y, 0) \in H_0$, (so that $y \in G$), $(p^n bx, p^n b) \in H_1$ (so that $b \in \mathbb{Z}, n \in \mathbb{Z}$ and $x \in G$ is the constant from $(*)$) and $(0, c_1/c_2) \in H_2$ (so that $c_1, c_2 \in \mathbb{Z}$ and p does not divide c_2) and integers a_0, a_1, a_2 such that

$$z = a_0(y, 0) + a_1(p^n bx, p^n b) + a_2(0, c_1/c_2),$$

which means that

$$z = (a_0 y + a_1 p^n bx, a_1 p^n b + a_2 c_1/c_2).$$

As $z \in \hat{G} \times \{0\}$ clearly $a_1 p^n b + a_2 c_1/c_2 = 0$; so as p does not divide c_2, necessarily $a_1 p^n b$ is an integer. Thus $a_1 p^n bx \in G$ and so as $y \in G$ clearly $a_0 y + a_1 p^n bx \in G$. Therefore $z \in G \times \{0\} = H_0$ as required.

It is easy to check now that H_0 is a pure subgroup of G'.

Let $y^* = (0, -1)$. Clearly $(x, 0) - y^*$ is divisible by p^k for every $k < \omega$ (as $(p^k x, p^k) \in H_1 \subseteq G'$ for every $k \in \mathbb{Z}$) and y^* is divisible by any integer b when b is not divisible by p (as $(1/b)y^* = (0, -1/b) \in H_2 \subseteq G'$).

Identifying $y \in G$ with $(y, 0) \in G$ we are done: G' is as required in $(*)$, with y^* witnessing that $p \in \mathbf{P}^-(x, G')$. $\square_{1.4}$

1.5 Claim. *Suppose that $G_1 \in \mathfrak{K}^{\mathrm{rtf}}$ is full and $G_2 \in \mathfrak{K}^{\mathrm{rtf}}$. If h is an embedding of G_1 into G_2 then*

$$\textit{for } x \in G_1 \backslash \{0\}, \quad \mathbf{P}(x, G_1) = \mathbf{P}(h(x), G_2).$$

Proof. Without loss of generality h is the identity; now we use 1.4(1), 1.4(2) and the definition of 'full'. $\square_{1.5}$

1.6 Conclusion. *Assume*

$(*)$ $2^{\aleph_0} < \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$.

Then there is no universal member in $\mathfrak{K}^{\mathrm{rtf}}_\lambda$.

Proof. Let

$$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0 \text{ and } \omega^2 \text{ divides } \delta\}$$

be stationary and $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$, where each η_δ is an increasing ω-sequence of ordinals $< \delta$ with limit δ such that $\eta_\delta(n) - n$ is well defined and divisible by ω; so if $\delta_1 \neq \delta_2$ then $\mathrm{Range}(\eta_{\delta_1}) \cap \mathrm{Range}(\eta_{\delta_2})$ is finite. Let $\{p_n^* : n < \omega\}$ list the primes in increasing order. Let $G^0_{\bar{\eta}}$ be the abelian group generated by

$$\{x_\alpha : \alpha < \lambda\} \cup \{y_\delta : \delta \in S\} \cup \{z_{\delta,n,l} : n, l < \omega\} \cup \{x_{\alpha,m,l} : \alpha < \lambda, m, l < \omega, \alpha \neq m \bmod \omega\}$$

freely except for the equations

$$p_n^* z_{\delta,n,l+1} = z_{\delta,n,l}, \quad y_\delta - x_{\eta_\delta(n)} = z_{\delta,n,0},$$
$$p_m^* x_{\alpha,m,l+1} = x_{\alpha,m,l}, \quad x_{\alpha,m,0} = x_\alpha, \quad \text{if } \alpha \neq m \bmod \omega.$$

It is easy to check that $G^0_{\bar\eta} \in \mathfrak{K}^{\mathrm{rtf}}_\lambda$, that $\mathbf{P}^-(y_\delta, G^0_{\bar\eta})$ is the set of all primes and that $\mathbf{P}(x_\alpha, G^0_{\bar\eta})$ is the set of primes $\neq p_n^*$ if $\alpha = n \bmod \omega$.

Let $G_{\bar\eta} \in \mathfrak{K}^{\mathrm{rtf}}_\lambda$ be a pure extension of $G^0_{\bar\eta}$ which is full (one exists by 1.4(3)). So

$(*)$ if h embeds $G_{\bar\eta}$ into $G \in \mathfrak{K}^{\mathrm{rtf}}_\lambda$ then

$$x \in G_{\bar\eta}\backslash\{0\} \Rightarrow \mathbf{P}(x, G_{\bar\eta}) = \mathbf{P}(h(x), G).$$

Hence the proof in [3] is valid here. $\square_{1.6}$

1.7 Remark. (1) Similarly, the results in [3] on λ singular (i.e. Case 4) hold for embedding (rather than for pure embedding).

(2) In Case 5, if there is a family $\mathscr{P} \subseteq \{C \subseteq \mu^+ : \mathrm{otp}(C) = \mu\}$ which guesses clubs (i.e. every club E of μ^+ contains one of them), and its cardinality is $< \mu^{\aleph_0}$ then the result of 1.6 holds for μ^+.

(3) Concerning the case when $\aleph_0 \leqslant \lambda < 2^{\aleph_0}$ see also 3.17.

2 The existence of universals for separable reduced abelian p-groups

Here we eliminate the very weak pcf assumption from the theorem showing that there is no universal in $\mathfrak{K}^{\mathrm{rs}(p)}_\lambda$ when $\lambda > \beth_\omega$. The class $\mathfrak{K}^{\mathrm{rs}(p)}$ is defined in 2.3(4).

In the first section we have eliminated the very weak pcf assumptions for the theorem concerning $\mathfrak{K}^{\mathrm{rtf}}_\lambda$ (though the condition that $\lambda = \mathrm{cf}(\lambda) > \mu^+$ remains, i.e. we assume that we are in Case 3). This was done using the 'infinitely many primes', so in the language of e.g. [3] the invariant refers to one element x. This cannot be generalized to $\mathfrak{K}^{\mathrm{rs}(p)}_\lambda$. However, in [8, Section 5] we use an invariant on e.g. suitable groups and related stronger 'combinatorial' ideals. We continue this, using combinatorial ideals closer to the algebraic ones to show that the algebraic ideal is non-trivial.

We rely on the 'GCH right version' provable from ZFC (see [11]); hence the condition '$\lambda > \beth_\omega$' is used.

2.1 Definition. (1) For $\bar\lambda = \langle \lambda_l : l < \omega \rangle$ and $\bar t = \langle t_l : l < \omega \rangle$ (with $1 < t_l < \omega$) we define $J^4_{\bar t, \bar\lambda}$ to be the family of subsets A of $\prod_{l<\omega} [\lambda_l]^{t_l}$ satisfying the following condition:

$(*)_A$ for every large enough $l < \omega$, for every $B \in [\lambda_l]^{\aleph_0}$ for some $k \in (l, \omega)$ we *cannot find*

$$\left\langle \nu_\eta : \eta \in \prod_{i \in [l,k)} [\omega]^{t_i} \right\rangle$$

such that

(a) $\nu_\eta \in A$;

(b) if $\eta_1, \eta_2 \in \prod_{i\in[l,k)}[\omega]^{t_i}$, $l \leqslant m \leqslant k$ and $\eta_1 \restriction [l,m) = \eta_2 \restriction [l,m)$ then $\nu_{\eta_1} \restriction m = \nu_{\eta_2} \restriction m$; hence $\nu_{\eta_1} \restriction l = \nu_{\eta_2} \restriction l$ for $\eta_1, \eta_2 \in \prod_{i\in[l,k)}[\omega]^{t_i}$;

(c) if $\eta_0 \in \prod_{i\in[l,k)}[\omega]^{t_i}$ and $l \leqslant m < k$ then for some $E \in [\lambda_m]^{\aleph_0}$ we have

$$[E]^{t_m} = \left\{\nu_\eta(\mu)\ \ \eta \in \prod_{\iota\in[\lambda,\kappa)}[\omega]^{\tau_\iota} \text{ and } \eta \restriction \mu = \eta_\varkappa \restriction \mu\right\}$$

and $m = l \Rightarrow E = B$.

(2) Let $J^4_{\bar t,\bar\lambda,<\theta}$ be the family of subsets A of $\prod_{l<\omega}[\lambda_l]^{t_l}$ such that for some $\alpha < \theta$ and $A_\beta \in J^4_{\bar t,\bar\lambda}$ for $\beta < \alpha$ we have $A \subseteq \bigcup_{\beta<\alpha} A_\beta$.

When $\theta = \kappa^+$, we may write κ instead of $<\theta$.

