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Yuri Gurevich, Menachem Magidor and Saharon Shelah

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THE MONADIC THEORY OF  $\omega_2$ <sup>1</sup>

YURI GUREVICH, MENACHEM MAGIDOR AND SAHARON SHELAH

**Abstract.** Assume ZFC + “There is a weakly compact cardinal” is consistent. Then:

- (i) For every  $S \subseteq \omega$ , ZFC + “ $S$  and the monadic theory of  $\omega_2$  are recursive each in the other” is consistent; and
- (ii) ZFC + “The full second-order theory of  $\omega_2$  is interpretable in the monadic theory of  $\omega_2$ ” is consistent.

**Introduction.** First we recall the definition of monadic theories. The monadic language corresponding to a first-order language  $L$  is obtained from  $L$  by adding variables for sets of elements and adding atomic formulas  $x \in Y$ . The monadic theory of a model  $M$  for  $L$  is the theory of  $M$  in the described monadic language when the set variables are interpreted as arbitrary subsets of  $M$ . Speaking about the monadic theory of an ordinal  $\alpha$  we mean the monadic theory of  $\langle \alpha, < \rangle$ .

Formal theories of order were studied very extensively. We do not review that study here. Our attention is restricted to the monadic theory of ordinals. The pioneer here was Büchi. He proved decidability of the monadic theory of  $\omega$ , the monadic theory of  $\omega_1$ , and the monadic theory of ordinals  $< \omega_2$ . See the strongest result in [Bu]. Note that the last of these theories is not the monadic theory of  $\omega_2$ , but the set of monadic statements true in every ordinal  $< \omega_2$ . As we will see below Büchi had a good reason to stop at  $\omega_2$ .

Shelah studied the monadic theory of  $\omega_2$  in [Sh1]. We shall use some of his results. Let  $U_i = \{\alpha < \omega_2: \text{cf } \alpha = \omega_i\}$  for  $i \leq 1$ , and  $I$  be the ideal of nonstationary sets. For  $X \subseteq U_0$  let  $D(X) = \{\alpha \in U_1: \alpha \cap X \text{ is stationary in } \alpha\}$ . We call  $D(X)$  the *derivative* of  $X$ . It is easy to see that  $D(X) = D(Y)$  modulo  $I$  if  $X = Y$  modulo  $I$ , thus  $D$  can be considered as a relation on the Boolean algebra  $\text{PS}(\omega_2)/I$  of subsets of  $\omega_2$  modulo  $I$ . ( $\text{PS}(X)$  denotes here the power set of  $X$  and the corresponding Boolean algebra.) Shelah proved:

(i) the monadic theory of  $\omega_2$  and the first-order theory of  $\langle \text{PS}(\omega_2)/I, D \rangle$  are recursive each in the other;

(ii) the monadic theory of  $\omega_2$  is decidable if for every stationary  $X \subseteq U_0$  and every  $Y_1, Y_2$  with  $D(X) = Y_1 \cup Y_2$  there are disjoint stationary  $X_1, X_2$  such that  $X_1 \cup X_2 = X$  and  $D(X_i) = Y_i$  modulo  $I$  for  $i = 1, 2$ .

He noted also that (Baumgartner and Jensen’s results imply that) “ $\omega_2 \models (DX \neq 0 \text{ for every } X \subseteq U_0)$ ” is independent in ZFC.

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<sup>1</sup>The results were obtained and the paper was written during the Logic Year in the Institute for Advanced Studies of Hebrew University.

Assuming ZFC + “There is a weakly compact cardinal” Magidor proved in [Ma] that ZFC + “ $D(X) = U_1$  modulo  $I$  for every stationary  $X \subseteq U_0$ ” is consistent. By (i) the monadic theory of  $\omega_2$  is decidable in Magidor’s universe. In [Sh2] Shelah proved that a certain combinatorial principle (the uniformization property for  $\omega_2$ ) implies the premise of (ii), and that the uniformization is consistent with ZFC + CH. Later he proved that the uniformization property is consistent even with ZFC + GCH, see [St & Ki]. By (ii) the monadic theory of  $\omega_2$  is decidable in the corresponding universes of Shelah. It is different however from the monadic theory of  $\omega_2$  in Magidor’s universe.

The first undecidability result on monadic theories of ordinals was presented by Magidor in Logic Colloquium 77 (Wroclaw, Poland, 1977). For  $n \geq 2$  let  $E_n$  say that for every stationary  $X \subseteq \omega_n$  consisting of ordinals of cofinality  $\omega$  there is  $\alpha < \omega_n$  of cofinality  $> \omega$  such that  $\alpha \cap X$  is stationary in  $\alpha$ . Assuming consistency of  $\omega$  supercompact cardinals Magidor proved that for every  $S \subseteq \omega - \{0, 1\}$  there is a world with  $\{n: E_n \text{ is true}\} = S$ . Shelah proved in this direction the following. Assume consistency of  $\omega$  Mahlo cardinals; then for every  $S \subseteq \{2n: 1 \leq n < \omega\}$  there is a world with  $\{2n: E_{2n} \text{ is true}\} = S$ . And for every  $S \subseteq \{2n: 1 \leq n < \omega\}$ , it is consistent with ZFC + GCH that  $\{2n: 1 \leq n < \omega \text{ and there is a stationary } X \subseteq \omega_{2n} \text{ such that for every } Y \subseteq X \text{ there is } Z \subseteq \omega_{2n} \text{ with } \{\alpha < \omega_{2n}: \text{cf}(\alpha) > \omega \text{ and } \alpha \cap Z \text{ is stationary in } \alpha\} = Y \text{ modulo nonstationary sets}\}$  is equal to  $S$ . None of these three results (one of Magidor and two of Shelah) is published.

In Part I of this article we prove in detail:

**THEOREM 1.** *Assume there is a weakly compact cardinal. Then there is an algorithm  $n \rightarrow \phi_n$  such that  $\phi_n$  is a sentence in the monadic language of order and for every  $S \subseteq \omega$  there is a generic extension of the ground world with  $\{n: \omega_2 \models \phi_n\} = S$ .*

Thus there are continuum many possible monadic theories of  $\omega_2$  (in different universes) and for every  $S \subseteq \omega$  there is a monadic theory of  $\omega_2$  (in some world) which is at least as complex as  $S$ .

The full second-order theory of a set  $X$  is the theory of  $X$  in the language with variables for elements, variables for arbitrary monadic predicates, variables for arbitrary dyadic predicates, etc. It depends on the cardinality of  $X$  only. It belongs more to set theory than to model theory and can be used as a standard of complexity. The monadic theory of  $\omega_2$  is easily interpretable in the full second-order theory of  $\omega_2$ . Thus the full second-order theory of  $\omega_2$  gives an upper bound of complexity of the monadic theory of  $\omega_2$ .

**THEOREM 2.** *Assume there is a weakly compact cardinal. Then there is a generic extension of the ground world where the full second-order theory of  $\omega_2$  is interpretable (therefore recursive) in the monadic theory of  $\omega_2$ .*

Theorem 2 is proved in Part II. It has actually the same proof as Theorem 1 with only a few alterations. Combining the technique of Part I and that of [Ma] we prove in Part III

**THEOREM 3.** *Assume GCH and existence of a weakly compact cardinal. For every  $S \subseteq \omega_2$  there is a generic extension of the ground universe where  $S$  and the monadic theory of  $\omega_2$  are recursive each in the other.*

If  $\kappa$  is weakly compact in a world  $V$  then it is weakly compact in the constructible part of  $V$  (see [Je]) where GCH holds. Hence Theorem 3 gives

**COROLLARY 4.** *Assume ZFC + “There is a weakly compact cardinal” is consistent. Then for every  $S \subseteq \omega$ , ZFC + “ $S$  and the monadic theory of  $\omega_2$  are recursive each in the others” is consistent.*

We use the book [Je] and the article [Sho] as sources of notation, terminology and information.

