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THE COMPLETE FINITELY AXIOMATIZED THEORIES OF ORDER ARE DENSE

BY

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ABSTRACT

We prove a conjecture of Lauchli and Leonard that every sentence of the theory of linear order which has a model, has a model with a finitely axiomatized theory.

0. Introduction

Ehrenfeuch [1] showed the decideability of the (first-order) theory of linear orders. Lauchli and Leonard [3], continuing an unpublished work of Galvin, published a proof there. They define a set of terms, and associate to each term τ a model $M(\tau)$ and show that every sentence which has a model, is satisfied by some $M(\tau)$, and the n-type of each $M(\tau)$ can be computed. So what we actually prove is that for any τ, n there is a σ, $M(\sigma) \equiv_n M(\tau)$, $M(\sigma)$ has a finitely axiomatized theory. In fact we prove:

THEOREM 0.1 *For each sentence ψ in the theory of order which has a model, we can effectively find ψ_1, such that $\psi \wedge \psi_1$ is a complete theory; and vaguely speaking, we have an algebraic characterization of the models of $\psi \wedge \psi_1$.*

We do not elaborate on the effectiveness. (We can check carefully the stages with trivial additions. But we can also note that in 2.9 we can effectively find all the possible relevant types of $M(\sigma)$, check which of them is suitable, and choose one; then the version of 2.10 will be easy.)

We can conclude:

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CONCLUSION 0.2. *The following questions are decideable*:

1) *Is a sentence complete?*
2) *Is a sentence* $\aleph_0$*-categorical? (already proved in Shelah* [7]).

We can prove

THEOREM 0.3. *The following questions are decideable*:

1) $M(\tau) \cong M(\sigma)$
2) $M(\tau) \equiv M(\sigma)$
3) $M(\tau)$ *has a finitely axiomatizable theory.*

The theorems (including 0.3) were proved by Shelah in 1972 and announced in Rubin [6], but the proof was very complicated: based on choosing canonical terms (up to isomorphism, elementary equivalence) using also the term $\sigma\omega + \sigma_1\omega^*$ as atomic. Amit, in his M.Sc. thesis, using a wider class of terms, gave a much simpler but less effective proof. This is a third proof of Shelah. Meanwhile, D. Myers in [4] announced that he had characterized the Boolean algebra of sentences of the theory of order; and, about the same time, Schmerl (private communications) also proved the Lauchli-Leonard conjecture. The first and second proofs try to reconstruct the direct component; this proof tries to reconstruct the smallest non-trivial components.

1. Definition of the classes

DEFINITION 1.1. The set of terms p_n^j will be defined by induction on n for every n, $n < \omega$ as follows:

1) $p_0^j = \{1_1, 1_2, \cdots, 1_j\}$

2) p_{2n+1}^j is the set of all terms of the form $\tau = \Sigma_{i=1}^k \tau_i$, $k \geqq 2$, where $\tau_i \in \bigcup_{l \leqq 2n} p_l^j$ and $\tau \notin \bigcup_{l \leqq 2n} p_l^j$.

REMARK. $\tau \cdot n$ is defined as $\tau + \cdots + \tau$, n times, and we assume that the addition is associative.

3) p_{2n+2}^j is the set of the following terms:

a) $\tau = \eta(\tau_1, \cdots, \tau_n)$ where $\tau_i \in \bigcup_{l<2n+2} p_l^j$ and $\tau \notin \bigcup_{l \leqq 2n} p_l^j (\eta(\tau_1, \cdots, \tau_n) = \eta(\sigma_1, \cdots, \sigma_l)$ if $\{\tau_1, \cdots, \tau_n\} = \{\sigma_1, \cdots, \sigma_l\})$

b) $\tau \cdot \omega$ where $\tau \in p_{2n}^j \cup p_{2n+1}^j$

c) $\tau \cdot \omega^*$ where $\tau \in p_{2n}^j \cup p_{2n+1}^j$

d) $\tau \cdot z$ where $\tau \in p_{2n}^j \cup p_{2n+1}^j$.

REMARKS.