2.2 Fact. (1) $J^4_{\bar t,\bar\lambda,\theta}$ is a θ^+-complete ideal.

(2) If $\lambda_l > \beth_{t_l-1}(\theta)$ for each $l < \omega$ then the ideal $J^4_{\bar t,\bar\lambda,\theta}$ is proper (where $\beth_0(\theta) = \theta$, $\beth_{n+1}(\theta) = 2^{\beth_n(\theta)}$, and for general α we have $\beth_\alpha(\theta) = \theta + \sum_{\beta<\alpha} 2^{\beth_\beta(\theta)}$).

Proof. Assertion (1) is trivial. To prove (2), for $l < \omega$ let

$$\begin{aligned}\mathrm{ERI}^{t_l}_{\lambda_l} = \{A \subseteq [\lambda_l]^{t_l} : {} & \text{for some } F : [\lambda_l]^{t_l} \to \theta \text{ there is no } B \in [\lambda_l]^{\aleph_0}\\ & \text{such that } F \restriction [B]^{t_l} \text{ is constant and } [B]^{t_l} \subseteq A\}.\end{aligned}$$

This is a θ^+-complete ideal and it is non-trivial by the Erdös–Rado theorem (which we used similarly in [10, Section 1]). Now we shall prove that the ideal $J^4_{\bar t,\bar\lambda,\theta}$ is proper. So we assume that $\prod_{l<\omega}[\lambda_l]^{t_l} = \bigcup_{i<\theta} X_i$ and $X_i \in J^4_{\bar t,\bar\lambda}$ for each $i < \theta$ and we shall obtain a contradiction. Let

$$X_i^+ = \left\{\eta \in \prod_{l<\omega}[\lambda_l]^{t_l} : \text{for every } l < \omega \text{ for some } \eta' \in X_i \text{ we have } \eta \restriction l = \eta' \restriction l\right\}$$

(i.e. X_i^+ is the closure of X_i). So

$$X_i^+ \subseteq \prod_{l<\omega}[\lambda_l]^{t_l} = \prod_{l<\omega}\mathrm{Dom}(\mathrm{ERI}^{t_l}_{\lambda_l})$$

is closed, and those ideals are θ^+-complete and $\prod_{l<\omega}\mathrm{Dom}(\mathrm{ERI}^{t_l}_{\lambda_i}) = \bigcup_{i<\theta} X_i^+$. Hence (see Rubin–Shelah [4], [9, Chapter XI, 3.5(2)] with $H_\alpha = X_i^+$) we can find T such that

(a) $T \subseteq \bigcup_{m<\omega}\prod_{l<m}[\lambda_l]^{t_l}$,

(b) T is closed under initial segments,

(c) $\langle\,\rangle \in T$,

(d) if $\nu \in T$ and $lg(\nu) = l$ then $\{u \in [\lambda_l]^{t_l} : \nu\hat{\ }\langle u\rangle \in T\} \in (\mathrm{ERI}^{t_l}_{\lambda_l})^+$,

(e) for some $i < \theta$, $\lim(T) \subseteq X_i^+$.

(Here, $\lim(T) = \{\nu \in \prod_{l<\omega}[\lambda_l]^{t_l} : (\forall m < \omega)\nu \restriction m \in T\}$.) Fix i from clause (e). We would like to prove $\neg(*)_{X_i^+}$; the definition of the ideal $\mathrm{ERI}^{t_l}_{\lambda_l}$ gives more than required (with 'for every l' instead of 'for arbitrarily large l' in the negation $(*)$ of Definition 2.1). $\square_{2.2}$

Remark. We note that we could have used the stronger ideal defined implicitly in 2.2, i.e. the family $J^5_{\bar{t},\bar{\lambda},\theta}$ of sets $X \subseteq \prod_{l<\omega}[\lambda_l]^{t_l}$ for which we can find $\alpha < \theta$ and $X_i \subseteq X$ for $i < \alpha$ such that $X = \bigcup_{i<\alpha} X_i$ and for each i and T satisfying (a)–(d) from the proof of 2.2 there is $T' \subseteq T$ satisfying (a)–(d) such that $\lim(T)$ is disjoint from the closure of X_i.

Of course, we can also replace $\mathrm{ERI}^{t_l}_{\lambda_l}$ by variants.

We recall the following definition from [8, 5.1].

2.3 Definition. (1) For $\bar{\mu} = \langle \mu_n : n < \omega\rangle$ let $B_{\bar{\mu}}$ be the following direct sum of cyclic p-groups. Let K^n_α be a cyclic group of order p^{n+1} generated by x^n_α and let $B^n_{\mu_n} = \bigoplus_{\alpha<\mu_n} K^n_\alpha$ and $B_{\bar{\mu}} = \bigoplus_{n<\omega} B^n_{\mu_n}$, i.e. $B_{\bar{\mu}}$ is the abelian group generated by $\{x^n_\alpha : n < \omega, \alpha < \mu_n\}$ freely except that $p^{n+1}x^n_\alpha = 0$.

Let

$$B_{\bar{\mu}\restriction n} = \bigoplus\{K^m_\alpha : \alpha < \mu_m, m < n\} \subseteq B_{\bar{\mu}}.$$

These groups are in $\mathfrak{K}^{\mathrm{rs}(p)}_{\leqslant \sum_n \mu_n}$.

Let $\hat{B}_{\bar{\mu}}$ be the p-torsion completion of $B_{\bar{\mu}}$ (i.e. from the completion under the norm $\|x\| = \min\{2^{-n} : p^n \text{ divides } x\}$ we take only the torsion elements; see Fuchs [1]. Note that $\hat{B}_{\bar{\mu}}$ is the torsion part of the p-adic completion of $B_{\bar{\mu}}$).

(2) Let $I^1_{\bar{\mu},<\theta} = I^1_{\bar{\mu},<\theta}[p]$ be the ideal on $\hat{B}_{\bar{\mu}}$ (depending on the choice of $\langle K^n_\alpha : \alpha < \mu_n,\ n < \omega\rangle$ or actually $\langle B_{\bar{\mu}\restriction n} : n < \omega\rangle$) consisting of unions of $< \theta$ members of $I^0_{\bar{\mu}}$, where

$$\begin{aligned} I^0_{\bar{\mu}} &= I^0_{\bar{\mu}}[p] \\ &= \{A \subseteq \hat{B}_{\bar{\mu}} : \text{for all large enough } n, \text{ we have } \mathrm{cl}_{\hat{B}_{\bar{\mu}}}(\langle A\rangle_{\hat{B}_{\bar{\mu}}}) \cap B_{\bar{\mu}} \subseteq B_{\bar{\mu}\restriction n}\}. \end{aligned}$$

(The definition of $\mathrm{cl}_{\hat{B}_{\bar{\mu}}}$ is given in (3) below.) When $\theta = \kappa^+$ we may write κ instead of $< \theta$. If $\mu_n = \mu$, we may write μ instead of $\bar{\mu}$.

(3) For $X \subseteq \hat{B}_{\bar{\mu}}$, recall that $\langle X\rangle_{\hat{B}_{\bar{\mu}}}$ is the subgroup of $\hat{B}_{\bar{\mu}}$ generated by X and that

$$\mathrm{cl}_{\hat{B}_{\bar{\mu}}}(X) = \{x : (\forall n)(\exists y \in X)(x - y \in p^n\hat{B}_{\bar{\mu}})\}.$$

(4) Let $\mathfrak{K}^{\mathrm{rs}(p)}$ be the family of pure subgroups of some $\hat{B}_{\bar{\mu}}$.

(5) If p is not clear from the context we may write $B_{\bar{\mu}}[p], \hat{B}_{\bar{\mu}}[p]$, etc.

2.4 Claim. *Assume that* $\bar{\mu} = \langle \mu_n : n < \omega \rangle$, $\bar{t} = \langle t_l : l < \omega \rangle$, $t_l = p$ *and that the ideal* $J^4_{\bar{t},\bar{\mu},\theta}$ *is proper* (*so that* $\mu_n \geqslant \beth_{p-1}(\theta)^+$ *is enough by* 2.2(2)). *Then the ideal* $I^1_{\bar{\mu},\theta}$ *is proper* (*and* $I^1_{\bar{\mu},\theta}$ *is a* θ^+*-complete ideal*).

Proof. We define a function h from $\prod_{l<\omega}[\lambda_l]^{t_l}$ into $\hat{B}_{\bar{\mu}}$. We let

$$h(\eta) = \Sigma\{p^n x^n_\beta : \beta \in \eta(n) \text{ and } n < \omega\} \in \hat{B}_{\bar{\mu}}[p].$$

Clearly h is one-to-one and it suffices to prove

$(*)$ if $X \in (J^4_{\bar{t},\bar{\mu},\theta})^+$ then $h''(X)$ belongs to $(I^1_{\bar{\mu},\theta})^+$.