### PART I. CONTINUUM POSSIBLE MONADIC THEORIES OF $\omega_2$

**§1. Coding.** Here a *graph* is a model  $\langle X, R \rangle$  where  $R$  is a reflexive symmetric binary relation on  $X$  such that for every different  $x, y$  in  $X$  there is  $z \in X$  with  $Rxz$  not equivalent to  $Ryz$ .

*Claim 1.* There is an algorithm  $n \rightarrow \varphi_n$  such that  $\varphi_n$  is a first-order graph sentence and for every  $S \subseteq \omega$  there is a graph  $\langle \omega, R \rangle$  with  $\{n: \langle \omega, R \rangle \text{ satisfies } \varphi_n\} = S$ .

**PROOF.** For every  $n \in \omega - S$  add to  $\omega$  an auxiliary element  $n'$ , for every  $n \in S$  add two auxiliary elements  $n'$  and  $n''$ . Let  $R$  be the least reflexive symmetric relation on the resulting set containing pairs  $(n, n')$ ,  $(n, n + 1)$  for  $n < \omega$  and pairs  $(n, n'')$  for  $n \in S$ . It is easy to see that  $x$  is auxiliary iff it is  $R$ -connected with at most two elements including itself. Hence  $0, 1, \dots$  are definable.  $\varphi_n$  says that  $n$  is  $R$ -connected with two auxiliary elements. It remains only to replace all elements by natural numbers.  $\square$

Given a graph  $\langle \omega, R \rangle$  and assuming existence of a weakly compact cardinal we define in §2 a forcing notion  $P$  and prove in §6:

**THEOREM 2.** *Suppose  $G$  is a  $P$ -generic filter over the ground world  $V$ . Then in  $V[G]$  there is a partition of  $\{\alpha < \omega_2: \text{cf } \alpha = \omega\}$  into stationary sets  $S_n, n < \omega$ , such that*

(i) *for every  $\alpha < \omega_2$  of cofinality  $\omega_1$  there is a pair  $(m, n) \in R$  such that  $S_m \cup S_n$  includes a club subset of  $\alpha$ , and*

(ii) *if  $(m, n) \in R, A_m \subseteq S_m, A_n \subseteq S_n$  and  $A_m, A_n$  are stationary in  $\omega_2$  then there are stationarily many ordinals  $\alpha < \omega_2$  of cofinality  $\omega_1$  with both  $A_m \cap \alpha$  and  $A_n \cap \alpha$  stationary in  $\alpha$ .*

Note that clause (i) of Theorem 2 implies  $D(S_m) \cap D(S_n) = \emptyset$  for  $(m, n) \in (\omega \times \omega) - R$ .

*Claim 3.* The first-order theory of graph  $\langle \omega, R \rangle$  is interpretable in the monadic theory of  $\omega_2$  if there is a partition described in Theorem 2.

**PROOF.**  $U_0, U_1$ , the ideal  $I$  and the derivative  $D$  are defined in the Introduction. Here are some more definitions in the monadic theory of  $\omega_2$ . Two subsets  $X, Y$  of  $U_0$  are *connected* if  $D(X) \cap D(Y)$  is stationary. (Note that  $X \subseteq U_0$  is connected with itself iff  $X$  is stationary.) A stationary  $X \subseteq U_0$  is an *atom* if there are no  $X_1, X_2 \subseteq X$  and  $Y \subseteq U_0$  such that  $X_1, X_2$  are stationary and  $Y$  is connected with one of sets  $X_1, X_2$  but not with the other. An atom  $X$  is *maximal* if  $X = Y$  modulo  $I$  for every atom  $Y$  including  $X$ .

For every  $m$  every stationary  $X \subseteq S_m$  is an atom. For, suppose  $X_1, X_2$  are stationary subsets of  $X$  and  $Y \subseteq U_0$ . Let  $Y_1 = \bigcup \{Y \cap S_n: (m, n) \in R\}$ ,  $Y_2 = Y - Y_1$ . If  $Y_1$  is stationary then some  $Y \cap S_n$  with  $(m, n) \in R$  is stationary and  $Y$  is connected with both  $X_1$  and  $X_2$ . Otherwise the derivative of  $Y$  coincides modulo  $I$  with the derivative  $Y_2$  which avoids even the derivative of  $X$ , thus  $Y$  is connected with neither  $X_1$  nor  $X_2$ .

If  $X$  is an atom then  $X \subseteq S_m$  modulo  $I$  for some  $m$ . For, let  $K = \{m: X \cap S_m \text{ is stationary}\}$ .  $K \neq \emptyset$  because  $X$  is stationary. If different  $m, n$  belong to  $K$  there is  $I$  such that  $Rlm$  is not equivalent to  $Rln$ . Set  $X_1 = X \cap S_m, X_2 = X \cap S_n, Y = S_I$  to contradict the assumption that  $X$  is an atom.

It is easy to see that an atom  $X$  is maximal iff  $X = S_m$  modulo  $I$  for some  $m$ . Now interpret variables of the first-order graph language as maximal atoms (equality is equality modulo  $I$ ) and  $R$  as connectedness.  $\square$

Claim 1, Theorem 2 and Claim 3 imply

**THEOREM 4.** *Assume there is a weakly compact cardinal. There is a recursive list  $\phi_0, \phi_1, \dots$  of monadic sentences such that for every  $S \subseteq \omega$  there is a generic extension of the ground world with  $\{n: \omega_2 \models \phi_n\} = S$ .*

**§2. Forcing notion.** Suppose  $\kappa$  is a weakly compact cardinal and  $R$  is a reflexive symmetric binary relation on  $\omega$ . We define a forcing notion  $P$  for collapsing  $\kappa$  onto  $\omega_2$  and creating stationary subsets  $S_m$  of  $\omega_2$  described in §1. A condition  $p$  is a triple  $(p_0, p_1, p_2)$  of countable functions.  $p_0$  gives a partial information about sets  $S_m$ , it is composed of pairs  $(\alpha, m)$  where  $\text{cf } \alpha = \omega, m < \omega$ ; the intended meaning is  $\alpha \in S_m$ .  $p_1$  assigns a pair  $(m, n) \in R$  to ordinals  $\alpha < \kappa$  of cofinality  $> \omega$ ; the intended meaning is:  $S_l \cap \alpha$  is stationary in  $\alpha$  iff  $l \in \{m, n\}$ . To assure our intentions  $p_2$  assigns a closed countable subset of  $\alpha$  to each  $\alpha \in \text{dom } p_1$  in such a way that  $p_2(\alpha) \subseteq \{\beta: p_0(\beta) \text{ is equal to } m \text{ or to } n\}$  where  $(m, n) = p_1(\alpha)$ ; the intended meaning is:  $p_2(\alpha)$  is an initial segment of a club subset of  $\alpha$  included in  $S_m \cup S_n$ . A condition  $p$  refines a condition  $q$  ( $p \leq q$ ) if  $q_0 \subseteq p_0, q_1 \subseteq p_1$  and for every  $\alpha \in \text{dom } p_1, p_2(\alpha)$  is an end extension of  $q_2(\alpha)$  and

$$\min(p_2(\alpha) - q_2(\alpha)) > \sup(\alpha \cap (\text{dom } p_0 \cup \text{dom } p_1)).$$

The last requirement is used to prove the following Claim 1. (It is convenient for us to treat elements of  $R$  as unordered pairs.)

