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On the Possible Number $no(M)$ = The Number of Nonisomorphic Models $L_{\infty,\lambda}$-Equivalent to M of Power λ, for λ Singular

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Introduction Let M be a model of power λ, with λ relations, each with $<\lambda$ places and of power $\leq\lambda$. What can be

$$no(M) = \{N/\cong : N \equiv_{\infty,\lambda} M,\ \|N\| = \lambda\}\ ?$$

We assume $V = L$ (otherwise there are independence results (by [8])). It is known that

(A) If $cf\,\lambda = \aleph_0$, it can be only 1 (by Scott [5] for $\lambda = \aleph_0$, and generally by Chang [1], essentially).

(B) If λ is regular uncountable and not weakly compact it can be 1 or 2^λ (it can be 2^λ, see [3]; cannot be $\neq 1, 2^\lambda$: for $\lambda = \aleph_1$ by Palyutin [4], for any λ by [6]).

(C) If λ is weakly compact $>\aleph_0$ then it can be any cardinal $\leq\lambda^+$ (by [7]).

We prove here

(D) If λ is singular of uncountable cofinality, $no(M)$ can be any cardinal $\chi < \lambda$ (and also $\chi = 2^\lambda$). (This follows by 3.18 here.)

So we answer the question from [7], bottom of p. 26. The second question there, top of p. 26, is answered trivially by 1.4.

Notation: We consider functions as relations.

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1 Introducing the notions

1.1 Definition

(1) Let for a model M of power λ, $no(M)$ be the cardinality of $\{N/\cong : N \equiv_{\infty,\lambda} M, \|N\| = \lambda\}$.
(2) $SP^{\lambda}_{\mu,\kappa} = \{no(M) : M \in K^{\lambda}_{\mu,\kappa}\}$ where $K^{\lambda}_{\mu,\kappa} = \{M : M$ is a model, $\|M\| = \lambda$ and M has μ relations each of $<\kappa$ places$\}$.
(3) Let $RK^{\lambda}_{\mu,\kappa} = \{M : M \in K^{\lambda}_{\mu,\kappa}, \Sigma\{|R^M| : R \in L(M)\} \leq \lambda\}$
$RSP^{\lambda,\kappa}_{\mu} = \{no(M) : M \in RK^{\lambda}_{\mu,\kappa}\}$.
(4) We always assume that λ, μ, κ are $\geq \aleph_0$, $\kappa \leq \lambda$ and that $\mu \geq cf\, \kappa$ or κ is a successor $\left(\text{otherwise } M \in K^{\lambda}_{\mu,\kappa} \Leftrightarrow M \in \bigcup_{\vartheta<\kappa} K^{\lambda}_{\mu,\vartheta}\right)$. So w.l.o.g. every $M \in K^{\lambda}_{\mu,\kappa}$ is an $L^{\lambda}_{\mu,\kappa}$-model with a fixed $L^{\lambda}_{\mu,\kappa}$, which has for a closed unbounded set of $\alpha < \kappa$ exactly μ α-place predicates when κ is a limit cardinal, and μ κ^- place relations when $\kappa = (\kappa^-)^+$.

Remark: Note that if $\lambda^{<\kappa} > \lambda$, then in a model $M \in K^{\lambda}_{\mu,\kappa}$ we can code an arbitrary model of $K^{\chi}_{\mu,\kappa}$, where $\chi = \lambda^{<\kappa}$. This is a point in favor of dealing with $RSP^{\lambda}_{\mu,\kappa}$.

1.2 Claim If $\mu \leq \mu_1$ and $\kappa \leq \kappa_1$, then $SP^{\lambda}_{\mu,\kappa} \subseteq SP^{\lambda}_{\mu_1,\kappa_1}$ and $RSP^{\lambda}_{\mu,\kappa} \subseteq RSP^{\lambda}_{\mu_1,\kappa_1}$.

Proof: Trivial.

1.3 Claim We assume $\mu \geq \kappa$.

(1) If $\lambda = \lambda^{<\kappa}$ then $SP^{\lambda}_{\mu,\kappa} = SP^{\lambda}_{\mu,\aleph_0}$.
(2) $RSP^{\lambda}_{\mu,\kappa} = RSP^{\lambda}_{\mu,\aleph_0}$ when $\lambda > \kappa \vee cf\,\lambda \geq \kappa$.

Proof: (1) For every $M \in K^{\lambda}_{\mu,\kappa}$ let M^* be the following model:

(i) $|M^*| = |M| \cup \bigcup_{\alpha<\kappa} {}^{\alpha}|M|$

(ii) for each $i < \alpha < \kappa$ let $R_{\alpha,i}$ be the two-place relation

$$R^M_{\alpha,i} = \{\langle a, \bar{b}\rangle : a \in M,\ \bar{b} \in {}^{\alpha}|M|, \alpha = \bar{b}[i]\}\ .$$

(iii) For every α-place relation R of M, a one-place relation R^*

$$(R^*)^{M^*} = \{\bar{b} \in {}^{\alpha}|M| : M \models R[\bar{b}]\}\ .$$

Clearly $no(M^*) = no(M)$, $M \in K^{\lambda}_{\mu,\kappa} \Rightarrow M^* \in K^{\lambda}_{\mu,\aleph_0}$, hence $SP^{\lambda}_{\mu,\kappa} \subseteq SP^{\lambda}_{\mu,\aleph_0}$. The other inclusion holds by Claim 1.2.

(2) The proof is similar: define $(R^*)^{M^*}$ as above, $|M^*| = |M| \cup \bigcup_R (R^*)^{M^*}$, and then

$$R_{\alpha,i} = \{\langle a, \bar{b}\rangle : a \in M,\ \bar{b} \in {}^{\alpha}|M| \cap |M^*|,\ a = \bar{b}[i]\}\ .$$

Why did we restrict λ? Because looking at $L_{\infty,\lambda}$-equivalence we want that for every subset A of M^* of power $<\lambda$, $(A \cap M) \cup \{Rang\ \bar{b} : \bar{b} \in A\}$ has power $<\lambda$.

1.4 Claim

(1) If $\mu \leq \lambda^{<\kappa}$ then $SP^{\lambda}_{\mu,\kappa} = SP^{\lambda}_{\kappa,\kappa}$.
(2) Moreover, if $\mu \leq \lambda$, then $SP^{\lambda}_{\mu,\kappa} = SP^{\lambda}_{cf\kappa,\kappa}$; if κ is a successor then $SP^{\lambda}_{\mu,\kappa} = SP^{\lambda}_{\aleph_0,\kappa}$ (really when κ is a successor or $\aleph_0$ $SP^{\lambda}_{\mu,\kappa} = SP^{\lambda}_{1,\kappa}$).
(3) Similar assertion holds for RSP.

Proof: (1) It is well known that $(\lambda, <)$ is isomorphic to any model $L_{\infty,\omega}$-equivalent to it; moreover each element of $(\lambda, <)$ is defined by a formula in $L_{\infty,\omega}$ (and we can replace $L_{\infty,\omega}$ by $L_{\infty,\lambda}$). Also $L_{\infty,\lambda}$ satisfies the Feferman-Vaught Theorem. So we can show that for any M

$$no(M) = no(M + (\lambda, <)) \ .$$

Now in $M + (\lambda, <)$ we can use the $\alpha < \lambda$ and even sequences of length $<\kappa$ to parametrize the relations.

(2) and (3): Left to the reader.

1.5 Claim

(1) If $\mu \geq \chi = \lambda^{<\kappa}$ then $SP^{\lambda}_{\mu,\kappa} = SP^{\lambda}_{\chi,\kappa}$.
(2) If $\mu \geq \chi = \lambda + \kappa$ then $RSP^{\lambda}_{\mu,\kappa} = RSP^{\lambda}_{\chi,\kappa}$.

Proof: (1) For every $\alpha < \kappa$ and M, on ${}^{\alpha}|M|$, we define an equivalence relation E_α, realizing the same atomic type. The number of classes is $\leq \lambda^{<\kappa} = \chi$ (if our hypothesis holds).