So assume that $X \in (J^4_{\bar{t},\bar{\lambda},\theta})^+$ is given and suppose for a contradiction that $h''(X) \in I^1_{\bar{\mu},\theta}$. So we can find $\langle Y_i : i < \theta\rangle$ such that for such $i < \theta$ we have $Y_i \in I^0_{\bar{\mu}}$ and $h(X) \subseteq \bigcup_{i<\theta} Y_i$. Let $X_i = h^{-1}(Y_i)$. So $h(X_i) \subseteq Y_i \in I^0_{\bar{\mu}}$ and hence $h(X_i) \in I^0_{\bar{\mu}}$. But as $J^4_{\bar{t},\bar{\lambda},\theta}$ is θ^+-complete and $X \in (J^4_{\bar{t},\bar{\lambda},\theta})^+$ we have $X_i \in (J^4_{\bar{t},\bar{\lambda},\theta})^+$ for some $i < \theta$, and so without loss of generality $h''(X) \in I^0_{\bar{\mu}}$. By the definition of $I^0_{\bar{\mu}}$, for some $n(*) < \omega$ we have

$(*)$ $B_{\bar{\mu}} \cap \mathrm{cl}_{\hat{B}_{\bar{\mu}}}(\langle h''(X)\rangle_{\hat{B}_{\bar{\mu}}}) \subseteq B_{\bar{\mu} \restriction n(*)}$.

On the other hand, as $X \in (J^4_{\bar{t},\bar{\mu},\theta})^+$ we have $X \notin J^4_{\bar{t},\bar{\mu}}$, and so from the definition of $J^4_{\bar{t},\bar{\mu}}$ in 2.1(1) we can find $\langle B_n : n \in \Gamma\rangle$ such that

(a) $\Gamma \in [\omega]^{\aleph_0}$ and $B_n \in [\lambda_n]^{\aleph_0}$, and

(b) for $n \in \Gamma$ and for every $k \in (n, \omega)$ we can find $\langle \nu^{n,k}_\eta : \eta \in \prod_{l\in[n,k)}[\omega]^{t_l}\rangle$ as in (a)–(c) of Definition 2.1(1), with n, B_n, k here standing for l, B, k there.

For $m \in (n,k]$ and $\eta \in \prod_{l\in[n,m)}[\omega]^{t_l}$ we let $\nu^{n,k}_\eta$ be $\nu^{n,k}_{\eta_1} \restriction m$ whenever we have $\eta \lhd \eta_1 \in \prod_{l\in[n,k)}[\omega]^{t_l}$; by clause (b) in $(*)$ of 2.1 this is well defined. We fix temporarily $n \in \Gamma$ and $k \in [n,\omega)$. Let $A_\eta = A^{n,k}_\eta \in [\lambda_m]^{\aleph_0}$ where $m = lg(\eta)$ be such that

$$\{\nu^{n,k}_{\eta\hat{}\langle u\rangle}(m) : u \in [\omega]^{t_m}\} = [A_\eta]^{t_m}$$

and without loss of generality (otp stands for 'the order type')

$(*)$ $\mathrm{otp}(A_\eta) = \omega$ and $\nu^{n,k}_{\eta\hat{}\langle u\rangle}(m) = \mathrm{OP}_{A_\eta,\omega}(u)$

(where $\mathrm{OP}_{A_\eta,\omega}(i) = \alpha$ if and only if $i = \mathrm{otp}(A_\eta \cap \alpha)$).

Now for $m \in (n,k]$ and $\eta \in \prod_{l\in[n,m)}[\omega]^{t_l}$ we define

$$\begin{aligned} y_\eta &= y^{n,k}_\eta \\ &= \sum\Big\{h(\nu^{n,k}_\rho) : \eta \trianglelefteq \rho \in \prod_{l\in[n,k)}[\omega]^{t_l} \text{ and } (\forall l)[lg(\eta) \leqslant l < k \rightarrow \rho(l) \subseteq [0,t_l]]\Big\} \end{aligned}$$

where $\trianglelefteq$ denotes being an initial segment. So $y_\eta \in \hat{B}_{\bar{\mu}}$ and we shall prove by downward induction on $m \in (n,k]$ that for every $\eta \in \prod_{l\in[n,m)}[\omega]^{t_l}$ we have (writing $\sum_{l<m}$

for $\sum_{l\in[n,m)}$)

$\boxtimes_m$ $y_\eta = (\prod_{l=m}^{k-1}(t_l+1)) \times (\sum_{l<m}\sum_{\alpha\in v_\eta^{n,k}(l)} p^l x_\alpha^l) \bmod p^k \hat{B}_{\bar{\mu}}$.

Case 1. $m = k$.

In this case the product $\prod_{l=m}^{k-1}(t_l+1)$ is just 1, so the equation becomes

$$y_\eta = \sum_{l<m}\sum_{\alpha\in v_\eta^{n,k}(l)} p^l x_\alpha^l \bmod p^k \hat{B}_{\bar{\mu}}.$$

Now the expression for y_η is

$$\begin{aligned}&\sum\Big\{h(v_\rho^{n,k}) : \eta \trianglelefteq \rho \in \prod_{l\in[n,k)}[\omega]^{t_l} \text{ and } (\forall l)[m \leqslant l < k \Rightarrow \rho(l) \subseteq [0,t_l]]\Big\}\\ &\quad = h(v_\eta^{n,k}) = \sum_{l<\omega}\sum_{\alpha\in v_\eta^{n,k}(l)} p^l x_\alpha^l\\ &\quad = \sum_{l<m}\sum_{\alpha\in v_\eta^{n,k}(l)} p^l x_\alpha^l + p^k\Big(\sum_{l\in[k,\omega)}\sum_{\alpha\in v_\eta^{n,k}(l)} p^{l-k} x_\alpha^l\Big)\end{aligned}$$

and so the equality is trivial.

Case 2. $n < m < k$.

Here (with equalities in the equation being in $\hat{B}_{\bar{\mu}}$, modulo $p^k\hat{B}_{\bar{\mu}}$), we have

$$\begin{aligned}y_\eta &= \sum\{y_{\eta\hat{}\langle u\rangle} : u \in [\{0,\dots,t_m\}]^{t_m}\} \quad (\text{by the definition of } y_\eta, y_{\eta\hat{}\langle u\rangle})\\ &= \sum\Big\{\Big(\prod_{l=m+1}^{k-1}(t_l+1)\Big)\Big(\sum_{l<m+1}\sum_{\alpha\in v_{\eta\hat{}\langle u\rangle}^{n,k}(l)} p^l x_\alpha^l\Big) : u \in [\{0,\dots,t_m\}]^{t_m}\Big\}\\ &\qquad (\text{by the induction hypothesis})\\ &= \sum\Big\{\Big(\prod_{l=m+1}^{k-1}(t_l+1)\Big)\Big(\sum_{l<m}\sum_{\alpha\in v_{\eta\hat{}\langle u\rangle}^{n,k}(l)} p^l x_\alpha^l\Big) : u \in [\{0,\dots,t_m\}]^{t_m}\Big\}\\ &\quad + \sum\Big\{\Big(\prod_{l=m+1}^{k-1}(t_l+1)\Big)\sum_{\alpha\in \mathrm{OP}_{\omega,A_\eta}(u)} p^m x_\alpha^m : u \in [\{0,\dots,t_m\}]^{t_m}\Big\}\\ &\qquad \Big(\text{dividing the sum } \sum_{l<m+1} \text{ into } \sum_{l<m} \text{ and } \sum_{l=m} \text{ and noting what } v_{\eta\hat{}\langle u\rangle}^{n,k}(m) \text{ is}\Big)\end{aligned}$$

$$= \sum \Big\{ \Big(\prod_{l=m+1}^{k-1} (t_l + 1) \Big) \Big(\sum_{l<m} \sum_{\alpha \in v_\eta^{n,k}(l)} p^l x_\alpha^l \Big) : u \in [\{0, \ldots, t_m\}]^{t_m} \Big\}$$

$$+ \sum \Big\{ \Big(\prod_{l=m+1}^{k-1} (t_l + 1) \Big) (p^m x_\alpha^m) |\{u : u \in [\{0, \ldots, t_m\}]^{t_m} \text{ and}$$

$$|\alpha \cap A_\eta| \in u\}| : \alpha \text{ is a member of } A_\eta, \text{ moreover } |\alpha \cap A_\eta| \leqslant t_m \Big\}$$

(collecting together terms with x_α^m in the second sum)

$$= \Big(\prod_{l=m+1}^{k-1} (t_l + 1) \Big) \Big(\sum_{l<m} \sum_{\alpha \in v_\eta^{n,k}(l)} p^l x_\alpha^l \Big) \times |\{u : u \in [\{0, \ldots, t_m\}]^{t_m}\}|$$

$$+ \sum \Big\{ \Big(\prod_{l=m+1}^{k-1} (t_l + 1) \Big) (p^m x_\alpha^m) \cdot ((t_m + 1) - 1) : \alpha \in A_\eta, |\alpha \cap A_\eta| \leqslant t_m \Big\}$$

$$= (t_m + 1) \Big(\prod_{l=m+1}^{k-1} (t_l + 1) \Big) \sum_{l<m} \sum_{\alpha \in v_\eta^{n,k}(l)} p^l x_\alpha^l + 0$$

(since $t_m = p$ and $p^{m+1} x_\alpha^m = 0$)

$$= \Big(\prod_{l=m}^{k} (t_l + 1) \Big) \Big(\sum_{l<m} \sum_{\alpha \in v_\eta^{n,k}(l)} p^l x_\alpha^l \Big).$$

Hence we have finished the proof of $\boxtimes_m$.