*Claim 1.*  $P$  is  $\omega_1$ -closed.

**PROOF.** Given  $p_0 > p_1 > \dots$  set  $x = \bigcup \{p_n 0: n < \omega\}, y = \bigcup \{p_n 1: n < \omega\}$  and  $z(\alpha)$  be the closure of  $\bigcup \{p_n 2(\alpha): \alpha \in \text{dom } p_n 1\}$  for  $\alpha \in \text{dom } y$ . Let  $d = \{\alpha \in \text{dom } y: \sup z(\alpha) \text{ does not belong to any } p_n 2(\alpha)\}$ . If  $d \neq \emptyset$  then  $(x, y, z) \notin P$  because  $\sup z(\alpha) \notin \text{dom } x$  for  $\alpha \in d$ . Select a function  $f = \{(\sup z(\alpha), m_\alpha): \alpha \in d, m_\alpha \in y(\alpha)\}$ ; this is possible because  $\alpha, \beta \in d, \alpha < \beta$  imply  $\sup z(\alpha) < \alpha < \sup z(\beta)$ .  $(x \cup f, y, z)$  belongs to  $P$  and refines any  $p_n$ .  $\square$

If  $p \in P$  then  $\text{dom } p_0 \cup \text{dom } p_1$  will be called the domain of  $p$  and denoted  $\text{dom}(p)$ .  $\sup \text{dom}(p)$  will be called the height of  $p$  and denoted  $h(p)$ .

*Claim 2.*  $P$  satisfies the  $\kappa$ -chain condition.

**PROOF.** By contradiction suppose that  $\{p_\alpha: \alpha < \kappa\}$  is a set of pairwise incompatible conditions. Define  $f(\alpha) = \sup(\alpha \cap \text{dom } p_\alpha)$ .  $f$  is regressive on  $\{\alpha: \text{cf } \alpha > \omega\}$ . By Fodor's Lemma there is a stationary  $A \subseteq \kappa$  with  $f(\alpha) = \delta$  for some  $\delta$  and any  $\alpha \in A$ . Let  $\lambda = |\delta|^\omega$ . There are at most  $\lambda$  possibilities for  $p_0 \upharpoonright \delta, p_1 \upharpoonright \delta, p_2 \upharpoonright \delta$ . Hence there is  $\alpha$  such that  $B = \{\beta \in A: p_\alpha \upharpoonright i \text{ and } p_\beta \upharpoonright i \text{ coincide on } \delta \text{ for } i \leq 2\}$  is of cardinality  $\kappa$ . There is  $\beta \in B$  exceeding  $h(p_\alpha)$ . Then  $p_\alpha, p_\beta$  are compatible, thus we have a contradiction.  $\square$

**COROLLARY 3.** For every  $P$ -name  $a$  there is a function  $f: \kappa \rightarrow \kappa$  such that for every  $\alpha < \kappa$  and every  $p$  forcing  $\alpha \in a$  there is  $q$  such that  $h(q) \leq f\alpha$ ,  $q$  is compatible with  $p$  and  $q$  forces  $\alpha \in a$ .

**PROOF.** Consider a maximal antichain  $C \subseteq P$  such that every element of  $C$  either forces  $\alpha \in a$  or forces  $\alpha \notin a$ . Set  $f\alpha = \sup\{h(p): p \in C\}$ .  $\square$

In the rest of this section  $G$  is a  $P$ -generic ultrafilter over the ground world  $V$ ,  $G_i = \bigcup\{p_i: p \in G\}$  for  $i \leq 1$ . It is easy to see that  $\text{dom } G_0 = \{\alpha < \kappa: \text{cf } \alpha = \omega\}$  and  $\text{dom } G_1 = \{\alpha < \kappa: \text{cf } \alpha > \omega\}$ . As  $P$  is  $\omega_1$ -closed,  $\omega_1^{V[G]} = \omega_1^V$ . Thus  $\text{cf } \alpha > \omega$  in  $V$  iff  $\text{cf } \alpha > \omega$  in  $V[G]$ . For  $\alpha < \kappa$  with  $\text{cf } \alpha > \omega$  let  $G_2(\alpha) = \bigcup\{p_2(\alpha): p \in G\}$ .

**Claim 4.** Suppose  $\alpha < \kappa$  and  $\text{cf } \alpha > \omega$ . Then  $G_2(\alpha)$  is a club subset of  $\alpha$  of cofinality  $\omega_1$ .

**PROOF.** If  $\beta \in G_2(\alpha)$  then  $\beta \in p_2(\alpha)$  for some  $p \in G$ , hence  $\text{cf } \beta = \omega$ . Let  $c = \{(\beta, p): \alpha \in \text{dom } p \text{ and } \beta \in p_2(\alpha)\}$ , so that  $G_2(\alpha)$  is the denotation of  $c$  in  $V[G]$ . For  $\beta < \alpha$  the set  $\{p: p \text{ forces } \gamma \in c \text{ for some } \gamma > \beta\}$  is dense, hence  $G_2(\alpha)$  is unbounded. Suppose  $\beta < \kappa$  is a limit point for  $G_2(\alpha)$ . There are  $\gamma > \beta$  and  $p \in G$  forcing  $\gamma \in c$ . Then  $G_2(\alpha) \cap \gamma = p_2(\alpha) \cap \gamma$ . As  $p_2(\alpha)$  is closed,  $\beta \in p_2(\alpha) \subseteq G_2(\alpha)$ . Thus  $G_2(\alpha)$  is a club subset of  $\alpha$  consisting of ordinals of cofinality  $\omega$ , hence  $\text{cf } G_2(\alpha) = \omega_1$ .  $\square$

**COROLLARY 5.**  $\kappa$  is  $\omega_2$  in  $V[G]$ .

**PROOF.** In  $V[G]: \kappa > \omega_1$  because  $\omega_1^{V[G]} = \omega_1^V$ ,  $\kappa \leq \omega_2$  by Claim 4,  $\kappa$  is a cardinal because  $P$  satisfies the  $\kappa$ -chain condition.  $\square$

**Claim 6.** Every new club subset of  $\kappa$  in  $V[G]$  includes an old club subset of  $\kappa$ .

Proof is well known and uses only the  $\kappa$ -chain condition. Suppose the empty condition  $(\emptyset, \emptyset, \emptyset)$  forces " $c$  is a club subset of  $\kappa$ ". Let  $C' = \{\alpha: \text{the empty condition forces } \alpha \in c\}$ . It is obvious that  $C'$  is closed. We prove that  $C'$  is unbounded. For  $\alpha < \kappa$  let  $A(\alpha)$  be the set of ordinals  $\beta < \alpha$  such that some  $p$  forces " $\beta$  is the least element of  $c$  above  $\alpha$ ".  $|A(\alpha)| < \kappa$  because  $P$  satisfies the  $\kappa$ -chain condition. Let  $f\alpha = \sup A(\alpha)$ . Now given  $\alpha_0$  let  $\alpha_{n+1} = f\alpha_n$ ,  $\alpha = \sup\{\alpha_n: n < \omega\}$ . The empty condition forces that  $c$  meets every interval  $(\alpha_n, \alpha_{n+1}]$ , hence it forces  $\alpha \in c$ , i.e.,  $\alpha \in C'$ .  $\square$