We define for every $M \in K^{\lambda}_{\mu,\kappa}$ a model M^*:

(i) $|M^*| = |M|$
(ii) for every $\alpha < \kappa$ and E_α-equivalence class A, let $R^{M^*}_A = \{\bar{a} \in {}^{\alpha}|M| : \bar{a} \in A\}$.

Clearly $M^* \in K^{\lambda}_{\chi,\kappa}$, $\|M^*\| = \lambda$ and $no(M) = no(M^*)$. Hence $SP^{\lambda}_{\mu,\kappa} \subseteq SP^{\lambda}_{\chi,\kappa}$, and the other inclusion follows by Claim 1.2.

(2) Similar proof.

1.6 Claim If $\lambda^{<\kappa} \geq \chi > \lambda$, then $Sup(SP^{\lambda}_{\mu,\kappa}) \geq Sup(SP^{\chi}_{\mu,\kappa})$.

Proof: Let $M \in K^{\chi}_{\mu,\kappa}$; for notational simplicity we assume that for some $\vartheta < \kappa$, $\lambda^{\vartheta} \geq \chi$, so w.l.o.g. $|M| \subseteq {}^{\vartheta}\lambda$. So we reinterpret the relations of M as relations on λ; i.e., we define a model M^*:

(i) $|M^*| = \lambda$
(ii) for $R \in L(M)$, R α-place.

$R^{M^*} = \{\langle a_i : i < \vartheta\alpha \rangle : a_i \in |M^*|$, and if we let for $\beta < \alpha$, $\bar{b}_\beta = \langle a_{\vartheta,\beta+i} : i < \vartheta \rangle$ then $\langle \bar{b}_\beta : \beta < \alpha \rangle \in R^M \rangle$.

It is easy to see that $M^* \in K^{\lambda}_{\mu,\kappa}$, and $no(M^*) \geq no(M)$ (we get $\geq$ and not necessarily equality, as in $no(M)$ we use a finer equivalence relation: $L_{\infty,\chi}$-equivalent and not $L_{\infty,\lambda}$-equivalence).

1.7 Claim

(1) If $\chi_i \in SP^\lambda_{\mu,\kappa}$ $(i < \alpha \le \lambda)$ then

$$\prod_{i<\alpha} \chi_i \in SP^\lambda_{\mu,\kappa} \ .$$

(2) Similarly for RSP.

Proof: (1) Let $M_i \in K^\lambda_{\mu,\kappa}$, $\chi_i = no(M_i)$ and $L = L(M_i)$is fixed (see Definition 1.1(4)). W.l.o.g. $|M_i| \cap |M_j| = \varnothing$ for $i \ne j$. We define a model M:

(i) $|M| = \bigcup_{i<\alpha} M_i$.

(ii) $R^M = \bigcup_{i<\alpha} R^{M_i}$ for each $R \in L$.

(iii) $\le^M = \{(a,b) : (\exists i \le j \le \alpha)[a \in M_i \wedge b \in M_j]\}$.

Clearly $M \in K^\lambda_{\mu,\kappa}$ and $no(M) = \prod_{i<\alpha} no(M_i) = \prod_{i<\alpha} \chi_i$, hence $\prod_{i<\alpha} \chi_i = no(M) \in SP^\lambda_{\mu,\kappa}$.

(2) The same proof.

1.8 Claim

(1) If $\chi \in SP^\lambda_{\mu,\kappa}$, ϑ a cardinal, $2 \le \vartheta \le \lambda$, then the cardinality of $\left\{\langle \vartheta_i : i < \chi\rangle : \sum_{i<\chi} \vartheta_i = \vartheta, \text{ each } \vartheta_i \text{ a cardinal, } 0 \le \vartheta_i \le \vartheta\right\}$ belongs to $SP^\lambda_{\mu,\kappa}$.

(2) Let $N_i \in K^{\le\lambda}_{\mu,\kappa}$ (may be even a finite model), for $i < \alpha$, $\alpha \le \lambda$, be pairwise nonisomorphic but $N_i \equiv_{\infty,\lambda} N_0$ and $\left[N \equiv_{\infty,\lambda} N_0 \wedge \|N\| < \lambda \Rightarrow \bigvee_{i<\alpha} N \cong N_i\right]$. Let G_i be the group of automorphisms of N_i and define $f \approx g$ *mod* G_i, if f,g are functions with domain N_i and $(\exists h \in G_i)(\forall a \in N_i)[f(a) = g(h(a))]$. Now $\approx$ is an equivalence relation, and let $\chi^{N_i}/G_i = 2f/\approx : fa$ function from N_i into χ. Now if $\chi \in SP^\lambda_{\mu,\kappa}$ then $\sum_i |\chi^{N_i}/G_i| \in SP^\lambda_{\mu,\kappa}$.

(3) Similarly for $RSP^\lambda_{\mu,\kappa}$ (and $N_i \in RK^{\le\lambda}_{\mu,\kappa}$).

Proof: (1) Let $M \in SP^\lambda_{\mu,\kappa}$, $\chi = no(M)$, and choose $M_i \cong M$, $|M_i| \cap |M_j| = \varnothing$ for $i < j < \vartheta$. Now define M as in the proof of Claim 1.7, except

(iii) $E^{M^*} = \{(a,b) : (\exists i < \vartheta)(a \in M_i \wedge b \in M_i]$.

Clearly $M^* \in K^\lambda_{\mu,\kappa}$, $no(M^*)$ is as required to exemplify the conclusion.

(2) and (3): Proved similarly.

In the following two sections we shall prove:

1.9 Theorem *If λ is singular of uncountable cofinality, $\aleph_0 \le \xi \le \lambda$ then* $\xi^{cf\lambda} \in RSP^\lambda_{\lambda,\lambda}$.

Proof: See 3.17.

1.10 Theorem *If λ is singular of uncountable cofinality, $\chi^{cf\lambda} < \lambda$ then* $\chi \in RSP^\lambda_{\lambda,\lambda}$.

Proof: See 3.18.

In a following paper (in a Springer lecture notes volume) we shall prove similar results for $SP^{\lambda}_{\aleph_0,\aleph_0}$. Let us summarize the known results:

1.11 Theorem

(1) *For every* λ, $1 \in SP^{\lambda}_{\aleph_0,\aleph_0}$.
(2) *If* $cf\,\lambda = \aleph_0$, *then* $SP^{\lambda}_{\mu,\aleph_0} = \{1\}$ *and when* $[\lambda > \kappa \vee cf\lambda \geq \kappa]$, $RSP^{\lambda}_{\mu,\kappa} = \{1\}$ (*by Scott* [5] *when* $\lambda = \aleph_0$ *and Chang* [1] *when* $\lambda > \aleph_0$)
(3) *If* $\lambda > \aleph_0$ *is regular or* $\lambda = \lambda^{\aleph_0}$ *then* $2^{\lambda} \in SP^{\lambda}_{\aleph_0,\aleph_0}$ (*see* [3] *for* λ *regular, and by Shelah* [8] *for* $\lambda = \lambda^{\aleph_0}$).
(4) $(V = L)$. *If* $\lambda > \aleph_0$ *is regular not weakly compact then* $SP^{\lambda}_{\mu,\lambda} = \{1, 2^{\lambda}\}$ (*by Palyutin* [4] *for* $\lambda = \aleph_1$ *by Shelah* [6] *generally*).
(5) *if* $\lambda > \aleph_0$ *is weakly compact then every* $\chi, 2 \leq \chi \leq \lambda$, *belong to* $SP^{\lambda}_{\aleph_0,\aleph_0}$ (*by Shelah* [7]).
(6) *If* λ *is singular*, $\chi^{cf\lambda} < \lambda$ *and* $cf\lambda > \aleph_0$ *then* $\chi \in RSP^{\lambda}_{\lambda,\lambda}$ (*by* 1.10).
(7) *If* $\lambda > cf\lambda > \aleph_0$ *and* $\chi \leq \lambda$ *then* $\chi^{cf\lambda} \in RSP^{\lambda}_{\lambda,\lambda}$ (*by* 1.9).
(8) *If* $\lambda^{<\kappa} > \lambda$ *then* $2^{\lambda} \in SP^{\lambda}_{\mu,\kappa}$ (*by* 1.6 *and* 1.7(1)).