Now as $t_l + 1 = p + 1$ and $pp^l x_\alpha^l = 0$ in $\hat{B}_{\bar{\mu}}$ we obtain

$\boxtimes'_m$ $y_\eta = \sum_{l<m} \sum_{\alpha \in v_\eta^{n,k}(l)} p^l x_\alpha^l \bmod p^k \hat{B}_{\bar{\mu}}$.

Note that for $m = n + 1$ the sum $\sum_{l<m}$ is just $\sum_{l=n}$. So, because for $n \in \Gamma$ the subset B_n serves for every $k \in (n, \omega)$, if $u_1, u_2 \in [B_n]^{t_n}$ are distinct, then for $k \in (n, \omega)$ we have letting $m = n + 1$

$$y_{\langle u_1 \rangle} - y_{\langle u_2 \rangle} = \sum_{l<m} \sum_{\alpha \in v_{\langle u_1 \rangle}^{n,k}(l)} p^l x_\alpha^l - \sum_{l<m} \sum_{\alpha \in v_{\langle u_2 \rangle}^{n,k}(l)} p^l x_\alpha^l \bmod p^k \hat{B}_{\bar{\mu}}.$$

As this holds for every $k \in (n, \omega)$ we get equality. By the demands on $v_\eta^{n,k}$ (see clause (b) above) we have $y_{\langle u_1 \rangle} - y_{\langle u_2 \rangle} \notin B_{\bar{\mu} \restriction n}$; but from the last sentence we have $y_{\langle u_1 \rangle} - y_{\langle u_2 \rangle} \in B_{\bar{\mu} \restriction (n+1)}$, contradicting $(*)$. $\square_{2.4}$

2.5 Definition. (1) Let I be an ideal on κ (or just $I \subseteq \mathscr{P}(\kappa)$ closed downwards, $I^+ = \mathscr{P}(\kappa) \backslash I$). Write

$$\mathbf{U}_I(\lambda) = \min\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^{\leqslant \kappa} \text{ and for every } f \in {}^\kappa\lambda$$
$$\text{for some } a \in \mathscr{P} \text{ we have } \{i < \kappa : f(i) \in a\} \in I^+\}.$$

(2) For $\sigma \leqslant \theta \leqslant \mu \leqslant \lambda$ let

$$\operatorname{cov}(\lambda,\mu,\theta,\sigma) = \min\{\lambda + |\mathscr{P}| : \mathscr{P} \text{ is a family of subsets of } \lambda \text{ each of cardinality} < \mu \text{ such that all } X \subseteq \lambda \text{ of cardinality} < \theta \text{ is contained in the union of} < \sigma \text{ members of } \mathscr{P}\}.$$

2.6 Claim. (1) *For every* $\lambda \geqslant \beth_\omega$, *for some* $\theta < \beth_\omega$ *for every* $\mu \in (\beth_{p-1}(\theta), \beth_\omega)$ *we have* (*writing* $\mu_n = \mu$), $\mathbf{U}_{I^1_{\mu,\theta}}(\lambda) = \lambda$ (*and hence* $\mathbf{U}_{I^0_{\bar\mu}}(\lambda) = \lambda$).

(2) *If* $\operatorname{cf}(\lambda) > \aleph_0$, *then for some* $\theta < \beth_\omega$, *for every* $\mu \in (\beth_{p-1}(\theta), \beth_\omega)$ *and* $\lambda' < \lambda$ *we have* $\mathbf{U}_{I^1_{\mu,\theta}}(\lambda') < \lambda$.

Proof. By 2.4, $I_{\mu,\theta}$ is a θ^+-complete proper ideal on a set of cardinality $\mu^{\aleph_0}$, for all μ, θ as in the assumptions. From [11], for each $\lambda' \leqslant \lambda$ for some $\theta = \theta[\lambda'] < \beth_\omega$ for every $\mu \in (\theta, \beth_\omega)$ we have $\operatorname{cov}(\lambda', \mu^+, \mu^+, \theta) = \lambda'$, i.e. there is $\mathscr{P}_\mu \subseteq [\lambda']^\mu$ of cardinality $\leqslant\lambda'$ such that if $Y \in [\lambda']^{\leqslant\mu}$ then Y is contained in the union of $< \theta$ members of $\mathscr{P}_\mu$. As $I^1_{\mu,\theta}$ is a θ^+-complete ideal on a set of cardinality μ it follows that $\mathbf{U}_{I^1_{\mu,\theta}}(\lambda') \leqslant \lambda'|\mathscr{P}_\mu| = \lambda'$ (and trivially $\mathbf{U}_{I^1_{\bar\mu,\theta}}(\lambda) \geqslant \lambda$). This proves (1).

In (2) we have $\operatorname{cf}(\lambda) > \aleph_0$, and so for some $\theta < \beth_\omega$, for arbitrarily large $\lambda' < \lambda$ we have $\theta[\lambda'] \leqslant \theta$; and clearly the result follows. $\square_{2.6}$

2.7 Conclusion. *If* $\beth_\omega \leqslant \mu^+ < \lambda = \operatorname{cf}(\lambda) < \mu^{\aleph_0}$, *then there is no universal member in* $\mathfrak{K}^{\mathrm{rs}(p)}_\lambda$.

Proof. This follows from 2.5 and [8, 5.9].

We also have the following result.

2.8 Claim. *Assume that the following conditions hold*:

(a) $\prod_{l<\omega} \kappa_l < \mu < \lambda = \operatorname{cf}(\lambda) \leqslant \lambda' < \mu^{\aleph_0}$;

(b) $\mu^+ < \lambda$ *or at least for some* $\mathscr{P}$ *we have*

$$(*)_{\mathscr{P}} \quad |\mathscr{P}| = \lambda \quad \& \quad (\forall a \in \mathscr{P})(a \subseteq \lambda \;\&\; \operatorname{otp}(a) = \mu) \quad \& \quad (\forall E)(E \text{ a club of } \lambda \to (\exists a \in \mathscr{P})(a \subseteq E));$$

(c) $\lambda' = \mathbf{U}_{I^0_{\bar\kappa}}(\lambda) < \mu^{\aleph_0}$ *where* $\bar\kappa = \langle \kappa_l : l < \omega \rangle$; *note that* $I^0_{\bar\kappa}$ *depends on the prime p*.

Then we can find reduced separable abelian p-groups $G_\alpha \in \mathfrak{K}^{\mathrm{rs}(p)}_\lambda$ *for* $\alpha < \mu^{\aleph_0}$ *such that for every reduced separable abelian p-group G of cardinality* λ *the following holds*: *some* G_α *is not embeddable into G*; *also the number of ordinals* $\alpha < \mu^{\aleph_0}$ *such that* G_α *is embeddable into G is* $\leqslant\lambda'$.

Moreover, each G_α *is slender, i.e. there is no homomorphism from* $\mathbb{Z}^\omega$ *into* G_α *with range of infinite rank.*

Proof. The proof is the same as that of [8, 5.9 and 7.5].

3. Non-existence of universals for $\aleph_1$-free abelian groups

The first section dealt with $\mathfrak{K}^{\mathrm{rtf}}_{\lambda}$ and improved results in [8]. But the groups used there are 'almost divisible'. Now we ask what happens if we replace $\mathfrak{K}^{\mathrm{rtf}}$ by a variant avoiding this. We propose to consider the $\aleph_1$-free abelian groups where type arguments like those in Section 1 break down. So the proof of [8] becomes relevant and it is natural to improve it as in Section 2 (which dealt with $\mathfrak{K}^{\mathrm{rs}(p)}$). For diversity we use a stronger ideal. We have not considered the problem for $\aleph_1$-free abelian groups of cardinality λ when $\aleph_0 < \lambda < 2^{\aleph_0}$. So we concentrate here on torsion-free (abelian) groups.