**§3. Decomposition of forcing.** First we recall the notion of quotient forcing. Suppose  $B$  is a partial ordering and  $A$  is a submodel of  $B$  satisfying the following conditions: if  $p \in A$  and  $p \leq q$  then  $q \in A$ ; if two elements of  $A$  are compatible in  $B$  then they are compatible in  $A$ ; and for each  $q \in B$  there is a unique  $p \in A$  (the projection of  $q$ ) such that  $q \leq p$  and  $p, q$  are compatible with exactly the same elements of  $A$ . Let  $c = \{(q, p): q \in B \text{ and } p \text{ is the projection of } q\}$ . For every  $A$ -generic filter  $G$  over the ground world  $V$ ,  $c^{V[G]}$  (i.e., the denotation of  $c$  in  $V[G]$ ) is equal to  $\{q \in B: q \text{ is compatible with any } p \in G\}$ . This denotation is the quotient forcing in the following sense:  $B$  is isomorphic to a dense subset of  $A * c$  and forcing with  $B$  is a composition of forcing with  $A$  and subsequent forcing with the denotation of  $c$ .

For  $0 < \alpha \leq \kappa$  let  $P(<\alpha)$  be the submodel of  $P$  comprising conditions of height  $< \alpha$ . For  $p \in P$  let  $p(<\alpha) = (p_0|\alpha, p_1|\alpha, p_2|\alpha)$ , it is the projection of  $p$  into  $P(<\alpha)$ . For every  $\alpha < \beta \leq \kappa$  with  $\text{cf } \alpha > \omega$  we get the quotient forcing completing forcing with  $P(<\alpha)$  to a forcing with  $P(<\beta)$ . Two cases are of special interest for us. Let  $\lambda < \kappa$  be of cofinality  $> \omega$  and  $P(\leq \lambda) = P(<\lambda + 1)$ .

*Case 1.*  $\alpha = \lambda$ ,  $\beta = \lambda + 1$ . Suppose  $G$  is a  $P(<\lambda)$ -generic over  $V$  filter and  $G0 = \bigcup\{p0: p \in G\}$ . The quotient forcing notion is the submodel of  $P(\leq\lambda)$  comprising conditions  $p \in P(\leq\lambda)$  such that  $p(<\lambda) \in G$  and  $p2(\lambda) \subset G0^{-1}\{m, n\}$  provided  $p1(\lambda) = \{m, n\}$ . Let  $P(\lambda, m, n, G)$  be the following forcing notion in  $V[G]$  (for shooting a club subset of  $\lambda$  in  $G0^{-1}\{m, n\}$ ): conditions are closed countable subsets of  $\lambda$  included in  $G0^{-1}\{m, n\}$ , a stronger condition means an end extension.

*Claim 1.* Suppose  $J$  is a  $P(\leq\lambda)$ -generic filter over  $V$ ,  $H = J \cap P(<\lambda)$ ,  $\{m, n\} = p1(\lambda)$  for some  $p \in J$ . Then  $H$  is a  $P(<\lambda)$ -generic filter over  $V$  and there is a  $P(\lambda, m, n, H)$ -generic filter  $I$  over  $V[H]$  such that  $V[H][I] = V[J]$ .

*Case 2.*  $\alpha = \lambda + 1$ ,  $\beta = \kappa$ . Suppose  $G$  is a  $P(\leq\lambda)$ -generic filter over  $V$ , and  $G0 = \bigcup\{p0: p \in G\}$ . The quotient forcing comprises conditions  $p \in P$  such that  $p(\leq\lambda) \in G$  and for each  $\gamma > \lambda$ , if  $\gamma \in \text{dom } p1$  and  $p1(\gamma) = \{m, n\}$  then  $\lambda \cap p2(\gamma) \subseteq G0^{-1}\{m, n\}$ . Let  $P(>\lambda)$  be the submodel of  $P$  comprising conditions  $p \in P$  such that  $p(\leq\lambda) = \emptyset$  and  $\lambda \cap p2(\lambda) = \emptyset$  for  $\gamma \in \text{dom } p1$ .

*Claim 2.* Suppose  $J$  is a  $P$ -generic filter over  $V$  and  $H = J \cap P(\leq\lambda)$ . Then  $H$  is a  $P(\leq\lambda)$ -generic filter over  $V$  and there is a  $P(>\lambda)$ -generic filter  $I$  over  $V[H]$  such that  $V[H][I] = V[J]$ .

Proof is easy.

Let us remark that one can go from  $H$  and  $I$  to  $J$  in Claims 1 and 2.

**§4. Preserving stationarity. I.** Suppose  $\lambda$  is a regular cardinal and  $A$  is a stationary subset of  $\lambda$  comprising ordinals of cofinality  $\omega$ . Let  $Q$  be the following forcing notion (for shooting a club subset of  $\lambda$  in  $A$ ): conditions are closed countable subsets of  $\lambda$  included in  $A$ , a stronger condition means an end extension. Suppose  $B \subseteq A$  is stationary in  $\lambda$ .

**THEOREM 1.** *Forcing with  $Q$  does not destroy stationarity of  $B$ .*

We prove Theorem 1 in the rest of this section. Suppose  $\Vdash_Q (c$  is a club subset of  $\lambda)$ . It suffices to prove that for every  $p_0 \in Q$  there are  $p \leq p_0$  and  $\mu \in B$  with  $p \Vdash \mu \in c$ . Let  $M$  be a model (i.e., a relational system) with universe  $H(\lambda)$  (that is the collection of sets hereditary of cardinality  $< \lambda$ ), unary relation  $x \in Q$ , and binary relations  $x \in y$ ,  $x \leq y$  (as conditions in  $Q$ ),  $x \Vdash y \in c$ . Let  $M'$  be a model obtained from  $M$  by adding Skolem functions. For  $\mu < \lambda$  let  $M(\mu)$  be the least submodel of  $M'$  including  $\mu$  (i.e., containing any  $\alpha < \mu$ ).

**LEMMA 2.**  $\{\mu: \lambda \cap M(\mu) = \mu\}$  is club in  $\lambda$ .

**PROOF OF LEMMA 2.** It is obvious that the set is closed. To prove unboundedness suppose  $\mu_0 < \lambda$  and consider the sequence  $\mu_0 \leq \mu_1 \leq \mu_2 \leq \dots$  where  $\mu_{n+1} = \text{sup}(\lambda \cap M(\mu_n))$ . If  $\mu = \text{sup}\{\mu_n: n < \omega\}$  then  $\lambda \cap M(\mu) = \mu$ .  $\square$

Given  $p_0$  take  $\mu \in B$  such that  $p_0 \in M(\mu)$  and  $\lambda \cap M(\mu) = \mu$ . Let  $\alpha_0 < \alpha_1 < \dots$  converge to  $\mu$ . Build  $p_0 > p_1 > \dots$  such that  $p_n \in M(\mu)$  and  $p_{n+1} \Vdash \beta_n \in c$  for some  $\alpha_n < \beta_n < \mu$ ; it is possible because  $M(\mu)$  is an elementary submodel of  $M$ . Then  $p = \bigcup\{p_n: n < \omega\} \cup \{\mu\} \in Q$  and  $p \Vdash \mu \in c$ .