1) Here z always denotes the ordered set of integers. In fact d) is not necessary

2) for $n \neq m$ it is easily seen that $p^j_n \cap p^j_m \neq \emptyset$

3) there may be different terms which have the same interpretation. Take for instance $(\sigma \cdot n)\omega$ and $\sigma \cdot \omega$ or $(\tau_1 + \cdots + \tau_n) \cdot z$ and $(\tau_i + \tau_{i+1} + \cdots + \tau_n + \tau_1 + \cdots + \tau_{i-1}) \cdot z$

4) if we write the term $\tau = \tau_1 + \cdots + \tau_n$ where $\tau_i = \tau_{i_1} + \cdots + \tau_{i_l}$, $\tau_{i_j} \in p^j_{2n}$, $j = 1, \cdots, l$, for instance, we always mean to the term τ written correctly as $\tau_1 + \cdots + \tau_{i-1} + \tau_{i_1} + \tau_{i_2} + \cdots + \tau_{i_l} + \tau_{i+1} + \cdots + \tau_n$

5) sometimes we informally write $\tau = 0$, and mean that $M(\tau)$ denotes the empty ordered set.

Definition 1.2 $p^j \stackrel{\text{def}}{=} \bigcup_{n<\omega} p^j_n$, $p^j_e \stackrel{\text{def}}{=} \bigcup_{n<\omega} p^j_{2n}$, $p_n = \bigcup_{j<\omega} p^j_n$, $p = \bigcup_{n<\omega} p_n$.

Definition 1.3. The rank of τ denoted $\delta(\tau)$ will be the n for which $\tau \in p_n$. The even rank of τ denoted $\delta_1(\tau)$ will be the $2n$ for which $\tau \in p_{2n} \cup p_{2n+1}$.

Let L^j be the language of linear order with j unary predicates $P_1, \cdots, P_j$. $L \stackrel{\text{def}}{=} \bigcup_{j<\omega} L^j$. An L^j-model will be a model satisfying the axioms of the linear order with j unary relations $R_1, \cdots, R_j$, corresponding to $P_1, \cdots, P_j$, which partition the model to j disjoint sets. By a model we mean an L^j-model for some j.

Definition 1.4. For every j and $\tau \in p^j$ we shall define an L^j-model $M(\tau)$ by induction on $\delta(\tau)$ as follows:

1) the model $M(\tau)$ is defined up to isomorphism

2) the sum of the models $\Sigma_{i\in I} M_i$ will be defined naturally

a) if $\tau = 1_i$ then $M(\tau)$ is a model of one element $\{x : R_i(x)\} = M(\tau)$, $\{x : R_l(x)\}_{l \neq i} = \emptyset$

b) if $\tau = \tau_1 + \cdots + \tau_n$ then $M(\tau) = \Sigma^n_{i=1} M(\tau_i)$

c) if $\tau = \tau_1 \cdot \omega$, $\tau_1 \cdot \omega^*$, $\tau_1 \cdot z$ then $M(\tau) = \Sigma_{i\in I} M_i$ where $M_i \cong M(\tau_1)$, $I = \omega, \omega^*, z$ respectively

d) if $\tau = \eta(\tau_1, \cdots, \tau_n)$ then $M(\tau) = \Sigma_{r\in Q} M_r$ where Q is a dense countable order without first and last elements, $Q = \bigcup^n_{i=1} Q_i$ where for every i, $1 \leqq i \leqq n$, Q_i is dense in Q, and $Q_i = \{r : r \in Q, M_r \cong M(\tau_i)\}$ $i = 1, \cdots, n$.

Definition 1.5. A convex set N of a model M will be called an initial (terminal) segment of M if there is $a \in M$ such that N includes the set $\{x : x < a\}$ $(\{x : x > a\})$, or if N is empty.