3.1 Definition. (1) Let $\bar{t} = \langle t_l : l < \omega \rangle$, $2 \leqslant t_l < \omega$. For an abelian group H, the $\bar{t}$-valuation is defined by

$$\|x\|_{\bar{t}} = \inf\Big\{2^{-m} : \prod_{l<m} t_l \text{ divides } x \text{ in } G\Big\}.$$

This is a semi-norm. We recall that $d_{\bar{t}}(x,y) = \|x-y\|_{\bar{t}}$. This semi-norm induces a topology, called the $\bar{t}$-adic topology. If $t_l = p$ for $l < \omega$, we may write p instead of $\bar{t}$.

(2) Let $\mathrm{cl}_{\bar{t}}(A,H)$ be the closure of A in H in the $\bar{t}$-adic topology.

Let $\mathrm{PC}_H(X)$ be the pure closure of X in H, that is

$$\mathrm{PC}_H(X) = \{x \in H : \text{for some } n > 0,\ nx \text{ belongs to } \langle x \rangle_H\}.$$

Let $\mathrm{PC}^p_H(X)$ be the p-adic closure in H of the subgroup of H generated by X.

(3) Let $\mathfrak{K}^{\mathrm{rtf}}[\bar{t}]$ be the class of $\bar{t}$-reduced torsion-free abelian groups, i.e. the groups $G \in \mathfrak{K}^{\mathrm{rtf}}$ such that $\bigcap_{n<\omega}(\prod_{i<n} t_i)G = \{0\}$; hence $\|-\|_{\bar{t}}$ induces a Hausdorff topology on these groups. (Conversely, if G is torsion-free and the $\bar{t}$-adic topology is Hausdorff, then $G \in \mathfrak{K}^{\mathrm{rtf}}[\bar{t}]$.)

(4) If the $\bar{t}$-adic topology is Hausdorff, then $G^{[\bar{t}]}$ is the completion of G with respect to $\|-\|_{\bar{t}}$. If $t_l = 2 + l$, this is the $\mathbb{Z}$-adic completion.

The following continues the analysis in [8, 1.1] (which dealt with $\mathfrak{K}^{\mathrm{rs}(p)}$) and [8, 1.5] (which dealt with $\mathfrak{K}^{\mathrm{rtf}}$).

3.2 Definition. We say that G has $\bar{t}$-density μ if it has a pure subgroup of cardinality $\leqslant \mu$ which is $\bar{t}$-dense, i.e. dense in the $\bar{t}$-adic topology, but has no such subgroup of cardinality $< \mu$.

3.1 Proposition. *Suppose that*

(α) $\mu \leqslant \lambda \leqslant \mu^{\aleph_0}$,

(β) *G is an $\aleph_1$-free abelian group with $|G| = \lambda$,*

(γ) *$\bar{t}$ is as in* 3.1 *such that* $(\forall l)(\exists m > l)$ *(t_l divides t_m).*

Then there is an $\aleph_1$-free group H such that $G \subseteq H$, $|H| = \lambda$ and H has $\bar{t}$-density μ.

Proof. Choose $\lambda_n < \mu$ for $n < \omega$ such that

$$\prod_{n<\omega} \lambda_n \geqslant \lambda, \quad \mu \geqslant \sum_{n<\omega} \lambda_n, \quad 2\lambda_n < \lambda_{n+1}$$

(so $\lambda_n > 0$ may be finite). Let $\{x_i : i < \lambda\}$ list the elements of G. Let $\lambda'_{n+1} = \lambda_{n+1}$, $\lambda'_0 = \mu$. Let $\eta_i \in \prod_{n<\omega} \lambda_n$ for $i < \lambda$ be pairwise distinct such that $\eta_i(n+1) \geqslant \lambda_n$ and such that

$$i \neq j \Rightarrow (\exists m)(\forall n)[m \leqslant n \Rightarrow \eta_i(n) \neq \eta_j(n)].$$

Without loss of generality $\mu = \{\eta_i(n) : i < \lambda, n < \omega\}$. Let H be generated by G, x_i^m for $i < \lambda'_m$, $m < \omega$, y_i^n for $i < \lambda$, $n < \omega$, freely except for the following relations:

(a) the equations of G;

(b) $y_i^0 = x_i$ $(\in G)$;

(c) $t_n y_i^{n+1} + x_{\eta_i(n)}^n = y_i^n$.

Fact A. H extends G and is torsion-free.

Proof. H can be embedded into the divisible hull of $G \times F$, where F is the abelian group generated freely by $\{x_\alpha^m : m < \omega \text{ and } \alpha < \lambda'_m\}$.

Fact B. H is $\aleph_1$-free and moreover H/G is $\aleph_1$-free.

Proof. Let K be a countable pure subgroup of H. Now, as we can increase K, without loss of generality K is generated by

(i) a pure subgroup $K_1 = \{x_i : i \in I\}$ of G, where I is some countably infinite subset of λ,

(ii) y_i^m, x_j^n for $i \in I$, $m < \omega$ and $(n, j) \in J$, where $J \subseteq \omega \times \lambda$ is countable and

$$i \in I, \quad n < \omega \Rightarrow (n, \eta_i(n)) \in J.$$

Moreover, the equations holding among those elements are deducible from the equations of the following form:

(a)$^-$ equations of K_1;

(b)$^-$ $y_i^0 = x_i$ for $i \in I$;

(c)$^-$ $t_n y_i^{n+1} + x_{\eta_i(n)}^n = y_i^n$ for $i \in I$, $n < \omega$.

We can find $\langle k_i : i \in I \rangle$ such that $k_i < \omega$ and

$$[i \neq j \ \& \ i \in I \ \& \ j \in I \ \& \ n \geqslant k_i \ \& \ n \geqslant k_j \ \& \ i \neq j \Rightarrow \eta_i(n) \neq \eta_j(n)].$$

Now we know that K_1 is free (being a countable subgroup of G), and it suffices to prove that K/K_1 is free. But K/K_1 is freely generated by

$$\{y_i^n : i \in I \text{ and } n > k_i\} \cup \{x_\alpha^n : (n,\alpha) \in J \text{ but for no } i \in I \text{ do we have } n > k_i,\ \eta_i(n) = \alpha\}.$$

So K is free.

Fact C. $H_0 = \langle x_i^n : n < \omega, i < \lambda_n' \rangle_H$ satisfies

(a) $i < \lambda \Rightarrow d_{\bar{t}}(x_i, H_0) = \inf\{d_{\bar{t}}(x_i, z) : z \in H_0\} = 0$,

(b) $x \in G \Rightarrow d_{\bar{t}}(x, H_0) = 0$,

(c) $x \in H \Rightarrow d_{\bar{t}}(x, H_0) = 0$.

Proof. First note that

$(*)_1$ $Y = \{x \in H : d_{\bar{t}}(x, H_0) = 0\}$ is a subgroup of H.

Also for every $i < \lambda$ and every n

$$(*)_2\ y_i^n = x_{\eta_i(n)}^n + t_n y_i^{n+1} = x_{\eta_i(n)}^n + t_n x_{\eta_i(n+1)}^{n+1} + t_n t_{n+1} y_i^{n+2} = \sum_{k=n}^m \left(\prod_{l=n}^{k-1} t_l\right) x_{\eta_i(k)}^k + \left(\prod_{l=n}^k t_l\right) y_i^{k+1}.$$

(This is proved by induction on $m \geqslant n$.) Note that as $(\forall l)(\exists m > l)$ (t_i divides t_m) we must have $(\forall l)(\exists^\infty m)$ (t_l divides t_m) and hence $(\forall k)(\exists^\infty m)$ ($\prod_{i \leqslant l} t_l$ divides $\prod_{i=k}^m t_i$). Now $(*)_2$ implies

$(*)_3$ $y_i^n \in Y$.

But $x_i = y_i^0$ and hence (a) holds, and so (b) holds too as $\{x_i : i < \lambda\}$ is dense in G. Therefore $G \subseteq Y$ (by (b)), and $x_\alpha^n \in Y$ (as $H_0 \subseteq Y$ and from the choice of H_0) and $y_i^n \in Y$ by $(*)_3$. By $(*)_1$ clearly $Y = H$, as required in (c).

Fact D. $|H| = \lambda$.

Fact E. The $\bar{t}$-density of H is μ.

Proof. The $\bar{t}$-density is at most μ as H_0 has cardinality μ and is $\bar{t}$-dense in H, and we now show that it is at least μ.