**§5. Preserving stationarity. II.**

*Claim 1.* Let  $\varepsilon$  be an ordinal of cofinality  $\omega_1$  and  $A$  be a stationary subset of  $\varepsilon$ . Let  $Q$  be an  $\omega_1$ -closed forcing notion. Then forcing with  $Q$  does not destroy stationarity of  $A$  i.e. the empty condition  $Q$ -forces “ $A$  is a stationary subset of  $\varepsilon$ ”.

This claim is a corollary of a stronger result of Baumgartner:  $\omega_1$ -closed forcing does not destroy stationarity of sets comprising ordinals of cofinality  $\omega$ , see [Ba]. For the reader's convenience we prove our claim.

PROOF. W.l.o.g.  $\varepsilon = \omega_1$ : there is club  $X \subseteq \varepsilon$  of type  $\omega_1$  and it suffices to prove that  $A \cap X$  remains stationary in  $X$ .

By contradiction suppose  $p \Vdash (c \text{ is a club subset of } \omega_1 \text{ avoiding } A)$ . Build (in the ground world) a sequence  $\langle p_\alpha : \alpha < \omega_1 \rangle$  of conditions and a sequence  $\langle \delta_\alpha : \alpha < \omega_1 \rangle$  of countable ordinals such that  $p_0 \leq p$ ,  $p_0 \Vdash \delta_0 \in c$ ,  $\delta_{\alpha+1} > \delta_\alpha$ ,  $p_{\alpha+1} \leq p_\alpha$ ,  $p_{\alpha+1} \Vdash \delta_{\alpha+1} \in c$ , and if  $\alpha$  is limit then  $\delta_\alpha = \sup\{\delta_\beta : \beta < \alpha\}$  and  $p_\alpha$  is stronger than  $p_\beta$  for  $\beta < \alpha$ .  $\{\delta_\alpha : \alpha < \omega_1\}$  is a club subset of  $\omega_1$  avoiding  $A$ , which is impossible.  $\square$

**§6. Standard stationary sets.** Suppose  $G$  is a  $P$ -generic filter over the ground world  $V$ . Let  $G_i = \bigcup\{p_i : p \in G\}$  for  $i \leq 2$ ,  $G_2(\alpha) = \bigcup\{p_2(\alpha) : p \in G\}$  for  $\alpha < \kappa$  with  $\text{cf } \alpha > \omega$ , and  $S_m = \{\alpha : G_0(\alpha) = m\}$ . The set  $s_m = \{(\alpha, p) : p_0(\alpha) = m\}$  is a name of  $S_m$ .

*Claim 1.* Every  $S_m$  is a stationary subset of  $\kappa$ .

PROOF. Given  $p_0 \Vdash (c \text{ is a club subset of } \kappa)$  build sequences  $p_0 > p_1 > \dots$  and  $\alpha_0 < \alpha_1 < \dots$  such that  $\alpha_i > h(p_i)$  and  $p_{i+1} \Vdash \alpha_i \in c$ . Let  $\alpha = \sup\{\alpha_i : i < \omega\}$ . By the proof of Claim 1 in §2 there is a condition  $p$  of height  $\alpha$  refining any  $p_i$ . Let  $q = (p_0 \cup \{(\alpha, m)\}, p_1, p_2)$ . Then  $p_0 > q \Vdash (\alpha \in c \cap s_m)$ .  $\square$

*Claim 2.* Sets  $S_m$  partition  $\kappa$ .

PROOF. For every  $p$  and ordinal  $\alpha$  of cofinality  $\omega$  there is  $q \leq p$  with  $\alpha \in \text{dom } q^0$ .  $\square$

By the same token every  $\alpha < \kappa$  with  $\text{cf } \alpha > \omega$  belongs to  $\text{dom}(G_1)$ .

*Claim 3.* Suppose  $\alpha < \omega_2$  and  $\text{cf } \alpha = \omega_1$  in  $V[G]$ . Then  $G_2(\alpha)$  is a club subset of  $\alpha$ . If  $G_1(\alpha) = (m, n)$  then  $G_2(\alpha) \subseteq S_m \cup S_n$ .

Proof is clear.

**THEOREM 4.** Suppose  $(m, n) \in R$ ,  $A_m \subseteq S_m$ ,  $A_n \subseteq S_n$ , and  $A_m, A_n$  are stationary subsets of  $\omega_2$  in  $V[G]$ . Then there are stationarily many ordinals  $\alpha < \omega_2$  of cofinality  $\omega_1$  such that both  $A_m \cap \alpha$  and  $A_n \cap \alpha$  are stationary in  $\alpha$ .

We prove Theorem 4 in the rest of this section. Without loss of generality  $m = 0$ ,  $n = 1$ . Suppose  $C \in V$  is the part of a club subset of  $\kappa$  comprising ordinals of cofinality  $> \omega$ . Since every new club subset of  $\kappa$  includes an old one it suffices to find  $\alpha \in C$  such that both  $A_0 \cap \alpha$ ,  $A_1 \cap \alpha$  are stationary in  $\alpha$ .

Let  $a'_i$  be a  $P$ -name for  $A_i$  and  $a_i = \{(\alpha, p) : p \text{ } P\text{-forces } \alpha \in a'_i\}$  for  $i \leq 1$ . There is  $p_0 \in G$  forcing " $a_i$  is a stationary subset of  $\kappa$ " for  $i \leq 1$ . It suffices to find  $\lambda \in C$  and  $p \leq p_0$  such that  $p$   $P$ -forces " $a_i \cap \lambda$  is stationary in  $\lambda$ " for  $i \leq 1$ .

By Corollary 3 of §2 there is in  $V$  a function  $f: \kappa \rightarrow \kappa$  such that for every  $\alpha < \kappa$  and  $p \in P$  forcing  $\alpha \in a_i$  for some  $i \leq 1$  there is  $q \in P$  of height  $\leq f\alpha$  compatible with  $p$  and forcing the same statement. Let  $M$  be the model with universe  $V_\kappa$  (the collection of sets of rank  $< \kappa$ ), distinguished element  $p_0$ , unary predicates  $x \in C$ ,  $x \in P$ , and binary predicates  $x \in y$ ,  $x \leq y$  (with respect to  $P$ ),  $x \Vdash y \in a_0$ ,  $x \Vdash y \in a_1$ ,  $(x, y) \in f$ . Let  $\sigma$  be a sentence in the language of  $M$  saying:  $\kappa$  is inaccessible and closed under  $f$ , and  $C$  is unbounded in  $\kappa$ , and for every club subset  $C'$  of  $\kappa$  and  $p_1 \leq p_0$  there are  $\alpha_0, \alpha_1 \in C'$  and  $p_2 \leq p_1$  with  $p_2 \Vdash \alpha_i \in a_i$  for  $i \leq 1$ . It is easy to see that  $\sigma$  is  $\Pi^1_1$ .



Any weakly compact cardinal is  $\aleph_1$ -indescribable. Hence there is  $\lambda < \kappa$  such that  $V_\lambda$  forms a submodel of  $M$  satisfying  $\sigma$ . Thus  $\lambda$  is inaccessible,  $h(p_0) < \lambda$ ,  $C \cap \lambda$  is unbounded in  $\lambda$  (therefore  $\lambda \in C$ ),  $\lambda$  is closed under  $f$  and for every  $p_1 \leq p_0$  of height  $< \lambda$  and every club subset  $C'$  of  $\lambda$  there are  $\alpha_0, \alpha_1 \in C'$  and  $p_2 \leq p_1$  of height  $< \lambda$  such that  $p_2$   $P$ -forces  $\alpha_i \in a_i$  for  $i \leq 1$ .