Theorem 1.6. (Cantor.) *If $M_i = \langle A^i, <, R^i_1, \cdots, R^i_n \rangle$, $i = 1, 2$, where A^i is denumerable, $<$ is a dense order on A^i without first and last elements, $A^i = \bigcup^n_{j=1} R^i_j$, and R^i_j, $j = 1, \cdots, n$, are disjoint sets dense in A^i, $i = 1, 2$, then $M_1 \cong M_2$.*

LEMMA 1.7. *Let $\tau \in p^j$, then for every convex set N in $M(\tau)$ there is $\sigma \in p^j$ such that $N = M(\sigma)$ and $\delta_1(\sigma) \leqq \delta_1(\tau)$.*

PROOF. By induction on $\delta(\tau)$.

a) If $\delta(\tau) = 0$ it is immediate.

b) If $\tau = \tau_1 + \cdots + \tau_n$ then either N is a convex set of $M(\tau_i)$, $1 \leqq i \leqq n$, and the proof follows from the induction hypothesis, or $\bigcup_{i_1<i<i_2} M(\tau_i) \subseteq N \subseteq \bigcup_{i_1 \leqq i \leqq i_2} M(\tau_i)$, $1 \leqq i_1 \leqq i_2 \leqq n$. In this case there is an initial segment of N which is a terminal segment of $M(\tau_{i_1})$ and a terminal segment of N which is an initial segment of $M(\tau_{i_2})$. By the induction hypothesis on $M(\tau_{i_1})$, $M(\tau_{i_2})$ we shall get τ'_{i_1}, τ'_{i_2}, $\delta_1(\tau'_{i_1}) \leqq \delta_1(\tau_{i_1})$, $\delta_1(\tau'_{i_2}) \leqq \delta_1(\tau_{i_2})$ such that for $\sigma = \tau'_{i_1} + \tau_{i_1+1} + \cdots + \tau_{i_2-1} + \tau'_{i_2}$, $M(\sigma) = N$ and $\delta_1(\sigma) \leqq \delta_1(\tau)$. (Note that τ'_{i_1}, τ'_{i_2} might be 0.)

c) If $\tau = \tau_1 \cdot \omega, \tau \cdot \omega^*, \tau_1 \cdot z$ the proof is similar. For example, if $\tau = \tau_1 \cdot \omega$ then either N is a convex set of $M(\tau_1 \cdot n)$ for $n < \omega$ and the case is clear or $M(\tau_1 \cdot \omega) \subseteq N \subseteq M(\tau_1) + M(\tau_1 \cdot \omega)$ by the induction hypothesis on $M(\tau_1)$ we shall get that $\sigma = \tau'_1 + \tau_1 \cdot \omega$ (τ'_1 might be 0).

d) If $\tau = \eta(\tau_1, \cdots, \tau_n)$ then if N is a convex set of $M(\tau_i)$, $1 \leqq i \leqq n$, the case is clear. If not, from Cantor's theorem (1.6) we shall get that $\sigma = \tau'_i + \tau + \tau'_j$ where τ'_i, τ'_j are obtained by the induction hypothesis on τ_i, τ_j (τ'_i, τ'_j might be 0).

THEOREM 1.8. (Lauchli and Leonard.) *For every j, and a sentence $\psi \in L^j$, which has an L^j-model, there is $\tau \in p^j$ such that $M(\tau) \models \psi$.*

2.

The beginning of this section is given with more details in [3] and the proofs of Lemmas 2.2, 2.5, 2.9 can easily be obtained from there. Let Φ_n denote the set of all sentences from L which are of quantifier depth n. (Note that $\bigcup_{n<\omega} \Phi_n = L$.) Let $\mathrm{Th}_n(M) = \{\psi \in \Phi_n : M \models \psi\}$.

DEFINITION 2.1. Two L^j-models M_1, M_2 will be called n-equivalent ($M_1 \equiv_n M_2$) iff $\mathrm{Th}_n(M_1) = \mathrm{Th}_n(M_2)$.

If $t = \mathrm{Th}_n(M)$ where M is an L^j-model, t will be called an L^j-n-type and $T(j, n)$ will denote the set of L^j-n-types.