Define a function h with domain the generators of H listed above, into H. Let
$h(x) = 0$ if $x \in G$;
$h(x_\alpha^m) = 0$ if $m > 0$ or $\alpha < \lambda_0$;
$h(x_\alpha^m) = x_\alpha^m$ if $m = 0$ and $\lambda_0 \leqslant \alpha < \lambda_0' (= \mu)$;
$h(y_i^m) = 0$ if $m < \omega$, $i < \lambda$.
This function preserves the equations defining H and hence induces a homomorphism $\hat{h}$ from H onto $\langle \mathrm{Range}(h) \rangle_H = \langle \{x_\alpha^0 : \alpha < \lambda_0', \alpha \geqslant \lambda_0\} \rangle_H$. Clearly $\hat{h}(h(x)) = \hat{h}(x)$ for the generators and hence $\hat{h} \circ \hat{h} = \hat{h}$. Therefore $\langle \{x_\alpha^n : \alpha < \lambda_0', \alpha \geqslant \lambda_0\} \rangle_H$ is a direct

summand of H and hence the $d_{\bar{t}}$-density of H is at least the $d_{\bar{t}}$-density of $\langle\{x^n_\alpha : \alpha \in [\lambda'_0, \lambda_0)\}\rangle_H$, which is $\lambda'_0 = \mu$. $\square_{3.3}$

We define variants of the ideal constructed in Definition 2.1.

3.4 Definition. For $\bar{\lambda} = \langle \lambda_l : l < \omega\rangle$, $\bar{t} = \langle t_l : l < \omega\rangle, 2 \leqslant t_l < \omega$, we let $J^5_{\bar{t},\bar{\lambda}}$ be the set of subsets X of $\prod_{l<\omega}[\lambda_l]^{t_l}$ for which we cannot find $m(*) < \omega$, $\bar{Y} = \langle Y_m : m < \omega$ and $m \geqslant m(*)\rangle$, and $\bar{A}^m = \langle A_\eta : \eta \in Y_m\rangle$ such that
(a) $Y_m \subseteq \prod_{l<m}[\lambda_l]^{t_l}$,
(b) $Y_{m(*)} \subseteq \prod_{l<m(*)}[\lambda_l]^{t_l}$ is a singleton,
(c) $\langle A_\eta : \eta \in Y_m\rangle$ is a sequence of pairwise disjoint subsets of λ_m each of order type ω,
(d) $Y_{m+1} = \{\eta\hat{\ }\langle u\rangle : \eta \in Y_m$ and $u \in [A_\eta]^{t_m}\}$,
(e) $Y_m \subseteq \{\nu \restriction m : \nu \in X\}$.

We define $J^6_{\bar{t},\bar{\lambda}}$ similarly but with $m(*) = 0$, and we define

$$J^l_{\bar{t},\bar{\lambda},<\theta} = \Big\{X : \text{for some } \alpha < \theta \text{ and } X_\beta \in J^l_{\bar{t},\bar{\lambda}} \text{ for } \beta < \alpha \text{ we have } X \subseteq \bigcup_{\beta<\alpha} X_\beta\Big\}.$$

Also let $J^l_{\bar{t},\bar{\lambda},\theta} = J^l_{\bar{t},\bar{\lambda},<\theta^+}$.

3.5 Claim. (1) $J^{i(1)}_{\bar{t},\bar{\lambda},<\theta_1} \subseteq J^{i(2)}_{\bar{t},\bar{\lambda},<\theta_2}$ *when* $\theta_1 \leqslant \theta_2$ *and* $4 \leqslant i(1) \leqslant i(2) \leqslant 6$.

(2) $J^i_{\bar{t},\bar{\lambda},\theta}$ *is a* θ^+*-complete ideal for* $i = 4, 5, 6$.

(3) *If* $\lambda_l \geqslant \beth_{t_l-1}(\theta)$ *then the ideal* $J^i_{\bar{t},\bar{\lambda},\theta}$ *is proper for* $i = 4, 5, 6$.

Proof. The proofs of (1), (2) are easy and (3) is proved as in 2.4. $\square_{3.5}$

3.6 Definition. Let $\bar{\lambda} = \langle \lambda_l : l < \omega\rangle$, $\bar{t} = \langle t_l : l < \omega\rangle$ and suppose that $2 \leqslant t_l < \omega$ and $(\forall n)(\exists m > n)(t_n | t_m)$. We define the following groups:

(A) $B^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}$ is the free (abelian) group generated by $\{x^m_\alpha : m < \omega, \alpha < \lambda_m\}$;

(B) $B^{\mathrm{rtf}}_{\bar{t},\bar{\lambda},n}$ is the subgroup of $B^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}$ generated by $\{x^m_\alpha : m < n$ and $\alpha < \lambda_m\}$;

(C) $G^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}$ is the pure closure in $(B^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}})^{[\bar{t}]}$ of the subgroup of $(B^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}})^{[\bar{t}]}$ generated by

$$B^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}} \cup \Big\{\sum_{m<\omega}\Big(\prod_{l<m} t_l\Big)(x^m_{(\eta(l))(1)} - x^m_{(\eta(l))(0)}) : \eta \in \prod_{l<\omega}[\lambda_l]^2\Big\}$$

(here our notation is that if e.g. $\eta(l) = \{\alpha, \beta\}$, $\alpha < \beta$ then $(\eta(l))(1) = \beta$, $(\eta(l))(0) = \alpha$);

(D) $\bar{B}^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}} = \langle B^{\mathrm{rtf}}_{\bar{t},\bar{\lambda},n} : n < \omega\rangle$.

To cover also the case $\neg(\forall n)(\exists m > n)(t_n | t_m)$ we can use

3.7 Definition. Let $\aleph_0 \leqslant \lambda_l \leqslant \lambda_{l+1}$ for $l < \omega$. Let $\bar{\lambda} = \langle \lambda_l : l < \omega \rangle$, $\bar{t} = \langle t_l : l < \omega \rangle$, $2 \leqslant t_l < \omega$, $\aleph_0 \leqslant \lambda_l \leqslant \lambda_{l+1}$, $\neg(\forall n)(\exists m > n)(t_n | t_m)$. Let (A), (B), (D) be as in Definition 3.6 and replace (C) by

(C)′ we choose $\bar{Y}^* = \langle Y^*_m : m < \omega \rangle$ such that $Y^*_m \subseteq \prod_{l<m} [\lambda_l]^2$, $Y^*_0 = \{\langle\,\rangle\}$ and for each m there is a sequence $\langle A^*_\eta : \eta \in Y^*_m \rangle$ of pairwise disjoint subsets of λ_m each of cardinality λ_m such that $Y^*_{m+1} = \bigcup\{[A^Y_\eta]^2 : \eta \in Y^*_m\}$. Let

$$Y^*_\omega = \Big\{\eta \in \prod_{l<\omega} [\lambda_l]^2 : \text{for every} < \omega \text{ we have } \eta \restriction m \in Y^*_m\Big\}.$$

Let $G^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}$ be the abelian group generated by

$$B^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}} \cup \{x_\eta, y_{\eta,l} : \eta \in Y^*_\omega, l < \omega\}$$

freely except for the equations which hold in $B^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}$ and

$$y_{\eta,0} = x_\eta, \quad t_l y_{\eta,l+1} - y_{\eta,l} = x^l_{(\eta(l))(1)} - x^l_{(\eta(l))(0)}.$$

3.8 Definition. Assume

$\boxtimes^{\bar{t}}_{H,\bar{H}}$ $\bar{H} = \langle H_n : n < \omega \rangle$ is an increasing sequence of abelian subgroups of H, such that $\bigcup_{n<\omega} H_n$ is dense in H with respect to the $\bar{t}$-adic topology.

Then write

$$I^{4,\bar{t}}_{H,\bar{H}} = \Big\{X \subseteq H : \text{for some } n < \omega, \text{ the intersection of the } \bar{t}\text{-adic closure of } \mathrm{PC}_H(X) \text{ in } H, \mathrm{cl}_{\bar{t}}(\mathrm{PC}_H(X), H) \text{ with } \bigcup_{l<\omega} H_l \text{ is a subset of } H_n\Big\},$$

$$I^{4,\bar{t}}_{H,\bar{H},<\theta} = \Big\{X \subseteq H : \text{for some } \alpha < \theta \text{ and } X_\beta \in I^{4,\bar{t}}_{H,\bar{H}} \text{ for } \beta < \alpha \text{ we have } X \subseteq \bigcup_{\beta<\alpha} X_\beta\Big\},$$

$$I^{4,\bar{t}}_{H,\bar{H},\theta} = I^{\bar{t}}_{H,\bar{H},<\theta^+}.$$

3.9 Definition. Assume $\bar{t} = \langle t_l : l < \omega \rangle$, $2 \leqslant t_l < \omega$, and

$\boxtimes^{\bar{t}}_{H,\bar{H}}$ H is Hausdorff in the $\bar{t} \restriction [k,\omega)$-topology for each $k < \omega$, where $t \restriction [k,\omega) = \langle t_{k+l} : l < \omega \rangle$; further $\bar{H} = \langle H_n : n < \omega \rangle$ is an increasing sequence of abelian groups, and $\bigcup_{n<\omega} H_n \subseteq H$ is dense in the $\bar{t} \restriction [k,\omega)$-adic topology for each $k < \omega$.