Let  $b_i = \{(\alpha, p) : \alpha < \lambda, h(p) < \lambda, p \text{ } P\text{-forces } \alpha \in a_i\}$ .

*Claim 5.* Suppose  $I$  is a  $P$ -generic filter over  $V$ ,  $p_0 \in I$ ,  $H = I \cap P(< \lambda)$ . Then

$$a_i^{V[I]} \cap \lambda = b_i^{V[H]} = b_i^{V[I]} \quad \text{for } i \leq 1.$$

**PROOF.** If  $\alpha \in b_i^{V[H]}$  then  $\alpha < \lambda$  and there is  $p \in H$   $P$ -forcing  $\alpha \in a_i$ , hence  $\alpha \in a_i^{V[I]}$ .

Suppose  $\alpha \in a_i^{V[I]} \cap \lambda$ . There is  $p \in I$   $P$ -forcing  $\alpha \in a_i$ . It suffices to prove that  $p(< \lambda)$   $P$ -forces  $\alpha \in a_i$ . Suppose the contrary. Then there is  $q \leq p(< \lambda)$   $P$ -forcing  $\alpha \notin a_i$ . As  $\lambda$  is closed under  $f$  there is  $q' \in P(< \lambda)$  compatible with  $q$  and forcing  $\alpha \notin a_i$ . But  $q'$  is compatible with  $p(< \lambda)$  hence with  $p$ , which is impossible.  $\square$

Let  $\tau$  be the statement " $b_0, b_1$  are stationary subsets of  $\lambda$ ".

*Claim 6.*  $p_0$   $P(< \lambda)$ -forces  $\tau$ .

**PROOF.**  $P(< \lambda)$  satisfies the  $\lambda$ -chain condition and the empty condition  $P(< \lambda)$ -forces that every new club subset of  $\lambda$  includes an old club subset of  $\lambda$ . (Just repeat the proof of the corresponding statements about  $P$  and  $\kappa$ .) For every  $C' \in V$  and  $p_1 \in P(< \lambda)$ , if  $C'$  is a club subset of  $\lambda$  and  $p_1 \leq p_0$  then there are  $\alpha_0, \alpha_1 \in C'$  and  $p_2 \in P(< \lambda)$  such that  $p_2 \leq p_1$  and  $p_2$   $P$ -forces  $\alpha_0 \in a_0, \alpha_1 \in a_1$ . By the previous claim  $p_2$   $P(< \lambda)$ -forces  $\alpha_i \in b_i$  for  $i \leq 1$ . Thus  $p_0$   $P(< \lambda)$ -forces  $\tau$ .  $\square$

Let  $p = (p_0^0, p_0^1 \cup \{(\lambda, \{0, 1\})\}, p_0^2 \cup \{(\lambda, \emptyset)\})$ .

*Claim 7.*  $p$  forces  $\tau$  with respect to  $P(\leq \lambda)$ .

**PROOF.** Suppose  $J$  is a  $P(\leq \lambda)$ -generic filter over  $V$  containing  $p$ ,  $H = J \cap P(< \lambda)$ . By §3 there is a  $P(\lambda, 0, 1, H)$ -generic filter  $I$  over  $V(H)$  such that  $V[J] = V[H][I]$ . By §4 (with  $Q = P(\lambda, 0, 1, H)$ ,  $A = \{\alpha < \lambda : q_0(\alpha) \in 2 \text{ for some } q \in H\}$ ) forcing with  $P(\lambda, 0, 1, H)$  does not destroy stationarity of  $b_i^{V[H]}$ .  $\square$

*Claim 8.*  $p$  forces  $\tau$  with respect to  $P$ .

**PROOF.** Use (Claim 2 of) §3 and §5.  $\square$

By Claims 5 and 8  $p$   $P$ -forces " $a_i \cap \lambda$  is a stationary subset of  $\lambda$ " for  $i \leq 1$ .

## PART II. INTERPRETING THE FULL SECOND-ORDER THEORY OF $\omega_2$ IN THE MONADIC THEORY OF $\omega_2$

**§7. Coding.** Recall that the full second-order theory of a set  $X$  is the theory of  $X$  in the language with variables for elements, monadic predicates, dyadic predicates, etc. This theory depends on the cardinality of  $X$  only. Speaking about the full second-order theory of  $\omega_2$  we mean the full second-order theory of the underlying set. Fix a pairing function  $\gamma = nu(\alpha, \beta)$  on  $\omega_2$  (so that  $nu$  gives a one-to-one correspondence from  $\omega_2 \times \omega_2$  onto  $\omega_2$ ). Define  $left(\alpha, \gamma)$  if  $\gamma = nu(\alpha, \beta)$  for some  $\beta$ , and  $right(\beta, \gamma)$  if  $\gamma = nu(\alpha, \beta)$  for some  $\alpha$ .

*Claim 1.* The full second-order theory is interpretable in the monadic theory of  $\langle \omega_2, left, right \rangle$ .

**Proof** is obvious.

As in §1, a graph is a model  $\langle X, R \rangle$  where  $R$  is a reflexive symmetric binary relation on  $X$  such that  $x \neq y \rightarrow \exists z (Rxz \text{ is not equivalent to } Ryz)$ .

*Claim 2.* There is a graph  $\langle \omega_2, R \rangle$  such that the monadic theory of  $\langle \omega_2, \text{left}, \text{right} \rangle$  is interpretable in the monadic theory of  $\langle \omega_2, R \rangle$ .

**PROOF.**  $5\alpha$  is the  $\alpha$ th ordinal divisible by 5. Let  $R$  be the least reflexive symmetric binary relation on  $\omega_2$  containing the following pairs:

- (i)  $(5\alpha + i, 5\alpha + i + 1)$  for  $\alpha < \omega_2, i < 4$ ,
- (ii)  $(5\alpha, 5\beta + 2)$  for  $(\alpha, \beta) \in \text{left}$ ,
- (iii)  $(5\alpha, 5\beta + 3)$  for  $(\alpha, \beta) \in \text{right}$ .

Equality is defined in  $\langle \omega_2, R \rangle$  as indistinguishability. Let  $w(x)$  be the cardinality of  $\{y: Rxy \text{ and } x \neq y\}$ . The statements  $w(x) = n, n < \omega$ , are expressible in the monadic theory of  $\langle \omega_2, R \rangle$ . It is easy to see that  $\alpha = 0$  modulo 5 iff  $w(\alpha) > 3$ ,  $\alpha = 1$  modulo 5 iff  $w(\alpha) = 2$ ,  $\alpha = 4$  modulo 5 iff  $w(\alpha) = 1$ ,  $\alpha = 3$  modulo 5 iff  $w(\alpha) = 3$  and  $\alpha$  is  $R$ -connected with some  $\beta = 4$  modulo 5,  $\alpha = 2$  modulo 5 iff  $w(\alpha) = 3$  and  $\alpha$  is not  $R$ -connected with any  $\beta = 4$  modulo 5. Furthermore,  $5\alpha + 1$  is the only element which equals 1 modulo 5 and is  $R$ -connected with  $5\alpha, 5\alpha + 2$  is the only element which equals 2 modulo 5 and is  $R$ -connected with  $5\alpha + 1, 5\alpha + 3$  is the only element which equals 3 modulo 5  $R$ -connected with  $5\alpha + 2$ .  $\langle \omega_2, \text{left}, \text{right} \rangle$  is now interpretable in  $\langle \omega_2, R \rangle$ , and the interpretation of the model gives rise to an interpretation of its monadic theory.  $\square$