In a subsequent paper we shall improve (6) for some λ, χ.

2 Constructing the example This section is dedicated to the proof of

2.1 Main Lemma *Suppose* λ *is strong limit singular*, $\kappa = cf\lambda$. *Also* M *is a model of power* $\leq \lambda$, *and*

(a) $|M| = \bigcup_{i<\kappa} P_i^M$, $P_i^M \cap P_j^M = \emptyset$ *for* $i \neq j$, $|P_i^M| < \kappa$, $\vartheta = no(M)$ P_i *a monadic* predicate of M, $\vartheta = no(M)$, *or even*
(b) $|M| = \bigcup_{i<\kappa} P_i^M$, $P_i^M \cap P_j^M = \emptyset$ *for* $i \neq j$, P_i^M *has power* $<\lambda$ *and the number of nonisomorphic* N *satisfying the following is* ϑ: $N \equiv_{L_{\infty,\kappa}} M$, *moreover in the following game* (*with* ω *steps*) *player II has a winning strategy*:

in stage $n(<\omega)$: *player I chooses* i_n, $\bigcup_{l<n} i_l < i_n < \kappa$; *player II chooses an isomorphism* g_n *from* $M \restriction \bigcup_{j<i_n} P_j^M$ *onto* $N \restriction \bigcup_{j<i_n} P_j^N$ *which extends* $\bigcup_{l<n} g_l$.

Then we can find a model M^*, *of cardinality* λ *such that*: $no(M^*) = \vartheta$ *and each nonlogical symbol of* M^**'s language has an arity smaller than* λ, *and power* $\leq 2^{\chi}$ *for some* $\chi < \lambda$, *and* $|L(M^*)| \leq \lambda \leq \lambda + |L(M)|$.

Remark: (1) We use hypothesis 2.1(b) only as 2.1(a) $\Rightarrow$ 2.1(b). (*Note* $\|M\| \leq \sum_{i<\kappa} |P_i^M| \leq \sum_{i<\kappa} \kappa = \kappa$; if $\|M\| < \kappa$ necessarily $\vartheta = 1$, in which case the conclusion is trivial, so $\|M\| = \kappa$.)

(2) In case (b) we can assume that the range of h_R (see below) is bounded (if we omit the R's with unbounded h_R the hypothesis is not changed).

In order to get this in case (a) we need every relation of M has arity $<\kappa$.

Proof: Let L be the language of M. W.l.o.g. L has no function symbols and for every α-place predicate R there is a function h_R from α to κ such that

$M \models (\forall x_0, \ldots, x_i, \ldots)_{i<\alpha}\left[R(x_0, \ldots, x_i, \ldots) \rightarrow \bigwedge_{i<\alpha} P_{h_R(i)}(x_i)\right]$. We let $\alpha = \alpha(R)$. We assume that there is $R \in L$, $\alpha(R) > 1$. Let $\lambda = \sum_{i<\kappa} \lambda_i$, $\kappa < \lambda_i < \lambda_j$ for $i < j < \kappa$, and for each i λ_i is a regular cardinal $> \sum_{j\le i} \lambda_j$.

2.2 Definition

(1) We define a class K of L-models: $\mathfrak{A} \in K$ iff $|\mathfrak{A}| = \bigcup_{i<\kappa} P_i^{\mathfrak{A}}$, for $i \neq j$ $P_i^{\mathfrak{A}} \cap P_j^{\mathfrak{A}} = \emptyset$, and for every predicate R, $\mathfrak{A} \models (\forall x_0, \ldots, x_i, \ldots)[R(x_0, \ldots, x_i, \ldots) \rightarrow \bigwedge_i P_{h_R(i)}(x_i)]$.

(2) We let $K^0 \subseteq K$ be the family of $N \in K$ such that player II wins the game described in 2.1(b).

(3) For each $\mathfrak{A} \in K$ we define an L^*-model $\mathfrak{A}^*$:

$|\mathfrak{A}^*| = \{\langle a, \xi\rangle : a \in \mathfrak{A}, \text{ and } a \in P_i^{\mathfrak{A}} \Rightarrow \xi < \lambda_i\}$.

$P_i^{\mathfrak{A}^*} = \{\langle a, \xi\rangle : a \in P_i^{\mathfrak{A}}, \text{ and } \xi < \lambda_i\}$.

For each $R \in L$ let $I_R = \{\langle\alpha, j\rangle : \alpha < \alpha(R) \text{ and } j < \lambda_{h_R(\alpha)}\}$, and let $R^{\mathfrak{A}^*}$ be the set of tuples

$$\langle x_{0,0}, x_{0,1}, \ldots, x_{0,j}, \ldots; x_{1,0}, x_{1,1}, \ldots, x_{1,j} \ldots; \ldots; x_{\alpha,0}, x_{\alpha,1}, \ldots, x_{\alpha,j} \ldots; \ldots)\rangle_{\langle\alpha,j\rangle \in I_R}$$

which satisfies: there are $a_\alpha \in \mathfrak{A}$ for $\alpha < \alpha(R)$ such that

(a) $\mathfrak{A} \models R[a_0, \ldots, a_\alpha, \ldots]$ hence $a_\alpha \in P_{h_R(\alpha)}^{\mathfrak{A}}$.
(b) for each α for all but $<\lambda_{h_R(\alpha)}$ ordinals $\gamma < \lambda_{h_R(\alpha)}$, $x_{\alpha,\gamma} = \langle a_\alpha, \gamma\rangle$
(c) the $x_{\alpha,\gamma} (\alpha < \alpha(R), \gamma < \lambda_{h_R(\alpha)})$ are distinct, and $x_{\alpha,\gamma} \in P_\alpha^{\mathfrak{A}^*}$.

(4) Let $K^* = \{\mathfrak{A} : \mathfrak{A} \text{ an } L^*\text{-model, } L_{\infty,\lambda}\text{-equivalent to } M^*\}$.

2.3 Fact If $\mathfrak{A} \in K^0$ then $\|\mathfrak{A}\| = \|M\|$, $|P_i^{\mathfrak{A}}| = |P_i^M|$ (for each i). Also $M \in K^0$.

Proof: Trivial.

2.4 Fact If $\mathfrak{B} \in K^*$ then $\|\mathfrak{B}\| = \lambda$ and $|P_i^{\mathfrak{B}}| = \lambda_i + |P_i^M| < \lambda$.

Proof: Trivial.

2.5 Fact If $N \in K^0$ then $N^* \in K^*$.

Proof: Call a set $A \subseteq M^*$ small if $|A \cap P_i| < \lambda_i$. Similarly for N. Call a partial isomorphism f from M^* to N^* good if some g induces it, which means:

(α) g is an isomorphism from $M \restriction \bigcup_{j<i} P_j^M$ onto $N \restriction \bigcup_{j<i} P_j^N$ (for some i) which is a winning position for player II in the game from 2.1(b).
(β) the set $\{\langle a, \xi\rangle : \langle g(a), \xi\rangle \neq f(\langle a, \xi\rangle)$, e.g., one is defined the other not$\}$ is a small subset of M^*.
(γ) f is one to one, preserving the predicates P_i, and it maps $\bigcup_{j<i} P_j^{M^*}$ onto $\bigcup_{j<i} P_j^{N^*}$.

It is easy to see that the family of good f's, exemplifies $M^* \equiv_{\infty,\lambda} N^*$.