Then we let

(1) $I^{5,\bar{t}}_{H,\bar{H}} = \{X \subseteq H$: for some $n(*) < \omega$, for every $n \in (n(*), \omega)$ there is no $y \in H_{n+1}$ such that: $d_{\bar{t}\restriction[n,\omega)}(y, \mathrm{PC}(\langle X \rangle_H)) = 0$ but $d_{\bar{t}\restriction[n,\omega)}(y, H_n) > 0\}$,
$I^{5,\bar{t}}_{H,\bar{H},<\theta} = \{X$: there are $\alpha < \theta$ and $X_\beta \in I^{5,\bar{t}}_{H,\bar{H}}$ for $\beta < \alpha$ such that $X \subseteq \bigcup_{\beta<\alpha} X_\beta\}$.
Moreover $I^{5,\bar{t}}_{H,\bar{H},\theta} = I^{5,t}_{H,\bar{H},<\theta^+}$.

(2) $I^{6,\bar{t}}_{H,\bar{H}}$ (and $I^{6,\bar{t}}_{H,\bar{H},<\theta}, I^{6,\bar{t}}_{H,\bar{H},\theta}$) are defined similarly except that we require that $n(*) = 0$.

(3) $I^{i,\mathrm{rtf}}_{\bar{t},\bar{\lambda}}$ stands for $I^{i,\bar{t}}_{G^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}, \bar{B}^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}}$ where $\bar{B}^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}} = \langle B^{\mathrm{rtf}}_{\bar{t},\bar{\lambda},n} : n < \omega \rangle$.

3.10 Claim. *Let $\bar{\lambda}, \bar{t}$ be as in* 3.4. *Then the following statements hold*:

(a) $\boxtimes^{\bar{t}}_{G^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}, \bar{B}^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}}$ (*from* 3.9);

(b) $G^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}$ *is $\aleph_1$-free*; *moreover $G^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}/B^{\mathrm{rtf}}_{\bar{t},\bar{\lambda},n}$ is $\aleph_1$-free for each $n < \omega$*;

(c) $I^{i,\mathrm{rtf}}_{\bar{t},\bar{\lambda},\theta}$ *are θ^+-complete ideals for $i = 4, 5, 6$*;

(d) *if $\boxtimes^{\bar{t}}_{H,\bar{H}}$* (*from* 3.9) *and $i \in \{4,5,6\}$ then $I^{i,\bar{t}}_{H,\bar{H},\theta}$ is a θ^+-complete ideal.*

Proof. This is straightforward; for $i = 6$ one uses an argument similar to that of 3.3. $\square_{3.10}$

The following lemma connects the combinatorial ideals defined above and the more algebraic ideals defined in 3.8.

3.11 Claim. (1) *Assume the following conditions*:

$\boxtimes_1$ $\bar{t} = \langle t_l : l < \omega \rangle$, $2 \leqslant t_l < \omega$;

$\boxtimes_2$ $\bar{\lambda} = \langle \lambda_l : l < \omega \rangle$, *and $\lambda_l > \beth_1(\theta)$ for $l < \omega$.*

Then the ideal $I^{i,\mathrm{rtf}}_{\bar{t},\bar{\lambda},\theta}$ is proper for $i = 4, 5, 6$.

(2) Assume $\boxtimes_1$ *and*

$\boxtimes'_2$ $\bar{\lambda} = \langle \lambda_l : l < \omega \rangle$, $\lambda_l = \aleph_0, \theta = \aleph_0$.

Then the ideal $I^{i,\mathrm{rtf}}_{\bar{t},\bar{\lambda},\theta}$ is proper.

Proof. (1) If this is not true, we can find $X_\alpha \subseteq L =: G^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}}$ for $\alpha < \theta$ such that $G^{\mathrm{rtf}}_{\bar{t},\bar{\lambda}} = \bigcup_{\alpha<\theta} X_\alpha$ and $X_\alpha \in I^{i,\mathrm{rtf}}_{\bar{t},\bar{\lambda}}$. For $\alpha \leqslant \omega$ and $\eta \in \prod_{l<\alpha} [\lambda_l]^2$ we let

$$x_\eta = \sum_{m<\alpha} \Big(\prod_{l<m} t_l \Big) (x^m_{(\eta(n))(1)} - x^m_{(\eta(n))(0)}).$$

As in the proof of 2.4, we can apply a partition theorem on trees (see [9, Chapter XI, 3.5]) for the ideal $J_l = \mathrm{ERI}^2_\theta(\lambda_l)$ (this ideal is, of course, θ^+-complete and non-trivial as $\lambda_l > 2^\theta$).

So we can find $\langle Y_m : m < \omega\rangle, \langle A_\eta : \eta \in Y_m\rangle$ and $\alpha(*) < \theta$ such that

(a) $Y_m \subseteq \prod_{l<m}[\lambda_l]^2$,

(b) Y_0 is a singleton,

(c) $A_\eta \in (J_{lg(\eta)})^+$ for $\eta \in Y_m$ (so that $A_\eta \subseteq [\lambda_{lg(\eta)}]^2$),

(d) $Y_{m+1} = \{\eta\hat{}\langle u\rangle : u \in A_\eta, \eta \in Y_m\}$,

(e) if $\eta \in Y_m$ then $\eta \in \{\nu \restriction m : x_\nu \in X_{\alpha(*)}\}$.

We now prove by induction on $k < \omega$ that

$(*)_k$ for any $m < \omega$, if $\eta \in Y_m$ and $A \subseteq A_\eta$ is from $(J_m)^+$ then for some infinite $A' \subseteq \lambda_m$ for any $\alpha < \beta$ from A' and $k < \omega$ we have

$$\bigotimes^k_{\alpha,\beta} (\textstyle\prod_{l<m} t_l)(x^m_\beta - x^m_\alpha) \in \mathrm{cl}_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + (\prod_{l<m+k} t_l)L.$$

For $k = 0$ this is trivial: the element $(\prod_{l<m} t_l)(x^m_\beta - x^m_\alpha)$ belongs to $(\prod_{l<m+k} t_l)L$.

For $k+1$, to prove $(*)_{k+1}$ we are given $m < \omega$, $\eta \in Y_m$ and $A' \subseteq A_\eta, A' \in (J_\eta)^+$, and we have to find $\{\alpha,\beta\} \in A'$ such that $\bigotimes^{k+1}_{\alpha,\beta}$ holds. For $l \in [m,\omega)$, as J_l is an ideal we can find $A''_\nu \in (J_l)^+$ for $\nu \in Y_l$ such that $A''_\nu \subseteq A_\nu$ and the statement $\bigotimes^k_{\alpha,\beta}$ holds for every $\{\alpha,\beta\} \in A''_\nu$ or for no $\{\alpha,\beta\} \in A''_\nu$ and $\nu = \eta$ implies $A''_\nu \subseteq A'_\nu$. Because $(*)_k$ holds, for $\{\alpha,\beta\} \in A''_\nu$ we have $\bigotimes^k_{\alpha,\beta}$. After renaming, without loss of generality we have $A''_\nu = A_\nu$. As $A_\eta \in (J_m)^+$, by the choice of J_m we can let $\gamma_0 < \gamma_1 < \gamma_2 < \cdots$ be in A_η. So for each $j < \omega$, let $\eta_j \in Y_{m+k+1}$ (yes, not $\eta_j \in Y_{m+1}$!) be such that $\eta_j \restriction m = \eta$, $\eta_j(m) = \{\gamma_j, \gamma_{j+1}\}$. By (e) above there are ν_j such that $\eta_j \lhd \nu_j \in \prod_{l<\omega}[\lambda_l]^2$ and

(i) $x_{\nu_j} \in X_{\alpha(*)}$.

Now from the definitions of x_{η_j}, x_{ν_j} we have the following statements:.