Let  $R$  be as in Claim 2. Assuming existence of a weakly compact cardinal we define in the next section a forcing notion  $P$  and prove

**THEOREM 3.** *Suppose  $G$  is a  $P$ -generic filter over the ground universe  $V$ . Then in  $V[G]$  there is a partition of  $\{\alpha < \omega_2: \text{cf } \alpha = \omega\}$  into stationary sets  $S_\alpha, \alpha < \omega_2$ , such that:*

- (0)  $\beta \in S_\alpha$  implies  $\alpha < \beta$ ;
- (1) for every ordinal  $\alpha$  of cofinality  $\omega_1$  there is a pair  $(\beta, \gamma) \in R$  such that  $S_\beta \cup S_\gamma$  includes a club subset of  $\alpha$ ; and
- (2) if  $R\alpha\beta$  holds,  $A_\alpha \subseteq S_\alpha, A_\beta \subseteq S_\beta$  and  $A_\alpha, A_\beta$  are stationary in  $\omega_2$  then there are stationarily many ordinals  $\gamma$  of cofinality  $\omega_1$  such that  $A_\alpha \cap \gamma$  and  $A_\beta \cap \gamma$  are stationary in  $\gamma$ .

*Claim 4.* The monadic theory of the graph  $\langle \omega_2, R \rangle$  is interpretable in the monadic theory of  $\omega_2$  if there is a partition described in Theorem 2.

**PROOF.** We use definitions of  $I, U_0, U_1$ , connected, atom, maximal atom given in §1.

If  $X$  is an atom then  $X \subseteq S_\alpha$  modulo  $I$  for some  $\alpha$ . For, suppose  $X$  is an atom. Then  $X$  is stationary and, by (0) and Fodor's Lemma  $X \cap S_\alpha$  is stationary for some  $\alpha$ . If  $X - S_\alpha$  is stationary then there are  $\beta \neq \alpha$  with stationary  $X \cap S_\beta$  and  $\gamma$  such that  $R\alpha\gamma$  holds and  $R\beta\gamma$  fails (or conversely), set  $X_1 = X \cap S_\alpha, X_2 = X \cap S_\beta, Y = S_\gamma$  to get a contradiction.

Every stationary  $X \subseteq S_\alpha$  is an atom. For suppose  $X_1, X_2$  are stationary subsets of  $X, Y \subseteq U_0, K_0 = \{\beta: R\alpha\beta \text{ holds}\}, K_1 = \omega_2 - K_0$  and  $Y_i = \{\beta \in Y: \beta \in S_\gamma \text{ for some } \gamma \in K_i\}$  for  $i \leq 1$ . If  $Y_0$  is stationary then, by (0) and Fodor's Lemma,  $Y_0 \cap S_\beta$  is stationary for some  $\beta \in K_0$ , in this case  $Y$  is connected with both  $X_1$  and  $X_2$ . Suppose  $Y_0 \in I$ . Then  $D(Y_0) \in I$ . If  $\alpha \in D(X) - D(Y_0)$  then, by (1), there is a club subset of  $\alpha$  avoiding  $Y_1$ . Hence  $D(X) \cap D(Y) \subseteq D(Y_0) \in I$  and  $Y$  is not connected even with  $X$ .

It is easy to see that  $X \subseteq U_0$  is a maximal atom iff  $X = S_\alpha$  modulo  $I$  for some  $\alpha$ . A subset  $X$  of  $U_0$  will be called *regular* if for every maximal atom  $Y$  either  $Y \subseteq X$  modulo  $I$  or  $Y \subseteq U_0 - X$  modulo  $I$ . Now, given a monadic graph sentence  $\varphi$

interpret individual variables as maximal atoms, set variables as regular subsets of  $U_0$ , equality as equality over  $I$ ,  $R$  as connectedness, and the containment relation as inclusion modulo  $I$ . The resulting monadic sentence  $\psi$  holds in  $\omega_2$  iff  $\varphi$  holds in  $\langle \omega_2, R \rangle$ .  $\square$

Claims 1, 2, 4 and Theorem 3 imply

**THEOREM 5.** *Assume there is a weakly compact cardinal. Then there is a generic extension of the ground universe where the full second-order theory of  $\omega_2$  is interpretable in the monadic theory of  $\omega_2$ .*

**§8. Forcing.** Suppose  $\kappa$  is a weakly compact cardinal and  $R$  is as in §7. We define a forcing notion  $p$  for collapsing  $\kappa$  onto  $\omega_2$  and creating a partition of  $\{\alpha < \omega_2: \text{cf } \alpha = \omega\}$  into stationary sets  $S_\alpha$ ,  $\alpha < \omega_2$ , described in §7. A condition  $p = (p_0, p_1, p_2)$  where:

$p_0$  is a countable function from a part of  $\{\alpha < \kappa: \text{cf } \alpha = \omega\}$  into  $\omega_2$  such that  $\beta = p_0(\alpha)$  implies  $\beta < \alpha$  (the intended meaning of  $\beta = p_0(\alpha)$  is:  $\alpha \in S_\beta$ ),

$p_1$  is a countable function from a part of  $\{\alpha < \kappa: \text{cf } \alpha > \omega\}$  into  $R$  (the intended meaning of  $(\beta, \gamma) \in p_1(\alpha)$  is  $\alpha \cap S_\delta$  is stationary in  $\alpha$  iff  $\delta = \beta$  or  $\delta = \gamma$ ),

$p_2$  is a countable function with  $\text{dom } p_2 = \text{dom } p_1$ , if  $\alpha \in \text{dom } p_1$  and  $p_1(\alpha) = (\beta, \gamma)$  then  $p_2(\alpha)$  is a closed countable subset of  $\alpha$  included in  $\{\delta: p_0(\delta) = \beta \text{ or } p_0(\delta) = \gamma\}$  (the intended meaning  $p_2(\alpha)$  is an initial segment of a club subset of  $\alpha$  included in  $S_\beta \cup S_\gamma$ ).

By definition,  $p$  refines  $q$  ( $p \leq q$ ) if  $q_0 \subseteq p_0$ ,  $q_1 \subseteq p_1$  and for every  $\alpha \in \text{dom } q_1$ ,  $p_2(\alpha)$  is an end extension of  $q_2(\alpha)$  in such a way that

$$\min(p_2(\alpha) - q_2(\alpha)) > \sup(\alpha \cap (\text{dom } p_0 \cup \text{dom } p_1)).$$

Suppose  $G$  is a  $P$ -generic filter over the ground universe  $V$ , and  $S_\alpha = \{\beta: p_0(\beta) = \alpha \text{ for some } p \in G\}$  for  $\alpha < \kappa$ . The clause (0) of Theorem 2 in §7 is obvious, the rest of the theorem is proved exactly as in Part I.

### PART III. MONADIC THEORY OF A GIVEN COMPLEXITY

**§9. Coding.** Given a sequence  $s_0 < s_1 < \dots$  of positive integers and assuming GCH and existence of a weakly compact cardinal, we define in the next section a forcing notion  $P$  and prove the following theorem. ( $U_0, U_1, D$  are defined in the introduction.)