2.6 Definition For each $\mathfrak{B} \in K^*$, we define $\mathfrak{B}^-$. For each $i < \kappa$ let

$$S_i = \{\langle a_\alpha : \alpha < \lambda_i \rangle : a_\alpha \in P_i^{\mathfrak{B}} \text{ for each } \alpha,\ a_\alpha \neq a_\beta \text{ for } \alpha < \beta < \lambda_i, \text{ and for some } R, \gamma, b,\ h_R(\gamma) = i,\ \mathfrak{B} \models R[\bar{b}_0, \ldots, \bar{b}_j, \ldots]_{j<\alpha(R)} \text{ and } \bar{b}_\gamma = \langle a_\alpha : \alpha < \lambda_i \rangle\}$$

(we allow to use equality for R).

Clearly S_i is a definable subset of $\mathfrak{B}$ (by a formula of $L_{\infty,\lambda}$ with no parameters). Now we define on S_i an equivalence relation E_i:

$$\langle a_\alpha^0 : \alpha < \lambda_i \rangle\ E_i \langle a_\alpha^1 : \alpha < \lambda_i \rangle \textit{ iff } \langle a_\alpha^0 : \alpha < \lambda_i \rangle \in S_i,\ \langle a_{\alpha_1} : \alpha < \lambda_i \rangle \in S_i \text{ and the symmetric difference of } \{a_\alpha^0 : \alpha < \lambda_i\},\ \{a_\alpha^1 : \alpha < \lambda_i\} \text{ has power } < \lambda_i.$$

Now we define $\mathfrak{B}^-$:

$$|\mathfrak{B}^-| = \{\bar{a}/E_i : \bar{a} \in S_i,\ i < \kappa\} \ .$$

$$P_i^{\mathfrak{B}^-} = \{\bar{a}/E_i : \bar{a} \in S_i\} \ .$$

$$R^{\mathfrak{B}^-} = \{\langle \bar{a}_0/E_{i(0)}, \ldots, \bar{a}_\alpha/E_{i(\alpha)}, \ldots \rangle_{\alpha<\alpha(R)} : \bar{a}_\alpha \in S_{h_R(\alpha)} \ . \ i(\alpha) = h_R(\alpha) \text{ and } \mathfrak{B} \models R^{\mathfrak{B}}[\bar{a}_0, \bar{a}_1, \ldots, a_\alpha, \ldots]_{\alpha<\alpha(R)}\} \ .$$

2.7 Fact If $N \in K^0$, then $(N^*)^-$ is isomorphic to N, and $P_i^{(N^*)^-} = \{\langle (a,\xi) : \xi < \lambda_i \rangle / E_i : a \in P_i\}$ and the isomorphism is the obvious one.

2.8 Fact If $\mathfrak{B} \in K^*$ then $\mathfrak{B}^- \in K^0$.

Proof: We call a partial isomorphism g from M to $\mathfrak{B}^-$ *good* if some f induces it, which means:

(α) f is an isomorphism from $M^* \restriction \bigcup_{j<i} P_j^M$ onto $\mathfrak{B} \restriction \bigcup_{j<i} P_j^{\mathfrak{B}}$ which preserve $L_{\infty,\lambda}$-equivalence, i.e.,

$$(M^*, c)_{c \in \bigcup_{j<i} P_j^{M^*}} \equiv_{\infty,\lambda} (\mathfrak{B}, f(c))_{c \in \bigcup_{j<i} P_j^{\mathfrak{B}}} \ .$$

(β) g is a function from $\bigcup_{j<i} P_j^M$ onto $\bigcup_{j<i} P_j^{\mathfrak{B}^-}$, where for $a \in P_j^M$

$$g(a) = \langle f\langle a, \xi \rangle : \xi < \lambda_j \rangle / E_j \ .$$

It is easy to see that the family of good g exemplifies $\mathfrak{B}^- \in K^0$.

2.9 Fact If $\mathfrak{B} \in K^*$ then $(\mathfrak{B}^-)^*$ is isomorphic to $\mathfrak{B}$.

Proof: As $\mathfrak{B} \in K^*$, $|P_i^{\mathfrak{B}}| < \lambda$ (see Fact 2.3). Now by the definition $\mathfrak{B} \equiv_{\infty,\lambda} M^*$, hence there is a partition of $P_i^{\mathfrak{B}}$, $P_i^{\mathfrak{B}} = \bigcup_{a \in M} \{t_{a,\xi} : \xi < \lambda_i\}$, the $t_{a,\xi}$ are distinct (for $a \in P_i^M$, $\xi < \lambda_i$) and $\{\langle t_{a,\xi} : \xi < \lambda_i \rangle / E_i : a \in M\}$ is a list of all E_i-equivalence classes. So $P_i^{\mathfrak{B}^-} = \{\langle t_{a,\xi} : \xi < \lambda_i \rangle / E_i : a \in M\}$, and

$$P_i^{(\mathfrak{B}^-)^*} = \{(\langle t_{a,\xi} : \xi < \lambda_i \rangle / E_i, \xi \rangle : a \in M,\ \xi < \lambda\} \ .$$

Now define $F : \mathfrak{B} \to (\mathfrak{B}^-)^*$, for $a \in M_i$

$$F(t_{a,\xi}) = (\langle t_{a,\xi} : \xi < \lambda_i \rangle / E_{i,\xi}) \ .$$

It is easy to check that F is an isomorphism from $\mathfrak{B}$ onto $(\mathfrak{B}^-)^*$.

Proof of Lemma 2.1: The series of facts above prove that the number of nonisomorphic models in K^0 and in K^* are equal: the map $N \to N^*$ is from K^0 into K^* (see Fact 2.5) and the map $\mathfrak{B} \to \mathfrak{B}^-$ is from K^* to K^0 (see Fact 2.8); those maps are each an inverse of the other (when we divide by isomorphism) (see Facts 2.7, 2.9). As by Definition 2.2(4) and Fact 2.4:

$$K^* = \{\mathfrak{A} : \mathfrak{A} \equiv_{\infty,\lambda} M^*,\ \|\mathfrak{A}\| = \lambda\}$$

clearly $no(M^*)$ is the number of nonisomorphic $M \in K$, which was assumed to be ϑ.

For λ not strong limit we use instead of Lemma 2.1:

2.10 Main Lemma *Suppose that in 2.1 we assume further that every relation of M, restricted to* $\bigcup_{j<i} P_j^M$ *(for* $i < \kappa$*) has power* $<\lambda$*, but* λ *is singular, not necessarily strong limit.*

Then $\vartheta \in RSP^\lambda_{\lambda,\lambda}$

Proof: As the proof is similar to that of Lemma 2.1, we shall only mention the required changes:

In Definition 2.2(3) we redefine $R^{\mathfrak{A}^*}$:

$$R^{\mathfrak{A}^*} = \Big\{ \langle x_{0,0}, x_{0,1}, \ldots, x_{0,j_0}, \ldots, x_{1,0}, x_{1,1}, \ldots, x_{1,j_1}, \ldots; \ldots; x_{\alpha,0}, x_{\alpha,1}, \ldots, x_{\alpha,j_\alpha} \ldots; \ldots \rangle_{\substack{\alpha<\alpha(R)\\ \langle\alpha,j\rangle\in I_R}} :$$

There are $a_\alpha \in \mathfrak{A}$ for $\alpha < \alpha(R)$ such that:

(a) $\mathfrak{A} \models R[a_0, \ldots, a_\alpha, \ldots]$ hence $a_\alpha \in P^{\mathfrak{A}}_{h_R(\alpha)}$;

(b) for each α there are n and $0 = \xi_0 < \xi_1 < \ldots < \xi_n < \lambda_{h_R(\alpha)}$ and $a_{\alpha,l} \in P_{h_R(\alpha)}$ for $l < n$, such that:

$$\xi_n \le \gamma < \lambda_{h_R(\alpha)} \Rightarrow x_{\alpha,\gamma} = \langle a_\alpha, \gamma\rangle$$

$$\xi_l \le \gamma < \xi_{l+1} \Rightarrow x_{\alpha,\gamma} = \langle a_{\alpha,l}, \gamma\rangle \Big\}.$$

In the proof of Fact 2.5 redefine "g induces f" by replacing (β) by:

$(\beta)_1'$ for each $j < i$, there is $\xi_j < \lambda_j$ such that for $a \in P_j^M$,

$$f(\langle a, \xi\rangle) = \begin{cases} \langle g_j, \xi\rangle & \text{if } \xi < \xi_j \\ \langle g(a), \xi\rangle & \text{if } \xi \ge \xi_j \end{cases}$$

$(\beta)_2'$ for each $j \ge i$ for some $\xi_j < \lambda_j$, $f(\langle g_j, \xi\rangle) = \langle a, \xi\rangle$ if $a \in P_j^M$, $\xi < \xi_j$, undefined otherwise.