(ii) $x_{\eta_j} = x_{\nu_j} \bmod (\prod_{l<m+k+1} t_l)L$;

(iii) if $l \in [m+1, m+k+1)$ and $j < \omega$ then

$$\begin{aligned} x_{\eta_j\restriction(l+1)} - x_{\eta_j\restriction l} &\in \mathrm{cl}_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + \Big(\prod_{i<l+k} t_i\Big)L \\ &\subseteq \mathrm{cl}_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + \Big(\prod_{i<m+k+1} t_i\Big)L \end{aligned}$$

(the first inclusion comes from the induction hypothesis as the difference is $(\prod_{i<m+l} t_i)(x^l_{(\eta_j(l))(1)} - x^l_{(\eta_j(l))(0)})$, and the second holds as $m+1 \leqslant l$);

(iv) $x_{\eta_j} - x_{\eta_j \restriction (m+1)} \in \mathrm{cl}_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + (\prod_{i<m+k+1} t_i)L$
(we use (iii) for $l = m+1, \dots, m+k$, noting that $lg(\eta_j) = m+k+1$);

(v) $x_{\eta_j \restriction (m+1)} \in \mathrm{cl}_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + (\prod_{i<m+k+1} t_i)L$
(from (i) + (ii) + (iv));

(vi) $\sum\{x_{\eta_j \restriction (m+1)} : j < \prod_{i<m+k+1} t_i\} \in \mathrm{cl}_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + (\prod_{i<m+k+1} t_i)L$
(from (v));

(vii) $(\prod_{i<m} t_i)(x^m_{\gamma_{j(*)}} - x^m_{\gamma_0}) \in \mathrm{cl}_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + (\prod_{i<m+k+1} t_i))L$ for $j(*) = \prod_{i<m+k+1} t_i$
(from (vi) because

$$\sum\Big\{x_{\eta_j \restriction (m+1)} : j < \prod_{i<m+k+1} t_i\Big\}$$

$$= \sum\Big\{x_{\eta_j \restriction m} + \Big(\prod_{i<m} t_i\Big)(x^m_{\gamma_{j+1}} - x^m_{\gamma_j}) : j < \prod_{i<m+k+1} t_i\Big\}$$

$$= \sum\Big\{x_{\eta_j \restriction m} : j < \prod_{i<m+k+1} t_i\Big\} + \Big(\prod_{i<m} t_i\Big)\sum\Big\{(x^m_{\gamma_{j+1}} - x^m_{\gamma_j}) : j < \prod_{i<m+k+1} t_i\Big\}$$

(as $\eta_j \restriction m$ does not depend on j and using obvious arithmetic)

$$= \Big(\prod_{i<m+k+1} t_i\Big) \cdot x_{\eta_{j(*)} \restriction m} + \Big(\prod_{i<m} t_i\Big)(x^m_{\gamma_{j(*)}} - x^m_{\gamma_0})$$

$$\in \Big(\prod_{i<m} t_i\Big)(x^m_{\gamma_{j(*)}} - x^m_{\gamma_0}) + \Big(\prod_{i<m+k+1} t_i\Big)L);$$

(viii) if $\rho \in Y_m$ and $\alpha < \beta$ are in A_η

then

$$\Big(\prod_{i<m} t_i\Big)(x^m_\beta - x^m_\alpha) \in \mathrm{cl}_{\bar t}(\langle X_{\alpha(*)}\rangle, L) + \Big(\prod_{i<m+k+1} t_i\Big)L$$

(from (vii) and the choice of the Y_m, A_η $(\eta \in Y_m, m < \omega)$).

So we have carried out the induction on k.

(2) The proof of this is easier and it will be omitted. $\square_{3.11}$

3.12 Claim. *Assume the following conditions*:

$\boxtimes_1$ $\bar t = \langle t_l : l < \omega\rangle$ *and* $2 \leqslant t_l < \omega$;

$\boxtimes_2$ $\lambda_l > \beth_1(\theta)$;

$\boxtimes_3$ $\mathrm{cov}(\lambda, (\prod_{l<\omega} \lambda_l)^+, (\prod_{l<\omega} \lambda_l)^+, \theta^+) \leqslant \lambda$.

Then $\mathbf{U}_{J^6_{\bar{t},\bar{\lambda},\theta}}(\lambda) = \lambda$ *and* $\mathbf{U}_{I^6_{\bar{t},\bar{\lambda},\theta}}(\lambda) = \lambda$.

Proof. This follows from the previous claims 3.10, 3.11 (and the relevant definitions 3.6–3.9).

3.13 Conclusion. *For every* $\lambda \geqslant \beth_\omega$ *for some* $\theta < \beth_\omega$, *for every* $\kappa \in (\beth_1(\theta), \beth_\omega)$ *for every* $\lambda_n \in [\beth_1(\theta), \kappa]$ *we have*

$$\mathbf{U}_{I^6_{\bar{t},\bar{\lambda},\theta}}(\lambda) = \lambda = \mathbf{U}_{J^6_{\bar{t},\bar{\lambda},\theta}}(\lambda).$$

Proof. This follows from the previous claim and [11] (and is similar to 2.5). $\square_{3.13}$

3.14 Claim. *Assume the following conditions*:

(a) $\prod_{l<\omega} \lambda_l < \mu < \lambda = cf(\lambda) \leqslant \lambda' \leqslant \lambda'' < \mu^{\aleph_0}$;

(b) $\mu^+ < \lambda$, *or at least for some* $\mathscr{P}$,

$(*)_{\mathscr{P}}$ $|\mathscr{P}| = \lambda$ *and* $(\forall a \in \mathscr{P})(a \subseteq \lambda \;\&\; \mathrm{otp}(a) = \mu)$
and $(\forall E)(E$ *a club of* $\lambda \rightarrow (\exists a \in \mathscr{P})(a \subseteq E))$;

(c) $\lambda'' = \mathbf{U}_{I^6_{\bar{t},\bar{\lambda}}}(\lambda') < \mu^{\aleph_0}$, *where* $t_m = \prod_{l<m} l!$ *or at least* $\lambda'' = \mathbf{U}_{J^6_{\bar{\lambda},\bar{t}}}(\lambda')$;

(d) $\mathrm{cov}(\lambda'', \lambda^+, \lambda^+, \lambda) < \mu^{\aleph_0}$ *or at least* $\mathbf{U}_{\mathrm{id}^a(\mathscr{P})}(\lambda'') < \mu^{\aleph_0}$, *where* $\mathscr{P}$ *satisfies the requirement* $(*)_{\mathscr{P}}$.

Then we can find $\aleph_1$*-free abelian groups* G_α *of cardinality* λ *for* $\alpha < \mu^{\aleph_0}$ *such that for every* $\aleph_1$*-free abelian group* G *of cardinality* λ' *or just* $G \in \mathfrak{K}^{\mathrm{rtf}}_{\lambda'}[\bar{t}]$ *the following holds*: *some* G_α *is not embeddable into* G; *also the number of ordinals* $\alpha < \mu^{\aleph_0}$ *for which* G_α *is embeddable into* G *is at most* $\mathrm{cov}(\lambda'', \lambda^+, \lambda^+, \lambda)$ (*or* $\leqslant \mathbf{U}_{\mathrm{id}^a(\mathscr{P})}(\lambda'')$ *at least*).

Proof. The proof is as in 2.8; note that '$\aleph_1$-free' implies that $\|-\|_{\bar{t}}$ is a norm.

3.15 Conclusion. *If* $\beth_\omega \leqslant \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$ *then in* $\mathfrak{K}^{\mathrm{rtf}}_\lambda$ *there is no member universal even just for* $\mathfrak{K}^{\aleph_1\text{-free}}_\lambda$.

Proof. This is straightforward.

3.16 Remark. In Section 2 we can use arguments parallel to those in 3.11.

3.17 Remark. For $\lambda = \aleph_0$ there is no universal member in $\mathfrak{K}^{\mathrm{rtf}}_\lambda$. In fact, for any $\mathbf{Q} \subset \mathbf{P}^*$ let $G_{\mathbf{Q}}$ be the subgroup of $\mathbb{Q}x \oplus \bigoplus_p \{\mathbb{Q}x_p : p \in \mathbf{P}^* \backslash \mathbf{Q}\}$ generated by

$$\{p^{-n}x : p \in \mathbf{Q}\} \cup \{q^{-n}x_p : p \in \mathbf{P}^* \backslash \mathbf{Q} \text{ and } n < \omega, \text{ and } q \in \mathbf{P}^* \backslash \{p\}\}$$

$$\cup \{p^{-n}(x - x_p) : n < \omega \text{ and } p \in \mathbf{Q}\}.$$

So $G_{\mathbf{Q}} \in \mathfrak{K}^{\mathrm{rtf}}_{\aleph_0}$, and (see Definition 1.3) $\mathbf{P}(x, G_{\mathbf{Q}}) = \mathbf{Q}$; and $\mathbf{P}^-(x, G_{\mathbf{Q}}) = \mathbf{P}^* \backslash \mathbf{Q}$ hence (see 1.4) if h embeds $G_{\mathbf{Q}}$ into $G \in K^{\mathrm{trf}}$ then $\mathbf{P}(h(x), G) = \mathbf{Q}$. As the number of possible subsets **Q** is $2^{\aleph_0}$ the statement follows easily. This argument gives an alternative proof to 1.2, but the proof there looks more amenable to generalization.

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[References of the form math.XX/$\cdots$ refer to the xxx.lanl.gov archive]

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