**THEOREM 1.** *Suppose  $G$  is a  $P$ -generic filter over the ground universe  $V$ . Then in  $V[G]$  there are*

(i) *a partition of  $U_0$  into stationary sets  $A_{ni}$  where  $n < \omega$ ,  $i < s_n$  and*

(ii) *a partition of  $U_1$  into stationary sets  $B_n, C_{ni}$  where  $n < \omega$ ,  $i < s_n$  such that  $D(X) = B_n \cup C_{ni}$  modulo  $I$  for every  $n < \omega$ ,  $i < s_n$  and stationary  $X \subseteq A_{ni}$ .*

It is clear that the sequence  $s_0, s_1, \dots$  is computable from the monadic theory of  $\omega_2$  in  $V[G]$ . The converse is true too.

**Claim 2.** Suppose  $\omega_2$  can be partitioned in the way described in Theorem 1. Then the monadic theory of  $\omega_2$  is computable from the sequence  $s_0, s_1, \dots$ .

**PROOF.** As was mentioned in the introduction the monadic theory of  $\omega_2$  is recursive in the first-order theory of  $M = \langle \text{PS}(\omega_2)/I, D \rangle$ . For every  $n$  let

$$E_n = (\bigcup \{A_{ni}: i < s_n\}) \cup B_n \cup (\bigcup \{C_{ni}: i < s_n\})$$

and  $I_n, D_n$  be the restrictions of  $I, D$ , respectively, on  $\text{PS}(E_n)$ . It is easy to see that  $M$  is the direct product of structures  $M_n = \langle \text{PS}(E_n)/I_n, D_n \rangle$ . The first-order theory of  $M_n$  is computable from  $s_n$ . Hence the first-order theory of  $\{M_n: n < \omega\}$  is computable from the sequence  $s_0, s_1, \dots$ . By the Feferman-Vaught Theorem (see [FV]) the first-order theory of  $M$  is computable from the first-order theory of  $\{M_n: n < \omega\}$ .  $\square$

Theorem 1 and Claim 2 imply

**THEOREM 3.** *Assume GCH holds and there is a weakly compact cardinal. Then for every sequence  $s_0 < s_1 < \dots$  of positive integers there is a generic extension of the ground world where the sequence  $s_0, s_1, \dots$  and the monadic theory of  $\omega_2$  are recursive each in the other.*

**§10. Forcing.** Suppose GCH holds and  $\kappa$  is a weakly compact cardinal and  $s_0 < s_1 < \dots$  is a sequence of positive integers. First we define a partial ordering  $Q$  for collapsing  $\kappa$  onto  $\omega_2$  and creating stationary sets  $A_{ni}, B_n, C_{ni}$  behaving in the way described in Theorem 1 of §9 with only one exception: instead of  $D(X) = B_n \cup C_{ni}$  we shall have only  $B_n \subset D(X) \subseteq B_n \cup C_{ni}$  for every  $n < \omega, i < s_n$  and stationary  $X \subseteq A_{ni}$ .  $Q$  is just a version of the forcing notion in §2.

A condition  $p = (p0, p1, p2)$  where

$p0$  is a countable function from a part of  $U_0$  into  $K = \{(n, i): n < \omega, i < s_n\}$  (the intended meaning of  $p0(\alpha) = (n, i)$  is  $\alpha \in A_{ni}$ ),

$p1$  is a countable function from a part of  $U_1$  into  $\omega \cup K$  (the intended meaning of  $p1(\alpha) = n$  is  $\alpha \in B_n$ ; the intended meaning of  $p1(\alpha) = (n, i)$  is  $\alpha \in C_{ni}$ ),

$p2$  is a countable function with  $\text{dom } p2 = \text{dom } p1$ , if  $\alpha \in \text{dom } p1$  and  $p1(\alpha) = n$ ,  $p2(\alpha)$  is a closed countable subset of  $\alpha$  included into  $\{\beta: p0(\beta) = (n, i) \text{ for some } i < s_n\}$  (the intended meaning is:  $p2(\alpha)$  is an initial segment of a club subset of  $\alpha$  included in  $\bigcup\{A_{ni}: i < s_n\}$ ); if  $\alpha \in \text{dom } p1(\alpha)$  and  $p1(\alpha) = (n, i)$  then  $p2(\alpha)$  is a closed countable subset of  $\alpha$  included in  $\{\beta: p0(\beta) = (n, i)\}$  (the intended meaning is  $p2(\alpha)$  is an initial segment of a club subset of  $\alpha$  included in  $A_{ni}$ ).

$p$  refines  $q$  if  $q0 \subseteq p0, q1 \subseteq p1$  and for every  $\alpha \in \text{dom } p1, p2(\alpha)$  is an end extension of  $q2(\alpha)$  in such a way that

$$\min(p2(\alpha) - q2(\alpha)) > \sup(\alpha \cap (\text{dom } p0 \cup \text{dom } p1)).$$

**THEOREM 1.** *Suppose  $G$  is a  $Q$ -generic filter over the ground world  $V$ ,  $A_{ni} = \{\alpha: p0(\alpha) = (n, i) \text{ for some } p \in G\}$ ,  $B_n = \{\alpha: p1(\alpha) = n \text{ for some } p \in G\}$ ,  $C_{ni} = \{\alpha: p1(\alpha) = (n, i) \text{ for some } p \in G\}$  for  $n < \omega, i < s_n$ . In  $V[G]$ :  $\kappa = \omega_2$ , and every  $A_{ni}, B_n, C_{ni}$  is stationary in  $\omega_2$ , the sets  $A_{ni}$  partition  $U_0$ , the sets  $B_n, C_{ni}$  partition  $U_1$  and  $D(X) \subseteq B_n \cup C_{ni}$  for every  $n < \omega, i < s_n$  and every stationary  $X \subseteq A_{ni}$ .*

Theorem 1 is proved in the same way Theorem 2 of §1 was proved.

We would like to have  $D(X) = B_n \cup C_{ni}$  for  $n < \omega, i < s_n$  and stationary  $X \subseteq A_n$ . In [Ma] Magidor, assuming GCH and existence of a weakly compact cardinal, forced a universe where the derivative of every stationary subset of  $U_0$  is equal to  $U_1$  modulo the ideal  $I$  of nonstationary subsets of  $\omega_2$ . First he collapsed  $\kappa$  onto  $\omega_2$ , then he started to shoot club subsets of  $\omega_2$  through  $U_0 \cup D(X)$  where  $X \subseteq U_0$  is stationary. We do here almost the same.

Let  $G, A_{ni}, B_n, C_{ni}$  be as in Theorem 1. Over  $V[G]$  start shooting club subsets of  $\omega_2$  through  $\omega_2 - (D(A_{ni}) - D(X))$  where  $X \subseteq A_{ni}$  is stationary. Iterate the procedure  $\omega_3$  times with support of cardinality  $\omega$ . The proof that the resulting model is the required one is just like that in [Ma, §2] with only one modification: use §§4,5 of our Part I instead of Magidor's Lemma 5 in the proof of claim which appears in the proof of Lemma 4 in §2 of [Ma].

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BEN-GURION UNIVERSITY  
BEER-SHEVA, ISRAEL  
UNIVERSITY OF MICHIGAN  
ANN ARBOR, MICHIGAN 48109

HEBREW UNIVERSITY  
JERUSALEM, ISRAEL