$(\beta)_3'$ g_j is a one-to-one function from P_j^M onto I_j^N.

Still the power of $L(M^*)$ is too large, but we can use Claim 1.4(1).

To get the desired conclusion we still have to find M as required in Lemma 2.1(b). We shall construct such M.

2.11 Conclusion If $\aleph_0 < \kappa = cf\,\lambda < \lambda$ then $2^\kappa \in RSP^\lambda_{\lambda,\lambda}$.

Proof: it is well known that there are two trees, with κ-levels, $L_{\infty,\kappa}$- equivalent:

one has a branch of order type κ, the other not. So each such tree is a model satisfying Lemma 2.1(a) for some $\vartheta \le 2^\kappa$, $\vartheta > 1$. In fact the hypothesis of Lemma 2.10 holds also. Hence, by 2.10, $(\exists \vartheta \le 2^\kappa)[\vartheta \in RSP^\lambda_{\lambda,\lambda} \wedge \vartheta > 1]$. By Claim 1.7(2) this implies that $2^\kappa \in RSP^\lambda_{\lambda,\lambda}$.

3 Building κ-Systems

3.1 Definition A κ-system will mean here a model of the form $\mathfrak{A} = \langle G_i, h_{i,j}\rangle_{i\le j<\kappa}$ where

(i) G_i is an Abelian group such that $(\forall x \in G_i)(x + x = 0)$, the G_i's are pairwise disjoint.
(ii) $h_{i,j}$ is a homomorphism from G_j into G_i when $i \le j$.
(iii) $h_{i_1,i_2} \circ h_{i_2,i_3} = h_{i_1,i_3}$ when $i_1 \le i_2 \le i_3$.
(iv) $h_{i,i}$ is the identity.

We denote κ-systems by $\mathfrak{A}, \mathfrak{B}$ and for a system $\mathfrak{A}$, we write $G_i = G_i^{\mathfrak{A}}$ $h_{i,j} = h_{i,j}^{\mathfrak{A}}$. Let $\|\mathfrak{A}\| = \sum_{i<\kappa} \|G_i\|$. Almost everything we prove holds for δ-systems, δ a limit ordinal and we shall use this.

Let $\mathfrak{A} \restriction \delta = \langle G_i^{\mathfrak{A}}, h_{i,j}^{\mathfrak{A}}\rangle_{i\le j<\delta}$.

3.2 Definition We say $\mathfrak{A} \le \mathfrak{B}$ if $G_i^{\mathfrak{A}}$ is a subgroup of $G_i^{\mathfrak{B}}$, $h_{i,j}^{\mathfrak{A}} \subseteq h_{i,j}^{\mathfrak{B}}$, and:

(*) for every $j < \kappa$, $a \in G_j^{\mathfrak{B}}$ there is a maximal $i \le j$ such that $h_{i,j}^{\mathfrak{B}}(a) \in G_i^{\mathfrak{A}}$.

3.3 Fact $\le$ is transitive reflexive and if $\mathfrak{A}_\alpha (\alpha < \delta)$ is increasing then

$$\bigwedge_{\alpha<\delta} \left[\mathfrak{A}_\alpha \le \bigcup_{\beta<\delta} \mathfrak{A}_\beta\right] .$$

3.4 Definition $Gr(\mathfrak{A}) = \{a = \langle a_{i,j} : i < j < \kappa\rangle : a_{i,j} \in G_i$, and if $\alpha < \beta < \gamma < \kappa$ then $a_{\alpha,\gamma} = a_{\alpha,\beta} + h_{\alpha,\beta}(a_{\beta,\gamma})\}$.

This is a group by coordinatewise addition.

3.5 Definition For $a = \langle a_i : i < \kappa\rangle \in \prod_{i<\kappa} G_i$, let fact $(a) = \langle a_{i,j} : i < j < \kappa\rangle$ where $a_{i,j} = a_i - h_{i,j}(a_j)$. Let Fact $(\mathfrak{A}) = \{fact(a) : a \in \Pi G_i^{\mathfrak{A}}\}$.

3.6 Claim The mapping $a \to fact(a)$ is from $\prod_{i<\kappa} G_i$ into $Gr(\mathfrak{A})$, and is a homorphism. So $Fact(\mathfrak{A})$ is a subgroup of $Gr(\mathfrak{A})$.

3.7 Definition

(1) $Gs(\mathfrak{A}) = \{\bar{a} \in Gr(\mathfrak{A})$: for every $\delta < \kappa$, $\langle a_{i,j} : i < j < \delta\rangle \in Fact(\mathfrak{A} \restriction \delta)\}$
(2) $E(\mathfrak{A}) = Gr(\mathfrak{A})/Fact(\mathfrak{A})$, $E^\circ(\mathfrak{A}) = Gs(\mathfrak{A})/Fact(\mathfrak{A})$.
(3) $\mathfrak{A}$ is called smooth if for every limit $\delta < \kappa$, $E^\circ(\mathfrak{A} \restriction \delta)$ has power 1.

Fact 3.7A Let $\mathfrak{A}$ be a κ-system:

(1) for every limit δ, Fact$(\mathfrak{A} \restriction \delta) \subseteq Gs(\mathfrak{A} \restriction \delta) \subseteq Gr(\mathfrak{A} \restriction \delta)$.
(2) If $\mathfrak{A}$ is smooth then for every limit $\delta < \kappa_1$, $E(\mathfrak{A} \restriction \delta)$ has power 1 and, i.e.,

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$Gr(\mathfrak{A} \restriction \delta) = \mathrm{Fact}(\mathfrak{A} \restriction \delta)$.
(3) $Gr(\mathfrak{A}) = Gs(\mathfrak{A})$.

Proof: (1) Easy.

(2) We prove this by induction on δ. For a given δ, by the induction hypotheses $Gr(\mathfrak{A} \restriction \delta) = Gs(\mathfrak{A} \restriction \delta)$. As $\mathfrak{A}$ is smooth, $E^\circ(\mathfrak{A} \restriction \delta) = Gs(\mathfrak{A} \restriction)\delta)/\mathrm{Fact}(\mathfrak{A} \restriction \delta)$ has power 1, hence $Gs(\mathfrak{A} \restriction \delta) = \mathrm{Fact}(\mathfrak{A} \restriction \delta)$; together with the previous sentence we get $Gr(\mathfrak{A} \restriction \delta) = \mathrm{Fact}(\mathfrak{A} \restriction \delta)$, hence $E(\mathfrak{A} \restriction \delta) = Gr(\mathfrak{A} \restriction \delta)/\mathrm{Fact}(\mathfrak{A} \restriction \delta)$ has power 1.

(3) Easy.

3.8 Claim There is $\mathfrak{A}$, $|\mathfrak{A}| = \mu + \kappa$ and $|E(\mathfrak{A})| \geq \mu$.

Proof: Let G_i be the free Abelian group of order two generated by $W_i = \{a^\xi_{i,j} : \xi < \mu,\ j < \kappa \text{ but } j > i\}$. So we can identify it with the family of finite subsets of W_i, with addition being the symmetric difference.