LEMMA 2.2. *For every $j, n < \omega$, there are only finite number of L^j-n-types.*

DEFINITION 2.3. For every convex equivalence relation E on an L^j-model M and $n < \omega$ let us define $M/_nE$ as follows: $M/_nE = \langle \{a/E, a \in M\}, <, \{R_t\}_{t \in T} \rangle$ where $T = T(j, n) a/E$ denotes the E-equivalence class of a, $<$ is the order on

$M/_nE$ induced by the order on M, and R_t is an unary relation on $M/_nE$, $R_t \stackrel{\text{def}}{=} \{a/E : \mathrm{Th}_n(M \restriction a/E) = t\}$ where t is some L^j-n-type. Clearly $M/_nE$ is an L^j-model for some j.

LEMMA 2.4. *Let $M^j = (\Sigma_{i \in I^j} M_i^j, E^{M^j})$, $j = 1,2$, where $\langle a, b \rangle \in E^{M^j}$ iff for some $i \in I$ $a, b \in |M_i^j|$ (we assume that the M_i^j's, $i \in I$ are pairwise disjoint L^k-models). Let $N^j = \langle I^j, P_t^{N^j} \rangle_{t \in T(k,n)}$ where $P_t^{N^j} = \{i \mid i \in I^j$ and $M_i^j \models t\}$, $j = 1,2$; suppose $N^1 \equiv_n N^2$ then $M^1 \equiv_n M^2$.*

PROOF. One proves by induction on n that if $\bar{a}_i^j \in |M_{l_i}^j|$, $i = 1, \cdots, m, j = 1,2, \langle l_1^1, \cdots, l_m^1 \rangle$ and $\langle l_1^2, \cdots, l_m^2 \rangle$ have the same n-type in N^1 and N^2 respectively, $i_1 \neq i_2$ implies $l_{i_1}^j \neq l_{i_2}^j$, $j = 1,2$ and for every $i = 1, \cdots, m$, $\bar{a}_i^1$ and $\bar{a}_i^2$ have the same n-type in $M_{l_i}^1$ and $M_{l_i}^2$ respectively, then $\bar{a}_1^1 {}^\wedge \bar{a}_2^1 {}^\wedge \cdots {}^\wedge \bar{a}_m^1$ and $\bar{a}_1^2 {}^\wedge \bar{a}_2^2 {}^\wedge \cdots {}^\wedge \bar{a}_m^2$ have the same n-type in M^1 and M^2 respectively.

DEFINITION 2.5. For every L^j model M and $n \in \omega$ we define a relation E_n^{1M}. (If no confusion may arise we write E_n^1 instead of E_n^{1M}.) $a' E_n^1 b'$ iff for $a = \min(\{a', b'\})$ and $b = \max(\{a', b'\})$ the following holds: for every $[a_1, b_1) \subseteq [a, b)$, $M \restriction [a_1, b_1)$ is n-equivalent to a finite model.

Clearly E_n^1 is a convex equivalence relation.

DEFINITION 2.6. For every L^j model M and $n \in \omega$ we define a relation E_n^M (abbreviated by E_n). $a E_n b$ iff one of the following conditions holds:

1) $a E_n^1 b$;

2) there is a convex subset $B \subseteq |M|$ and $T \subseteq T(j, n)$ such that $a, b \in B$, for every $t \in T$, t is the n-type of some finite model and for every $c, d \in B$ and $t \in T$: if $c/E_n^{1M} < d/E_n^{1M}$, then M has submodels N_1, N_2, N_3 such that $B \supseteq |N_1| < c/E_n^{1M} < |N_2| < d/E_n^{1M} < |N_3| \subseteq B$, $N_i \models t$ and $|N_i| = a_i/E_n^{1M}$ for some a_i, $i = 1,2,3$.

LEMMA 2.7. *Let M be an L^j model, $j \in \omega$, then for every $n \in \omega$ and $a \in |M|$ there is a finitely axiomatizable model which is n-equivalent to $M \restriction (a/E_n)$.*

PROOF. Let N_a be the submodel of M whose universe is a/E_n.

a) Suppose a/E_n is not a dense order of E_n^1 equivalence classes. If N_a is n-equivalent to a finite model N, then certainly N_a is finitely axiomatizable and we are through. Otherwise a/E_n has no last element or has no first element. Suppose a/E_n has a first element x_0 and let $\{x_i \mid i \in \omega\}$ be an unbounded strictly increasing sequence in a/E_n. By the Ramsey theorem there is a subsequence

$\{x'_i \mid i \in \omega\}$ and an L^j-n-theory t of a finite model N^1, such that if N_i is a submodel of M whose universe is $[x'_i, x'_{i+1})$, then for every $i > 0$, $N_i \models t$. Let N^0 be a finite model n-equivalent to N_0 and let $N = N^0 + N^1 \cdot \omega$ then certainly $N \equiv_n N_a \cdot |N^0|$ is definable in N and since it is finite its theory is finitely axiomatizable. So in order to show that $\mathrm{Th}(N)$ is finitely axiomatizable it suffices to prove that $\mathrm{Th}(N^1 \cdot \omega)$ is finitely axiomatizable. Suppose $|N^1| = m$ then the following set of axioms is a finite axiomatization for $\mathrm{Th}(N^1 \cdot \omega)$:

1) Every element has a successor and there is a first element.

2) Every element is contained in a convex set with m elements isomorphic to N^1.

3) For every convex set A of cardinality m isomorphic to N^1 there is a convex set B isomorphic to N^1 such that $A < B$ and for no x, $A < x < B$. If in addition $|\{x \mid x < A\}| \geqq m$ there is a convex set C isomorphic to N^1 such that $C < A$ and for no x, $C < x < A$.

The case when N has a last element and when N has no first and no last elements are similar.

b) If a/E_n is a dense order of E^1_n-equivalence classes then N_a is n-equivalent to $M(\eta(\sigma_1, \cdots, \sigma_k))$ where each $M(\sigma_i)$ is finite. $\mathrm{Th}_n(M(\eta(\sigma_1, \cdots, \sigma_k)))$ is finitely axiomatizable, so the lemma is proved.

LEMMA 2.8.

a) *Let $\tau = \tau_1 \cdot \omega$ and $n \in \omega$ then either $M(\tau)/E_n$ has a last element or there are τ'_0, τ'_1, N_0 and M_i such that $\delta_1(\tau'_0), \delta_1(\tau'_1) < \delta_1(\tau)$, $N_0 \cong M(\tau'_0)$, for every $i \in \omega$, $M_i \cong M(\tau'_1)$, $M(\tau) = N_0 + \Sigma_{i \in \omega} M_i$ and every element of $M(\tau)/E_n$ is included either in $|N_0|$ or in some $|M_i|$, $i \in \omega$. Analogous claims hold for $\tau_1 \cdot \omega^*$, $\tau_1 \cdot z$ and for E^1_n replacing E_n.*

b) *For every τ there is $n(\tau) \in \omega$ such that for every $n \geqq n(\tau)$, $E^{1M(\tau)}_n = E^{1M(\tau)}_{n(\tau)}$ and for every N_1, N_2 such that $|N_1|, |N_2| \in M(\tau)/E^1_{n(\tau)}$, if $N_1 \equiv_{n(\tau)} N_2$ then $N_1 \equiv_n N_2$.*

c) *Let $n_0(\tau) = \max(n(\tau), 4)$, then for every $n \geqq n_0(\tau)$, $E^{M(\tau)}_n = E^{M(\tau)}_{n_0(\tau)}$.*

PROOF.

a) Suppose $M(\tau)$ has not a last E_n-equivalence class. W.l.o.g. we can assume that $M(\tau) = \Sigma_{i \in \omega} M'_i$ where $|M'_i| = |M(\tau_1)| \times \{i\}$ and $a \to \langle a, i \rangle$ is an isomorphism between $M(\tau_1)$ and M'_i. Let $f_i : |M(\tau)| \to |M(\tau)|$ be defined by $f_i(\langle a, l \rangle) = \langle a, l + i \rangle$. No E_n-equivalence class of M contains some $|M'_i|$ properly for then it is easily seen that this class is a last element of $M(\tau)/E_n$. So every equivalence class intersects at most two consecutive $|M'_i|$'s. It follows easily that if