$h_{\alpha,\beta} : G_\beta \to G_\alpha$ is defined by

[1] $h_{\alpha,\beta}(a^\xi_{\beta,\gamma}) = a^\xi_{\alpha,\gamma} - a^\xi_{\alpha,\beta}$.

Check: For $\alpha < \beta < \gamma$ $h_{\alpha,\gamma} = h_{\alpha,\beta} \circ h_{\beta,\gamma}$ as

$$\begin{aligned} h_{\alpha,\beta}(h_{\beta,\gamma}(a^\xi_{\gamma,i})) &= h_{\alpha,\beta}(a^\xi_{\beta,i} - a^\xi_{\beta,\gamma}) = (a^\xi_{\alpha,i} - a^\xi_{\alpha,\beta}) - (a^\xi_{\alpha,\gamma} - a^\xi_{\alpha,\beta}) \\ &= a^\xi_{\alpha,i} - a^\xi_{\alpha,\gamma} = h_{\alpha,\gamma}(a^\xi_{\gamma,i}) \ . \end{aligned}$$

Let $a^\xi = \langle a^\xi_{i,j} : i < j < \kappa \rangle$. Clearly $a^\xi \in Gr(\mathfrak{A})$. We want to show $a^\xi - a^\zeta \notin \mathrm{Fact}(\mathfrak{A})$ for $\xi \neq \zeta$.

If not there are $w_i \in G_i$

[2] $a^\xi_{i,j} - a^\zeta_{i,j} = w_i - h_{i,j}(w_j)$.

Clearly w_i is nothing but a finite subset of W_i.

Let $G^*_i = \langle \{a^\epsilon_{i,j} : \epsilon \neq \xi,\ i < j < \kappa\} \rangle$. We can define a projection g_i onto $G^*_i : g_i(x) = x \cap \{a^\xi_{i,j} : j < \kappa, j > i\}$. It is easy to check that for $i < j < \kappa$, $h_{i,j} \circ g_j = g_i \circ h_{i,j}$ and $h_{i,j}$ maps G^*_j into G^*_i. Applying g_i on the equations [2] we get $a^\zeta_{i,j} = w^0_i - h_{i,j}(w^0_j)$ when $w^0_i = g_i(w_i)$. So we get that for some $w_i (i < \kappa)$

[3] $a^\xi_{i,j} = w_i - h_{i,j}(w_i)$.

So there are $n < \omega$ and S, an unbounded subset of κ such that $(\forall i \in S) |w_i| = n$.

Let $\alpha < \beta < \gamma$ be in S, by [3] $a^\xi_{\beta,\gamma} = w_\beta - h_{\beta,\gamma}(w_\gamma)$; apply $h_{\alpha,\beta}$ and get $a^\xi_{\alpha,\gamma} - a^\xi_{\alpha,\beta} = h_{\alpha,\beta}(w_\beta) - h_{\alpha,\gamma}(w_\gamma)$.

So

$$a^\xi_{\alpha,\gamma} + h_{\alpha,\gamma}(w_\gamma) = a^\xi_{\alpha,\beta} + h_{\alpha,\beta}(w_\beta) \ .$$

So for some $c_\alpha \in G_\alpha$ for every β, $\alpha < \beta$

[4] $a^\xi_{\alpha,\beta} + h_{\alpha,\beta}(w_\beta) = c_\alpha$.

Let $U_\alpha = \{i : a^\xi_{\alpha,i} \text{ appear in } c_\alpha\}$, remember c_α is a finite subset of W_α, so U_α is a finite subset of κ.

W.l.o.g. $\alpha \in S \wedge \beta \in S \wedge \alpha < \beta \Rightarrow \beta > \mathit{Max}\ U_\alpha$. So if $\alpha < \beta$ are in S, by the equation [4], $h_{\alpha,\beta}(w_\beta)$ has elements of the form $a^\xi_{\alpha,\beta}$ or $a^\xi_{\alpha,\gamma} : (\gamma < \beta)$ only.

(Clearly $a^{\xi}_{\alpha,\beta}$ does not appears in c_α, so it appears in $h_{\alpha,\beta}(w_\beta)$.) Hence (by $h_{\alpha,\beta}$'s definition) some $a^{\xi}_{\alpha,\gamma}(\gamma > \beta)$ appears in $h_{\alpha,\beta}(w_\beta)$, but this contradicts the equality.

3.9 Fact Assume $cf\ \kappa > \aleph_0$. If $\mathfrak{A}_\alpha(\alpha < \delta)$ is $\leq$-increasing continuous, $\boldsymbol{a} \in Gr(\mathfrak{A}_0) \subseteq Gr(\mathfrak{A}_\alpha)$, $\boldsymbol{a} \notin Fact(\mathfrak{A}_\alpha)$ (for $\alpha < \delta$) then $\boldsymbol{a} \notin \mathrm{Fact}\left(\bigcup_{\alpha<\delta} \mathfrak{A}_\alpha\right)$.

Proof: Suppose $\boldsymbol{a} = fact(\bar{b})$ $\bar{b} = \langle b_i : i < \kappa \rangle$. For each i there is a minimal $\alpha = \alpha(i) < \delta$, $b_i \in G_i^{\mathfrak{A}_{\alpha(i)}}$.

Now $i < j \Rightarrow \alpha(i) \leq \alpha(j)$, because $a_{i,j} = b_i - h_{i,j}(b_j)$ hence $b_i = a_{i,j} + h_{i,j}(b_j)$ but $a_{i,j} \in G_i^{\mathfrak{A}_0} \subseteq G_i^{\mathfrak{A}_{\alpha(j)}}$, and $b_j \in G_i^{\mathfrak{A}_{\alpha(j)}}$. So $b_i \in G_i^{\mathfrak{A}_{\alpha(j)}}$ hence $\alpha(i) \leq \alpha(j)$. If $\langle \alpha(i) : i < \kappa \rangle$ has a bound $\alpha^* < \delta$ then $\boldsymbol{a} \in Fact(\mathfrak{A}_\alpha)$ contradiction.

Hence $\langle \alpha(i) : i < \kappa \rangle$ converge to δ. So $cf\ \delta = cf\ \kappa > \aleph_0$.

Hence for some $\vartheta < \kappa$, $cf\ \vartheta = \aleph_0$, $\langle \alpha(i) : i < \vartheta \rangle$ is not eventually constant and let $\beta = \bigcup_{i<\vartheta} \alpha(i)$.

However, look at 3.2(*), apply to $\mathfrak{A} = \mathfrak{A}_\beta$, $\mathfrak{B} = \mathfrak{A}_{\alpha(\vartheta)}$, $j = \beta$, $a = b_\beta$, and get contradication.

3.10 Fact There is a smooth $\mathfrak{A}$, $|\mathfrak{A}| = \mu^\kappa$ with $|E(\mathfrak{A})| = \mu$ such that every $h_{i,j}^{\mathfrak{A}}$ is onto $G_i^{\mathfrak{A}}$.

Proof: By 3.8 there is $\mathfrak{A}_0$, $\|\mathfrak{A}_0\| \leq \mu^\kappa$, $|E(\mathfrak{A})| \geq \mu$. Let $\boldsymbol{a}_\xi + Fact(\mathfrak{A}_0) \in Gr(\mathfrak{A}_0)/Fact(\mathfrak{A}_0)$ be distinct for $\xi < \mu$. We define by induction on $\alpha < \mu^\kappa \times \kappa^+$ (ordinal multiplication) $\mathfrak{A}_\alpha$, $\leq$-increasing, continuous $\|\mathfrak{A}_\alpha\| \leq \mu^\kappa$, such that $\boldsymbol{a}_\xi - \boldsymbol{a}_\zeta \notin Fact(\mathfrak{A}_\alpha)$ for $\xi \neq \zeta$. Clearly it is enough to prove [1], [2], [3] below (see later):

[1] if $\boldsymbol{b} \in Gr(\mathfrak{A}_\alpha) - \langle Fact(\mathfrak{A}_0), \ldots, \boldsymbol{a}_\xi, \ldots \rangle_{\xi<\mu}$, *then* we can define $\mathfrak{A}_{\alpha+1}$ such that: $\boldsymbol{b} \in Fact(\mathfrak{A}_{\alpha+1})$.