$C \in |M(\tau)/E_n|$ and $C \cap |M_0'| = \emptyset$ then for every i, $f_i(C) \in M(\tau)/E_n$. Let $C \in M(\tau)/E_n$ and $C \cap |M_0'| = \emptyset$. For every $i \in \omega$ let $D_i = \{x \mid x \in |M(\tau)|$ and for some $y \in f_i(C)$, $y \leqq x < f_{i+1}(C)\}$ and $D = \{x \mid x \in |M(\tau)|$ and $x < C\}$. Let M_i and N_0 be the submodels of $M(\tau)$ whose universes are D_i and D respectively, then for some τ_1' and τ_0', $M_i \cong M(\tau_1')$, $N_0 \cong M(\tau_0')$ and $\tau_0', \tau_1', N_0, M_i$ fulfill the requirements of a).

b) We prove b) by induction on $\delta(\tau)$. When $\delta(\tau) = 0$ there is nothing to prove. Suppose b) is true for every σ such that $\delta(\sigma) < \delta(\tau)$. It suffices to prove b) in the following cases:

i) $\tau = \tau_1 + \tau_2$;

ii) $\tau = \eta(\tau_1, \cdots, \tau_k)$;

iii) $\tau = \tau_1 \cdot \omega$. The case $\tau = \Sigma_{i=1}^k \tau_i$ follows from i) by induction on k, and the cases $\tau = \tau_1 \cdot \omega^*$, $\tau = \tau_1 \cdot z$, are similar to iii).

It is worthwhile to notice that if N is a convex submodel of M then for every $n \in \omega$, $E_n^{1N} = E_n^{1M} \restriction |N|$.

i) Let $n_0 = \max(n(\tau_1), n(\tau_2))$. Suppose first that for some $n_1 \in \omega$ every element of $M(\tau)/E_{n_1}^1$ is included either in $M(\tau_1)$ or in $M(\tau_2)$. W.l.o.g. $n_1 \geqq n_0$. Then it is easily seen that for every $n \geqq n_1$, $M(\tau)/E_{n_0}^1 = M(\tau_1)/E_{n_0}^1 + M(\tau_2)/E_{n_0}^1$. By induction hypothesis on $n(\tau_i)$ there is $n_2 \geqq n_1$ such that if $N_i \subseteq M(\tau_i)$, $|N_i| \in M(\tau_i)/E_{n_2}^1$ and $N_1 \equiv_{n_2} N_2$ then for every $n \geqq n_2$, $N_1 \equiv_n N_2$. It is easily seen that $n(\tau) = n_2$ is as desired.

Suppose now that for every n there are $a_i \in M(\tau_i)/E_n^1$, $i = 1, 2$, such that $\langle a_1, a_2 \rangle \in E_n^{1M(\tau)}$. It is easily seen that for every $n \geqq n_0$, $E_n^{1M(\tau)} = E_{n_0}^{1M(\tau)}$. Let $N_0 \subseteq M(\tau)$ be the $E_{n_1}^{1M(\tau)}$-equivalence class which intersects both $M(\tau_1)$ and $M(\tau_2)$. Let $n_1 \geqq n_0$ be such that for every N_1, N_2 if $|N_i| \in M(\tau_i)/E_{n_1}^1 \cup \{|N_0|\}$, $i = 1, 2$, and $N_1 \equiv_{n_1} N_2$ then for every $n \geqq n_1$, $N_1 \equiv_n N_2$. It is easily seen that such n_1 exists and that $n(\tau) = n_1$ is as desired.

ii) $\tau = \eta(\tau_1, \cdots, \tau_k)$. It is evident that either for every n $\|M(\tau)/E_n^1\| = 1$ or there is n_0 such that for every $n \geqq n_0$, $M(\tau)/E_n^1 = \eta(M(\tau_1)/E_n^1, \cdots, M(\tau_k)/E_n^1)$. Both cases are easily dealt with.

iii) $\tau = \tau_1 \cdot \omega$. It is easily seen that if $M(\tau)/E_n^1$ has a last element then $\|M(\tau)/E_{n_0}^1\| = 1$. So suppose there is n_0 such that $M(\tau)/E_{n_0}^1$ has no last element. So by a) and i) we can assume that $M(\tau)/E_{n_0}^1 = M(\tau)/E_{n_0}^1 \cdot \omega$. $n(\tau) = \max(n_0, n(\tau_1))$ is easily seen to fulfill the requirements of b).

c) Let $|N| \in M(\tau)/E_{n_0(\tau)}$. We show that for every $n \geqq n_0(\tau)$, $|N| \in M(\tau)/E_n$. We assume that $|N| \not\in M(\tau)/E_{n_0(\tau)}^1$ since the other case is trivial. Since $n_0(\tau) \geqq 4$ every element of $N/E_{n_0(\tau)}^1$ has a first and a last element, and so since also $n_0(\tau) \geqq n(\tau)$, for every $n \geqq n_0(\tau)$, $|N/E_{n_0(\tau)}^1| = |N/E_n^1|$ and every element of

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$N/E^1_{n_0(\tau)}$ is n-equivalent to a finite model. Since $n_0(\tau)$-equivalence of $E^1_{n_0(\tau)}$-equivalence classes implies n-equivalence for every n it follows easily that for every $n \geqq n_0(\tau)$, $|N| \in M(\tau)/E_n$. Q.E.D.

THEOREM 2.9. *For every m, n, j and $\tau \in p^j$ there is a term σ such that $M(\sigma) \equiv_n M(\tau)/E_n$, $\delta_1(\sigma) < \delta_1(\tau)$ or $\delta_1(\tau) = 0$. (Note that $\sigma \in p^{j'}$, $j' \neq j$.)*

PROOF. By induction on $\delta_1(\tau)$, for every detail m, n, j.

a) If $\delta_1(\tau) = 0$ then τ is a finite sum of terms from the set $\{1_1, \cdots, 1_k\}$ so $\delta(\sigma) = 0$.

b) If $\delta_1(\tau) = 2$ it is easily seen that there is a finite number of equivalence classes and the case is clear.

c) Let us first show that if $\tau = \tau_1 + \tau_2$, $n \in \omega$ and for every $m \in \omega$ there are σ_1 and σ_2 such that $M(\tau_i)/E_n \equiv_m M(\sigma_i)$ and $\delta_1(\sigma_i) < \delta_1(\tau)$ or $\delta_1(\sigma) = 0$. Then for every $m \in \omega$ there is a σ such that $M(\tau)/E_n \equiv_n M(\sigma)$ and $\delta_1(\sigma) < \delta_1(\tau)$.

This is true since there are τ^j_i, $1 \leqq i, j \leqq 2$, such that $M(\tau_i) \cong M(\tau^1_i + \tau^2_i)$, $M(\tau_i)/E_n \cong M(\tau^1_i)/E_n + M(\tau^2_i)/E_n$, $i = 1, 2$, $\| M(\tau^2_1)/E_n \|$, $\| M(\tau^1_2)/E_n \| \leqq 2$ and $M(\tau)/E_n \cong M(\tau^1_1)/E_n + M(1_t) + M(\tau^2_2)/E_n$. (The τ^i_j might be 0.) If $m \in \omega$ is given, it is easy to show (by the induction hypothesis on τ_1, τ_2, n and $m + 2$) that there are σ^1, σ^2 such that $M(\tau^i_i)/E_n \equiv_m M(\sigma^i)$ and $\delta_1(\sigma^i) < \delta_1(\tau_i)$ or $\delta_1(\sigma^i) = 0$, $i = 1, 2$ so $\sigma = \sigma^1 + 1_t + \sigma^2$ is as desired.