We take care of smoothness similarly. This is done as follows: let $\mathfrak{A}_{\alpha+1} = \langle G_i^{\alpha+1}, h_{i,j}^{\alpha+1} \rangle_{i<j<\kappa}$, where

$G_i^{\alpha+1} = \langle G_i^\alpha, x_i \rangle$-free extension (among Abelian satisfying $x + x = 0$)

$h_{i,j}(x_j) = x_i - b_{i,j}$

[2] if $i < j < \kappa$, $x \in G_i^{\mathfrak{A}_\alpha} - Range\ h_{i,j}^{\mathfrak{A}_\alpha}$ we can define $\mathfrak{A}_{\alpha+1}$ such that $x \in Rang\ h_{i,j}^{\mathfrak{A}_{\alpha+1}}$

We let

$$G_\xi^{\mathfrak{A}_{\alpha+1}} = \begin{cases} G_\xi^{\mathfrak{A}_\alpha} & \text{if } \xi \leq i \text{ or } \xi > j \\ \langle G_\xi^{\mathfrak{A}_\alpha}, x_\xi \rangle & \text{if } i < \xi \leq j \ . \end{cases}$$

$h_{\zeta,\xi}(x_\xi) = x_\zeta$ when $i < \zeta < \xi \leq j$, $h_{i,\xi}(x_\xi) = x$.

[3] if $\delta < \kappa$, $b \in Gr(\mathfrak{A}_\alpha \restriction \delta)$ then we can define $\mathfrak{A}_{\alpha+1}$ such that $b \in \mathrm{Fact}(\mathfrak{A}_{\alpha+1} \restriction \delta)$.

This is similar to [1].

Why are [1], [2], [3] enough?

As we can define the $\mathfrak{A}_\alpha$'s such that if $\epsilon = \mu^\kappa \times \gamma$, $\epsilon(1) = \mu^\kappa \times (\gamma + 1)$ $\epsilon(*) = \mu^\kappa \times \kappa^+$:

(a) for $b \in Gr(\mathfrak{A}_{\mu^\kappa \times \gamma})$ for some $\beta < \epsilon(1)$, $b \in \langle Fact(\mathfrak{A}_\beta), \dots, a_\xi, \dots \rangle_{\xi<\mu}$ (use [1]) hence:

$$b \in \langle Fact(\mathfrak{A}_{\epsilon(1)}), \dots, a_\xi, \dots \rangle$$

(b) for every $x \in P_i^{\mathfrak{A}_\xi}$, $i < j < \kappa$, for some $\beta < \epsilon(1)$

$$x \in \mathrm{Rang}(h_{i,j}^{\mathfrak{A}_\beta}) \text{ (use [2]) (hence } x \in \mathrm{Rang}(h_{i,j}^{\mathfrak{A}_{\epsilon(*)}}))$$

(c) for every limit $\delta < \kappa$, if $b \in Gr(\mathfrak{A}_\epsilon \restriction \delta)$ then for some $\beta < \epsilon(1)$, $b \in Fact(\mathfrak{A}_\beta \restriction \delta)$ (see [3]) hence $b \in Fact(\mathfrak{A}_{\epsilon(*)} \restriction \delta)$.

As $cf\ \epsilon(*) > \kappa$, $Gr(\mathfrak{A}_{\epsilon(*)}) = \bigcup_{\beta<\kappa^+} Gr(\mathfrak{A}_{\lambda^\kappa \times \beta})$ and $Gr(\mathfrak{A}_{\epsilon(*)} \restriction \delta) = \bigcup_{\beta<\kappa^+} Gr(\mathfrak{A}_{\lambda^\kappa \times \beta} \restriction \delta)$, so $\mathfrak{A}_{\epsilon(*)}$ is as required.

3.11 Claim For every κ-system $\mathfrak{A}$ where the $h_{i,j}^{\mathfrak{A}}$ are onto, there is M, $\|M\| = \|\mathfrak{A}\|$, as in Lemma 2.1(b), and we get for M, $\vartheta = |E^\circ(\mathfrak{A})|$.

Proof: We concentrate on $\vartheta \geq \aleph_0$.

For every $a \in Gr(\mathfrak{A})$ we define a model M_a:

(i) $|M_a| = \bigcup_{i<\kappa} G_i^{\mathfrak{A}}$.
(ii) $P_i^{M_a} = G_i^{\mathfrak{A}}$.
(iii) for every $i < \kappa$, $c \in G_i$ we have a partial function $F_c : P_i^{M_a} \to P_i^{M_a}$:

$$F_c(x) = c + x$$

(iv) for every $i < j$, we have a partial function $H_{i,j} : P_j^{M_a} \to P_i^{M_a}$

$$H_{i,j}(x) = h_{i,j}(x) + a_{i,j} \ .$$

The following series of Facts will prove Claim 3.11.

3.12 Fact $M_a \cong M_b$ iff $a - b \in Fact(\mathfrak{A})$ (the subtraction is in $Gr(\mathfrak{A})$).

Proof: Suppose $b - a = fact(d)$ where $d = \langle d_i : i < \kappa \rangle$. We define an isomorphism $g = g_d$ from M_a onto M_b:

for $x \in G_i^{\mathfrak{A}}$ let $g(x) = x + d_i$.

Clearly g maps each $P_i^{M_a}$ onto $P_i^{M_b}$ hence it maps $|M_a|$ onto $|M_b|$. Also g is one-to-one.

Now for each $i < \kappa$, $c \in G_i^{\mathfrak{A}}$, $x \in P_i^{M_a} = G_i^{\mathfrak{A}}$

$$g(F_c^{M_a}(x)) = g(c + x) = c + x + d_i = c + g(x) = F_c^{M_b}(g(x))$$

Lastly for $i < j$, $x \in P_j^{M_a} = G_j^{\mathfrak{A}}$

$$y(H_{i,j}^{M_a}(x)) = g(h_{i,j}(x) + a_{i,j}) = h_{i,j}(x) + a_{i,j} + d_i =$$
$$h_{i,j}(x) + h_{i,j}(d_j) + b_{i,j} = h_{i,j}(x + d_j) + b_{i,j} = H_{i,j}^{M_b}(x + d_j) = H_{i,j}^{M_b}(g(x))$$

(the third equality is as $b - a = fact(d)$ and $fact(d)$'s definition.

For the other direction suppose g is an isomorphism from M_a onto M_b. We let

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$d_i = g(x) - x$ for any (some) $x \in P_i^{M_a}$ and $d = \langle d_i : i < \kappa \rangle$, and can check that $b - a = fact(d)$.

3.13 Fact For any M_a, $M_b (a, b \in Gs(\mathfrak{A}))$ player II wins the game of 2.1(b).

Proof: We let (using the notation from the proof of Fact 3.12)

$$\mathfrak{P}_\alpha = \{g_d : d \in \prod_{i \in \alpha} G_i^{\mathfrak{A}},\ a \restriction \alpha - b \restriction \alpha = fact(d)\} \ .$$

By 3.12 and the hypothesis, $\mathfrak{P}_\alpha \neq \emptyset$, and by the proof of 3.12, $\mathfrak{P}_\alpha$ is a set of isomorphisms from $M_a \restriction \bigcup_{i<\alpha} G_i^{\mathfrak{A}}$ onto $M_b \restriction \bigcup_{i<\alpha} G_i^{\mathfrak{A}}$. The strategy of player II is to use partial isomorphisms from $\bigcup_{\alpha<\kappa} \mathfrak{P}_{\alpha+1}$. The only missing point is: for successor $\alpha < \beta < \kappa$, $g \in \mathfrak{P}_\alpha$, there is $g' \in \mathfrak{P}_\beta$, $g \subseteq g'$; equivalently, for $d_0 \in \prod_{i<\alpha} G_i^{\mathfrak{A}}$, satisfying $a \restriction \alpha - b \restriction \alpha = fact(d_0)$ there is $d \in \prod_{i<\beta} G_i$, $d_0 = d \restriction \alpha$, and $a \restriction \beta - b \restriction \beta = fact(d)$. We know that for some $d_1, d_2 \in \prod_{i<\beta} G_i^{\mathfrak{A}}$, $a \restriction \beta = fact(d_1)$, $b \restriction \beta = fact(d_2)$.