Let $\delta_1(\tau) > 0$ and assume that the claim of the theorem is true for every τ' such that $\delta_1(\tau') < \delta_1(\tau)$. By the preceding claim we can assume that τ has one of the following forms:

i) $\tau = \tau_1 \cdot \omega$;

ii) $\tau = \eta(\tau_1, \cdots, \tau_k)$;

iii) $\tau = \tau_1 \cdot \omega^*$ or $\tau = \tau_1 \cdot z$.

i) By 2.8(a) there are τ'_0 and τ'_1 such that $M(\tau) \cong M(\tau'_0 + \tau'_1 \cdot \omega)$, $\delta_1(\tau'_0)$, $\delta_1(\tau'_1) < \delta_1(\tau)$, and $M(\tau)/E_n \cong M(\tau'_0)/E_n + M(\tau'_1)/E_n \cdot \omega$. Let $m \in \omega$ and σ_0 and σ_1 be the terms the existence of which is assured by the induction hypothesis for τ'_0, τ'_1 respectively and m. Then $\sigma_0 + \sigma_1 \cdot \omega$ fulfills the requirement of the theorem for τ and m.

ii) Either $\| M(\tau)/E_n \| = 1$ or $M(\tau)/E_n \cong \eta(M(\tau_1)/E_n, \cdots, M(\tau_k)/E_n)$. In both cases the existence of an appropriate σ is easily proved. Case (iii) is proved in the same way as case (i). Q.E.D.

THEOREM 2.10. *For every term $\tau \in p^j$ and every $m \in \omega$ there is a model M such that $M \equiv_m M(\tau)$ and $Th(M)$ is finitely axiomatizable.*

PROOF. By induction on $\delta_1(\tau)$. If $\delta_1(\tau) = 0$ then $M(\tau)$ is finite, so there is nothing to prove. Suppose the claim of the theorem is true for every $k < \delta_1(\tau)$. Let n_0 be such that for every $n \geqq n_0$ and every $a, b \in M(\tau)$ $a E_n b$ iff $a E_{n_0} b$. Suppose that the formula $\varphi(x, y)$ which says in every L^j model, that $x E_{n_0} y$ is of depth n_1, and let $n = \max(m, n_1 + 2)$. Let $M(\sigma) \equiv_n M(\tau)/E_n$ and $\delta_1(\sigma) < \delta_1(\tau)$. By the induction hypothesis there is $M^1 \equiv_n M(\sigma)$ such that $\mathrm{Th}(M^1)$ is finitely axiomatizable. Let $\{t_1, \cdots, t_k\} = T(j, n)$ be the set of all n-types of E_n-equivalence classes of $M(\tau)$. (So the language of M^1 has a unary predicate P_i corresponding to every $t_i \in T(j, n)$.) For every $1 \leqq i \leqq k$ let M_i be the model chosen in 2.7 such that $\mathrm{Th}(M_i) = t_i$, and let $M = (\Sigma_{a \in |M^1|} M^a, E^M)$ where $M^a \cong M_i$ iff $a \in P_i^{M^1}$ and $\langle b, c \rangle \in E^M$ iff $a \in |M^1|$ for some $a \in |M^1|$ $b, c \in |M^a|$. By Lemma 2.4 $M \equiv_n (M(\tau), E_n^{M(\tau)})$.

In order to show that $M \restriction L^j$ is finitely axiomatizable we first prove that for every $a \in |M^1|$, $|M^a|$ is an E_n-equivalence class of $M \restriction L^j$. By the particular choice of the M_i's it is clear that every $|M^a|$ is included in some E_n-equivalence class of $M \restriction L^j$. By the choice of n, the sentence φ, which tells that all E_{n_0}-equivalent elements are E_n-equivalent belongs to $\mathrm{Th}(M(\tau), E_n^{M(\tau)})$, so it is true in M. So if $b_i \in |M^{a_i}|$, $i = 1, 2$, $a_1 \neq a_2$ and $\langle b_1, b_2 \rangle \in E_n^{M \restriction L^j}$ then $\langle b_1, b_2 \rangle \in E_{n_0}^{M \restriction L^j}$, contradicting φ is M. Thus every $|M^a|$ is an E_n-equivalence class of $M \restriction L^j$. Since $\mathrm{Th}(M^1)$ is finitely axiomatizable and $E_n^{M \restriction L^j}$ is definable in $M \restriction L^j$, $M \restriction L^j$ is finitely axiomatizable and the theorem is proved.

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