Let $d_0 = \langle d_i^0 : i < \alpha \rangle$, $d_1 = \langle d_i^1 : i < \beta \rangle$, $d_2 = \langle d_i^2 : i < \beta \rangle$.

As $a \restriction \alpha = fact(d_1 \restriction \alpha)$, $b \restriction \alpha = fact(d_2 \restriction \alpha)$ and $a \restriction \alpha - b \restriction \alpha = fact(d_0)$ clearly for every $i < j < \alpha$

$$(d_i^1 - h_{i,j}(d_j^1)) - (d_i^2 - h_{i,j}(d_j^2)) = d_i^0 - h_{i,j}(d_j^0) \ ;$$

hence,

(a) $d_i^1 - d_i^2 - d_i^0 = h_{i,j}(d_j^1 - d_j^2 - d_j^0)$.

As $h_{\beta-1,\alpha-1}$ is from $G_{\beta-1}^{\mathfrak{A}}$ onto $G_{\alpha-1}^{\mathfrak{A}}$ (remember α, β are successor ordinals) for some $x \in G_{\beta-1}^{\mathfrak{A}}$:

(b) $h_{\alpha-1,\beta-1}(x) = d_{\alpha-1}^1 - d_{\alpha-1}^2 - d_{\alpha-1}^0$.

By (a) for every $i < \alpha$:

(c) $h_{i,\beta-1}(x) = d_i^1 - d_i^2 - d_i^0$.

Now define for i, $i < \beta$.

(d) $d_i = d_i^1 - d_i^2 - h_{i,\beta-1}(x)$.

By (c) for $i < \beta$:

(e) $d_i = d_i^0$.

Let $d = \langle d_i : i < \beta \rangle$, so $d \restriction \alpha = d_0$. We shall show that $a \restriction \beta - b \restriction \beta = fact(d)$ thus finishing the proof of 3.13. For $i < j < \beta$

$$\begin{aligned} a_{i,j} - b_{i,j} &= (d_i^1 - h_{i,j}(d_j^1)) - (d_i^2 - h_{i,j}(d_j^2)) \\ &= (d_i^1 - d_i^2) - h_{i,j}(d_j^1 - d_j^2) \\ &= (d_i + h_{i,\beta-1}(x)) - h_{i,j}(d_j + h_{j,\beta-1}(x)) \\ &= (d_i - h_{i,j}(d_j)) + (h_{i,\beta-1}(x) - h_{i,j} \circ h_{j,\beta-1}(x)) = d_i - h_{i,j}(d_j) \end{aligned}$$

So $\boldsymbol{d}$ is as required and we finish.

3.14 Fact If in the game for $(M_{\boldsymbol{a}},M)$ player II wins then $(\exists \boldsymbol{b})[M \cong M_{\boldsymbol{b}} \wedge \boldsymbol{b}-\boldsymbol{a} \in Gs(\mathfrak{A})]$.

Proof: We can use a weaker hypothesis:

(*) For every α, $M \restriction \bigcup_{i\leq\alpha} P_i^M$ is isomorphic to $M_{\boldsymbol{a}} \restriction \bigcup_{i\leq\alpha} G_i^{\mathfrak{A}}$ and let the isomorphism be g_α^{-1} and prove $M \cong M_{\boldsymbol{b}}$ for some $\boldsymbol{b} \in Gr(\mathfrak{A})$;

by 3.12 (applied to the various $\mathfrak{A} \restriction \delta$), $\boldsymbol{b}$ will be as required.

For any $i<j<\kappa$, $(g_i^{-1}g_j) \restriction P_i^{M_a}$ is necessarily an automorphism of $M_{\boldsymbol{a}} \restriction P_i^{M_{Ra}}$. So using the functions $F_c(c \in G_i^{\mathfrak{A}})$ clearly for some $d_{i,j} \in G_i^{\mathfrak{A}}$ $g_i^{-1}g_j(x) = x + d_{i,j}$ for every $x \in G_i^{\mathfrak{A}}$).

Using the functions $H_{i,j}^{M_\alpha}$ we can check that $d_{\alpha,\gamma} = d_{\alpha,\beta} + h_{\alpha,\beta}(d_{\beta,\gamma})$ for $\alpha<\beta<\gamma<\kappa$ hence $\boldsymbol{d} = \langle d_{\alpha,\beta} : \alpha<\beta<\kappa\rangle \in Gr(\mathfrak{A})$. It is also easy to check that $M_{\boldsymbol{d}+\boldsymbol{a}} \cong M$ (the isomorphism takes $x \in G_i$ to $g_i(x)$), so we finish.

* * *

What about finite μ? The proof is O.K. for powers of 2. Similarly we can use Abelian group of order p to get power of p, and then sum of models gives us any product.

Alternatively use 1.8.

3.15 Claim For every ϑ, there is a κ-system $\mathfrak{A}$, $\|\mathfrak{A}\| = \kappa+\vartheta$, $\vartheta \leq |E^\circ(GA)|$.

Proof: Just like the proof of 3.10.

3.16 Fact For every κ-system $\mathfrak{A}$, $|E^\circ(\mathfrak{A})| \leq |E(\mathfrak{A})| \leq \|\mathfrak{A}\|^\kappa$.

Proof: As $|Gr(\mathfrak{A})| \leq \|\mathfrak{A}\|^\kappa$.

3.17 Conclusion If $\aleph_0 < \kappa = cf\,\lambda < \lambda$ and $\vartheta \leq \lambda$ then $\vartheta^\kappa \in RSP_{\lambda,\lambda}^\lambda$.

Proof: First assume $\vartheta < \lambda$. Let $\mathfrak{A}$ be a system as provided by 3.15. So by 3.16 $\vartheta \leq |E^\circ(\mathfrak{A})| \leq \vartheta^\kappa$. By 3.11 there is a model M of power $\|\mathfrak{A}\| = \kappa+\vartheta \leq \lambda$, satisfying the conditions of 2.1(b), 2.10 for $\vartheta = |E^\circ(\mathfrak{A})|$. So by 2.1, 2.10 $|E^\circ(\mathfrak{A})| \in RSP_{\lambda,\lambda}^\lambda$; hence, by 1.7, $|E^\circ(\mathfrak{A})|^\kappa \in RSP_{\lambda,\lambda}^\lambda$. However, $\vartheta^\kappa \leq |E^\circ(\mathfrak{A})|^\kappa \leq \vartheta^\kappa)^\kappa = \vartheta^\kappa$. So

$$\vartheta^\kappa \in RSP_{\lambda,\lambda}^\lambda\ .$$

We are left with the case $\vartheta = \lambda$. Let $\lambda = \sum_{i<\kappa} \lambda_i$, $\lambda_i < \lambda$. By what we have already proved $\lambda_i^\kappa \in RSP_{\lambda,\kappa}^\lambda$ for each $i<\kappa$. By 1.7 $\prod_{i<\kappa} \lambda_i^\kappa \in RSP_{\lambda,\kappa}^\lambda$ but by easy cardinal arithmetic $\vartheta^\kappa = \lambda^\kappa = \prod_{i<\kappa} \lambda_i^\kappa$.

3.18 Conclusion If $\aleph_0 < \kappa = cf\,\lambda < \lambda$, $\vartheta^\kappa \leq \lambda$, then $\vartheta \in RSP_{\lambda,\lambda}^\lambda$.

Proof: Like the proof of 3.17, using 3.10 instead of 3.15.

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